Photon bremsstrahlung and diffusive broadening of a hard jet

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The photon bremsstrahlung rate from a quark jet produced in deep-inelastic scattering (DIS) off a large nucleus is studied in the collinear limit. The leading medium-enhanced higher twist corrections that describe the multiple scattering of the jet in the nucleus are re-summed to all orders of twist. The propagation of the jet in the absence of further radiative energy loss is shown to be governed by a transverse momentum diffusion equation. We compute the final photon spectrum in the limit of soft photons, taking into account the leading and next-to-leading terms in the photon momentum fraction y. In this limit, the photon spectrum in a physical gauge is shown to arise from two interfering sources: one where the initial hard scattering produces an off-shell quark, which immediately radiates the photon and then undergoes subsequent soft rescattering, and an alternative in which the quark is produced on-shell and propagates through the medium until it is driven off-shell by rescattering and radiates the photon. Our result has a simple formal structure as a product of the photon splitting function, the quark transverse momentum distribution coming from a diffusion equation, and a dimensionless factor that encodes the effect of the interferences encountered by the propagating quark over the length of the medium. The destructive nature of such interferences in the small-y limit is responsible for the origin of the Landau-Pomeranchuck-Migdal (LPM) effect. Along the way we also discuss possible implications for quark jets in hot nuclear matter.

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I. INTRODUCTION

The modification of high-transverse-momentum (high-p_T) jets [1–3] as they pass through dense matter has achieved center stage in the experimental program at the Relativistic Heavy-Ion Collider (RHIC) at Brookhaven National Laboratory (BNL) [4] and is poised for a more comprehensive exploration at the upcoming Large Hadron Collider (LHC). The central goal is an quantitative exploration of certain properties of the dense matter produced, based on the modification encountered by a jet while passing through the excited medium. The benchmark for such observables is provided by the modification encountered by a hard jet in cold nuclear matter [5]. The basic partonic processes that lead to the modification of the jet in cold or excited media are identical and lead to a formulation in terms of the same correlation functions. Experimental observations of the modification in cold and excited matter provide numerical values for these correlation functions and as such allow comparisons between these two environments.

The most noticeable modification observed is the loss of forward momentum and energy of the hard partons, believed to result from a combination of elastic [6,7] and radiative [1] mechanisms. The experimentally observed effect is the depletion of high-momentum hadrons produced in the fragmentation of these partons after they escape the medium [5,8]. For the light partons, the energy loss by induced gluon radiation is believed to be the dominant mechanism. Calculations based on this are by now considerably developed. There currently exist four different schemes of radiative energy loss: the Arnold-Moore-Yaffe (AMY) [9], the Armesto-Salgado-Wiedemann (ASW) [10], the Gyulassy-Levai-Vitev (GLV) [11], and the higher twist (HT) [12] approach, each incorporating a slightly different approximation scheme in the calculation of the radiative gluon spectrum. There have also been attempts to calculate both the broadening of a jet and its energy loss in a dipole approach [13]. The different schemes also utilize different definitions of the medium-modified fragmentation function and, in most cases, apparently different models of the medium.

The AMY and ASW approaches, based on the previous work of Baier *et al.* (usually referred to as BDMPS) [2], are cast in the thick-medium approximation, in the sense that the hard parton as well as the radiated gluon scatters multiple times in the medium and an infinite series of such scatterings has to be re-summed [14].[1] In contrast, the GLV and HT approaches have traditionally been cast in the thin-medium approximation, where the hard parton and radiated gluon scatter a finite number of times in the medium, with the effect of each subsequent scattering being incorporated order by order in opacity or twist [15].

Some time ago, it was pointed out that the higher twist formalism may also be extended to the thick-medium limit [16]. In a recent effort, the formulation of jet modification by multiple scattering within a re-summed higher twist approach was begun [17]. In that paper, contributions from all twists, which are enhanced by the length of the medium, were re-summed to calculate the transverse broadening experienced by a hard parton as it traversed a dense medium without radiative emission. The resulting *all-twist* expression assumed the physically transparent form of a two-dimensional transverse momentum diffusion equation. The diffusion tensor was

[1]There is a particular case of the ASW approach that deals with the possibility that the hard parton and radiated gluon scatter once in the medium [14].

shown to be related, up to overall constant factors, to the well-known jet transport coefficient $\hat{q}$ [2]. It is the object of the current article to extend this formulation of *all-twist* jet modification. The specific problem being addressed here is the spectrum of real photons radiated from a partonic jet multiply scattering off a dense medium. Although the photon is assumed to escape the medium without interaction, very unlike the case of a radiated gluon, the problem does encode many of the characteristic features of gluon radiation such as the Landau-Pomeranchuck-Migdal (LPM) effect, which suppresses very collinear radiation.

Besides its role as an intermediate step to the problem of energy loss by gluon radiation, hard photon bremsstrahlung from a high-p_T jet is a problem of sufficient phenomenological relevance in it own right. Traditionally, thermal photons and dileptons have been considered as ideal probes of the plasma as they escape the plasma after their formation without any further interaction, directly conveying information regarding conditions at their origin [18]. Jet-medium interactions, where a hard jet undergoes Compton scattering off a constituent of the medium, leading to direct photon [19] and dilepton [20] production, have also been proposed as deep probes of the matter produced: These measure the partonic density of the medium at the scale of the hard parton. Recently, photon-hadron correlations [21–23], where a hard photon is produced in the course of the multiple forward scattering of a hard parton, have emerged as an electromagnetic window to the energy loss transport parameter $\hat{q}$. There now exist a series of measurements of photon and high-p_T hadron correlations at RHIC. These are meant to be used in tandem with single inclusive high-p_T hadron suppression [24] and hadron-hadron correlations [25] to characterize the jet transport parameters of the medium. The results of the current article are directly relevant to all near-side measurements of such correlations and represent an extra source of jet broadening in away-side correlations between a trigger photon and a leading hadron.

The results presented here may also be easily ported into the calculation of hard photon production from a hot partonic medium. Such calculations, nowadays, are mostly carried out within the finite temperature formalism of AMY [9] by assuming that the medium exists at asymptotically high temperature and invoking the hard thermal loop (HTL) approximation [26] (for a complete discussion see Ref. [27] and references therein). Our reformulation of this problem in the HT approach achieves several goals. The AMY scheme invokes the assumption that the medium is a thermalized plasma of quarks and gluons at asymptotically high temperature T. The HT scheme makes no such approximation regarding the medium; the only approximation made is that the medium have a short-distance color correlation length. This is a property of a multitude of media including the HTL plasma. Hence, the results derived in the current article should be applicable to cases both where the HTL approximation is applicable and where it is not (for instance, in the confined environment of cold nuclear matter). This conjecture may be easily tested by calculating the diffusion tensor in a hot partonic medium and promoting the hard parton from a single probe to an ensemble with a thermal distribution. In such a comparison, contributions that occur because the initial parton is off-shell owing to its formation in a violent collision should be neglected.

Such comparisons and the associated phenomenology will be deferred to a future publication. Here, the focus will lie on a derivation of the all-twist re-summed expression for photon bremsstrahlung from a hard parton traversing a dense medium. Although the remaining sections specifically treat photon production from a hard jet produced in deep-inelastic scattering (DIS) off a large nucleus, it should be pointed out that this is merely an instrument of convenience and the results may be readily generalized to photon production off a hard jet in a different environment. To allow for such extensions in the future, our derivations are carried out in a factorized form, almost independent of the initial source of production of the hard quark.

The article is organized as follows: In Sec. II the simplest calculation of photon bremsstrahlung at leading twist is carried out. This is solely performed to familiarize the reader with the notation, frame, and particular choice of gauge used. In Sec. III, the problem of photon radiation from a hard parton that scatters a finite number of times in the medium is analyzed and general expressions are derived. In Sec. IV, sums over the various photon production points in the medium are carried out. The soft photon approximation is invoked and the twist expansion in cold nuclear matter is carried out in terms of a transverse momentum gradient expansion. In Sec. V, the resulting expressions for soft photon production from a hard parton that scatters n times in the medium are re-summed to obtain the all-twist photon spectrum. Concluding discussions appear in Sec. VI.

II. LEADING TWIST AND COLLINEAR PHOTON RADIATION

In this section, the calculation of the photon production rate from a hard parton, produced in the DIS off a large nucleus, which subsequently undergoes multiple scattering in the confined nuclear environment is begun. The photon, not having scattered in the medium after its production, escapes the medium unscathed, revealing the nature of the matter at the location of its production. The interference between the photon produced at different locations in the hard parton's diffusive history lead to a modulation of the factorized rate of photon production. In this vein, we focus on an all-twist evaluation of the differential photon production rate from a hard jet. Consider the semi-inclusive process of DIS off a nucleus in the Breit frame where one jet with a transverse momentum $l_{q_\perp}$ and a bremsstrahlung photon with transverse momentum $l_\perp$ are produced,

$$\mathcal{L}(L_1) + A(p) \longrightarrow \mathcal{L}(L_2) + J(l_{q_\perp}) + \gamma(l_\perp) + X. \quad (1)$$

In this equation, L_1 and L_2 represent the momentum of the incoming and outgoing leptons. The incoming nucleus of atomic mass A is endowed with a momentum Ap. In the final state, all high-momentum hadrons $(h_1, h_2, \ldots)$ with momenta $p_1, p_2, \ldots$ are detected and their momenta are summed to obtain the jet momentum and X denotes that the process is semi-inclusive.

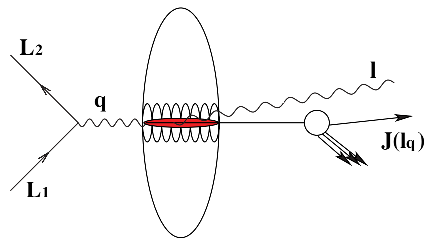

FIG. 1. (Color online) The Lorentz frame chosen for the process where a nucleon in a large nucleus is struck by a hard spacelike photon, leading to the production of an outgoing parton and a radiated photon.

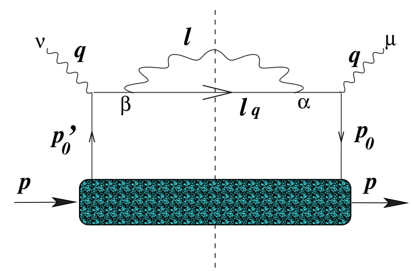

FIG. 2. (Color online) The Lowest order and leading twist contribution to $W^{\mu\nu}$ with a real photon in the final state.

Throughout this article, the light-cone component notation for four-vectors ($p \equiv [p^+, p^-, \vec{p}_\perp]$) will be used, where

$$p^+ = \frac{p^0 + p^3}{2}, \quad p^- = p^0 - p^3. \tag{2}$$

The kinematics is defined in the Breit frame as sketched in Fig. 1. In such a frame, the incoming virtual photon γ^* and the nucleus have momentum four-vectors q, P_A given as

$$q = L_2 - L_1 \equiv \left[\frac{-Q^2}{2q^-}, q^-, 0, 0\right], \quad P_A \equiv A[p^+, 0, 0, 0].$$

In this frame, the Bjorken variable is defined as $x_B = Q^2/2p^+q^-$. The radiated photon has a transverse momentum of $l_\perp$ and carries a fraction y of the forward momentum q^- of the quark originating in the hard scattering, that is,

$$y = \frac{l^-}{q^-}. \tag{3}$$

The double differential cross section of the semi-inclusive process with a jet with transverse momentum $l_{q_\perp}$ and a final-state photon with transverse momentum $l_\perp$ may be expressed as

$$\frac{E_{L_2} d\sigma}{d^3L_2 d^2 l_{q_\perp} d^2 l_\perp dy} = \frac{\alpha_{\rm em}^2}{2\pi s Q^4} L_{\mu\nu} \frac{dW^{\mu\nu}}{d^2 l_{q_\perp} d^2 l_\perp dy}, \tag{4}$$

where $s = (p + L_1)^2$ is the total invariant mass of the lepton-nucleon system. The reader may have already surmised the form of the leptonic tensor as

$$L_{\mu\nu} = \tfrac{1}{2}\mathbf{Tr}[\not{L}_1\gamma_\mu \not{L}_2\gamma_\nu]. \tag{5}$$

In the notation used in this article, $|A; p\rangle$ represents the spin-averaged initial state of an incoming nucleus with A nucleons with a momentum p per nucleon. The general final hadronic or partonic state is defined as $|X\rangle$. As a result, the semi-inclusive hadronic tensor in the nuclear state $|A\rangle$ may be defined as

$$\begin{aligned} W^{A\mu\nu} &= \sum_X (2\pi)^4 \delta^4(q + P_A - p_X) \\ &\quad \times \langle A; p|J^\mu(0)|X\rangle\langle X|J^\nu(0)|A; p\rangle \\ &= 2\,\mathrm{Im}\left[\int d^4y_0 e^{iq\cdot y_0}\langle A; p|J^\mu(y_0)J^\nu(0)|A; p\rangle\right]. \end{aligned} \tag{6}$$

In this equation, the sum ($\sum_X$) runs over all possible hadronic states and $J^\mu = Q_q\bar{\psi}_q\gamma^\mu\psi_q$ is the hadronic electromagnetic current, where Q_q is the charge of a quark of flavor q in units of the positron charge e. It is understood that the quark operators are written in the interaction picture, and two factors of the electromagnetic coupling constant $\alpha_{\rm em}$ have already been extracted and included in Eq. (4). The leptonic tensor will not be discussed further and we will focus exclusively on the hadronic tensor. This tensor will be expanded order by order in a partonic basis with one photon in the final state and leading twist; maximally length enhanced higher twist contributions will be isolated.

The leading twist contribution is obtained by expanding the products of currents at leading order in α_s and next-to-leading order in $\alpha_{\rm em}$ to account for the radiated photon. This contribution may be expressed diagrammatically as in Fig. 2. This represents the process in which a hard quark, produced from one nucleon in a deep-inelastic scattering on a nucleus, proceeds to radiate a hard photon and then exits the nucleus without further interaction. Other diagrams are suppressed in our choice of light-cone gauge. In the following, we analyze this contribution in some detail. Indeed, there is no new information presented in the current section and the discussion of such contributions is now well established [28]. The approximations made in this section will form the basis for the analysis of photon production at all twist. The semi-inclusive hadronic tensor may be expressed as

$$\begin{aligned} W_0^{A\mu\nu} &= AC_p^A W_0^{\mu\nu} \\ &= AC_p^A \int d^4y_0 \langle p|\bar{\psi}(y_0)\gamma^\mu \widehat{\mathcal{O}^{00}}\gamma^\nu\psi(0)|p\rangle \\ &= C_p^A \int d^4y_0 \mathbf{Tr}\left[\frac{\gamma^-}{2}\gamma^\mu\frac{\gamma^+}{2}\gamma^\nu\right] F(y_0)\mathcal{O}^{00}(y_0). \end{aligned} \tag{7}$$

where C_p^A expresses the probability of finding a nucleon state with momentum p inside a nucleus with A nucleons.

