CONTROL SYSTEMS: NEW APPROACHES TO ANALYSIS AND DESIGN

A Dissertation

by

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ABSTRACT

This dissertation deals with two *open* problems in control theory. The first problem concerns the synthesis of fixed structure controllers for Linear Time Invariant (LTI) systems. The problem of synthesizing fixed structure/order controllers has practical importance when simplicity, hardware limitations, or reliability in the implementation of a controller dictates a low order of stabilization. A new method is proposed to simplify the calculation of the set of fixed structure stabilizing controllers for any given plant. The method makes use of computational algebraic geometry techniques and sign-definite decomposition method. Although designing a stabilizing controller of a fixed structure is important, in many practical applications it is also desirable to control the transient response of the closed loop system. This dissertation proposes a novel approach to approximate the set of stabilizing Proportional-Integral-Derivative (PID) controllers guaranteeing transient response specifications. Such desirable set of PID controllers can be constructed upon an application of Widder's theorem and Markov-Lukacs representation of non-negative polynomials.

The second problem explored in this dissertation handles the design and control of linear systems without requiring the knowledge of the mathematical model of the system and directly from a small set of measurements, processed appropriately. The traditional approach to deal with the analysis and control of complex systems has been to describe them mathematically with sets of algebraic or differential equations. The objective of the proposed approach is to determine the design variables directly from a small set of measurements. In particular, it will be shown that the functional dependency of any system variable on any set of system design parameters can be

determined by a small number of measurements. Once the functional dependency is obtained, it can be used to extract the values of the design parameters.

DEDICATION

To my mother Shamsozzoha,

my father Mohammad Farid,

and my brother Mehrdad,

for their love and support.

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1. INTRODUCTION

This dissertation is organized in 9 chapters. Chapters 2 and 3 are dedicated to the study of fixed structure controller synthesis based on mathematical model of the plant. The problem of synthesizing stabilizing controllers of a fixed structure arises in many practical applications and has been open for about five decades. Without any restriction on the structure of the controller, the problem of controller design can be handled through various techniques of modern control theory. However, the constraint on structure yields non-convex constraints and the corresponding set of stabilizing controller parameters is often non-convex and at times, is disconnected. While the attempts on solving this problem have been numerous, in this dissertation, we will restrict our study to a subset of these bodies of work and focus on methods that deal with approximating the set of fixed structure controllers using algebraic techniques such as elimination theory. The problem of deciding the existence of a stabilization with a fixed structure/order controller reduces to the problem of deciding the feasibility of a system of polynomial inequalities and this can be shown to be decidable using a plethora of techniques such as Quantifier Elimination (QE) [1] or using Groebner bases [2]. In [3], a method is proposed using sign-definite condition and a special Quantifier Elimination (QE) technique to design robust controllers of a fixed structure. Recently, the problem of optimal decentralized controller synthesis using Groebner bases has been studied in [4, 5]. Parametric control design techniques are well suited for approximating the set of stabilizing controllers and earlier work concerning these techniques can be found in [6]. Recently, in [7] a systematic method is provided for constructing the set of PID controllers. PID controllers are fixed structure controllers that are widely being used in industrial applications. This work exploits the specific structure of PID controllers. A systematic method for arbitrarily tight inner and outer approximation of the set of stabilizing controllers of a fixed order for a single-input or a single-output system is presented in [8]. In [9], the properties of positive polynomials have been used to obtain a convex inner approximation of the set of stabilizing controllers in the space of controller parameters.

In Chapter 2, we plan to construct an approximation of the set of stabilizing controllers for Linear Time Invariant (LTI) control systems. We use elimination theory in polynomial rings, Groebner bases and sign-definite decomposition method [10, 11] to construct an inner approximation of the set of stabilizing controllers [12]. Chapter 3 deals with the problem of controlling transient response of a system which is a fundamental and open problem with a lot of practical applications. Typical transient response specifications require the response of the closed loop system to lie within a specified envelope. For instance, a transient specification is that the overshoot in the response of a system to a unit step input be less than a specified amount. The problem of controlling transient response of a system involves the synthesis of a controller that guarantees such transient specifications. The problem of achieving non-overshooting step response has been studied in [13, 14, 15, 16, 17, 18, 19, 20, 21]. For the discrete-time systems a non-overshooting step response can be achieved based on the results provided in [22]. Applying the results in [22] to the class of continuous-time Linear Time Invariant (LTI) systems results in controllers with irrational transfer functions. However, in [23], it is shown that a non-overshooting response can be achieved by proper, rational two parameter controllers. For the class of discrete-time LTI systems, the problem of controlling the transient response is studied in [24]. In [25] the problem of achieving transient specifications for LTI systems using fixed order and PID controllers is considered. In this dissertation, we show that the transient response specifications can be guaranteed using the non-negativity of polynomials whose coefficients are polynomial functions of the controller parameters. We present a novel method to construct an outer approximation for the set of PID controllers guaranteeing stability and transient response specifications [26, 27]. We also provide a technique to tighten such outer approximation by which we will be able to refine any outer approximation arbitrarily.

Chapters 4 though 9 of this dissertation explore a new approach to the design and control of linear systems without requiring a mathematical model of the system which instead can determine the design parameters directly from a small set of measurements. In many fields of science and engineering such as control, signal processing, communication networks, genomics, one has to deal with increasingly complex systems. In many complex systems, one may isolate a few set of design variables interacting with the complex system whose values are to be controlled or determined. For the class of linear systems, this dissertation proposes a new approach which is able to determine the design parameters without the knowledge of the mathematical model of the system. The proposed approach is developed based on the properties of parameterized solution of linear system of equations and uses a specific type of parameter dependence. It shows that a functional dependency between system variables and design parameters holds which can be determined by solving a set of linear equations obtained by taking few measurements. This mapping function can be inverted to impose performance specifications and extract the design variables. Some recent related results on this problem are as follows. In [28], a data-based method is proposed for stability analysis of discrete-time LTI systems. The method presented in [29] uses data space to find the control input which minimizes a quadratic performance index. A procedure based on Qualitative Robust Control (QRC) technique is proposed in [30] to synthesize controllers from a qualitative model of the plant. Quantitative Feedback Theory (QFT) is used in [31] to impose robustness bounds at different frequencies which are related to loop shaping. Three term controllers can be designed directly from frequency response data [32]. A Bode plot characterization of all stabilizing controllers is given in [33], and followed in [34] to synthesize stabilizing controllers of fixed structure.

