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Mechanical study of a modern yo-yo

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Abstract
This paper presents the study of a modern yo-yo having a centrifugal clutch allowing the free rolling. First, the mechanical parts of the yo-yo are measured, allowing us to determine analytically its velocity according to its height of fall. Then, we are more particularly interested in the centrifugal device constituted by springs and small masses. The physics of this toy is suitable for undergraduate students, illustrating the concepts of dynamics of rigid bodies and of potential energy.

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Having succumbed to the whims of my children who wanted a yo-yo, I have been myself taken with the game, and after some express learning and a little training, I finally settled some questions on the physics of this toy.

The yo-yo question studied here involves a modern yo-yo provided with a centrifugal device which allows us to obtain the free rolling when rotation velocity is high enough. We return later to this device, which will be studied in detail in section 5. Reference [1] presents the elementary physics of the yo-yo with a very good and clear description of the device allowing the *free rolling*, without entering into quantitative details.

Modern yo-yos are easy to find at low prices. The measurements presented here were made with a yo-yo costing about 2 euros, allowing us to sacrifice one without remorse to measure the characteristics of its constituent elements (springs, masses, . . .). Through experience, we note that it is rather difficult to replace the small elements of an unsettled yo-yo.

2. Description—list of notations

The yo-yo used in this study is represented in figure 1. The thread is placed on a ring perfectly adjusted on the central hub of the yo-yo. The device allowing free rolling is made up of four masses and four identical springs (called *A* in figure 1). When the rotation velocity of the

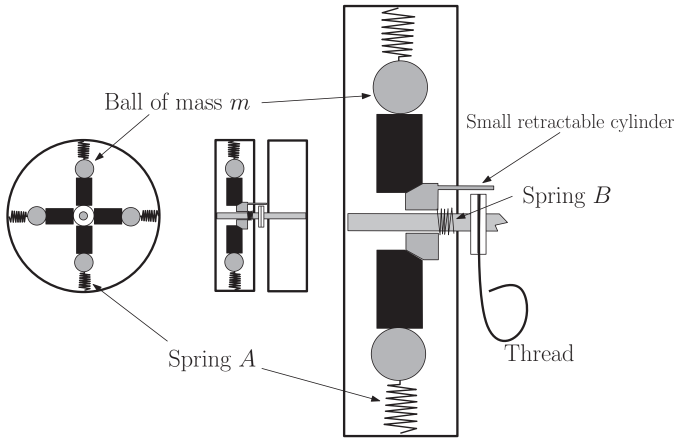

Figure 1. Description of the yo-yo used in this study. On the right-hand side, we have presented a magnification of the central picture.

Table 1. Mechanical parameters used in this study.

$M = (51.30\pm0.05)$ g	Total mass of the yo-yo
$m = (2.30\pm0.05)$ g	Mass of a ball
$r_0 = (12.0\pm0.5)$ mm	Radius for the position at rest of the balls relative to the axis of rotation
$R_0 = (24.0\pm0.5)$ mm	Inside radius of the yo-yo
$R_e = (28.0\pm0.5)$ mm	Outside radius of the yo-yo
$k = (444 \pm 14)$N/m	Spring constant
$l_0 = (10.0\pm0.5)$ mm	Length of the springs at rest
$J_{\text{tot}} = (5.4 \pm 0.4)10^{-6}$ kg.m^2	Inertia moment of the yo-yo relative to its rotation axis
$g = 9.81$ m/s^2	Acceleration of gravity
$\epsilon = (1.0\pm0.2)$ mm	Diameter of the thread of negligible mass
$r_m = (4.0\pm0.5)$ mm	Diameter of the central hub
$l = (1.00\pm0.01)$ m	Total length of the thread
J	Inertia moment of the yo-yo without the masses m
$\dot{\varphi}$	Angular velocity of the yo-yo

yo-yo reaches a critical value, the centrifugal force compresses the springs A and the central part is freed by the action of the spring B (see figure 1), which produces a retreat of the cylindrical point in the inside of the yo-yo. Then, the thread cannot wind any more around the central hub, and the yo-yo is in free rolling.

The yo-yo and its centrifugal device are described in a way slightly different from that in [1], but the physical principle remains the same. The study presented in this paper can make the object of an experiment in the practical mechanics class more fun than traditional experiments on free fall, springs and the mechanics of the solid bodies [2].