In the collinear limit, the incoming parton is assumed to be endowed with very high forward momentum (${p_0}^+ = x_0 p^+$, $p_0^- \to 0$) with negligible transverse momentum $p_{0\perp} \ll p_0^+$. Within the kinematics chosen, the incoming virtual photon also has no transverse momentum. As a result, the produced final-state parton also has a vanishingly small transverse momentum [i.e., with a distribution $\delta^2(\vec{p}_\perp)$]. In this limit, the leading spin projection of the pieces that represent the initial state and final state may be taken. The factors

$$\gamma^+ = \frac{\gamma^0 + \gamma^3}{2}, \quad \gamma^- = \gamma^0 - \gamma^3, \tag{8}$$

are used to obtain the spin projections along the leading momenta of the outgoing state and the incoming state.

The coefficients of these projections are the two functions

$$F(y_0) = A\langle p|\bar{\psi}(y_0)\frac{\gamma^+}{2}\psi(0)|p\rangle \quad (9)$$

and (in a notation where the superscript on the operator $\mathcal{O}^{00}$ implies that the quark undergoes no scattering in the initial or final state)

$$\begin{aligned}\mathcal{O}^{00} &= \mathbf{Tr}\left[\frac{\gamma^-}{2}\widehat{\mathcal{O}^{00}}\right] \\ &= \int \frac{d^4l}{(2\pi)^4}d^4zd^4z'\frac{d^4l_q}{(2\pi)^4}\frac{d^4p_0}{(2\pi)^4}\frac{d^4p_0'}{(2\pi)^4} \\ &\times \mathbf{Tr}\left[\frac{\gamma^-}{2}\frac{-i(\not{p}_0+\not{q})}{(p_0+q)^2-i\epsilon}i\gamma^\alpha \not{l}_q 2\pi\delta\left(l_q^2\right)\right. \\ &\times \left. G_{\alpha\beta}(l)2\pi\delta(l^2)(-i\gamma^\beta)\frac{i(\not{p}_0'+\not{q})}{(p_0'+q)^2+i\epsilon}\right] \\ &\times e^{iq\cdot y_0}e^{-i(p_0+q)\cdot(y_0-z)}e^{-il\cdot(z-z')}e^{-il_q\cdot(z-z')} \\ &\times e^{-i(p_0'+q)\cdot z'}e_q^2. \end{aligned} \quad (10)$$

Integrating over z and z' yields two four-dimensional δ functions: $\delta^4(p_0+q-l-l_q)$ and $\delta^4(p_0-p_0')$.

The erudite reader will have noticed that we have ignored various projections such as those which arise from the $(\perp)$-components of the γ matrices, i.e.,

$$C_P^A\mathbf{Tr}\left[\frac{\gamma^{\perp_i}}{2}\gamma^\mu\frac{\gamma^{\perp_j}}{2}\gamma^\nu\right]F_{\perp_i}(y_0)\mathcal{O}^{00}_{\perp_j}. \quad (11)$$

This approximation may be justified in the high-energy, collinear limit $l_\perp^2/y \ll Q^2$ where such contributions are suppressed compared to those of Eq. (7). In so doing, the focus of the remainder of this article has been limited to projections where the incoming virtual photon is transverse. As our primary interest is a description of the effect of final-state scattering on the outgoing quark leading to final-state photon bremsstrahlung, we defer the discussion of the slightly different process, where the initial virtual photon is longitudinal, to a separate effort.

The on-shell δ function over l is used to set $l^+ = l_\perp^2/2l^-$. The other on-shell δ function instills the condition

$$\begin{aligned}\delta\left(l_q^2\right) &= \delta[(p_0+q-l)^2] \\ &\simeq \delta[-Q^2+2p_0^+(q^- - l^-) - 2q^+l^- - 2q^-l^+] \\ &= \frac{1}{2p^+q^-}\delta\left[x_0(1-y) - x_B(1-y) - \frac{l_\perp^2}{2p^+q^-y}\right] \\ &= \frac{\delta[x_0-x_B-x_L]}{2p^+q^-(1-y)},\end{aligned} \quad (12)$$

where the collinear condition that $p_{0_\perp}\to 0$ has been used to simplify the final equation. The new *momentum fraction* x_L has been introduced:

$$x_L = \frac{l_\perp^2}{2p^+q^-y(1-y)} = \frac{1}{p^+\tau_f}, \quad (13)$$

where y has already been defined as the momentum fraction of the radiated photon (l^-/q^-) and τ_f is the formation time of the radiated photon.

The factor $G_{\alpha\beta}(l)$ in Eq. (10) represents the radiated photon's spin sum. In this effort, the light-cone gauge ($A^-=0$) will be assumed, that is,

$$G_{\alpha\beta}(l) = -g^{\mu\nu} + \frac{l^\mu n^\nu + l^\nu n^\mu}{n\cdot l}, \quad (14)$$

where we have introduced the light-cone vector $n \equiv [1,0,0,0]$, which yields $l\cdot n = l^-$. Note that, with this choice of gauge, the largest component of the vector potentials from the initial states may still be regarded as the $(+)$-components.

Substituting these simplifications into Eq. (10) leads to the simplified form for the final-state projection:

$$\begin{aligned}\mathcal{O}^{00} &= \int\frac{d^4p_0}{(2\pi)^4}\frac{dl^-d^2l_\perp}{(2\pi)^3 2l^-}e_q^2 e^{-ip_0\cdot y_0} \\ &\times \mathbf{Tr}\left[\frac{\gamma^-}{2}\frac{\gamma^+q^-}{2p^+q^-(x_0-x_B-x_{D0}-i\epsilon)}\right. \\ &\times \left\{\gamma_\perp^\alpha\gamma^-([x_0-x_B]p^+ - l^+)\gamma_\perp^\beta(-g_{\perp\alpha\beta})\right. \\ &- \frac{\gamma_\perp\cdot l}{l^-}\gamma_\perp\cdot l\gamma^- - \gamma^-\gamma_\perp\cdot l\frac{\gamma_\perp\cdot l}{l^-} \\ &\times\left.\gamma^-\gamma^+(q^- - l^-)\gamma^-\frac{2l^+}{l^-}\right\} \\ &\times \left.\frac{\gamma^+q^-}{2p^+q^-(x_0-x_B-x_{D0}+i\epsilon)}\right] \\ &\times 2\pi\frac{\delta[x_0-x_B-x_L]}{2p^+q^-(1-y)}.\end{aligned} \quad (15)$$

The δ function may be used to carry out the $p_0^+ = x_0p^+$ integral; the absence of p_0^- and $p_{0_\perp}$ from the integrands allows for these integrals to be carried out and constrains the locations y_0^+, $y_{0_\perp}$ to the origin. Further simplifications may be carried out by noting that $\gamma^\pm$ anticommutes with $\gamma_\perp$, and $\{\gamma^+,\gamma^-\} = 2\mathbf{1}$ (i.e., twice the unit matrix in spinor space) and $\{\gamma^\pm,\gamma^\pm\}=0$. Replacing $l^- = q^-y$ in Eq. (15), one obtains

$$\begin{aligned}\mathcal{O}^{00} &= \delta(y_0^+)\delta^2(y_{0_\perp})\int\frac{dyd^2l_\perp}{(2\pi)^3 2y}e^{-i(x_B+x_L)p^+y_0^-}p^+ \\ &\times \frac{e^2Q_q^2 4(q^-)^2}{(2p^+q^-)^2 4p^+q^-(1-y)x_L^2} \\ &\times \left[2(x_Lp^+ - l^+) + \frac{2l_\perp^2}{l^-} + 2(1-y)\frac{2l^+}{y}\right] \\ &= \delta(y_0^+)\delta^2(y_{0_\perp})\frac{Q_q^2\alpha_{\rm em}}{2\pi}\int\frac{dydl_\perp^2}{l_\perp^2}\frac{2-2y+y^2}{y}.\end{aligned} \quad (16)$$

Reintroduction of the final-state projection $\mathcal{O}^{00}$ in Eq. (7) produces the well-known and physically clear formula for the differential semi-inclusive hadronic tensor with single photon emission in the final state,

$$\begin{aligned}\frac{dW_0^A\mu\nu}{dydl_\perp^2} &= C_p^A 2\pi\sum_q Q_q^4 f_q^A(x_B+x_L)(-g_\perp^{\mu\nu}) \\ &\times \frac{\alpha_{\rm em}}{2\pi}\frac{1}{l_\perp^2}P_{q\to q\gamma}(y).\end{aligned} \quad (17)$$

In this equation, $f_q^A(x_B)$ represent the parton distribution function of a quark with flavor q and electric charge Q_q in units of the electron charge e, in a nucleus with momentum Ap, that is,

$$f_q^A(x_B + x_L) = A \int \frac{dy_0^-}{2\pi} e^{-i(x_B+x_L)p^+y_0^-} \times \frac{1}{2}\langle p|\bar{\psi}(y_0^-)\gamma^+\psi(0)|p\rangle. \quad (18)$$

In Eq. (17), the factor $P_{q\to q\gamma}(y) = (2 - 2y + y^2)/y$ is the quark-to-photon splitting function; it represents the probability that a quark will radiate a photon that will carry away a fraction y of its forward momentum. The projection $g_\perp^{\mu\nu} = g^{\mu\nu} - g^{\mu-}g^{\nu+} - g^{\mu+}g^{\nu-}$.

As the parton produced immediately after the hard scattering has a vanishingly small transverse momentum, the transverse momentum of final produced quark is simply the negative of the photon's transverse momentum (i.e., $\vec{l}_{q_\perp} = -\vec{l}_\perp$). As a result, the differential hadronic tensor for the transverse momentum distribution of the final quark is given as

$$\frac{dW_0^{\mu\nu}}{dydl_\perp^2 d^2l_{q\perp}} = \frac{dW_0^{\mu\nu}}{dydl_\perp^2}\delta^2(\vec{l}_\perp + \vec{l}_{q_\perp}). \quad (19)$$

III. PHOTON RADIATION FROM MULTIPLE SCATTERING

In this section, the calculation of the single-photon production rate from diagrams that include the multiple scattering of the quark on the gluon field within the various nucleons inside the nucleus will be carried out. The focus will be on a generic diagram of the type that will be encountered in the evaluation of the photon differential rate at all twist. In another language, this implies the inclusion of higher twist corrections to Eq. (17). Higher twist contributions are obtained from diagrams that include expectation values of products containing more partonic operators in the medium [29] e.g., the gluon field strength operator product $F^{+\nu}(y)F_\nu^+(0)$]. Although contributions from the expectation of such operators are suppressed by powers of the hard scale Q^2, a subclass of these are enhanced in extended media (such as an atomic nucleus) owing to the longitudinal extent that must be traveled by the struck quark. As in Ref. [17], only such length-enhanced higher twist contributions will be isolated and eventually re-summed. Issues relating to the generalized factorization [30,31] of such contributions will not be dealt with in this effort; the focus will be on deriving the effect of multiple scattering of the hard parton in the nucleus on its ability to radiate a collinear photon. To obtain higher twist contributions, higher orders need to be included. A diagram with $2n$ gluon insertions may contribute to twist $m \leqslant 2n$. In all calculations, the high-energy and hence small-g limit will be assumed; as a result all diagrams containing the four-gluon vertex will be ignored. Within this choice of kinematics and gauge there exists a class of diagrams that contains three-gluon vertices and receives similar length enhancements. These diagrams involve radiated gluons in the final state and contribute to the energy loss of the parent parton. Such length-enhanced contributions will be considered separately in an upcoming publication.

The aim of this effort is to isolate the higher twist contributions at a given order in coupling and twist that carry the largest multiple of length $L \sim A^{1/3}$ [32]. In the language of power counting, one looks at the combination $\alpha_s^m L^n$ (usually $n \leqslant m$) and focuses on diagrams with the maximum n. The multiples of L are obtained by insisting on conditions that lead to the largest number of propagators going close to their on-shell conditions. This is easily achieved by the hard parton having the maximum number of spacelike exchanges with the medium. The photon is radiated on-shell and final-state scatterings of the photon with the rest of the medium, which involve additional powers of $\alpha_{\rm em}$, are ignored.

The generic contributions being considered thus have the form of Fig. 3. A hard virtual photon strikes a hard quark in the nucleus with momentum p_0' (p_0 in the complex conjugate) at location $y_0' = 0$ (y_0 in the complex conjugate) and sends it back through the nuclear medium. At this stage, the quark has momentum q_1' in the amplitude and q_1 in the complex conjugate. In the course of its propagation, the hard parton scatters off the gluon field within the medium at locations y_l' with $0 < l < m$ (at locations y_i with $0 < i < n$ in the complex conjugate) wherein the hard parton picks up momenta p_l' (p_i in the complex conjugate), changing its momentum to q_{l+1}' (q_{i+1} in the complex conjugate). The photon is radiated between locations q and $q+1$ in the amplitude and between p and $p+1$ in the complex conjugate. The quark line immediately after the photon radiation has momentum q_{q+2}' in the amplitude and q_{p+2} in the complex conjugate. Following the notation of the previous section, the photon is assigned a momentum l and the final cut quark propagator is assigned a momentum l_q. The ellipses in between gluon lines in Fig. 3 are meant to indicate

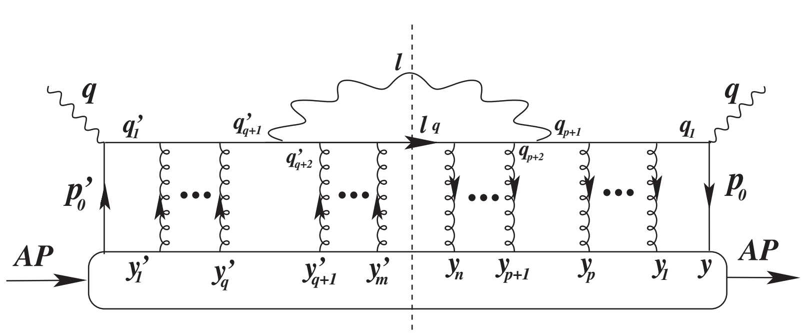

FIG. 3. An order $m+n$ contribution to the single-photon production rate. There are n outgoing gluons attached to the quark line on the right-hand side of the cut and m incoming gluon lines on the left-hand side of the cut. The photon attaches between the pth and $(p+1)$th location on the right and between the qth and $(q+1)$th location on the left. This contributes to twist $k \leqslant m+n$.