Chapter 4 presents the mathematical preliminaries required to develop such measurement based approach. It describes some basic results on the parametrized solution of linear equations. These results will be used in Chapter 5 to show how measurements can be directly used to design and control linear DC circuits [35, 36]. It will be described that any circuit variable, such as current or power, can be expressed as a function of any set of circuit design variables, such as resistors, gyrators and sources, and this function can be obtained by taking few measurements. The extension of these results to linear AC circuits will be discussed in Chapter 6. In Chapter 7, an application of this new measurement based approach to linear mechanical systems, truss structures and linear hydraulic networks will be studied [37]. Chapter 8 explores the synthesis of fixed structure controllers, satisfying closed loop frequency response specifications, based on the proposed method. An extension of this method to adaptive control and biological systems can be found in [38] and [39], respectively. Chapter 9 concentrates on the class linear systems containing real parameters with interval uncertainties and presents an extremal result. The problem of analyzing and controlling interval systems is important for practical applications and has been open for the last few decades. Several results concerning robustness analysis of systems with real parametric uncertainty can be found in the early works presented in [6, 40, 41, 42, 43]. Kharitonov's theorem [44], later generalized in [45], provided a means to evaluate the stability of an interval plant by testing a finite number of polynomials for stability. An extension of the Kharitonov's theorem, known as the edge theorem, provided in [46], states that the stability of a polytope of polynomials is equivalent to the stability of its one-dimensional exposed edge polynomials. The sign-definite decomposition method can be used to decide the robust positivity (or negativity) of a polynomial over a box of uncertain parameters by evaluating the sign of the decomposed polynomials at the vertices of the box [10, 11]. Also, recent results on the robust control of linear systems are provided in [7]. In this dissertation, we show that if in an unknown linear system the uncertain parameters appear with rank one dependency in the system characteristic matrix, then the extremal values of any system variable over a box in the parameter space occur at the vertices of that box [47]. This enables us to evaluate the performance of an unknown interval system over a box of uncertain parameters by checking the respective performance index at the vertices. Finally, Chapter 10 summarizes concluding remarks.

2. FIXED STRUCTURE CONTROLLER SYNTHESIS USING GROEBNER BASES AND SIGN-DEFINITE DECOMPOSITION

This chapter presents a new method for computing stabilizing fixed structure controllers using Groebner bases and sign-definite decomposition. An application of Routh-Hurwitz stability condition results in a system of polynomial inequalities that must be satisfied by the parameters of any stabilizing controller. We use positive slack variables to convert the original system of polynomial inequalities to a system of polynomial equations. This system of equations can be simplified using elimination theory and Groebner bases which finally facilitates the computation of the set of of stabilizing controllers using the sing-definite decomposition method.

Section 2.1 introduces some mathematical preliminaries and proposes our method. In Section 2.2, we provide some illustrative examples to show how this method can be applied to a given problem. Finally, we summarize our conclusions in Section 2.3.

2.1 Main Results

2.1.1 Routh-Hurwitz Criterion and Groebner Bases

Consider a unity feedback control system with a known plant transfer function $P(s) = \frac{N_p(s)}{D_p(s)}$ and a controller transfer function $C(s) = \frac{N_c(s,\mathbf{K})}{D_c(s,\mathbf{K})}$, where $\mathbf{K} = [k_1, k_2, \ldots, k_m]^T$ is the vector of controller design parameters. The set of stabilizing controllers is all vectors $\mathbf{K}$ for which the closed loop characteristic polynomial is Hurwitz stable. The closed loop characteristic polynomial can be expressed as

$$\delta(s,\mathbf{K}) = N_p(s)N_c(s,\mathbf{K}) + D_p(s)D_c(s,\mathbf{K}), \tag{2.1}$$

or in the following general form

$$\delta(s, \mathbf{K}) = a_n(\mathbf{K})s^n + a_{n-1}(\mathbf{K})s^{n-1} + \cdots + a_0(\mathbf{K}) = 0, \tag{2.2}$$

where $a_n(\mathbf{K}) \neq 0$. From Routh-Hurwitz stability criterion, the number of RHP roots of the closed loop characteristic polynomial is equal to the number of changes in sign of the elements of the first column of Routh-Hurwitz table. This means that

$$f_0(\mathbf{K}) > 0\ ,\ f_1(\mathbf{K}) > 0\ , \ldots ,\ f_n(\mathbf{K}) > 0, \tag{2.3}$$

where f_i's represent the elements in the first column of the Routh-Hurwitz table and define the boundaries of the stability region in the space of the controller parameters. In general, (2.3) is a set of multivariate polynomial inequalities in terms of the controller design parameters $\mathbf{K} = [k_1, k_2, \ldots, k_m]$, which is hard to solve and in some cases practically impossible. The set of inequalities in (2.3) can be converted to set of equalities by introducing strictly positive slack variables $s_0, s_1, \ldots, s_n$, so that

$$\begin{aligned} h_0(\mathbf{K}, s_0) &= f_0(\mathbf{K}) - s_0 = 0, \\ h_1(\mathbf{K}, s_1) &= f_1(\mathbf{K}) - s_1 = 0, \\ &\vdots \\ h_n(\mathbf{K}, s_n) &= f_n(\mathbf{K}) - s_n = 0. \end{aligned} \tag{2.4}$$

In this set, the slack variables are dependent variables and expressed in terms of the independent variables which are the controller parameters. The controller parameters are coupled in (2.4). An approach that can decouple these parameters, expressing them in terms of the slack variables, is now desirable. Such a decoupling can be accomplished using elimination theory on polynomial rings and Groebner bases [2].