We will use the notation in table 1 to describe the physics of the yo-yo. The numerical values of some of these quantities are measured (see section 3).

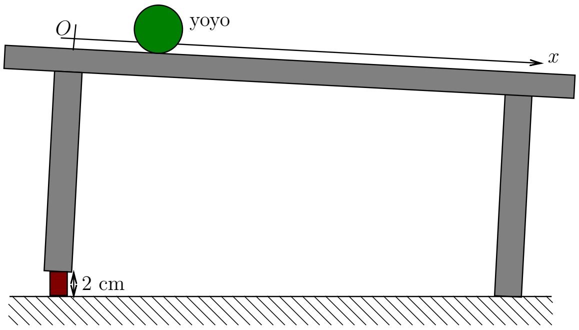

Figure 2. Experiment to measure the moment of inertia of the yo-yo.

3. Measurement of the mechanical parameters of the yo-yo

This section is dedicated to the measurement of the mechanical quantities useful in describing the movement of the yo-yo. The masses of the various elements ($M, m, \ldots$) were determined by weighing. We verified that the thread and the springs had negligible masses in relation to the total mass of the yo-yo.

3.1. Moment of inertia

The moment of inertia J_{tot} around the rotation axis was measured by studying the rolling motion of the yo-yo on an inclined plane (table of 1.50 m in length inclined using two pieces of wood 2 cm high (figure 2)).

By neglecting the friction, the total mechanical energy E of the yo-yo is conserved, and can be written according to the distance x crossed by the yo-yo, time t and the angle α of the inclined plane with respect to the horizontal:

$$E = \frac{1}{2}\left(M + \frac{J_{\text{tot}}}{R_e^2}\right)\dot{x}^2 - Mg\sin\alpha\ x. \tag{1}$$

Taking the time derivative of equation (1) we obtain a motion equation of the yo-yo that it is easy to integrate to obtain the trajectory $x(t)$:

$$x(t) = \frac{1}{2}\frac{Mg\sin\alpha}{\left(M + J_{\text{tot}}/R_e^2\right)}t^2. \tag{2}$$

Equation (2) allows us to obtain the expression for J_{tot}:

$$J_{\text{tot}} = MR_e^2\left(\frac{g\sin\alpha\ t^2}{2x(t)} - 1\right). \tag{3}$$

The time t corresponding to the length $x = 1.5$ m was measured using a chronometer. We obtain $t = (5.1 \pm 0.2)$ s, and after application of relation (3) we obtain $J_{\text{tot}} = (5.4\ 10^{-6} \pm 0.4)$ 10^{-6} kg m^2. Let us note that the main error on this measurement is given by the response time of the manipulator.

3.2. Spring constants

By hypothesis, springs are supposed to follow Hooke's law. The spring constant of springs A (see figure 1) is measured by determining the elongation produced by a known strength (weight of masses of 100–400 g, 100 g stepped). To increase the precision of the measure, the four springs were mounted in series, which corresponds to an equivalent spring of constant $k/4$. The elongation measured as a function of the applied strength was treated using a least-squares procedure. We obtain $k = (444 \pm 14)$ N m^{-1}.

4. The free fall of the yo-yo

We can take two approaches to the yo-yo in free fall (i.e. the yo-yo dropped without initial velocity, keeping its thread in the hand); the first one will be dedicated to an 'academic yo-yo' for which the thread has a null diameter, and the second will deal with a real yo-yo with a thread of diameter ϵ.

The vertical axis being positively directed downwards and the yo-yo being subjected only to conservative forces (we neglect friction), its total mechanical energy E is the sum of the kinetic energies of translation and rotation, and the potential energy of gravity $U = -Mgz$, with the arbitrary choice $U = 0$ for $z = 0$:

$$E = \tfrac{1}{2}M\dot{z}^2 + \tfrac{1}{2}J_{\text{tot}}\dot{\varphi}^2 - Mgz, \tag{4}$$

where $\dot{z}$ is the velocity of the mass centre of the yo-yo and $\dot{\varphi}$ its angular velocity around its mass centre. In equation (4), we have supposed that the moment of inertia J_{tot} was constant: this is experimentally justified because the rotational velocity of the yo-yo is too low to action the centrifugal clutch and the four balls of mass m stay in their rest position. When the yo-yo is dropped without initial velocity in $z = 0$, its total mechanical energy is null.