an arbitrary number of insertions. The various momentum equalities outlined in this paragraph may be now be expressed succinctly as

$$\begin{aligned} q_{i+1} &= q + \sum_{j=0}^{i} p_i \ \forall[0 \leqslant i \leqslant p], \\ q'_{k+1} &= q + \sum_{l=0}^{k} p'_k \ \forall\,[0 \leqslant k \leqslant q], \\ q_{i+1} &= q + \sum_{j=0}^{i} p_i - l \ \forall[p+1 \leqslant i \leqslant n], \\ q'_{k+1} &= q + \sum_{l=0}^{k} p'_k - l \ \forall[q+1 \leqslant k \leqslant m]. \end{aligned} \tag{20}$$

The general hadronic tensor for such a contribution may be written as

$$\left[W^{n,m}_{p,q}\right]^{\mu\nu}_A = AC^A_{p_1} \int d^4 y_0 \langle p_1|\bar{\psi}(y_0)\gamma^\mu \langle A|\widehat{\mathcal{O}^{nm}_{p.q}}|A\rangle \gamma^\nu \psi(0)|p_1\rangle. \tag{21}$$

In this equation, we have lumped all operators and propagators except those of the first struck quark in the amplitude and complex conjugate within the multiple operator product $\widehat{\mathcal{O}^{nm}_{p.q}}$. We have also suggestively factored out the one-nucleon state denoted as $|p_1\rangle$, which is the nucleon containing the struck quark. The remaining nucleons are contained in the state $|A\rangle$,[2] which is acted upon by the multiple operator product $\widehat{\mathcal{O}^{nm}_{p.q}}$. The coefficient $C^A_{p_1}$ is meant to indicate the correlation between the first nucleon and the rest of the nucleus. We will defer issues related to such a factorization of the nuclear state to Sec. IV, where the collinear expansion of the multiple operator product will be carried out and the nuclear state decomposed into an ensemble of nucleon states. The hadronic tensor may be further decomposed in terms of the leading spin projections of the two operator products as

$$\left[W^{n,m}_{p,q}\right]^{\mu\nu}_A = C^A_{p_1} \int d^4 y_0 \mathbf{Tr}\left[\frac{\gamma^-}{2}\gamma^\mu \frac{\gamma^+}{2}\gamma^\nu\right] F(y_0)\mathcal{O}^{n,m}_{p.q}, \tag{22}$$

[2]A more correct notation would be to refer to this state as $|A-1\rangle$, ignoring the 1 is appropriate for large A. These issues are dealt with in more rigor in Sec. IV.

where $F(y_0)$ is defined as in Eq. (9) with $|p\rangle$ replaced with $|p_1\rangle$ and $\mathcal{O}^{n,m}_{p.q}$ is the component of the leading spin projection of the final-state operators:

$$\mathcal{O}^{n,m}_{p,q} = \mathbf{Tr}\left[\frac{\gamma^-}{2}\langle A|\widehat{\mathcal{O}^{n,m}_{p,q}}|A\rangle\right]. \tag{23}$$

In the remainder of this section, we will focus exclusively on the component $\mathcal{O}^{n,m}_{p,q}$.

The expression for $\mathcal{O}^{n,m}_{p,q}$ may now be written down by applying the Feynman rules to the diagram of Fig. 3. The large rectangular blob is meant to indicate the nuclear state. The lines connecting the quark propagators to the blob are not propagators themselves; these are simply quark and gluon operator insertions; they introduce a certain momentum, spin, and color into the diagram. Note that the quark operator insertions that couple with the initial incoming virtual photon have already been extracted from the diagram and placed in Eq. (21). Note also that there is no particular ordering of the gluon lines. As these are not propagators there is no meaning associated with crossed gluon lines. The entire set of $n+m$ vertex insertions (with the gluon vector potentials contracted with the nucleus) may then be connected by quark propagators in $(n+m)!$ ways. [This overall combinatorial factor is removed by the $(n+m)!$ that appears in the denominator from the perturbation expansion.] All the Feynman propagators are written in position space as Fourier transforms of their momentum-space expressions; for example, for the case of a propagator in the complex conjugate amplitude between the ith and $(i+1)$th insertion, where $i < p$, we obtain

$$\begin{aligned} \mathcal{T}\left[\psi(y_i)\bar{\psi}(y_{i+1})\right] &= \int \frac{d^4 q_{i+1}}{(2\pi)^4} \frac{i \not{q}_{i+1} e^{-iq_{i+1}\cdot(y_i - y_{i+1})}}{q^2_{i+1} - i\epsilon} \\ &= \int \frac{d^4 p_i}{(2\pi)^4} \frac{i\left(\not{q} + \sum_{j=0}^{i} \not{p}_j\right)}{\left(q + \sum_{j=0}^{i} p_j\right)^2 - i\epsilon} \\ &\quad \times e^{-i(q+\sum_{j=0}^{i} p_j)\cdot(y_i - y_{i+1})}. \end{aligned} \tag{24}$$

By using the remaining relations among the various momenta as outlined in Eq. (20) the expression for the leading spin component of the final-state operator product may be expressed as

$$\begin{aligned} \mathcal{O}^{nm}_{pq} = &\prod_{i=1}^{n}\prod_{k=1}^{m} d^4 y_i d^4 y'_k \frac{d^4 p_{i-1}}{(2\pi)^4}\frac{d^4 p'_{k-1}}{(2\pi)^4}\frac{d^4 l}{(2\pi)^4}\frac{d^4 l_q}{(2\pi)^4}\mathbf{Tr}\left[\frac{1}{2}\prod_{i=0}^{p}\left\{\gamma^- \frac{\left(\sum_{j=0}^{i} \not{p}_j\right) + \not{q}}{\left[\left(\sum_{j=0}^{i} p_j\right) + q\right]^2 - i\epsilon}\right\}\right. \\ &\times \gamma^\alpha \prod_{i=p}^{n-1}\left\{\frac{\left(\sum_{j=0}^{i} \not{p}_j\right) + \not{q} - \not{l}}{\left[\left(\sum_{j=0}^{i} p_j\right) + q - l\right]^2 - i\epsilon}\gamma^-\right\} 2\pi\delta(l^2) 2\pi\delta\left(l_q^2\right) G_{\alpha\beta}(l)\, l_q \\ &\times \prod_{k=m-1}^{q}\left\{\gamma^- \frac{\left(\sum_{l=0}^{k} \not{p}'_l\right) + \not{q} - \not{l}}{\left[\left(\sum_{l=0}^{k} p'_l\right) + q - l\right]^2 + i\epsilon}\right\}\gamma^\beta \prod_{k=q}^{1}\left\{\frac{\left(\sum_{l=0}^{k} \not{p}'_l\right) + \not{q}}{\left[\left(\sum_{l=0}^{k} p'_l\right) + q\right]^2 + i\epsilon}\gamma^-\right\} \left.\frac{\not{p}'_0 + \not{q}}{(p'_0 + q)^2 + i\epsilon}\right]\prod_{i=0}^{n-1} e^{-ip_i\cdot y_i} \\ &\times \prod_{k=0}^{m-1} e^{ip'_k\cdot y'_k} e^{-iy_n\cdot\{l_q - (\sum_{i=0}^{n-1} p_i) + q - l\}} e^{iy'_m\cdot\{l_q - (\sum_{k=0}^{m-1} p'_k) + q - l\}} \langle A|\prod_{i=1}^{n} t^{a_i} A^+_{a_i}(y_i) \prod_{k=m}^{1} t^{b_k} A^+_{b_k}(y'_k)|A\rangle \end{aligned} \tag{25}$$

The expression for $\mathcal{O}^{nm}_{pq}$ in Eq. (25) may be decomposed into a formal convolution of three terms,

$$\mathcal{O}^{mn}_{pq} = \int d^4l d^4l_q \mathfrak{D}y\mathfrak{D}y'\mathfrak{D}p\mathfrak{D}p' T(\mathbf{p},\mathbf{p}',q,l,l_q) \times \Gamma(\mathbf{p},\mathbf{p}',\mathbf{y},\mathbf{y}')M(\mathbf{y},\mathbf{y}'), \tag{26}$$

where T denotes the trace in Eq. (25), which is solely a function of the momenta, $M(\mathbf{y},\mathbf{y}') = \langle A|\cdots|A\rangle$ is the pure position-dependent multioperator matrix element and $\Gamma(\mathbf{p},\mathbf{p}',\mathbf{y},\mathbf{y}')$ is the phase factor that contains both positions and momenta. The boldface quantities $\mathbf{p}$, $\mathbf{y}$, $\mathbf{p}'$, and $\mathbf{y}'$ represent an array of momenta and positions:

$$\mathbf{p} \equiv [p_0,\ldots,p_{n-1}], \quad \mathbf{p}' \equiv [p'_0,\ldots,p'_{m-1}],$$
$$\mathbf{y} \equiv [y_0,\ldots,y_n], \quad \mathbf{y}' \equiv [y'_1,\ldots,y'_m].$$

The range of locations in the complex conjugate amplitude start from y_0, which is the location of the initial struck quark, whereas the location of the same parton in the amplitude is at the origin. Hence $y'_0 = 0$ is not explicitly written in the equations presented. The integration measures $\mathfrak{D}p, p'$ and $\mathfrak{D}y, y'$ denote a product of integrals over the different four-vectors contained in the arrays mentioned above.

In an effort to simplify the writing of the matrix element and the momentum-dependent part, a certain aspect of collinear dynamics has been already been introduced. As calculations will be carried out in the light-cone gauge ($A^- = 0$) in the Breit frame at very high energy, the dominant components of the vector potential are the forward or (+)-components [32] (i.e., $A^\sigma \sim g^{-\sigma}A^+$). As a result, the corresponding γ matrices have solely a (−)-component.

To simplify the phase factor, an nth momentum (p_n) may be introduced, with

$$1 = \int d^4p_n \delta^4\left(l_q - \sum_{k=0}^{n} p_k - q + l\right). \tag{27}$$

This leads to a considerable simplification of the phase factor as

$$\Gamma = \exp\left[-\sum_{i=0}^{n} ip_i\cdot y_i + \sum_{j=0}^{m-1} ip'_j\cdot y'_j + iy'_m\cdot\left(\sum_{i=0}^{n}p_i - \sum_{j=0}^{m-1}p'_j\right)\right]. \tag{28}$$

Note the complete absence of the initial hard photon momentum q or the final-state momenta l, l_q from the phase factor. The integrations over the light-cone components of the cut line l_q have been performed by using two of the four δ functions introduced in Eq. (27). The remaining two components of the transverse momentum integration ($d^2l_{q_\perp}$) are constrained by the δ function

$$\delta^2(\vec{l}_{q\perp} + \vec{l}_\perp - \vec{K}_\perp) = \delta^2\left(\vec{l}_{q_\perp} + \vec{l}_\perp - \sum_{i=0}^{n}\vec{p}^{\,i}_\perp\right), \tag{29}$$

where $\vec{K}_\perp$ is a representative of the sum of the transverse momenta brought in by the n gluon insertions.

The approximations stemming from collinear dynamics may now be instituted to further simplify the momentum-dependent coefficient T in Eq. (26). The calculation is carried out in the Breit frame at very high energy. As a result, all momentum lines that originate in the target are dominated by the large (+)-components of their momentum, followed by their transverse coordinates; that is,

$$p_i^+ \gg p_{i\perp} \gg p_i^-. \tag{30}$$

In most cases this condition will allow us to practically drop all (−)-components of momentum from the hadronic tensor and focus solely on the (+)- and (⊥)-components. It should be pointed out that the (−)-components are only being dropped from locations where they appear in addition to the larger (+, ⊥)-components. These are not dropped from the phase factors in Γ.

The simplification of the numerator of T begins by isolating the largest components of the momentum. In most cases, this essentially reduces to retaining the sole factor γ^+q^- in the numerator of each of the propagators. A complication arises from the presence of a radiated photon in the final state. The structure of the sum over polarizations controls which terms are to be retained. It should be pointed out that unlike the sum over polarizations in covariant gauge,

$$G^{+-} = G^{-+} = 0, \quad G^{++} = \frac{2l^+}{l^-}, \quad G^{\perp+} = G^{+\perp} = \frac{l^\perp}{l^-}, \tag{31}$$

but $G^{\alpha\beta}_\perp = -g^{\alpha\beta}_\perp$ as in the covariant gauge. Here, and in what follows, the ⊥ tensor notation is introduced (i.e., $A^{\alpha\beta\cdots}_\perp$ indicates that the only nonzero components of the tensor A involve its transverse components). The Dirac trace in the numerator of Eq. (25) denoted by $N(T)$ may be separated from its propagation structure involving the denominator of the propagators and the δ functions denoted as $D(T)$. These may then be evaluated separately, as in the following.

The approximation is begun by ignoring all appearances of the (−)-components of the momentum that originate in the nuclear state (i.e., $p_i^-, p_l'^-$) from all terms in the purely momentum dependent component of the integrand T. These momenta now appear solely in the phase factors, which allows for the p^- and p'^- integrations to be done, resulting in the localization of the process on the negative light cone, that is,

$$\Gamma^- = \prod_{i=0}^{n}\prod_{l=0}^{m-1}\int dp'^-_l dp^-_i e^{-\sum_{i=0}^{n} ip_i^-(y_i^+ - y'^+_m)} e^{\sum_{l=0}^{m-1} ip_l'^-(y_l'^+ - y'^+_m)} = \prod_{i=0}^{n}\delta(y_i^+ - y'^+_m)\prod_{l=0}^{m-1}\delta(y'^+_l - y'^+_m). \tag{32}$$

In this equation, $y'_0 = 0$ and, as may be noted from the definition of the location arrays, it is not being integrated over. As a result, this constrains all the positive light-cone locations in this equation to the origin.