If we were to choose a *Lexicographic ordering* $k_m > k_{m-1} > k_{m-2} > \cdots > k_1 > s_n > s_{n-1} > \cdots > s_1 > s_0$, then the variable k_m is eliminated first, followed by k_{m-1} and so on. Thus, the resulting reduced set of polynomial equations will involve one less variable every time a variable is eliminated as is the case in a Gaussian elimination, i.e. the system of polynomial equations will be triangular. Let the system of polynomial equations after the elimination process be

$$\begin{aligned} g_0(\mathbf{S}) &= 0, \\ g_1(\mathbf{S}) &= 0, \\ &\vdots \\ g_p(\mathbf{S}) &= 0, \end{aligned} \tag{2.5}$$

$$\begin{aligned} g_{p+1}(\mathbf{K}, \mathbf{S}) &= 0, \\ g_{p+2}(\mathbf{K}, \mathbf{S}) &= 0, \\ &\vdots \\ g_t(\mathbf{K}, \mathbf{S}) &= 0. \end{aligned} \tag{2.6}$$

We observe that equations (2.5) may not have the triangular structure because the number of variables is $m + n + 1$, including the slack variables, and is more than the number of polynomial equations, which are only $n + 1$ in number. By specifying the Lexicographic ordering in the above mentioned manner, we want to treat m of the slack variables to be independent variables, which is given by equations (2.5), and the rest of them, including the controller parameters, can be determined for any given value of the m independent variables through the system of equations in (2.5) and the triangular system of equations in (2.6). It is possible that for the same set of m independent variables, there may be more than one set of control parameters. The equations (2.5), referred to as the slack constraints, define an algebraic variety in the space of the slack variables. Since the slack variables are strictly positive, this variety is confined to the first orthant of the space of the slack variables. All the vectors $\mathbf{S} = [s_0, s_1, \ldots, s_n]^T$, where $s_i > 0, \; i = 0, 1, \ldots, n$, satisfying the equations (2.5) represent the stability region in the space of the slack variables. The computation of the stability region, via the sign-definite decomposition, is simpler in the space of the slack variables than in the space of the controller parameters because the slack variables take positive values; however, the controller parameters take positive and negative values. For a specific vector $\mathbf{S} = [s_0, s_1, \ldots, s_n]^T$ satisfying the equations (2.5), one can sequentially find $k_1, k_2, \ldots, k_m$, using equations (2.6). Therefore the procedure described above can be summarized as

1. Write the Routh-Hurwitz stability inequalities for the closed loop characteristic polynomial,

2. Convert inequalities to equalities by introducing slack variables,

3. Find the Groebner bases of the system of polynomials obtained above, which involve controller parameters and slack variables, using Lexicographic ordering.

It should be noted that the necessary condition for the Routh-Hurwitz stability criterion is that all the coefficients of the characteristic polynomial must be non-zero and must have the same sign. These conditions can be embedded into the set of equations (2.4). This will induce more slack variables which will increase the number of slack constraints in (2.5), but may simplify the equations (2.5) and (2.6).

2.1.2 Sign-definite Decomposition in Determining Positivity (Negativity) of Polynomials

A method has been proposed in [10], and followed in [11], to determine the robust positivity (negativity) of a real function $f(\mathbf{x})$ as the real vector $\mathbf{x}$ varies over a box $X \in R^n$ by only checking a finite number of specially constructed points. Let $f(\mathbf{x})$ with $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be a real function of $\mathbf{x}$ and consider the problem of determining if $f(\mathbf{x})$ is positive over the box

$$X = \{\mathbf{x} : x_i^- \leq x_i \leq x_i^+, \text{ for all } i\}.$$

The function $f(\mathbf{x})$ can be decomposed as

$$f(\mathbf{x}) = f^+(\mathbf{x}) - f^-(\mathbf{x}), \tag{2.7}$$

where $f^+(\mathbf{x}) \geq 0$, $f^-(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in X$. Now, assume that x_i's take only positive values. Defining $\mathbf{x}^+$ and $\mathbf{x}^-$ as

$$\mathbf{x}^+ = (x_1^+, x_2^+, \ldots, x_n^+),$$
$$\mathbf{x}^- = (x_1^-, x_2^-, \ldots, x_n^-),$$

such that

$$\begin{aligned} f^+(\mathbf{x}^+) &= \max_{\mathbf{x} \in X} f^+(\mathbf{x}), \\ f^-(\mathbf{x}^+) &= \max_{\mathbf{x} \in X} f^-(\mathbf{x}), \\ f^+(\mathbf{x}^-) &= \min_{\mathbf{x} \in X} f^+(\mathbf{x}), \\ f^-(\mathbf{x}^-) &= \min_{\mathbf{x} \in X} f^-(\mathbf{x}). \end{aligned} \tag{2.8}$$

Therefore

$$
\begin{aligned}
&f^+(\mathbf{x}^-) \le f^+(\mathbf{x}) \le f^+(\mathbf{x}^+), \\
&f^-(\mathbf{x}^-) \le f^-(\mathbf{x}) \le f^-(\mathbf{x}^+).
\end{aligned} \tag{2.9}
$$

Now, consider the rectangle formed by the following four points in the (f^-, f^+) plane

$$
\begin{aligned}
A &= \left(f^-(\mathbf{x}^-), f^+(\mathbf{x}^-)\right), \\
B &= \left(f^-(\mathbf{x}^-), f^+(\mathbf{x}^+)\right), \\
C &= \left(f^-(\mathbf{x}^+), f^+(\mathbf{x}^+)\right), \\
D &= \left(f^-(\mathbf{x}^+), f^+(\mathbf{x}^-)\right).
\end{aligned} \tag{2.10}
$$

It can be shown that for all $\mathbf{x} \in X$ (see Fig. 2.1)

$$
f(\mathbf{x}) \begin{cases} \ge 0, \text{ if } \; f^+(\mathbf{x}^-) - f^-(\mathbf{x}^+) \ge 0, \\ \le 0, \text{ if } \; f^+(\mathbf{x}^+) - f^-(\mathbf{x}^-) \le 0. \end{cases} \tag{2.11}
$$

This relation can be used recursively to construct the robustly positive regions. For more details see [10, 11]. We use (2.11) later to plot the stability region in the space of the (free) slack variables.