4.1. The 'academic yo-yo'

The academic yo–yo has a thread with a null diameter. We then have the obvious relation $z = r_m\varphi$ between z and φ. In this case, the expression for E becomes

$$E = \frac{1}{2}M\dot{z}^2 + \frac{1}{2}J_{\text{tot}}\frac{\dot{z}^2}{r_m^2} - Mgz = 0.$$

We can deduce the expression of the velocity of the yo-yo as a function of z:

$$\dot{z} = \sqrt{\frac{2Mgz}{M + J_{\text{tot}}/r_m^2}}. \tag{5}$$

4.2. The real yo-yo

The real yo-yo has a thread with a diameter ϵ, and we can write the variation of the radius ρ of the central hub (whose value depends on the number of turns of the thread) as a function of the angle φ (figure 3).

We have

$$d\rho = -\frac{\epsilon}{2\pi}d\varphi. \tag{6}$$

Noting that $\rho = R_0$ when $\varphi = 0$ in $z = 0$, we have

$$\rho = R_0 - \frac{\epsilon}{2\pi}\varphi. \tag{7}$$

In polar coordinates (ρ, φ), the length $\mathrm{d}z$ of the unrolled thread is given by $\mathrm{d}z = \sqrt{\rho^2(\mathrm{d}\varphi)^2 + (\mathrm{d}\rho)^2}$, so, taking into account relations (6) and (7),

$$\mathrm{d}z = \sqrt{\left(R_0 - \frac{\epsilon}{2\pi}\varphi\right)^2 + \frac{\epsilon^2}{4\pi^2}}\,\mathrm{d}\varphi. \tag{8}$$

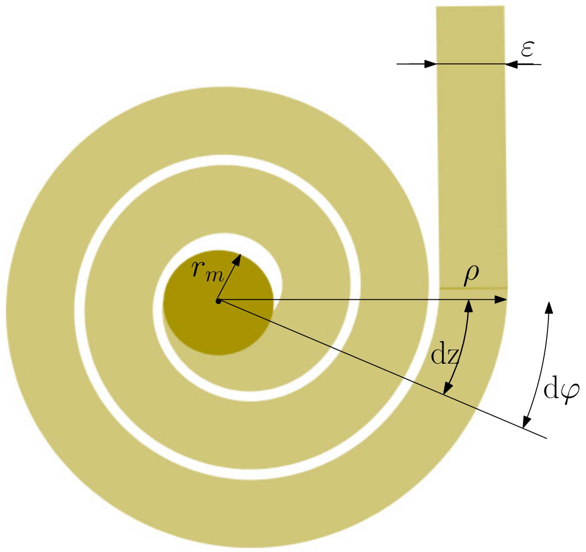

Figure 3. Variation of the radius of the central hub of the real yo-yo as a function of the rotation angle φ.

The term $\epsilon^2/4\pi^2$ being negligible in relation to the other term, we can admit with very good precision,

$$dz = \left(R_0 - \frac{\epsilon}{2\pi}\varphi\right)d\varphi,$$

and after integration,

$$z(\varphi) = R_0\varphi - \frac{\epsilon}{4\pi}\varphi^2. \tag{9}$$

The expression of φ as a function of z is obtained by searching the root (in φ) of equation (9). Considering the roots allowing us to have $\varphi = 0$ for $z = 0$, we have

$$\varphi = \frac{2\pi}{\epsilon}\left(R_0 - \sqrt{R_0^2 - \frac{\epsilon z}{\pi}}\right). \tag{10}$$

From the expression of φ as a function of z, we can operate as above, and from the expression of E, obtain the fall velocity of the yo-yo. We obtain

$$\dot{z} = \sqrt{\frac{2Mgz}{M + J_{\text{tot}}/(R_0^2 - \epsilon z/\pi)}} \tag{11}$$

and

$$\dot{\varphi} = \frac{\dot{z}}{\sqrt{R_0^2 - \epsilon z/\pi}}. \tag{12}$$

For $z = l$, corresponding to the total length of the thread and a rotation angle equal to $\varphi_{\max}$, we have the following relations:

$$r_m = R_0 - \frac{\epsilon}{2\pi}\varphi_{\max}$$

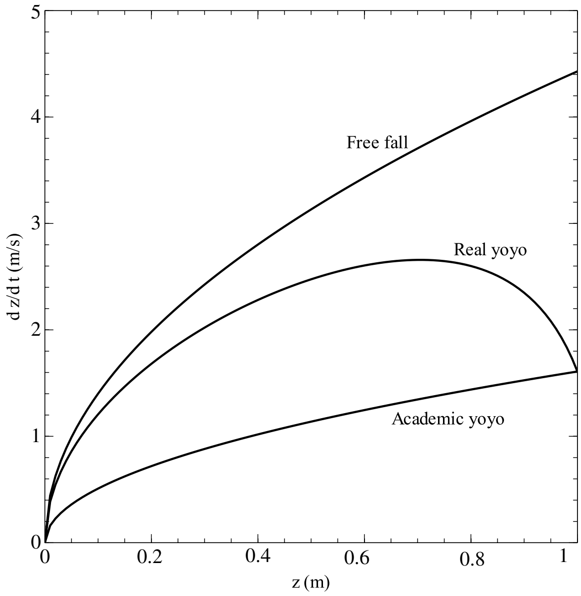

Figure 4. Plot of the linear velocity ($\dot{z}$) of an 'academic yo-yo' and a real yo-yo as a function of the falling distance. The free-fall motion has also been plotted for comparison.

$$\varphi_{\max} = \frac{2\pi}{\epsilon}\left(R_0 - \sqrt{R_0^2 - \frac{\epsilon l}{\pi}}\right).$$

We can see that $r_m = \sqrt{R_0^2 - \epsilon l/\pi}$. When $z = l$, the two expressions for $\dot{z}$ are the same for the real yo-yo and the academic yo-yo. Relation (11) can be transformed as

$$\dot{z} = \sqrt{\frac{2Mgz}{M + J_{\mathrm{tot}}\big/\left(r_m^2 + \epsilon(l - z)\big/\pi\right)}}. \tag{13}$$

The plots giving the velocity $\dot{z}$ and the angular velocity $\dot{\varphi}$ as a function of the falling length are presented in figures 4 and 5, respectively. The plots are in agreement with the plots presented in [1]. We can note that the real yo-yo has a greater velocity than the academic yo-yo. For the real yo-yo, the velocity has a maximum value. Lastly, the final velocities are the same. The angular velocity of the yo-yo at the end of its free fall is 400 rad s^{-1} (3822 rpm).

4.3. Falling times

Knowing the expressions for the velocities $\dot{z}$, it is easy to calculate time Δt taken by the yo-yo dropped without initial speed to reach the full stop of its free fall. We have $\int_0^l \mathrm{d}z/f(z) = \Delta t$ where the functions $f(z)$ are the expressions given by equations (5) and (13). In the case of the academic yo-yo, the integration is immediate and gives $\Delta t = 1.2$ s. For the real yo-yo, the integration is made numerically (Simpson's method) and gives $\Delta t = 0.56$ s. These values were compared to measures of falling times with a thread of diameter ϵ and with a very thin thread. For the academic yo-yo (thin thread) we obtain $\Delta t = 1.5$ s, and for the real yo-yo, we measure $\Delta t = 0.72$ s. Even if the calculated values are a little bit different, we find

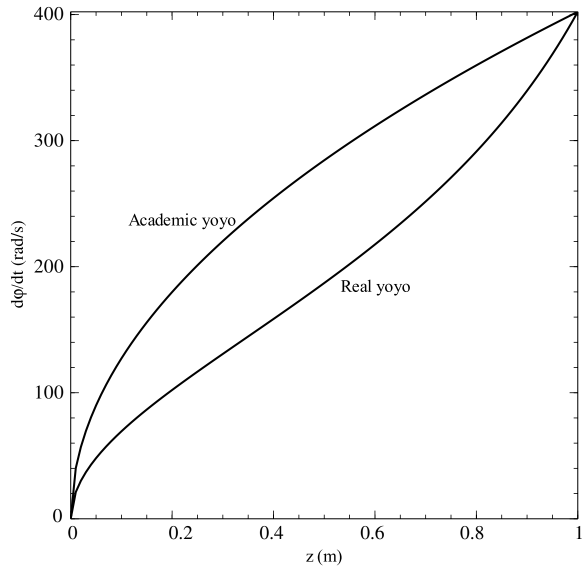

Figure 5. Plot of the rotational velocity ($\dot{\varphi}$) of an 'academic yo-yo' and a real yo-yo as a function of the falling distance.