A set of new quantities may now be introduced by organizing the various factors that appear in the denominators of Eq. (25). There remain the obvious longitudinal momentum fractions

$$Q^2 = 2x_B p^+ q^-, \quad p_i^+ = x_i p^+, \quad p'^+_j = x'_j p^+. \quad (33)$$

As in Ref. [17], we introduce two sets of momentum fractions that are dependent on the transverse momenta imparted to the struck quark from the medium:

$$x_{Di} = \frac{\sum_{j=0}^{i} 2\vec{p}_\perp^{\,i} \cdot \vec{p}_\perp^{\,j} + |\vec{p}_\perp^{\,i}|^2}{2p^+q^-}, \quad (34)$$

$$x'_{Dk} = \frac{\sum_{l=0}^{k} 2\vec{p'}_\perp^{k} \cdot \vec{p'}_\perp^{l} + |\vec{p'}_\perp^{k}|^2}{2p^+q^-}. \quad (35)$$

Two new sets of momentum fractions that depend on both the transverse momentum imparted from the medium as well as that carried out by the radiated photon are also introduced:

$$x_{Li} = \frac{|\vec{l}_\perp|^2 - y\sum_{j=0}^{i} 2\vec{l}_\perp \cdot \vec{p}_\perp^{\,i}}{2y(1-y)p^+q^-}, \quad (36)$$

$$x'_{Lk} = \frac{|\vec{l}_\perp|^2 - y\sum_{l=0}^{k} 2\vec{l}_\perp \cdot \vec{p'}_\perp^{l}}{2y(1-y)p^+q^-}. \quad (37)$$

In this equation, i, the index of the unprimed momenta (both for the longitudinal and transverse components), runs from 0 to n, whereas k, the index of the primed momenta, runs from 0 to $m-1$ (i.e., one less than maximum). The other indices, j and l, denote smaller ranges of sums, with $0 < j < i$ and $0 < l < k$. The reader will note the obvious role played by these factors in the limit that the $(-)$-components of all momenta that originate in the nucleus have been ignored and one expands around the dominant $(+)$-components along with the vectors q and l, which have large $(-)$-components. In this limit one may factor out the large components of the momentum (i.e., p^+, q^-) and express the remaining components such as combinations and products of $\vec{l}_\perp$, $\vec{p}_\perp^{\,i}$, and $\vec{p'}_\perp^{k}$ as fractions of the large product p^+q^- such as x_{Li}, x_{Di}, and x'_{Dk}.

Invoking the notion of the collinear approximation, we can use the various definitions in Eqs. (33)–(37) to simplify the integrations that need to be performed in Eq. (25). The factors appearing in $D(T)$ may be written in a simplified form along with the longitudinal part of the phase factor [Eq. (28)] as

$$\begin{aligned} D(T) = {} & \prod_{i=0}^{p}\left[2p^+q^-\left(\sum_{j=0}^{i} x_j - x_{Di} - x_B - i\epsilon\right)\right]^{-1} \prod_{i=p}^{n-1}\left[2p^+q^-(1-y)\left(\sum_{j=0}^{i} x_j - \frac{x_{Dj}}{1-y} - x_B - x_{Li} - i\epsilon\right)\right]^{-1} \\ & \times \frac{\delta\left(\sum_{j=0}^{n} x_j - \frac{x_{Dj}}{1-y} - x_B - x_{Ln}\right)}{2p^+q^-(1-y)} \prod_{k=m-1}^{q}\left[2p^+q^-(1-y)\left(\sum_{l=0}^{k} x'_l - \frac{x'_{Dl}}{1-y} - x_B - x'_{Lk} + i\epsilon\right)\right]^{-1} \\ & \times \prod_{k=q}^{0}\left[2p^+q^-\left(\sum_{l=0}^{k} x'_l - x'_{Dl} - x_B + i\epsilon\right)\right]^{-1} \exp\left[-\sum_{i=0}^{n} i x_i p^+ \cdot y_i^- + \sum_{l=0}^{m-1} i x'_l p^+ \cdot y'^-_l \right. \\ & \left. + i y'^-_m \cdot p^+\left(\sum_{i=0}^{n} x_i - \sum_{l=0}^{m-1} x'_l\right)\right]. \end{aligned} \quad (38)$$

In this equation, the integration over x_n may be performed with the δ function, which denotes the cut quark line, to obtain

$$x_n = x_B + x_{Ln} + \sum_{i=0}^{n} x_{Di} - \sum_{i=0}^{n-1} x_i. \quad (39)$$

This may be used to rearrange the longitudinal part of the phase factor as

$$\begin{aligned} \Gamma^+ = {} & \exp\left[-i\left(x_B + x_{Ln} + \frac{\sum_{i=0}^{n} x_{Di}}{1-y}\right)p^+(y_n^- - y'^-_m)\right] \\ & \times \prod_{i=0}^{n-1} e^{[-i x_i p^+(y_i^- - y_n^-)]} \prod_{l=0}^{m-1} e^{[i x'_l p^+(y'^-_l - y'^-_m)]}, \end{aligned} \quad (40)$$

where the first product in the equation above involves only momentum fractions and locations from the left-hand side of the cut and the last product involves momentum fractions and locations from the right-hand side of the cut. The integrations over the remaining longitudinal momentum fractions may now be performed by starting from the propagators adjacent to the cut and proceeding to the propagators adjacent to the photon vertices.

The first such integration involves the propagator from the third line of Eq. (38). Isolating the piece that depends on the fraction x_{n-1} yields the integral, which may be performed by closing the contour of x_{n-1}^- with a counterclockwise semicircle in the upper half of the complex plane of

$$\int \frac{dx_{n-1}}{2\pi} \frac{e^{-ix_{n-1}\cdot p^+(y^-_{n-1}-y^-_n)}}{x_{n-1}+\sum_{i=0}^{n-2}\left(x_i-\frac{x_{Di}}{1-y}\right)-\frac{x_{Dn-1}}{1-y}-x_B-x_{Ln-1}-i\epsilon} = i\theta(y^-_n-y^-_{n-1})\, e^{-i[\frac{x_{Dn-1}}{1-y}+x_B+x_{Ln-1}-\sum_{i=0}^{n-2}(x_i-\frac{x_{Di}}{1-y})]p^+(y^-_{n-1}-y^-_n)}. \tag{41}$$

The effect of performing this integration is the incorporation of both the real and imaginary parts of this propagator into the overall expression of the hadronic tensor. It has the physical effect of propagating the quark from y^-_{n-1} to y^-_n. Similarly, the integration over the propagator to the immediate right of the cut line may be carried out by closing the contour of x'_{m-1} with a clockwise semicircle in the lower complex plane,

$$\int \frac{dx'_{m-1}}{2\pi} \frac{e^{ix'_{m-1}p^+(y'^-_{m-1}-y'^-_m)}}{x'_{m-1}+\sum_{j=0}^{m-2}\left(x'_i-\frac{x'_{Di}}{1-y}\right)-\frac{x'_{Dm-1}}{1-y}-x_B-x'_{Lm-1}+i\epsilon}$$

$$= -i\theta(y'^-_m-y'^-_{m-1})e^{i[\frac{x'_{Dm-1}}{1-y}+x_B+x'_{Lm-1}-\sum_{j=0}^{m-2}(x'_i-\frac{x'_{Di}}{1-y})]p^+(y'^-_{m-1}-y'^-_m)}. \tag{42}$$

Incorporation of the results of these two integrals into the longitudinal phase factors leads to the expression

$$\Gamma^+ = \exp\left[-i\left(\frac{x_{Dn}-\delta x_{Ln}}{1-y}\right)p^+y^-_n + i\left(\frac{x'_{Dm}-\delta x'_{Lm}}{1-y}\right)p^+y'^-_m\right]\exp\left[-i\left(x_B+x_{Ln-1}+\frac{\sum_{i=0}^{n-1}x_{Di}}{1-y}\right)p^+y^-_{n-1}\right.$$

$$\left. +i\left(x_B+x'_{Lm-1}+\frac{\sum_{l=0}^{m-1}x'_{Dl}}{1-y}\right)p^+y'^-_{m-1}\right]\prod_{i=0}^{n-2}\exp[-ix_ip^+(y^-_i-y^-_{n-1})]\prod_{l=0}^{m-2}\exp[ix'_lp^+(y'^-_l-y'^-_{m-1})], \tag{43}$$

where the notion of overall momentum conservation was invoked to define the new variable $\vec{p}'^{\,m}_\perp = \sum_{i=0}^n \vec{p}^{\,i}_\perp - \sum_{k=0}^{m-1}\vec{p}'^{\,k}_\perp$ and, as a result,

$$x'_{Dm} = \sum_{i=0}^{n} x_{Di} - \sum_{j=0}^{m-1} x'_{Dj}. \tag{44}$$

In this equation, the quantities labeled x_{Di} for $0 < i < n-1$ are defined in Eq. (34), and x'_{Dj} for $0 < j < m-1$ are defined in Eq. (35). The nth transverse fraction x_{Dn} is set by the δ function arising from the cut line and is given as in Eq. (39). The extra momentum fractions δx_{Ln} and $\delta x'_{Lm}$ arise from the fact that there is a small difference between x_{Ln} and x_{Ln-1}, that is,

$$x_{Ln} - x_{Ln-1} = \frac{-2\vec{l}_\perp\cdot\vec{p}^{\,n}_\perp}{2p^+q^-(1-y)} = \frac{-\delta x_{Ln}}{1-y} \tag{45}$$

and, similarly,

$$x'_{Lm} - x'_{Lm-1} = \frac{-2\vec{l}_\perp\cdot\vec{p}'^{\,m}_\perp}{2p^+q^-(1-y)} = \frac{-\delta x'_{Lm}}{1-y}. \tag{46}$$

It should be pointed out that δx_{Ln} or the primed momentum fraction has no dependence on the momentum fraction of the radiated gluon, y. Thus such contribution will be retained even in the small-y approximation.

The evaluation of the remaining longitudinal momentum fraction integrals is analogous up to the outgoing photon vertices. Therefore, the general result for the integrations over the remaining momentum fractions in the phase factor may be carried out up to the integral involving x_{p+1} and x_{q+1}. Around the photon insertion there are two propagators, either of which may be evaluated to obtain the conditions on x_p and x'_q. We focus on the x_p integral, but the case for x'_q is completely analogous. The integral in question is

$$\int \frac{dx_p}{2\pi} \frac{e^{-ix_p p^+(y_p^- - y_{p+1}^-)}}{\left[x_p + \sum_{i=0}^{p-1}\left(x_i - \frac{x_{Di}}{1-y}\right) - \frac{x_{Dp}}{1-y} - x_B - x_{Lp} - i\epsilon\right]\left[x_p + \sum_{i=0}^{p-1}\left(x_i - x_{Di}\right) - x_{Dp} - x_B - i\epsilon\right]}$$

$$= \frac{i\theta(y_{p+1}^- - y_p^-)}{x_{Lp} + \frac{y}{1-y}\sum_{j=0}^{p} x_{Dj}}\left[e^{-i\left\{\frac{x_{Dp}}{1-y}+x_B+x_{Lp}-\sum_{i=0}^{p-1}\left(x_i-\frac{x_{Di}}{1-y}\right)\right\}p^+(y_p^- - y_{p+1}^-)} - e^{-i\left\{x_{Dp}+x_B-\sum_{i=0}^{p-1}\left(x_i-x_{Di}\right)\right\}p^+(y_p^- - y_{p+1}^-)}\right]. \tag{47}$$

The origin of two separate contributions lies in the fact that the leading length enhancement arises when at most one of the propagators is off-shell. This necessarily has to be one of the propagators adjacent to the outgoing photon line. The remaining integrations over the propagators between the photon insertion and the hard scattering vertex are similar to the case of transverse broadening [17].

The reader will have noticed that these integrals are over the momentum fractions x_i and x'_l, whereas, in Eq. (25), the integrals are over the momenta p_i^+ and p'^+_l. We have substituted the definitions of Eq. (33) and hence focus on the momentum fractions instead. Thus each integral produces a dimensionful factor p^+. There being $n+m+1$ integrals over the $(+)$-components of the momentum produces an overall factor of $(p^+)^{n+m+1}$. This factor will not be explicitly written out in the remaining simplifications of the different parts of $\mathcal{O}^{n,m}_{p,q}$ that follow. They will be reintroduced at the end of the section, when the different pieces are recombined to write down the simplified structure of $\mathcal{O}^{n,m}_{p,q}$.

Invoking this simplifications, the factor $D(T)$ [i.e., the part of $\mathcal{O}^{n,m}_{p,q}$ which depends solely on the denominators of the propagators of the hard parton and the longitudinal phase factors in Eq. (38)] takes the form

$$D(T) = \frac{\theta(y_n^- > y_{n-1}^- > \cdots > y_0^-)}{(2p^+q^-)^{n+1}(1-y)^{n-p}} \prod_{i=0}^{p} e^{-ix_{Di}p^+y_i^-} \prod_{i=p+1}^{n} e^{-i\left(\frac{x_{Di}-\delta x_{Li}}{1-y}\right)p^+y_i^-} \frac{1}{x_{Lp} + \frac{y}{1-y}\sum_{j=0}^{p} x_{Dj}}\Big[e^{-i\left(x_{Lp}+\frac{y}{1-y}\sum_{i=0}^{p}x_{Di}\right)p^+y_p^-}$$

$$- e^{-i\left(x_{Lp}+\frac{y}{1-y}\sum_{i=0}^{p}x_{Di}\right)p^+y_{p+1}^-}\Big] \frac{\theta(y'^-_m > y'^-_{m-1} > \cdots > y'^-_0)}{(2p^+q^-)^{m+1}(1-y)^{m-q}} \prod_{k=0}^{q} e^{ix'_{Dk}p^+y'^-_k} \prod_{k=q+1}^{m} e^{i\left(\frac{x'_{Dk}-\delta x'_{Lk}}{1-y}\right)p^+y'^-_k} \frac{1}{x'_{Lq} + \frac{y}{1-y}\sum_{l=0}^{q} x'_{Dl}}$$

$$\times \Big[e^{i\left(x'_{Lq}+\frac{y}{1-y}\sum_{l=0}^{q}x'_{Dl}\right)p^+y'^-_q} - e^{i\left(x'_{Lq}+\frac{y}{1-y}\sum_{l=0}^{q}x'_{Dl}\right)p^+y'^-_{q+1}}\Big]\frac{1}{2p^+q^-(1-y)}. \tag{48}$$

In this equation, the expression $\theta(y_n^- > y_{n-1}^- > \cdots)$ is meant to indicate a product of n θ functions:

$$\theta(y_n^- > y_{n-1}^- > \cdots) = \prod_{i=0}^{n-1}\theta(y_{i+1}^- - y_i^-). \tag{49}$$

These θ functions organize the longitudinal locations of the scatterings, starting from the hard incoming virtual photon vertex and moving toward the cut line (and the same is true for the primed locations in the complex conjugate amplitude).