2.2 Illustrative Examples

2.2.1 SISO: A Second-order Plant and a First-order Controller

Consider a general second-order plant and a general first-order controller in a unity feedback control system. The corresponding transfer functions for the plant

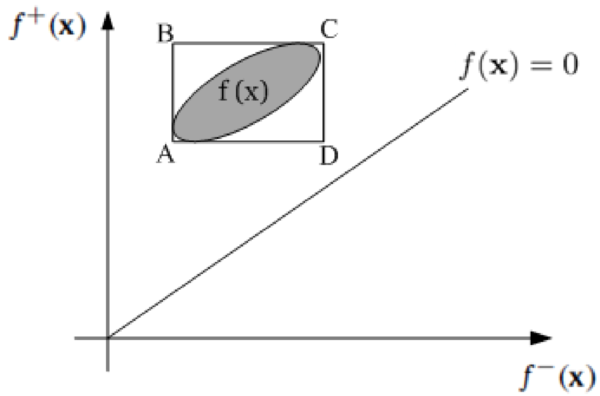

Figure 2.1: Condition for positivity of $f(\mathbf{x})$

and the controller are

$$\begin{aligned} P(s) &= \frac{q_1 s + q_0}{s^2 + p_1 s + p_0}, \\ C(s) &= \frac{k_1 s + k_2}{s + k_3}, \end{aligned} \tag{2.12}$$

where the plant parameters p_0, p_1, q_0, q_1 are known and the controller parameters $\mathbf{K} = [k_1, k_2, k_3]^T$ are unknown. The closed loop characteristic polynomial in this case will be

$$\begin{aligned} \delta(s, \mathbf{K}) &= s^3 + (q_1 k_1 + p_1 + k_3) s^2 \\ &\quad + (p_0 + q_0 k_1 + q_1 k_2 + p_1 k_3) s \\ &\quad + (p_0 k_3 + q_0 k_2). \end{aligned} \tag{2.13}$$

The elements of the first column of the Routh-Hurwitz array must be strictly positive in order to have a stable closed loop system, therefore

$$\begin{aligned}
f_0(\mathbf{K}) &= q_1 k_1 + p_1 + k_3 > 0, \\
f_1(\mathbf{K}) &= q_0 q_1 k_1^2 + p_1 k_3^2 + q_1^2 k_1 k_2 + (p_1 q_1 + q_0) k_1 k_3 \\
&\quad + q_1 k_2 k_3 + (p_1 q_0 + p_0 q_1) k_1 + (p_1 q_1 - q_0) k_2 \\
&\quad + p_1^2 k_3 + p_0 p_1 > 0, \\
f_2(\mathbf{K}) &= p_0 k_3 + q_0 k_2 > 0,
\end{aligned} \tag{2.14}$$

where the term $f_1(\mathbf{K})$ represents only the numerator of the 3rd element in the Routh-Hurwitz array because its denominator, $f_0(\mathbf{K})$, is already assumed to be positive. Also the first element of the array is 1 which is positive and is not included in (2.14). Defining slack variables $s_0 > 0,\ s_1 > 0,\ s_2 > 0$, one can generate h_0, h_1, h_2 as

$$\begin{aligned}
h_0(\mathbf{K}, s_0) &= f_0(\mathbf{K}) - s_0 = 0, \\
h_1(\mathbf{K}, s_1) &= f_1(\mathbf{K}) - s_1 = 0, \\
h_2(\mathbf{K}, s_2) &= f_2(\mathbf{K}) - s_2 = 0.
\end{aligned} \tag{2.15}$$

The Groebner bases of the polynomials in (2.15) with respect to Lexicographic ordering $k_1 > k_2 > k_3 > s_2 > s_1 > s_0$ are

$$\begin{aligned}
g_0(k_3, \mathbf{S}) &= -q_0^2 s_1^2 - q_1^2 s_0 s_1 + q_0 q_1 s_0 \\
&\quad +(q_0^2 p_1 - q_0 p_0 q_1) s_1 + q_0 q_1 s_2 \\
&\quad +(p_0 q_1^2 - q_0 p_1 q_1 + q_0^2) s_1 k_3, \\
g_1(k_2, k_3, \mathbf{S}) &= q_0 k_2 + p_0 k_3 - s_0, \\
g_2(k_1, k_3, \mathbf{S}) &= q_1 k_1 + k_3 - s_1 + p_1. \qquad (2.16)
\end{aligned}$$

None of the above Groebner bases are in terms of only the slack variables, i.e. there is no constraint on choosing slack variables, therefore the entire first orthant in the space of (s_0, s_1, s_2) is the stability region for this example. The set (2.16) can be solved for the controller parameters k_1, k_2, k_3 as

$$k_3 = \frac{q_0^2 s_1^2 + q_1^2 s_0 s_1 - q_0 q_1 (s_0 + s_2) + (q_0 p_0 q_1 - q_0^2 p_1) s_1}{(p_0 q_1^2 - q_0 p_1 q_1 + q_0^2) s_1}, \qquad (2.17)$$

$$\begin{aligned}
k_2 &= \frac{-1}{q_0}(p_0 k_3 - s_0), \qquad (2.18) \\
k_1 &= \frac{-1}{q_1}(p_1 + k_3 - s_1). \qquad (2.19)
\end{aligned}$$

This example shows a special case of our method where there is no restriction on choosing slack variables, i.e. the entire first orthant in the space of the slack variables is the stability region. This is analogous to the pole placement problem where the number of controller parameters is the same as the number of closed loop poles.

2.2.2 SISO: A Third-order Plant and a First-order Controller

In this example we show a case where a constraint on slack variables exists. Consider the following third-order plant and a general first-order controller as

$$
\begin{aligned}
P(s) &= \frac{s^2+s-1}{s^3+2s^2+s-1}, \\
C(s) &= \frac{k_1 s + k_2}{s + k_3},
\end{aligned} \tag{2.20}
$$

where the controller parameters $\mathbf{K} = [k_1, k_2, k_3]^T$ are unknown. The closed loop characteristic polynomial is

$$
\begin{aligned}
\delta(s, \mathbf{K}) &= s^4 + (k_3 + 2 + k_1)s^3 \\
&\quad +(k_1 + k_2 + 1 + 2k_3)s^2 \\
&\quad +(k_2 - k_1 + k_3 - 1)s - k_3 - k_2.
\end{aligned} \tag{2.21}
$$