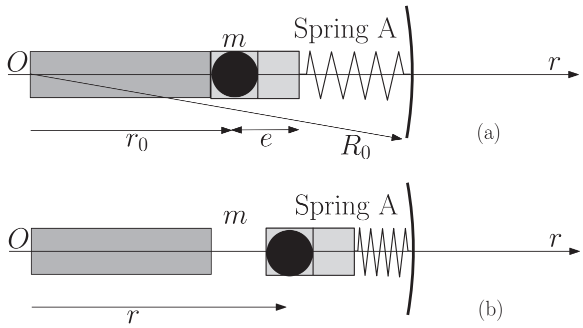

Figure 6. Mechanical model of the centrifugal device allowing free rolling of the yo-yo. Figure (a) corresponds to the case when the rotational velocity is null or lower than a critical value, and figure (b) to the case when the rotational velocity is high (see the text).

experimentally a ratio of 2 between the falling times, which shows the importance of thread diameter in the motion of the yo-yo.

5. The free-rolling device

The device allowing free rolling of the yo-yo when the angular velocity is high enough can be easily modelled using a Lagrangian approach. The notation used is that of figure 6.

Let us only consider the movement of rotation of the yo-yo at the angular velocity $\dot{\varphi}$, at the centre of mass reference. If J is the moment of inertia of the yo-yo without the four masses m, we can write the kinetic energy as

$$T = \frac{1}{2}J\dot{\varphi}^2 + 4\left(\frac{1}{2}mr^2\dot{\varphi}^2\right) + 4\left(\frac{1}{2}m\dot{r}^2\right) \tag{14}$$

where r is the distance between the position of the masses m and the centre of the yo-yo. Let us note that the masses m are put in a small plastic box of dimension $e = 6$ mm (see figure 6). The potential energy U of the system is the sum of the potential energies of the springs and of the potential energy of the weight U_{weight} of the masses m; it is clear that U_{weight} must only be taken into account when the yo-yo spins in a vertical plane:

$$U = 4\left(\frac{1}{2}k[(R_0 - e - r) - l_0]^2\right) + U_{\text{weight}}. \tag{15}$$

The masses m being small, the term U_{weight} can be ignored in relation to the potential energy of the springs. The Lagrangian $L = T - U$ of the system is

$$L = \frac{1}{2}J\dot{\varphi}^2 + 2m\dot{r}^2 + 2mr^2\dot{\varphi}^2 - 2k\,[(R_0 - e - r) - l_0]^2$$

and can be written in the form $L = T^* - U^*$ with

$$\begin{aligned} T^* &= \frac{1}{2}J\dot{\varphi}^2 + 2m\dot{r}^2 \\ U^* &= -2mr^2\dot{\varphi}^2 + 2k[(R_0 - e - r) - l_0]^2. \end{aligned} \tag{16}$$

The system is then reduced to a planar rotator with a moment of inertia J subjected to the effective potential energy U^*. The analysis of U^* allows us to plot the potential energy of the springs: it is a parabola with positive concavity, null for the value $r = R_0 - e - l_0$ (with $r < r_0$ because the springs are mounted in compression), and the centrifugal potential energy that is a parabola with negative concavity.

Figure 7 presents the plot of the potential energy of the springs, and the plot of the effective potential energy U^* evaluated for different values of the angular velocity (from 200 rad s^{-1} up to 500 rad s^{-1}). In figure 7, we have also indicated the lower limit r_0 corresponding to the rest position of the masses m when $\dot{\varphi} = 0$.

5.1. Analysis of the potential curves

To have a motion of the masses m from their position r_0 under the effect of the centrifugal force, it is necessary for U^* to have a minimum value for r_0. Considering the null value of the first derivative of relation (16) as a function of r in r_0 allows us to find the expression of a critical angular velocity $\dot{\varphi}_{\text{cr}}$:

$$\dot{\varphi}_{\text{cr}} = \sqrt{\frac{k[l_0 - (R_0 - e - r_0)]}{mr_0}} \tag{17}$$

whose numerical value is $\dot{\varphi}_{\text{cr}} = 254$ rad s^{-1}.