The numerator of the momentum-dependent part of Eq. (25), which is denoted as $N(T)$, may be decomposed, as in the previous section, into four parts [i.e., $N(T) = \sum_{i=1}^{4} N(T)_i$] depending on the components of the photon polarization tensor chosen. These are denoted in the same order as in Eq. (7): $N(T)_1$ denotes the term arising from the projection $G^{\perp\perp}$. Using this combination of the components α, β, we can simplify the leading part of the numerator as (in the remaining equations in this section, we will ignore the vector symbol on the transverse momenta $\vec{l}_\perp$, $\vec{p}^{\,i}_\perp$, and $\vec{p}'^{\,k}_\perp$; they should, however, always be understood to be two-dimensional vectors)

$$N(T)_1 = \mathbf{Tr}\left[\frac{1}{2}\left(\prod_{i=0}^{p-1}\gamma^-\gamma^+q^-\right)\gamma^-\left\{\gamma^+q^- + \gamma^\perp\sum_{j=0}^{p}p_\perp^j\right\}\gamma^\perp{}_\alpha\left\{\gamma^+(q^- - l^-) + \gamma^\perp\left(\sum_{j=0}^{p}p_\perp^j - l_\perp\right)\right\}\right.$$
$$\times\left(\prod_{i=p+1}^{n-1}\gamma^-\gamma^+(q^- - l^-)\right)\gamma^-\gamma^+l_q^-(-g_\perp^{\alpha\beta})\left(\prod_{k=m-1}^{q+1}\gamma^-\gamma^+(q^- - l^-)\right)$$
$$\left.\times\,\gamma^-\left\{\gamma^+(q^- - l^-) + \gamma^\perp\left(\sum_{l=0}^{q}p'^l_\perp - l_\perp\right)\right\}\gamma^\perp{}_\beta\left\{\gamma^+q^- + \gamma^\perp\sum_{l=0}^{q}p'_\perp\right\}\left(\prod_{k=q-1}^{0}\gamma^-\gamma^+q^-\right)\right]. \quad (50)$$

In Eq. (50), the transverse components of the propagators are only retained for those propagators in the immediate vicinity of the photon insertion points, identified by the $\gamma_\alpha^\perp, \gamma_\beta^\perp$ in Eq. (50). Only the transverse components in these locations give leading contributions. The transverse components in the remaining propagators yield suppressed contributions compared to the longitudinal components of the momenta, q^- and $q^- - l^-$. Using the well-known relations between the various γ matrices, such as $\gamma^\pm\gamma^\pm = 0$, $\{\gamma^\pm, \gamma^\mp\} = 2\mathbf{1}$ (where $\mathbf{1}$ is the unit matrix in spinor space), and the anticommutation rule $\{\gamma^\pm, \gamma^\perp\} = 0$, the entire numerator term in Eq. (50) may be readily simplified as

$$N(T)_1 = (2q^-)^{n+m+1}(1-y)^{n+m-p-q-1} \times 2\left[\{1+(1-y)^2\}\left(\sum_{j=0}^{p}p_\perp^j\right)\cdot\left(\sum_{l=0}^{q}p'^l_\perp\right) \times l_\perp^2 - l_\perp\cdot\left(\sum_{j=0}^{p}p_\perp^j + \sum_{l=0}^{q}p'^l_\perp\right)\right]. \quad (51)$$

Using a similar procedure, as above, we can write the terms originating from the projection $G^{+\perp}$ and $G^{\perp+}$ in combination as

$$N(T)_2 + N(T)_3 = (2q^-)^{n+m+1}\frac{(1-y)^{n+m-p-q}}{y}2\left[2l_\perp^2 - l_\perp \cdot\left(\sum_{i=0}^{p}p_\perp^i + \sum_{l=0}^{q}p'^l_\perp\right)(2-y)\right], \quad (52)$$

(where we have refrained from presenting the full derivation of these terms as it is rather straightforward). Note that these factors appear solely from the use of the light-cone gauge. In a covariant gauge such as Feynman gauge there are no photon polarization tensor components which connect the $+$ and $\perp$-components. The last contribution originates from the projection G^{++},

$$N(T)_4 = (2q^-)^{m+n+1}(1-y)^{m+n-p-q+1}2\frac{2l_\perp^2}{y^2}. \quad (53)$$

The sum of these four terms has the rather simple and physical form

$$N(T) = (2q^-)^{n+m+1}\frac{(1-y)^{n+m-p-q-1}}{y}2P_\gamma(y) \times\left[|l_\perp|^2 - y\left\{l_\perp\cdot\left(\sum_{i=0}^{p}p_\perp^i + \sum_{k=0}^{q}p'^k_\perp\right)\right\} + y^2\left\{\left(\sum_{j=0}^{p}p_\perp^j\right)\cdot\left(\sum_{l=0}^{q}p'^l_\perp\right)\right\}\right], \quad (54)$$

where, $P_\gamma(y)$ represents the photon splitting function.

The full contribution to the term $\mathcal{O}_{p,q}^{n,m}$ of Eq. (25), post integration over all longitudinal fractions, may now be reconstituted by combining the various parts as stated in Eq. (26):

$$\mathcal{O}_{p,q}^{m,n} = e^2\int\frac{d^2l_\perp dl^-}{(2\pi)^3 2l^-}\frac{d^2l_{q\perp}}{(2\pi)^2}\prod_{i=0}^{n}dy_i^-d^2y_\perp^i\prod_{j=1}^{m}dy'^-_j d^2y'^j_\perp\prod_{i=0}^{n}\frac{d^2p_\perp^i}{(2\pi)^2}\prod_{j=0}^{m-1}\frac{d^2p'^j_\perp}{(2\pi)^2}(2\pi)^2\delta^2(\vec{l}_{q\perp} + \vec{l}_\perp - \vec{K}_\perp)$$
$$\times\frac{2P_\gamma(y)e^{-ix_Bp^+y_0^-}}{y(2p^+q^-)^2(1-y)^2}\left[|l_\perp|^2 - y\left\{l_\perp\cdot\left(\sum_{i=0}^{p}p_\perp^i + \sum_{k=0}^{q}p'^k_\perp\right)\right\} + y^2\left\{\left(\sum_{j=0}^{p}p_\perp^j\right)\cdot\left(\sum_{l=0}^{q}p'^l_\perp\right)\right\}\right]$$
$$\times\prod_{i=0}^{p}e^{-ix_{Di}p^+y_i^-}e^{ip_\perp^i\cdot y_\perp^i}\prod_{i=p+1}^{n}e^{-i(\frac{x_{Di}-\delta x_{Li}}{1-y})p^+y_i^-}e^{ip_\perp^i\cdot y_\perp^i}\prod_{i=n}^{1}\theta(y_i^- - y_{i-1}^-)\prod_{k=0}^{q}e^{ix'_{Dk}p^+y'^-_k}e^{-ip'^k_\perp\cdot y'^k_\perp}$$

$$\times \prod_{k=q+1}^{m} e^{i(\frac{x'_{Dk}-\delta x'_{Lk}}{1-y})p^+ y'^-_k} e^{-i p'^k_\perp \cdot y'^k_\perp} \prod_{k=m}^{1} \theta(y'^-_k - y'^-_{k-1}) \frac{1}{x_{Lp} + \frac{y}{1-y}\sum_{j=0}^{p} x_{Dj}} \left[e^{-i(x_{Lp}+\frac{y}{1-y}\sum_{i=0}^{p} x_{Di})p^+ y^-_p} - e^{-i(x_{Lp}+\frac{y}{1-y}\sum_{i=0}^{p} x_{Di})p^+ y^-_{p+1}}\right] \frac{1}{x'_{Lq} + \frac{y}{1-y}\sum_{l=0}^{q} x'_{Dl}} \left[e^{i(x'_{Lq}+\frac{y}{1-y}\sum_{l=0}^{q} x'_{Dl})p^+ y'^-_q} - e^{i(x'_{Lq}+\frac{y}{1-y}\sum_{l=0}^{q} x'_{Dl})p^+ y'^-_{q+1}}\right]$$
$$\times \langle A;p|g^{m+n}\mathbf{Tr}\left[\prod_{i=1}^{n} t^{a_i} A^+_{a_i}(y^-_i, y^i_\perp) \prod_{j=n'}^{1} t^{a_j} A^+_{a_j}(y'^-_j, y'^j_\perp)\right]|A;p\rangle. \quad (55)$$

To simplify the discussion of the remaining sections, the soft photon radiation approximation (i.e., $y \to 0$) will be introduced. Only the leading behavior in y will be retained. This leads to a considerable simplification of the various expressions and also represents a well-studied limit. Two of the energy-loss formalisms, those of Refs. [10] and [11], are cast in this limit. In the limit of the soft radiation approximation, all factors of $1-y$ in the denominators may be replaced with $1+y$ in the numerators and the leading and next-to-leading contributions in y are retained. As a result of this approximation, only the contributions from $N(T)_4$ and $N(T)_2 + N(T)_3$ need to be retained as they contain the leading and subleading negative power of y and are thus dominant in the limit $y \ll 1$. In this limit,

$$x_{Li} \simeq \frac{|l_\perp|^2(1+y) - y\sum_{j=0}^{i} 2l_\perp \cdot p^i_\perp}{2p^+q^-y} \gg y x_{Di}$$
$$= y\frac{|p^i_\perp|^2 + \sum_{j=0}^{i} 2p^i_\perp \cdot p^j_\perp}{2p^+q^-}. \quad (56)$$

As a result, retaining solely the leading and next-to-leading behavior in y we obtain $\mathcal{O}^{m,n}_{p,q}$ in the soft radiation limit as

$$\mathcal{O}^{m,n}_{p,q} = e^2 \int \frac{d^2l_\perp dl^-}{(2\pi)^3 2l^-} \frac{d^2 l_{q\perp}}{(2\pi)^2} \prod_{i=0}^{n} dy^-_i d^2 y^i_\perp \prod_{j=1}^{m} dy'^-_j d^2 y'^j_\perp \prod_{i=0}^{n} \frac{d^2 p^i_\perp}{(2\pi)^2} \prod_{j=0}^{m-1} \frac{d^2 p'^j_\perp}{(2\pi)^2} (2\pi)^2 \delta^2(\vec{l}_{q\perp} + \vec{l}_\perp - \vec{K}_\perp) \frac{2P_\gamma(y) e^{-ix_B p^+ y^-_0}}{y(2p^+q^-)^2(1-y)^2}$$
$$\times \left[|\vec{l}_\perp|^2 - y\vec{l}_\perp \cdot \left(\sum_{i=0}^{p} \vec{p}^i_\perp + \sum_{k=0}^{q} \vec{p}'^k_\perp\right)\right] \prod_{i=0}^{p} e^{-ix_{Di}p^+ y^-_i} e^{ip^i_\perp \cdot y^i_\perp} \prod_{i=p+1}^{n} e^{-i(x_{Di}-\delta x_{Li})(1+y)p^+ y^-_i} e^{ip^i_\perp \cdot y^i_\perp} \prod_{k=0}^{q} e^{ix'_{Dk}p^+ y'^-_k}$$
$$\times e^{-ip'^k_\perp \cdot y'^k_\perp} \prod_{k=q+1}^{m} e^{i(x'_{Dk}-\delta x'_{Lk})(1+y)p^+ y'^-_k} e^{-ip'^k_\perp \cdot y'^k_\perp} \prod_{i=n}^{1} \theta(y^-_i - y^-_{i-1}) \prod_{k=m}^{1} \theta(y'^-_k - y'^-_{k-1})$$
$$\times \frac{1}{x_{Lp}}\left[e^{-ix_{Lp}p^+ y^-_p} - e^{-ix_{Lp}p^+ y^-_{p+1}}\right] \frac{1}{x'_{Lq}}\left[e^{ix'_{Lq}p^+ y'^-_q} - e^{ix'_{Lp}p^+ y'^-_{q+1}}\right] \langle A;p|g^{m+n}\mathbf{Tr}\left[\prod_{i=1}^{n} t^{a_i} A^+_{a_i}(y^-_i, y^i_\perp)\right.$$
$$\left.\times \prod_{j=n'}^{1} t^{a_j} A^+_{a_j}(y'^-_j, y'^j_\perp)\right]|A;p\rangle. \quad (57)$$

The expression derived here is completely general, in the sense that no assumption regarding the nature of the nuclear state has been made. In the next section, a re-summation over the different locations of where the photon is radiated will be carried out. We then carry out a factorization of the hadronic tensor into a part that is solely dependent on hard momenta and a part dependent on soft momenta and make simplifying assumptions regarding the nuclear state. This will then be followed by the re-summation over number of scatterings.

IV. SUM OVER PHOTON PRODUCTION POINTS, FACTORIZATION, AND GRADIENT EXPANSION

In the preceding section, a general expression for single-photon bremsstrahlung from a multiply scattering hard quark parton was derived. The expression in Eq. (57) represents the case where the quark scatters n times in the amplitude and m times in the complex conjugate. The photon is produced after the pth scattering in the amplitude and after the qth scattering in the complex conjugate. To obtain the full differential cross section to produce a photon with momentum fraction y and transverse momentum $l_\perp$, sums have to be carried out over the various quantities p, q and n, m. A few comments are in order: In analogy with the case of transverse momentum broadening of a hard quark propagating in a nuclear medium [17], cases where $n \neq m$ represent unitarity corrections to terms where the produced quark scatters $\min[n, m]$ times. The cases where $p = q$ represent squares of the amplitude where the photon is produced in the pth location. The cases where $p \neq q$ represent interference terms that lead to the well-known LPM suppression of the photon production rate.

In this section, the sum over the various locations of the radiated photon in the amplitude and the complex conjugate amplitude (i.e., p, q) will be carried out. By invoking the small y approximation, the $l_\perp$-dependent momentum fractions may be expanded as

$$x_{Lp} = x_L(1+y) - \sum_{i=0}^{p} \delta x_{Li} = \frac{l_\perp^2(1+y)}{2p^+q^-y} - \frac{\sum_{i=0}^{p} 2l_\perp \cdot p_\perp^i}{2p^+q^-}. \quad (58)$$

To simplify notation, we set $\Delta x_L^i = \sum_{j=0}^{i} \delta x_{Lj}$. As most of the factors in $\mathcal{O}_{p,q}^{m,n}$ [Eq. (57)] are independent of p, q, this contribution may be written compactly as

$$\mathcal{O}_{pq}^{mn} = \int \frac{d^2 l_\perp dl^-}{(2\pi)^3 2l^-} \frac{d^2 l_{q\perp}}{(2\pi)^2} \mathfrak{D}(y^-, y_\perp)\mathfrak{D}(y'^-, y'_\perp)\mathfrak{D}p_\perp \mathfrak{D}p'_\perp \times M(\mathbf{y}, \mathbf{y}')(2\pi)^2 \delta^2\left(\vec{l}_{q\perp} + \vec{l}_\perp - \vec{K}_\perp\right) \frac{2e^{-ix_B p^+ y_0^-}}{y(2p^+q^-)^2(1-y)^2} \times P_\gamma(y) \prod_{i=0}^{n} e^{-ix_D^i p^+ y_i^- + ip_\perp^i \cdot y_\perp^i} \times \prod_{j=0}^{m} e^{-ix_D'^i p^+ y_i'^- + ip_\perp'^i \cdot y_\perp'^i} f_p g_q, \quad (59)$$

where the terms that depend on the location where the photon is produced are denoted separately. The product of these factors is given, in the small-y limit, as

$$f_p g_q = \frac{1-y+\frac{\Delta x_L^p}{x_L}}{x_L}\left[e^{-i[(x_L(1+y)-\Delta x_L^p]p^+y_p^- + i\sum_{i=p+1}^{n}\delta x_{Li}p^+y_i^-} - e^{-i[x_L(1+y)-\Delta x_L^p]p^+y_{p+1}^- + i\sum_{i=p+1}^{n}\delta x_{Li}p^+y_i^-}\right] \times \left(\vec{l}_\perp - y\sum_{i=0}^{p}\vec{p}^i{}_\perp\right) \frac{1-y+\frac{\Delta x'^q_L}{x_L}}{x_L}\left[e^{+i[x_L(1+y)-\Delta x'^q_L]p^+y'^-_q - i\sum_{j=q+1}^{m}\delta x'_{Lj}p^+y'^-_j} - e^{+i[x_L(1+y)-\Delta x'^q_L]p^+y'^-_{q+1} - i\sum_{j=q+1}^{m}\delta x'_{Lj}p^+y'^-_j}\right] \times \left(\vec{l}_\perp - y\sum_{k=0}^{q}\vec{p}'^k_\perp\right).$$

In both the denominators and in the exponent, the leading contribution is from the term $x_L \sim 1/y$. The remaining terms Δx_L^p, δx_{Li}, and x_D^i are all independent of (hence subleading in) y. Therefore, the factor of y in products such as $\delta x_{Li}(1+y)$ may be dropped.