The Routh-Hurwitz array corresponding to the characteristic polynomial (2.21) can be constructed easily. In this example embedding the positivity of the coefficients of the characteristic polynomial simplifies the Groebner bases equations. Although this increases the number of the slack variables and the slack constraints, the number of the *free slack variables*, introduced later, does not change and therefore the stability region in the space of the free slack variables can still be plotted in a 3-dimensional space. Therefore there are 6 inequalities in this case. Defining strictly positive slack variables $s_0, s_1, \ldots, s_5$, one can construct $h_0, h_1, \ldots, h_5$ as

$$\begin{aligned}
h_0 &= k_3 + 2 + k_1 - s_0 = 0, \\
h_1 &= 3k_1k_3 + 4k_1 + {k_1}^2 + k_2k_3 + k_2 + k_2k_1 \\
&\quad +4k_3 + 3 + 2{k_3}^2 - s_1 = 0, \\
h_2 &= -3 - 7k_1 + 6k_2 + 3k_3 + k_1k_3 - 5{k_1}^2 \\
&\quad +8k_2k_3 + 6k_2k_1 + 6{k_3}^2 + {k_2}^2 + {k_2}^2k_3 \\
&\quad +{k_2}^2k_1 + 4k_2{k_3}^2 - {k_1}^2k_3 + 3k_1{k_3}^2 \\
&\quad +k_2{k_1}^2 + 5k_2k_3k_1 - {k_1}^3 + 3{k_3}^3 - s_2 = 0, \\
h_3 &= -k_3 - k_2 - s_3 = 0, \\
h_4 &= k_2 - k_1 + k_3 - 1 - s_4 = 0, \\
h_5 &= k_1 + k_2 + 1 + 2k_3 - s_5 = 0.
\end{aligned} \tag{2.22}$$

The Groebner bases of the polynomials in (2.22) with respect to Lexicographic ordering $k_1 > k_2 > k_3 > s_2 > s_1 > s_3 > s_0 > s_4 > s_5$ are

$$\begin{aligned}
g_1 &= 1 + s_3 - s_0 + s_5 = 0, \\
g_2 &= -s_5s_0 + s_4 + s_1 = 0, \\
g_3 &= {s_4}^2 - {s_0}^2 - s_5s_4s_0 - s_5{s_0}^2 \\
&\quad +s_2 + {s_0}^3 = 0,
\end{aligned} \tag{2.23}$$

$$\begin{aligned}
g_4 &= 2 - 2s_0 - s_4 + s_5 + k_3 = 0, \\
g_5 &= -3 + 3s_0 + s_4 - 2s_5 + k_2 = 0, \\
g_6 &= s_0 + s_4 - s_5 + k_1 = 0.
\end{aligned} \tag{2.24}$$

Equations (2.24) involve the controller parameters and they are decoupled. These equations can be solved to obtain the controller parameters k_1, k_2 and k_3 as

$$k_3 = -2 + 2s_0 + s_4 - s_5, \tag{2.25}$$

$$k_2 = 3 - 3s_0 - s_4 + 2s_5, \tag{2.26}$$

$$k_1 = -s_0 - s_4 + s_5. \tag{2.27}$$

In this example s_0, s_4 and s_5 are the *free slack variables*. Equations (2.23) can be solved for s_3, s_1 and s_2 respectively as (recall that the slack variables are strictly positive)

$$s_3 = -1 + s_0 - s_5 > 0, \tag{2.28}$$

$$s_1 = s_5 s_0 - s_4 > 0, \tag{2.29}$$

$$s_2 = -{s_4}^2 + {s_0}^2 + s_5 s_4 s_0 + s_5 {s_0}^2 - {s_0}^3 > 0. \tag{2.30}$$

Now, s_1, s_2 and s_3 are the *constrained slack variables* and the inequalities (2.28)-(2.30) define the stability region in the first orthant of the space of the free slack variables s_0, s_4 and s_5. Any vector (s_0, s_4, s_5) satisfying the above inequalities will guarantee the positivity of s_1, s_2 and s_3 and can be mapped into the space of the controller parameters by (2.25)-(2.27). As mentioned earlier, one important advantage of this approach is that the slack variables are positive. This simplifies the computations involving sign-definite decomposition because all the variables are positive and therefore the approximation boxes should be constructed only in the first orthant of the space of the free slack variables; however, on the other hand, applying the sign-definite decomposition directly to the Routh-Hurtwitz inequalities requires consideration of all orthants because the controller parameters can take negative values as well. For this example we define the following polynomials

$$\begin{aligned}
s_3^+ &= s_0, \\
s_3^- &= 1 + s_5, \\
s_1^+ &= s_5 s_0, \\
s_1^- &= s_4, \\
s_2^+ &= s_0^2 + s_5 s_4 s_0 + s_5 s_0^2, \\
s_2^- &= s_4^2 + s_0^3.
\end{aligned} \tag{2.31}$$

Now, each pair of $(s_i^+, s_i^-),\ i = 1, 2, 3$, are treated as $f^+(\mathbf{x}), f^-(\mathbf{x})$, introduced earlier, and the approximation boxes are defined as

$$S = \{\mathbf{s} : s_i^- \leq s_i \leq s_i^+,\ i = 0, 4, 5\}.$$

The stability region defined by (2.28)-(2.30) in the space of the free slack variables is plotted in Fig. 2.2 via the sign-definite decomposition method. Each vector $\mathbf{S} = [s_0, s_4, s_5]^T$ in the plot of Fig. 2.2 corresponds to a vector $\mathbf{K} = [k_1, k_2, k_3]^T$ by (2.25)-(2.27). Fig. 2.3 shows the stability region in the space of the controller parameters (k_1, k_2, k_3).