For rotational velocities higher than $\dot{\varphi}_{\text{cr}}$, the point describing the mechanical system is located at the bottom of the potential well (corresponding to a stable position), only if it is located between r_0 and R_0. It is obvious that the compression of the spring will never be complete, with a reduction in its length to zero, and the stable position will never be located in $r = R_0$.

To obtain the free rolling, the masses must move to a length of 3 mm from the position r_0, leading to a retractation of the small cylinder inside the body of the yo-yo. Considering that the bottom of the potential well is located in $r = r_0 + 5$ mm, we obtain the critical rotational velocity for the free rolling $\dot{\varphi}_{rl} = 320$ rad s^{-1} (3057 rpm).

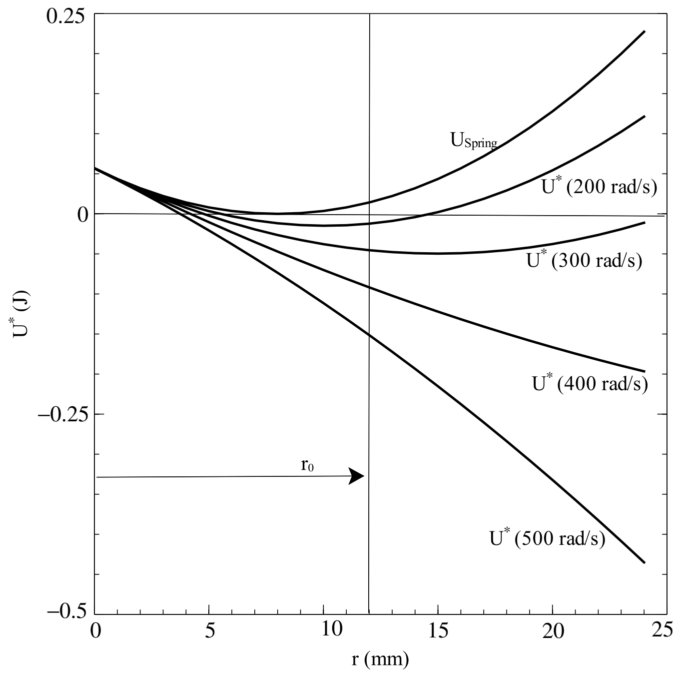

Figure 7. Plot of the potential energy of the springs, and of the effective potential energy U^* evaluated for different rotational velocities of the yo-yo.

5.2. Measurement of the the critical rotational velocity for the free rolling $\dot{\varphi}_{rl}$

To directly determine the critical rotational velocity for the free rolling $\dot{\varphi}_{rl}$, we have fixed the axis of the yo-yo in the chuck of a drill with a speed variator. A thin thread fixed on the yo-yo intercepts a light curtain sensor connected to a frequency counter. A first test realized with the drill with a maximum velocity of 2500 rpm did not trigger the centrifugal device. We have used a drill for the scale model with a maximum velocity of 100 00 rpm. The critical rotational velocity needed to trigger the free rolling device is $\dot{\varphi}_{rl} = 470$ rad s^{-1} (about 4500 rpm).

This value is higher than the theoretical value found above; this difference can be explained by our assumptions: all the frictions between the different parts of the yo-yo have been neglected. However, the high critical value found is quite surprising.

This value is in agrement with the fact that for the yo-yo in free fall for which the maximum rotation velocity is 400 rad s^{-1}, there is no activation of the free-rolling device. The yo-yo player knows this, and throws its yo-yo, giving it an initial kinetic energy, in order to increase the value of the total mechanical energy of the system. By considering the equations presented in section 4, it is easy to check that the rotation velocity of the yo-yo is increased. When the yo-yo is in free roll, its rotation velocity decreases because of the effects of friction forces; when the critical rotation velocity is reached, the small cylinder of the free-rolling device goes out of the body of the yo-yo and the yo-yo then goes upwards.

6. Conclusion

This study of the modern yo-yo with a centrifugal device using springs and masses is very rich, and deals with different fundamental concepts of classical mechanics [3]. Other configurations of centrifugal devices may be found, but they are all built using masses and springs, and use the centrifugal force effect to trigger the free rolling of the yo-yo.

Acknowledgments

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