The sums to be carried out are those over $0 < p < n-1$ and $0 < q < m-1$ restricted to the terms $f_p g_q$. The reader will note that these two sums are independent of each other and as a result may be carried out rather easily. The sum over p is

$$\sum_{p=0}^{n-1} \frac{1-y+\frac{\Delta x_L^p}{x_L}}{x_L}\left[e^{-i[x_L(1+y)-\Delta x_L^p]p^+y_p^- + i\sum_{i=p+1}^{n}\delta x_{Li}p^+y_i^-} - e^{-i[x_L(1+y)-\Delta x_L^p]p^+y_{p+1}^- + i\sum_{i=p+1}^{n}\delta x_{Li}p^+y_i^-}\right]\left(\vec{l}_\perp - y\sum_{i=0}^{p}\vec{p}^i{}_\perp\right). \quad (60)$$

Note that the second phase factor in the pth term is identical to the first phase factor in the $(p+1)$th term. As a result, many terms cancel in the sum if the small correction Δx_L^p in the coefficient of the phases is neglected. The phases that never cancel are the first phase of the term $p = 0$ and the second phase of the term $p = n-1$. The result of the sum over p, up to next-to-leading terms in y, is

$$\sum_{p=0}^{n-1} f_p = \frac{1}{x_L}\Bigg[(1-y)\vec{l}_\perp\left\{e^{-ix_L(1+y)p^+y_0^-} e^{i\sum_{i=0}^{n}\delta x_{Li}p^+y_i^-} - e^{-ix_L(1+y)p^+y_n^-} e^{i\sum_{i=0}^{n}\delta x_{Li}p^+y_n^-}\right\} + \sum_{p=0}^{n-1}\left(\frac{\vec{l}_\perp \delta x_{Lp}}{x_L} - y\vec{p}_\perp^{\,p}\right) \times e^{-i[x_L(1+y)-\Delta x_L^p]p^+y_p^- + i\sum_{i=p+1}^{n}\delta x_{Li}p^+y_i^-} \times \sum_{i=0}^{n-1}\left(\frac{\vec{l}_\perp \delta x_{Li}}{x_L} - y\vec{p}_\perp^{\,i}\right) e^{-i[x_L(1+y)-\Delta x_L^n]p^+y_n^-}\Bigg]. \quad (61)$$

In the limit that the photon is extremely soft, one may retain solely the leading contribution in y (i.e., only the first two terms in the brackets). This corresponds to the "deep-LPM" limit: the regime where the radiated photon cannot resolve the various scatterings and treats the entire nucleus (or the entire process of n scatterings) coherently as one single-scattering event. The next-to-leading term in y, which is the third term in the equation, represents the first correction to this limit where the photon just begins to resolve the different scatterings of the hard quark in the medium. The terms obtained from the sum $\sum_{q=0}^{m-1}$ in the complex conjugate amplitude yield a near identical expression with n replaced by m and the phase factors complex conjugated.

We may now add the terms f_n and g_m (i.e., the cases where the photon is produced after all the n or m scatterings). Such contributions necessarily have an on-shell, cut quark line on one side of the photon vertex. As a result, these terms do not contain the difference of two phase factors as in Eq. (47) and only have the first factor that occurs when the line after the photon emission is taken on-shell (in this case taken by cutting the line). For the sum over p this yields

$$\sum_{p=0}^{n-1} f_p = \frac{1}{x_L}\Bigg[(1-y)\vec{l}_\perp e^{-ix_L(1+y)p^+y_0^-} e^{i\sum_{i=0}^{n}\delta x_{Li}p^+y_i^-} + \sum_{p=0}^{n-1}\left(\frac{\vec{l}_\perp \delta x_{Lp}}{x_L} - y\vec{p}_\perp^{\,p}\right) \times e^{-i[x_L(1+y)-\Delta x_L^p]p^+y_p^- + i\sum_{i=p+1}^{n}\delta x_{Li}p^+y_i^-} + \left(\frac{\vec{l}_\perp \delta x_{Ln}}{x_L} - y\vec{p}_\perp^{\,n}\right) e^{-i[x_L(1+y)-\Delta x_L^n]p^+y_n^-}\Bigg]. \quad (62)$$

We will also ignore the next-to-leading terms in y that occur in the phases as these will not play a major role in what follows. With these simplifications, we obtain

$$\sum_{p,q=0}^{n,m} f_p g_q = \frac{e^{-ix_Lp^+y_0^-}}{x_L^2} l_\perp^2 \Bigg[1 - 2y + y\sum_{p=1}^{n}\frac{p_\perp^p\cdot l_\perp}{l_\perp^2} \times e^{-ix_Lp^+(y_p^- - y_0^-)} y\sum_{q=1}^{m}\frac{p'^q_\perp\cdot l_\perp}{l_\perp^2} e^{ix_Lp^+y'^-_q}\Bigg]. \quad (63)$$

Up to this point, the collinear approximation has been used to simplify the expressions for the hadronic tensor, without the introduction of factorization. The separation of the hadronic tensor into a hard short-distance piece and a soft long-distance contribution may now be accomplished. All factors in Eq. (57) which contain the hard scales p^+, q^- constitute the hard part. All factors which depend solely on the soft $\perp$ momenta and distances along with the matrix element constitute the long-distance element. All phase factors which contain a factor $x_D^i p^+$ (as well as its primed counterparts) or $x_L p^+$ as part of their arguments belong in the hard part. The purely transverse phase factors such as $\exp[i\vec{p}_\perp^{\,i}\cdot\vec{y}_\perp^{\,i}]$ (which do not contain any factor of $l_\perp$) belong in the soft part along with the matrix elements. Thus, we may decompose the general contribution to the hadronic tensor as

$$\sum_{p,q}^{n,m}\mathcal{O}_{p,q}^{n,m} = \int \mathfrak{D}y\mathfrak{D}p_\perp H(p^+,q^-,p_\perp,l_\perp,y)S(p_\perp,y). \quad (64)$$

In this equation, y and $p_\perp$ are representative of the entire set of distances and transverse momentum that appear in Eq. (57).

The soft part, which contains the matrix elements of the gluon vector potentials in the nuclear state, may be simplified first. The simplifications arise from approximations made regarding the nature of the nuclear state $|A;p\rangle$. In this article, the nucleus is approximated as a weakly interacting homogeneous gas of nucleons. Such an approximation is only sensible at very high energy, where, because of time dilation, the nucleons appear to travel in straight lines almost independent of each other over the interval of the interaction of the hard probe. In a sense, all forms of correlators between nucleons (spin, momentum, etc.) are assumed to be rather suppressed. As a result, the expectation of the $n+n'+2$ operators in the nuclear state may be decomposed as

$$\langle A;p|\bar{\psi}(y^-,y_\perp)\gamma^+\psi(0)\prod_{i=1}^{n+n'}A_{a_i}^+(y_i)|A;p\rangle = AC_{p_1}^A\langle p_1|\bar{\psi}(y^-,y_\perp)\gamma^+\psi(0)\prod_{i=1}^{n+n'}A_{a_i}^+(y_i)|p_1\rangle + C_{p_1,p_2}^A\langle p_1|\bar{\psi}(y^-,y_\perp)\gamma^+\psi(0)|p_1\rangle \times\langle p_2|\prod_{i=1}^{n+n'}A_{a_i}^+(y_i)|p_2\rangle + \cdots, \quad (65)$$

where the factor $C_{p_1}^A$ represents the probability of finding a nucleon in the vicinity of the location $\vec{y}$, which is a number of order unity (being the probability of finding one of A nucleons distributed in a volume of size cA within a nucleon size sphere centered at $\vec{y}$). The remaining coefficients $C_{p_1,\ldots}^A$ represent the weak position correlations between different nucleons. The overall factor of A arises from the determination of the origin [the location 0 in Eq. (65)] in the nucleus, which may be situated on any of the A nucleons. Solely for the current discussion, we reintroduce the quark operators $\bar{\psi}(y^-,y_\perp)$ and $\psi(0)$ in this equation.

It is clear from the above mentioned decomposition that the largest contribution arises from the term where the expectation of each partonic operator is evaluated in separate nucleon states as the $\vec{y}_i$ integrations may be carried out over the nuclear volume. As a nucleon is a color singlet, any combination of quark or gluon field strength insertions in a nucleon state must itself be restricted to a color singlet combination. As a result, the expectation of single partonic operators in nucleon states is vanishing. The first (and hence largest) nonzero contribution emanates from the terms where the quark operators in the singlet color combination are evaluated in a nucleon state and the $n+n'$ gluons are divided into pairs of singlet combinations, with each singlet pair evaluated in a separate nucleon state. This requires that $n+n'$ is even and may lead to a maximum overall factor of

$$C_{p_1,p_2,\ldots}^A \sim A^{[(n+n'+2)/2]}, \quad (66)$$

in the large-A limit. It should be pointed out that large contributions may also arise, in principle, when $n+n'$ is odd. In this case, the two quarks and a gluon are considered in the singlet combination with the remaining gluons evaluated in singlet pairs in the remaining nucleons. Here we institute the experimental observation that (n)-parton observables are much smaller than ($n-1$)-parton observables. This is only true outside the saturation regime [33,34]. In this effort, the focus remains exclusively outside this region; as a result we ignore all terms with more than two quarks or two gluons per nucleon.

Further simplifications arise in the evaluation of gluon pairs in a singlet combination in the nucleon states by carrying out the $y_\perp$ integrations. The basic object under consideration is

$$\int d^2y_\perp^i d^2y'^j_\perp \langle p|A^+(\vec{y}^i_\perp)A^+(\vec{y}'^j_\perp)|p\rangle e^{-ix^i_D p^+ y_i^-} e^{ip^i_\perp\cdot y^i_\perp} \times e^{ix'^j_D p^+ y'^-_j} e^{-ip'^j_\perp \cdot y'^j_\perp} = (2\pi)^2\delta^2(\vec{p}^{\,i}_\perp - \vec{p}'^{\,j}_\perp)\int d^2y_\perp e^{-ix^i_D p^+(y^-_i - y'^-_j)} \times e^{ip_\perp\cdot y_\perp}\langle p|A^+(\vec{y}_\perp/2)A^+(-\vec{y}_\perp/2)|p\rangle, \quad (67)$$

where $y_\perp$ is the transverse gap between the two gluon insertions and $p_\perp = (p^i_\perp + p^j_\perp)/2$ (and where we have ignored the longitudinal positions and color indices on the vector potentials). The physics of this equation is essentially the transverse translation symmetry of the two-gluon correlator in a very large nucleus. One will note that the two-dimensional delta function over the transverse momenta has removed an integration over the transverse area of the nucleus, thus reducing the overall A enhancement that may be obtained. This is then used to equate the transverse momenta emanating from the two-gluon insertions in the amplitude and complex conjugate amplitude. This also simplifies the longitudinal phase factors, which now depend solely on the difference of the longitudinal positions of the two gluon insertions. To simplify further discussion, we consider the specific case where there are n gluon insertions in the amplitude and $m = n + 2\Delta n$ insertions in the complex conjugate (where Δn is a positive integer). Under the assumptions of restricting each nucleon state to be acted upon by only two gluon operators, we find that $n + \Delta n$ nucleons are involved. The n path-ordered gluon insertions in the amplitude are matched up by a series of path-ordered insertions in the complex conjugate amplitude on the same set of nucleons. The remaining Δn nucleons with two gluon insertions each may be distributed between the n nucleons. The equating of transverse momenta emanating from and being injected into these Δn nucleons insists that they impart vanishing net transverse momentum to the propagating quark. As such, the transverse momenta emanating from such nucleons does not contribute to the overall transverse broadening of the quark and so does not have any appearance in the overall transverse momentum delta function. Thus the overall transverse momentum δ function only contains n different transverse momentum variables from $p^1_\perp$ to $p^n_\perp$.

One now invokes the collinear approximation in expanding the hard part as a Taylor expansion in transverse momenta around the origin $p^i_\perp \to 0$. In the case of a cut with $m = n + 2\Delta n$ there are $2n$ gluon insertions that produce transverse momentum broadening and, as a result, as many derivatives, which involve n different transverse momenta. All terms of the form

$$\prod_{i=1}^{p}\frac{1}{2}\frac{\partial^2}{\partial p^{i\,\alpha}_\perp \partial p^{i\,\beta}_\perp} H|_{p_\perp=0}\, p^{i\,\alpha}_\perp p^{i\,\beta}_\perp,$$

where, $p < n$, yield gauge corrections for the contributions with $2p$ gluon insertions and as many derivatives [32]. The genuine $2n$ twist correction at this order has $p = n$. Expanding to this order, one obtains the generic term

$$\prod_{i=1}^{n}\frac{1}{2}\frac{\partial^2}{\partial p^{i\,\alpha}_\perp \partial p^{i\,\beta}_\perp}\; H\big(p^+, q^-, p^i_\perp,\big)\big|_{p^i_\perp=0} \times p^{i\,\alpha}_\perp p^{i\,\beta}_\perp A^+_{a_i}\big(y_i^-, y^i_\perp/2\big)A^+_{a'_j}\big(y'^-_j, -y^i_\perp/2\big), \quad (68)$$

where we have assumed the result of Eq. (67) and reintroduced the color indices and longitudinal locations.