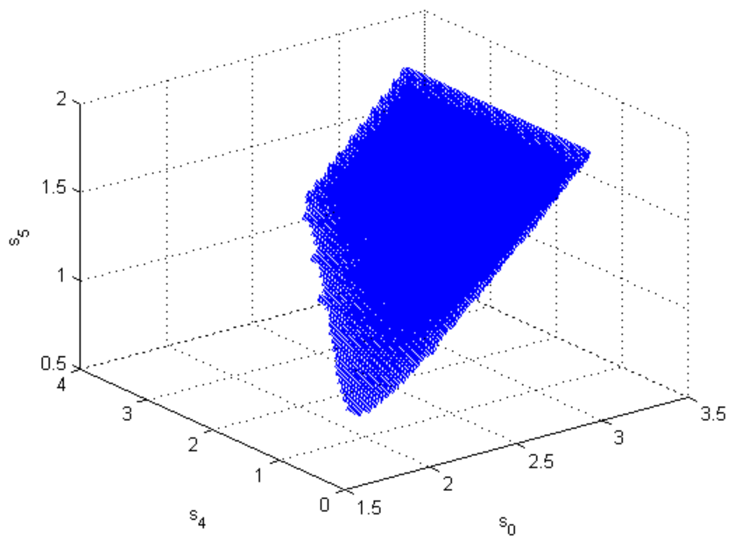

Figure 2.2: The stability region in the space of the free slack variables for the feedback system (2.20)

2.2.3 MIMO: A Feedback Control System

Consider the following characteristic polynomial corresponding to a MIMO feedback system. Here, the controller parameters are k_1 and k_2.

$$\begin{aligned}\delta(s, \mathbf{K}) \quad = \quad & s^4 + (k_1 - 2 + k_2)\, s^3 \\ & + (k_1 k_2 + 2k_2 + k_1 - 3)\, s^2 \\ & + (4 - 5k_2 - 4k_1 + 5k_1 k_2)\, s \\ & + 4 - 6k_2 + 6k_1 k_2 - 4k_1. \end{aligned} \tag{2.32}$$

The stability inequalities from the Routh-Hurwitz array and the coefficients of the characteristic polynomial are

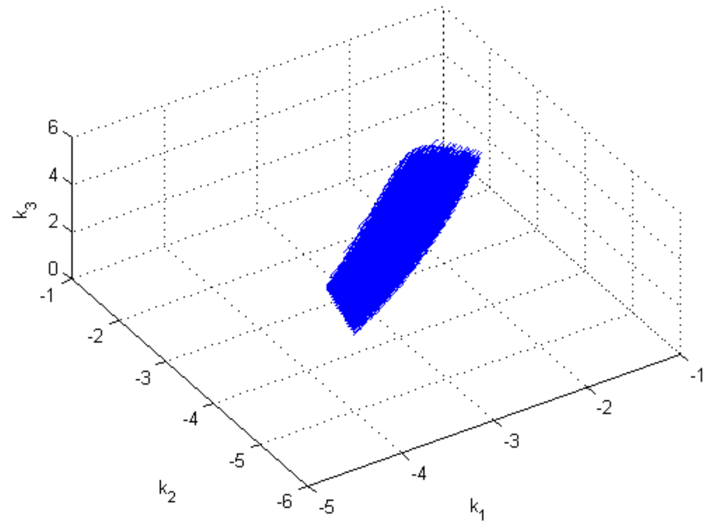

Figure 2.3: The stability region in the space of the controller parameters for the feedback system (2.20)

$$
\begin{aligned}
f_0 &= k_1 - 2 + k_2 > 0, \\
f_1 &= {k_1}^2 k_2 - 4k_1k_2 + k_1{k_2}^2 - 2k_2 \\
&\quad +2{k_2}^2 + {k_1}^2 - k_1 + 2 > 0, \\
f_2 &= -8 + 22k_2 - 65k_1k_2 + 48{k_1}^2k_2 + 46k_1{k_2}^2 \\
&\quad -10{k_2}^2 - 12{k_1}^2 - 41{k_1}^2{k_2}^2 - k_1{k_2}^3 \\
&\quad -5{k_1}^3k_2 + 5{k_1}^3{k_2}^2 + 5{k_1}^2{k_2}^3 \\
&\quad -4{k_2}^3 + 20k_1 > 0, \\
f_3 &= 4 - 6k_2 + 6k_1k_2 - 4k_1 > 0, \\
f_4 &= 4 - 5k_2 - 4k_1 + 5k_1k_2 > 0, \\
f_5 &= k_1k_2 + 2k_2 + k_1 - 3 > 0. \qquad (2.33)
\end{aligned}
$$

Defining strictly positive slack variables $s_0, s_1, \ldots, s_5$, the Groebner bases for this example are

$$\begin{aligned} g_0 &= 20 + 120s_0 + 20s_3 - 56s_5 + 180{s_0}^2 + 28s_0s_3 \\ &\quad +{s_3}^2 - 168s_5s_0 - 12s_5s_3 + 36{s_5}^2 = 0, \\ g_1 &= 2 + 6s_0 - 3s_3 - 2s_5 + 4s_4 = 0, \\ g_2 &= -2 - 6s_0 + 3s_3 + 2s_5 + 4s_1 - 4s_5s_0 = 0, \\ g_3 &= 8 + 68s_0 - 64s_3 - 8s_5 + 192{s_0}^2 - 156s_0s_3 \\ &\quad +40{s_3}^2 - 44s_5s_0 + 60s_5s_3 + 180{s_0}^3 + 100s_3{s_0}^2 \\ &\quad +s_0{s_3}^2 - 60{s_0}^2s_5 - 66s_0s_3s_5 + 72s_2 = 0, \end{aligned} \tag{2.34}$$

$$\begin{aligned} g_4 &= -2 + 8k_2 + 10s_0 + s_3 - 6s_5 = 0, \\ g_5 &= -14 - 18s_0 - s_3 + 6s_5 + 8k_1 = 0. \end{aligned} \tag{2.35}$$

Equations (2.35) can be solved for the controller parameters k_2 and k_1 in terms of the free slack variables s_0, s_3 and s_5. Solution to the first 3 equations in (2.34) for s_4, s_1 and s_2, respectively, yields (recall that the slack variables are strictly positive)

$$s_4 = -\frac{1}{2} - \frac{3}{2}s_0 + \frac{3}{4}s_3 + \frac{1}{2}s_5 > 0, \tag{2.36}$$

$$s_1 = \frac{1}{2} + \frac{3}{2}s_0 - \frac{3}{4}s_3 - \frac{1}{2}s_5 + s_5s_0 > 0, \tag{2.37}$$

$$\begin{aligned} s_2 &= -\frac{1}{9} - \frac{17}{18}s_0 + \frac{8}{9}s_3 \\ &\quad +\frac{1}{9}s_5 - \frac{8}{3}{s_0}^2 + \frac{13}{6}s_0s_3 - \frac{5}{9}{s_3}^2 + \frac{11}{18}s_5s_0 \\ &\quad -\frac{5}{6}s_5s_3 - \frac{5}{2}{s_0}^3 - \frac{25}{18}s_3{s_0}^2 - \frac{1}{72}s_0{s_3}^2 \\ &\quad +\frac{5}{6}{s_0}^2s_5 + \frac{11}{12}s_0s_3s_5 > 0. \end{aligned} \tag{2.38}$$

Equation (2.34), for g_3, involves only the free slack variables s_0, s_3 and s_5, thus is an algebraic variety in the first orthant of the space of the free slack variables. Inequalities (2.36)-(2.38) and the equation for g_3 in (2.34) define the stability region in the space of the free slack variables (s_0, s_3, s_5). Fig. 2.4 shows this stability region plotted using the sign-definite decomposition. The stability region in the space of the controller parameters (k_1, k_2) can be plotted using the equations (2.35) (see Fig. 2.5).

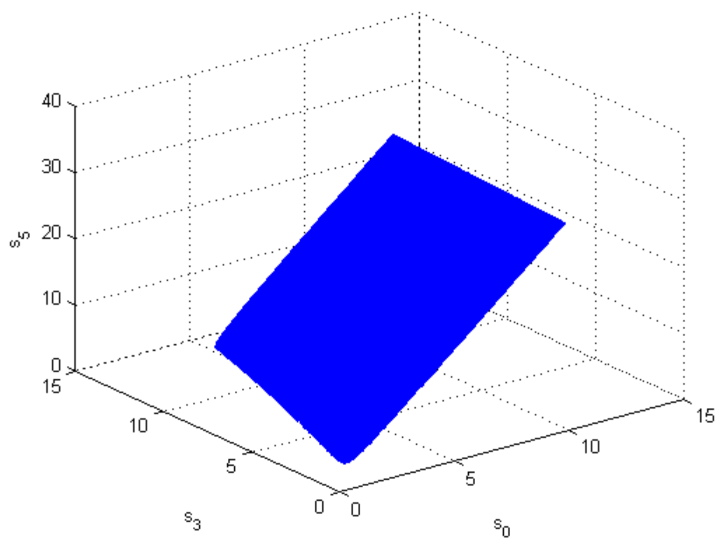

Figure 2.4: The stability region for the MIMO example in the space of the free slack variables

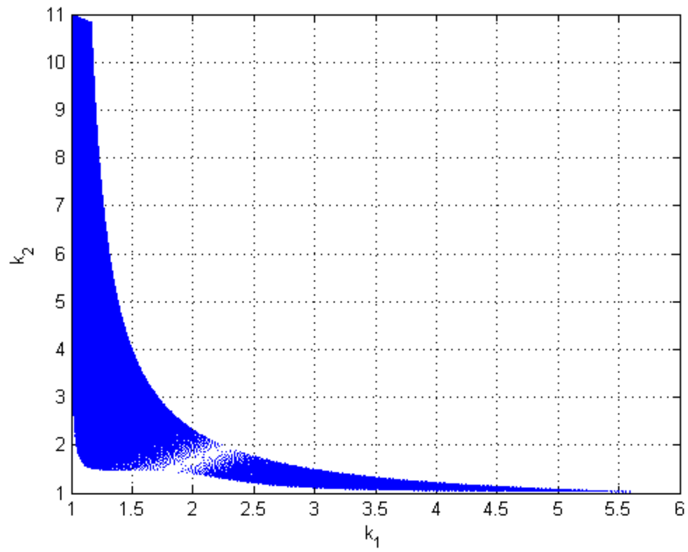

Figure 2.5: The stability region for the MIMO example in the space the controller parameters

2.3 Concluding Remarks

In this chapter we proposed a method to construct the set of stabilizing controllers of fixed structure using strictly positive slack variables. This is accomplished through a systematic use of elimination theory on the Routh-Hurwitz stability inequalities which allows for the computation of controller parameters in a sequential manner. The presence of strictly positive slack variables in the equations simplifies the computations of the stability region via the sign-definite decomposition method. Also by introducing free slack variables and constrained slack variables, we showed that the stability region can be plotted in the space of the free slack variables.

It is also possible to add performance to the problem. The performance requirements can be embedded to the initial set of stability inequalities by additional corresponding polynomial inequalities. In this case the region obtained in the space of the slack variables will satisfy both the stability and the performance of the closed loop system.

3. APPROXIMATING THE SET OF STABILIZING PID CONTROLLERS WITH GUARANTEED TRANSIENT RESPONSE

This chapter presents a new approach to approximating the set of stabilizing continuous-time and discrete-time PID controllers for satisfying a class of transient specifications. A typical transient specification requires the response of a closed loop system be within a specified envelope. We show that this task can be carried out as a problem of guaranteeing the impulse response of appropriate closed loop error transfer functions to be non-negative. The set of stabilizing PID controllers for Linear Time Invariant (LTI) systems can be constructed as a union of convex polygons in $k_i - k_d$ space, for k_p's lying in a specific range. Widder's theorem, and its discrete-time counterpart developed in this chapter, provide necessary and sufficient conditions for the impulse response of a transfer function to be non-negative. These conditions require a sequence of transfer functions, derivable from the given transfer function, to have no zeros in specific intervals of real axis. An application of these theorems yields a sequence of polynomials, whose coefficients are polynomial functions of the controller parameters, which must be non-negative. For a specified k_p, an application of Markov-Lukacs theorem to every polynomial in the sequence gives a polynomial inequality in k_i and k_d that must be satisfied by every controller satisfying the desired transient specification.

In Section 3.1 we provide prerequisites which we will use to develop our approach. In Sections 3.2 and 3.3 we propose our method for continuous-time systems and discrete-time systems, respectively. Section 3.4 provides some illustrative examples. Finally, in Section 3.5 we provide concluding remarks.

3.1 Mathematical Preliminaries

In this section we present useful theorems and previous related results which will help us to develop our approach to the problem of designing stabilizing PID controllers guaranteeing transient response specifications.