Using integration by parts over the transverse distance $y^i_\perp$ one may convert the product $p^{i\,\alpha}_\perp A^+_{a_i}(y^-_i, y^i_\perp/2) \to \frac{1}{2}\partial^\alpha_\perp A^+_{a_i}(y^-_i, y^i_\perp/2)$. In the extreme collinear limit, in the presence of a hard scale such that g is small, one may make the approximation

$$\partial_\perp A^+_a \simeq F_{a\,\perp}^{\;+}, \quad (69)$$

where, $F_{a\,\perp}^{\;+}$ represents the gluon field strength. Carrying this out consistently on the two-gluon operator in the nucleon state of Eq. (67) and ignoring derivatives of the field strength $[\partial^\alpha F^{+\beta} \sim g(m^2)g^{\alpha\beta}j^+ \to 0]$ we obtain

$$\int d^2y_\perp p^{i\,\alpha}_\perp p^{i\,\beta}_\perp e^{ip_\perp\cdot y_\perp}\langle p|A^+(\vec{y}_\perp/2)A^+(-\vec{y}_\perp/2)|p\rangle = \int d^2y_\perp e^{ip_\perp\cdot y_\perp}\frac{1}{2}\langle p|F^{+\alpha}(\vec{y}_\perp/2)F^{+\beta}(-\vec{y}_\perp/2)|p\rangle = \int d^2y_\perp e^{ip_\perp\cdot y_\perp}\frac{-g^{\alpha\beta}_\perp}{4}\langle p|F^{+\rho}(\vec{y}_\perp/2)F^{+}_{\rho,}(-\vec{y}_\perp/2)|p\rangle. \quad (70)$$

In the last line of this equation we have averaged over the spins in the two field strength expectations in the nucleon state with the constraint that the operator being evaluated in the nucleon be a spin singlet. The nucleon states are always assumed to be spin singlets or in spin-averaged states.

With the derivative expansion (at vanishing transverse momenta $p^i_\perp \to 0$) imposed on the hard part H, it no longer has any functional dependence on the transverse momenta. The integrations over the transverse momenta may now be included completely into the soft part. As in the case of transverse broadening [17], the action of the transverse momentum derivatives on the phase factors will not be considered. All such derivatives necessarily extract a factor of spatial separation y_j (with $0 < j < n, m$) as

$$\frac{\partial}{\partial p^j_\perp}e^{-ix_{Dj}p^+y^-_j} = -iy^-_j\frac{\partial x_{Dj}}{\partial p^j_\perp}p^+e^{-ix_{Dj}p^+y^-_j}. \quad (71)$$

These result in spatial moments of the two-gluon matrix elements such as $\langle p|F^{+\rho}(y^-_j, y^j_\perp)y^-_j F_{\rho,}^{\;+}(0)|p\rangle$, which will be ignored in this effort.

The action of the $2n$ derivatives is thus restricted to the overall transverse-momentum-conserving delta function and the transverse momentum factors of Eq. (63). The remnant transverse momentum dependence lies in the soft part, which contains the matrix elements of the gluon field strengths, and δ functions, which equate pairs of transverse momentum. We point out that, although we are considering cases where $n = m - 2\Delta n$, the final results for $n = m + 2\Delta n$ are identical. The entire structure of the term with $n + m$ gluon insertions and $2n$ transverse derivatives may be expressed as

$$\mathcal{O}^{n,m} = \sum_{p,q}^{n,m} \mathcal{O}^{n,m}_{p,q}$$

$$= \int \frac{d^2 l_\perp dy}{(2\pi)^3 y} \frac{d^2 l_{q\perp}}{(2\pi)^2} \mathfrak{D}y^- \mathfrak{D}y'^- \mathfrak{D}p_\perp \mathfrak{D}p'_\perp \frac{P_\gamma(y)}{y(2p^+q^-)^2} e^{-i(x_B+x_L)p^+y_0^-} \left\{ \prod_{j=1}^{n} \frac{\partial^2}{\partial p_{j\perp}^{\alpha_j} \partial p_{j\perp}^{\beta_j}} \right\}$$

$$\times \left[\left\{ \prod_{i=1}^{n} \theta(y_i^- - y_{i-1}^-) e^{-ix_{Di}p^+y_i^-} \right\} \left\{ \prod_{k=1}^{m} \theta(y'^-_k - y'^-_{k-1}) e^{ix_{Dk}p^+y'^-_k} \right\} \frac{l_\perp^2}{x_L^2} \left\{ 1 + y \sum_{i=1}^{n} \frac{p_{i\perp} \cdot l_\perp}{l_\perp^2} e^{-ix_Lp^+(y_i^- - y_0^-)} \right.\right.$$

$$\left.\left. + y \sum_{k=1}^{m} \frac{p'_{k\perp} \cdot l_\perp}{l_\perp^2} e^{ix_Lp^+y'^-_k} \right\} \delta^2 \left(l_\perp + l_{q\perp} - \sum_{q=0}^{n} p_{q\perp} \right) \right] C^A_{p_1,\ldots,p_{n-\Delta n}} \, {}^{n-1}P_{\Delta n}$$

$$\times \left\{ \prod_{i}^{n} (2\pi)^2 \delta^2 (p^i_\perp - p'^i_\perp) \int d^2 y^i_\perp \langle p_i | F^{+\mu_i} F^+_{\mu_i} | p_i \rangle e^{i\bar{p}^i_\perp \cdot y^i_\perp} \frac{(-g_\perp^{\alpha_i\beta_i})}{8} \right\} \left\{ \prod_{l=0,2,4,\ldots}^{2\Delta n}{}' (2\pi)^2 \delta^2 (p'^l_\perp + p'^{l+1}_\perp) \right.$$

$$\left. \times \int d^2 y'^l_\perp \langle p_l | A^+(y'^-_l, y^l_\perp/2) A^+(y'^-_{l+1}, -y^l_\perp/2) | p_l \rangle e^{i\delta p'^l_\perp \cdot y^l_\perp} \right\} \qquad (72)$$

(where we also approximate $1/(1-y)^2 \sim 1 + 2y$). In this equation, ${}^{n-1}P_{\Delta n}$ represents the number of permutations of Δn pairs of consecutive gluon insertions in $n - 1$ locations. The *barred* transverse momenta $\bar{p}_\perp = (p_\perp + p'_\perp)/2$, whereas $\delta p'_{l\perp} = p'_{l\perp} - p'_{l+1\perp}$ represent the consecutive gluon pair insertions with two gluons per nucleon. The transverse momentum δ functions in the soft part that correspond to the second type of insertions require that the momentum brought in by one such gluon is taken out immediately by the other. As a result, such insertions play no role in the double differential transverse momentum distribution being calculated and may be ignored. They however provide unitarity corrections to the total cross section. Alternatively, one may state that such insertions induce two exactly opposing currents in close proximity and thus produce a vanishing net transverse broadening and completely destructive radiative contributions. In the remainder of this article we will consider only symmetric contributions, where $n = m$ or $\Delta n = 0$.

In the symmetric case, one may carry out $p'^k_\perp$ integrations using the transverse momentum δ functions in the soft part. As a result, there is a pairwise equality between the transverse momentum brought in by the gluon insertions on the left- and right-hand sides of the cuts. Using integration by parts one may also replace $\partial/\partial p'^k_\perp \to -\partial/\partial p^i_\perp$. By setting $n = m$ in Eq. (72), the action of the derivatives $\partial/\partial p_{i\perp}$ (where $1 < i < n$) on the transverse momentum dependence in the hard part may be simplified as

$$\prod_{i=1}^{n} \left(-g_\perp^{\alpha_i\beta_i} \right) \frac{\partial}{\partial p^{i\,\alpha_i}_\perp} \frac{\partial}{\partial p^{i\,\beta_i}_\perp} \left[\delta^2 \left(l_\perp + l_{q\perp} - \sum_{i=0}^{n} p^i_\perp \right) \right.$$
$$\times \frac{e^{-ix_Lp^+y_0^-}}{x_L^2} \left\{ 1 + y \sum_{p=1}^{n} \frac{p^p_\perp \cdot l_\perp}{l_\perp^2} \left(e^{-ix_Lp^+(y_p^- - y_0^-)} \right.\right.$$
$$\left.\left.\left. + \, e^{ix_Lp^+y'^-_p} \right) \right\} \right], \qquad (73)$$

where the transverse derivatives may act entirely on the transverse momentum δ function or one or two derivatives may act on the terms within the curly brackets above and the remaining may act on the δ function. The $p^i_\perp$ derivatives acting on the transverse momentum δ function may be replaced as

$$\frac{\partial}{\partial p^j_\perp} \delta^2 \left(l_\perp + l_{q_\perp} - \sum_{i=1}^{n} p_{i_\perp} \right)$$
$$= -\frac{\partial}{\partial l_{q,\perp}} \delta^2 \left(l_\perp + l_{q_\perp} - \sum_{i=1}^{n} p_{i_\perp} \right). \qquad (74)$$

By using this equation, the action of the $2n$ derivatives on the combination of the two-dimensional transverse-momentum-dependent δ function and the phase factor $\sum_{p,q} f_p g_q$ followed by an imposition of the limit of very small transverse momenta $p_\perp^i \to 0$ may be simply expressed as

$$\prod_i \frac{\partial^2}{\partial {p_\perp^i}^2} \delta^2(\vec{l}_\perp + \vec{l}_{q\perp} - \vec{K}_\perp) \sum_{p,q} f_p g_q \Bigg|_{p_\perp=0}$$
$$= \frac{e^{-ix_L p^+ y_0^-}}{x_L^2} l_\perp^2 \Bigg[\left(\nabla^2_{l_{q\perp}}\right)^n - y \sum_{p=1}^{n} \left(e^{-ix_L p^+ (y_p^- - y_0^-)}\right.$$
$$\left. + e^{ix_L p^+ y'^-_p}\right) \frac{l_\perp \cdot \nabla_{l_{q\perp}}}{l_\perp^2} \left(\nabla^2_{l_{q\perp}}\right)^{n-1} \Bigg] \delta^2(\vec{l}_\perp + \vec{l}_{q\perp} - \vec{K}_\perp). \tag{75}$$

The action of the derivatives on the transverse δ function or on the factors of $p_\perp^i$ that appear in $f_p g_q$ are the only nonvanishing contributions from Eq. (57). The longitudinal integrals, owing to color confinement, yield the requirement that the longitudinal locations of the two gluons that act on the same nucleon state be in close proximity. The factor of δy_p^- represents the small gap between the longitudinal positions of the gluon insertions in a single nucleon. One now tries to identify the most length enhanced term by isolating the maximum number of unconstrained dy^- integrals. Note that, because of the assumption of short-distance color correlation (ignoring color and spin indices),

$$\int dy^- dy'^- \langle p|F(y^-)F(y'^-)|p\rangle \simeq \int dy^- \langle FF\rangle y_c^-, \tag{76}$$

where $\langle FF\rangle$ is the gluon expectation at the mean location $y^- \pm y_c^-$ and y_c^- represents the color correlation length in the medium. In a nucleus, this is equivalent to the confining distance, whereas in a quark gluon plasma it would be related to the Debye length.

Each such integral yields a factor of $L^- \sim A^{1/3}$ from the unconstrained y^- integration. Equating the pairs of transverse momenta that appear in each two-gluon correlation, as well as using the relation between the longitudinal momenta from the θ functions in Eq. (57), requires that the largest length enhancement arises from the terms where the gluon correlations are built up in a mirror symmetric fashion (i.e., where the gluon insertion at y^i is contracted with that at y'^i). One may now let the transverse momenta $p_\perp^i$ (for all i) in the hard part tend to zero and integrate over the remaining $p_\perp^i$ integrals in the soft part, setting the corresponding transverse distances between the gluon insertions in a nucleon to zero (i.e., $y_\perp^i \to 0$).

One may average colors of the quark and gluon field operators,

$$\langle p|F^a F^b|p\rangle = \frac{\delta^{ab}}{\left(N_c^2 - 1\right)} \langle p|F^a F^a|p\rangle \tag{77}$$

$$\langle p|\bar{\psi}_i \gamma^+ \psi_j|p\rangle = \frac{\delta_{ij}}{N_c} \langle p|\bar{\psi}\gamma^+\psi|p\rangle. \tag{78}$$

This reduces the overall trace over color factors to

$$\frac{1}{N_c\left(N_c^2-1\right)^n} \mathbf{Tr}\left[\prod_{i=1}^{n} t^{a_i} \prod_{j=n}^{1} t^{a_j}\right] = \frac{C_F^n}{\left(N_c^2-1\right)^n} = \frac{1}{(2N_c)^n}. \tag{79}$$

The remaining n longitudinal position integrals for the gluon insertions may be simplified as

$$\int \prod_{i=1}^{n} dy_i^- \theta(y_i^- - y_{i-1}^-) = \frac{1}{n!} \int \prod_{i=1}^{n} dy_i^-. \tag{80}$$

By invoking these simplifications, the leading length enhanced contribution at order $2n$ to the term $\mathcal{O}^{nn}$ is obtained as

$$\mathcal{O}^{n,n} = \frac{\alpha_{\rm em}}{2\pi} \int \frac{dl_\perp^2 dy}{l_\perp^2} d^2 l_{q\perp} P_\gamma(y) e^{-i(x_B + x_L)p^+ y_0^-}$$
$$\times C^A_{p_0,\ldots,p^n} \Bigg\{ \frac{\left(\bar{D}L^- \nabla^2_{l_{q\perp}}\right)^n}{n!} - y[\bar{E}^+(x_L) + \bar{E}^-(x_L)]$$
$$\times \frac{l_\perp \cdot \nabla_{l_{q\perp}}}{l_\perp^2} \frac{\left(\bar{D}L^- \nabla^2_{l_{q\perp}}\right)^{n-1}}{(n-1)!} \Bigg\} \delta^2(\vec{l}_\perp + \vec{l}_{q\perp}). \tag{81}$$

In this equation, the function $\bar{D}$ represents the following expectation value in a nucleon state:

$$\bar{D} = \frac{\pi^2 \alpha_s}{2N_c} \int \frac{dy^-}{2\pi} \langle p|F^{a+\alpha}(y^-) F^{a}{}_{\alpha,}{}^{+}(0)|p\rangle, \tag{82}$$

Although no limits have been specified in the integration over longitudinal separation y^-, it should be understood that because of the assumption of a short-distance color correlation length in the medium [as in Eq. (76)], the integral in this equation only receives strength form the region with $|y^-| \leqslant y_c$. The *nuclear* functions $\bar{E}^\pm$ represent the following expectation value in the nuclear state:

$$\bar{E}^\pm(x_L) = \frac{\pi^2\alpha_s}{2N_c} \int \frac{dy^- dy_p^-}{2\pi} e^{\pm i x_L p^+ (y_p^- - y_0^-)}$$
$$\times c_{\rm OF} \langle p|F^{a+\alpha}(y_p^- + y^-) F^{a}{}_{\alpha,}{}^{+}(y_p^-)|p\rangle. \tag{83}$$

In the above equation $C_{\rm OF}$ is a coefficient (where ‘OF’ stands for Off-Forward) specific to the E-functions and is described in the next paragraph. As in the case of the correlator $\bar{D}$, the integration over the longitudinal separation y^- is limited by the color correlation length in the medium or the confining distance in a nucleus. The integration over the mean location of the two gluon field strength insertions (i.e., y_p^-) has its upper bound constrained only by the size of the medium L (in the case of a nucleus $L \sim A^{1/3}$). The lower bound of y_p^-, in the case of E^+, is given by y^-: the location of the $\bar{\psi}$ operator insertion in Eq. (65). In the case of E^-, the lower bound is $y_p^- = 0$, given by the location of the ψ operator in Eq. (65). It should be pointed out the two lower bounds are separated by at most one unit of the color correlation length (the size of a nucleon in the case of DIS on a large nucleus) and, as such, this small distinction will be ignored in what follows.

Although the two nucleon states on either side of this equation are denoted as carrying the same momentum p, this may not strictly be the case. Throughout this calculation, the hard scattering piece, which includes the scattering of the initial virtual photon with the incoming quark (including the nucleon that contained the quark), was factored out [see Eq. (22)]. In the case of the first term in Eq. (81), which represents the case that the photon is radiated immediately after the hard scattering, such a factorization along with the deconvolution of the nucleus into nucleon states is well justified as there is no longitudinal momentum that is picked up in the multiple scattering that leads to the transverse broadening. In the case that the photon is radiated at a later scattering, the nucleon state that is struck by the hard virtual photon and the nucleon state where the struck quark radiates the final outgoing photon may exchange longitudinal momentum differently between them in the amplitude and in the complex conjugate. In such a case, these two nucleon states represent the convolution of two generalized parton distribution (GPD) functions [35], where the excess momentum from one such distribution is balanced by the other. The momentum correlation between these two states, no doubt, depends on the distance y_p^- between them. The overall constant $c_{\rm OF}$ in Eq. (83) accounts for both the average momentum correlation in a large nucleus and corrections resulting from the approximation of replacing a generalized parton distribution by a regular diagonal parton distribution. A phenomenological discussion on $c_{\rm OF}$ and its value in large nuclei may be found in Ref. [36]. In the next section, a re-summation of all such contributions of arbitrary order n will be carried out.

V. RE-SUMMATION AND SOFT PHOTON PRODUCTION FROM MULTIPLE SCATTERING

In the preceding section, the length-enhanced contributions of twist $2n$ to the soft photon double differential rate were extracted. It turned out that the photon rate at twist $2n$ (including the leading and next-to-leading contributions in y) depended on two different correlation functions $\bar{D}$ and $\bar{E}$ evaluated in the nuclear state, as well as on the multinucleon combinatorial factor $C^A_{p_0,\ldots,p_n}$. In the case where terms such as these are not very small compared to the leading twist contribution, all such terms need to be re-summed.

To sum over all n, this last factor has to be simplified. In a sense, re-summation requires that this coefficient have a formally multiplicative structure. In the preceding section, a model of the nucleus as a weakly interacting homogeneous gas of nucleons was used. The formal expression of this assumption is hidden within the dimensionful parameter $C^A_{p_0,\ldots,p^n}$. The precise evaluation of such combinatorial coefficients is rather complicated, even for the case of next-to-leading twist [36]. From general dimensional arguments, in the case of noninteracting nucleons, this factor may be approximated as

$$C^A_{p_0,\ldots,p^n} \simeq C^A_p \left(\frac{\rho}{2p^+}\right)^n, \tag{84}$$

where ρ is the nucleon density inside the nucleus and $1/2p^+$ originates in the normalization of the nucleon state. The remaining unknown coefficient C^A_p is now considered to be independent of the order n and is included in the leading twist hadronic tensor of Eq. (7). This decomposition is also identical to that used in the case of transverse broadening. It should be pointed out that there may still be an extra normalization factor for each nucleon and this may be included with the factor of $\rho/2p^+$. We will ignore any such normalization factors, as they are not essential to the re-summation. It should however be pointed out that the correlation between the nucleons off which the hard parton scatters and the original nucleon struck by the hard virtual photon is different from that between the original nucleon and that where the outgoing photon is emitted. As previously mentioned, these constitute off-forward distributions where there is momentum shared between the two nucleon states.

The decomposition of the combinatorial factor allows for the definition of the new quantities

$$D = \frac{\rho}{2p^+}\bar{D} = \frac{\pi^2\alpha_s}{2N_c}\rho\int\frac{dy^-}{2\pi 2p^+}\langle p|F^{a+\alpha}F^{a}{}_{\alpha,}{}^{+}|p\rangle \tag{85}$$

and, similarly,

$$E^{\pm}(x_L) = \frac{\rho}{2p^+}\bar{E}^{\pm}. \tag{86}$$

Note that D is exactly the diffusion tensor for the transverse broadening experienced by a hard parton traversing the nucleus without radiation. This is directly proportional to the well-known jet transport parameter $\hat{q}$ [17,37]:

$$\hat{q} = \frac{2\langle l_\perp^2\rangle L^-}{L^-} = 8D. \tag{87}$$

The new quantities $E^{\pm}$ may be related to D as

$$E^{\pm}(x_L) = \int dy_p^- c_{\rm OF} e^{\pm i x_L p^+(y_p^- - y_0^-)} D(y_p^-). \tag{88}$$

It should be pointed out that D is a local quantity evaluated at particular location y_p and could very well change with time in a nonstatic medium, whereas the E's are intergrated over the space-time path of the jet as it traverses the medium.

In the following, we will only present the re-summation of the quantity $\mathcal{O}^{n,n}$, that is,

$$\mathcal{O} = \sum_{n=0}^{\infty}\mathcal{O}^{n,n}. \tag{89}$$

The re-summed differential hadronic tensor and the differential cross section may be easily obtained by incorporation of this quantity into Eqs. (4) and (7). The final-state operator matrix element for a parton undergoing $1 < n < \infty$ scatterings may be re-expressed as

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{O}^{n,n} &= \frac{\alpha_{\rm em}}{2\pi}\int\frac{dl_\perp^2 dy}{l_\perp^2}d^2l_{q\perp}P_\gamma(y)e^{-i(x_B+x_L)p^+y_0^-}\\ &\quad\times\sum_{n=0}^{\infty}\left[\left\{\frac{\left(DL^-\nabla^2_{l_{q_\perp}}\right)^n}{n!}\right\}\delta^2(\vec{l}_\perp+\vec{l}_{q\perp})\right.\\ &\quad - c_p\left\{y\left[E^+(x_L)+E^-(x_L)\right]\frac{l_\perp\cdot\nabla_{l_{q_\perp}}}{l_\perp^2}\right\}\\ &\quad\left.\times\frac{\left(DL^-\nabla^2_{l_{q_\perp}}\right)^{n-1}}{(n-1)!}\delta^2(\vec{l}_\perp+\vec{l}_{q\perp})\right].\end{aligned} \tag{90}$$

In this equation, it is understood that the second term in the square bracket does not receive contributions from the case of $n = 0$. The reader may immediately verify that for the case of $n = 0$, which corresponds to the case of no scattering, the first term by itself reproduces the result of Eq. (16) for the case of $y \to 0$. The new coefficient c_p accounts for the weak correlation between the nucleon struck by the hard virtual photon and the nucleon from which the outgoing photon is radiated. In principle, c_p may depend on the shared momentum fraction x_L.

Both the terms in this equation may be re-summed with the observation that the sum

$$\phi(l_{q_\perp}, L^-) = \sum_{n=0}^{\infty} \frac{\left(DL^- \nabla^2_{l_{q_\perp}}\right)^n}{n!} \delta^2(\vec{l}_\perp + \vec{l}_{q\perp})$$

obeys the diffusion equation

$$\frac{\partial}{\partial L^-}\phi(l_{q_\perp}, L^-) = D\nabla^2_{l_{q_\perp}} \phi(l_{q_\perp}, L^-), \tag{91}$$

with the initial condition discerned from Eq. (19) as

$$\phi(L^- = 0, \vec{l}_{q_\perp}) = \delta^2(\vec{l}_{q_\perp} + \vec{l}_\perp). \tag{92}$$

As demonstrated in the case of transverse broadening, contributions such as those from Eq. (72) lead to a unitarization of the cross sections and, as a result, normalize the solutions of this diffusion equation. The general normalized solution to Eq (91) is given as [38]

$$\phi(L^-, \vec{l}_\perp) = \frac{1}{4\pi DL^-} \exp\left\{ -\frac{|\vec{l}_\perp + \vec{l}_{q\perp}|^2}{4DL^-} \right\}. \tag{93}$$

The reader will note that $\phi(L^-, l_\perp)$ reconverts back to the two-dimensional delta function in the limit of $L^- \to 0$ and the solution is unitary in the sense that

$$\frac{\partial}{\partial L^-} \int d^2 l_\perp \phi(l_\perp, L^-) \simeq 0.$$

Substitution of this solution back into Eq. (90) yields the very simple expression for the final-state re-summed operator matrix element $\mathcal{O}$:

$$\begin{aligned}\mathcal{O} = {} & \frac{\alpha_{\rm em}}{2\pi} \int \frac{dl_\perp^2 dy}{l_\perp^2} d^2 l_{q\perp} P_\gamma(y) e^{-i(x_B + x_L)p^+ y_0^-} \\ & \times \left[1 + y c_p \frac{\{E^+(x_L) + E^-(x_L)\}}{2DL^-} \frac{l_\perp^2 + \vec{l}_\perp \cdot \vec{l}_{q\perp}}{l_\perp^2} \right] \\ & \times \phi(L^-, l_{q_\perp}).\end{aligned} \tag{94}$$

Substitution of this final-state operator matrix element into Eq. (7) yields the full differential hadronic tensor and, as a result, the differential cross section for photon production at all twist:

$$\begin{aligned}\frac{dW^{A\mu\nu}}{dy dl_\perp^2 d^2 l_{q\perp}} = {} & C_p^A 2\pi \sum_q Q_q^4 \left(-g_\perp^{\mu\nu}\right) \frac{\alpha_{\rm em}}{2\pi} \frac{P_{q\to q\gamma}(y)}{l_\perp^2} \\ & \times \int \frac{dy_0^-}{2\pi} e^{-i(x_B + x_L)p^+ y_0^-} F_q(y_0^-) \\ & \times \left[1 + y c_p \frac{\{E^+(x_L) + E^-(x_L)\}}{2DL^-} \right. \\ & \left. \times \frac{l_\perp^2 + \vec{l}_\perp \cdot \vec{l}_{q\perp}}{l_\perp^2} \right] \phi(L^-, l_{q_\perp}).\end{aligned} \tag{95}$$

This equation represents the main result of this article. The unintegrated expectation value of the two-quark operator $F(y_0)$ is defined in Eq. (9). The differential spectrum of photons radiated from a hard parton undergoing multiple scattering in the medium has been expressed as a factorized product of the photon splitting function, the two-dimensional transverse momentum distribution of the propagating parton, and a multiplicative factor [factor in square brackets in Eq. (95)], which encodes the phases picked up by the particles as they traverse the medium. It is this factor that has to be evaluated for each path taken by the hard quark in the medium, weighted by the expectation of the two-gluon field strength operator product along the path. The unknown coefficients, C_p^A and c_p, have to be determined by a phenomenological analysis. The decomposition of this equation into quark and gluon structure functions in the nucleus and its associated phenomenological implications will be discussed in an upcoming publication.

VI. CONCLUSIONS AND DISCUSSIONS

The study of the modification of hard jets in dense matter is now approaching a rather sophisticated stage of its development. The wide variety of experimental measurements of the modification of jets and jetlike correlations require the emergence of a single formalism capable of describing the modification of hard jets over a wide range of energies and a variety of different media. In this article, the higher twist mechanism of jet modification is extended in an effort to make it applicable to both the *thin*-medium and *thick*-medium limits. The limits of thick and thin refer to the number of scatterings that a hard parton will encounter on its path through the medium. Whereas previous efforts [12,15] have focused on the modification from a few scatterings, the current article furthers the development of the formalism required to describe the modification from an infinite number of scatterings.

The focus here has been on computing the spectrum of single-photon bremsstrahlung from a propagating parton undergoing multiple scattering in an extended medium. The entire process is cast in a deep-inelastic scattering framework, where a virtual photon (with virtuality Q^2) strikes a hard parton in a nucleon, which is itself situated in a large nucleus. The parton, on its way through the nucleus, scatters multiple times and radiates a real photon. Energy loss from real gluon emission was ignored. Each scattering of the hard parton is suppressed by powers of Q^2, but it is enhanced by the length of the nucleus ($\propto A^{1/3}$) over which the scattering may occur. For large A, an infinite class of such contributions need to be re-summed.

The photon may be radiated from the vicinity of any of the scattering locations. The sum over such contributions leads to the destructive interference for very forward radiation known as the LPM effect. In the extreme case of an on-shell parton entering such a medium and radiating a very soft photon with momentum fraction $y \ll 1$, it is shown that the destructive interference is complete. This is referred to as the deep-LPM limit. In the current effort, the first two corrections in terms of initial virtuality and expansion in powers of y to this limit are computed. The first correction is due to the fact that the initial parton that enters the medium is not on shell and may indeed be very virtual. As a result, it may radiate the photon even without the need for further rescattering. The second contribution originates from not assuming the very soft limit $y \to 0$ for the photon and instead keeping the first leading corrections in y. This is the reason our results differ from the case of no rescattering by a factor proportional to y.

As in the case of transverse momentum broadening, it is demonstrated that the infinite series of power corrections may indeed be re-summed. The result has the simple and intuitive structure of a product of the photon splitting function, the two-dimensional transverse momentum distribution of the propagating parton, and a multiplicative factor, which encodes the phases picked up by the particles as they traverse the medium. As in the case of multiple scattering without radiation, a Gaussian profile in transverse momentum centered around the origin was obtained for the propagating parton [17]. However, the Gaussian obtained in this case was found centered around the momentum $-\vec{l}_\perp$ needed to balance the transverse momentum of the photon.

The results of such a computation are crucial to the understanding of both the near-side and away-side correlations between a trigger hadron and an associated photon or vice versa. The phenomenological comparisons with experiment and other theoretical approaches, based on the expressions calculated here, will appear in a future publication. The derived results also bear considerable relevance as a secondary source of single hard photon production.

An obvious next step is the evaluation of gluon radiation in the same process. Ultimately this might lead the way to a unified description of multiple scattering in the medium and radiative energy loss. Although the photon does not scatter in the medium, its formation incorporates many similar physics issues such as the destructive interference of the LPM effect. Such effects will reappear in the future calculation of gluon radiation and will have to be dealt with in the case of energy loss of hard partons. However, the various scales that will be encountered in that problem as well as the approximation scheme used will be nearly identical to those in the current article.

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