3.1.1 Calculation of the Stabilizing Set

Consider the continuous-time unity feedback control system depicted in Fig. 3.1. The plant transfer function is denoted by $P(s) = N_p(s)/D_p(s)$ and the PID controller is represented as $C(s) = (k_d s^2 + k_p s + k_i)/s$. Let $\mathbf{K} = [k_p, k_i, k_d]^T$ denotes the vector of controller parameters.

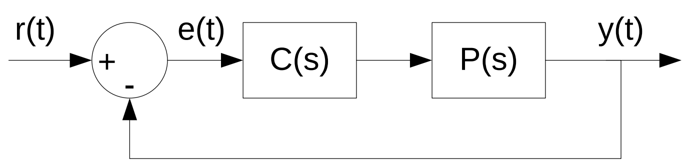

Figure 3.1: A continuous-time unity feedback control system

The set of all stabilizing continuous-time PID controllers in the $k_i - k_d$ space, for k_p values lying in an admissible range, can be constructed as union of interior of convex polygons [7], where each polygon is the feasible solution of a set of linear inequalities in k_i and k_d, at a specified k_p. The entire stabilizing set $\mathcal{S}_{stb}$, can be generated by sweeping k_p values over an admissible range. The stabilizing set at

$k_p = k_p^*$, denoted by $\mathcal{S}_{stb}(k_p^*)$, can then be expressed as

$$\mathcal{S}_{stb}(k_p^*) = \{(k_i, k_d) \text{ s.t. } (k_p^*, k_i, k_d) \in \mathcal{S}_{stb}\}.$$

The set of stabilizing digital PID controllers, for discrete-time systems, can be constructed as follows. Consider the discrete-time unity feedback control system in Fig. 3.2 where the plant transfer function is $P(z) = N_p(z)/D_p(z)$ and $C(z) = (k_2 z^2 + k_1 z + k_0)/(z^2 - z)$ represents a digital PID controller.

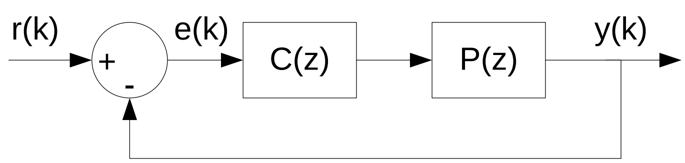

Figure 3.2: A discrete-time unity feedback control system

Let us define

$$k_3 := k_2 - k_0, \tag{3.1}$$

and denote the vector of controller parameters by $\mathbf{K} = [k_1, k_2, k_3]^T$. Based on the results in [7], for a given value of k_3 in an admissible range, the stability set in the $k_1 - k_2$ space can be constructed as union of interior of convex polygons, where each polygon is the feasible solution of a set of linear inequalities in k_1 and k_2. The entire stabilizing set $\mathcal{S}_{stb}$, can be obtained by sweeping k_3 values over an admissible range.

Denoting by $\mathcal{S}_{stb}(k_3^*)$, the stabilizing set at $k_3 = k_3^*$, we have

$$\mathcal{S}_{stb}(k_3^*) = \{(k_1, k_2) \text{ s.t. } (k_1, k_2, k_3^*) \in \mathcal{S}_{stb}\}.$$

Once the stabilizing set, $\mathcal{S}_{stb}$, is calculated, we restrict the set further to find an outer approximation $\mathcal{S}_{outer}$, of the desired set $\mathcal{S}_{des}$, which guarantee the transient specifications of the closed loop system. Also, a method will be provided to refine the outer approximation arbitrarily.

3.1.2 Transient Response Specification

A typical transient response specification requires the response, to a given input, be within an envelope. This can be satisfied by guaranteeing the impulse response of appropriate closed loop error transfer functions to be non-negative. Berstein and Widder [48] provide necessary and sufficient conditions for the impulse response of a continuous-time rational, proper transfer function to be non-negative in terms of the derivatives of the transfer function.

Theorem 3.1. *Given $D(s, \mathbf{K})$ is Hurwitz, denote the impulse response of $H(s, \mathbf{K}) = \frac{N(s,\mathbf{K})}{D(s,\mathbf{K})}$ by $h(t)$. Then, $h(t) \geq 0$ for all $t \geq 0$ if and only if*

$$H_k(s, \mathbf{K}) = (-1)^k \frac{d^k H(s, \mathbf{K})}{ds^k} \geq 0, \ \forall k \geq 0, \ \forall s \geq 0. \tag{3.2}$$

The necessity of this statement can be verified by recalling that the Laplace transform of $th(t)$ is $-\frac{dH(s)}{ds}$, and furthermore for any integer $k \geq 0$, $t^k h(t) \geq 0$ if and only if $h(t) \geq 0$. Since (3.2) holds for all $k \geq 0$, by considering a finite number of derivative terms, one can construct an outer approximation to the desired set. The sufficiency part of the above statement will be used to propose a procedure to arbitrarily tighten the outer approximation of interest.

An application of Theorem 3.1 yields a sequence of polynomials, whose coefficients are polynomial functions of the control design parameters $\mathbf{K} = [k_p, k_i, k_d]^T$, which are required to be non-negative for all $s \geq 0$.

For discrete-time systems, the counterpart of the Widder's theorem is useful in characterization. Denote by $H(z)$, the Z-transform of the impulse response $h(k)$, of a discrete-time LTI system. Let us define $\{H_k(z, \mathbf{K})\}_{k=0}^{\infty}$, the sequence of transfer functions associated with $H(z, \mathbf{K})$ as follows

$$H_0(z, \mathbf{K}) := H(z, \mathbf{K}),$$
$$H_{k+1}(z, \mathbf{K}) := -z \, \frac{dH_k(z, \mathbf{K})}{dz}, \ \forall k \geq 0.$$

The following lemma and theorem (see [24]) are useful.

Lemma 3.1. *Let $G(z)$ be a rational, proper transfer function with a decaying impulse response, $g(k)$. If $G(z_0) = 0$ for some $z_0 \geq 1$, then $g(k)$ changes sign at least once.*

Proof. Since $g(k)$ is decaying and $z_0 > 1$, then

$$\sum_{k=0}^{\infty} g(k) z_0^{-k},$$

converges and $\sum_{k=0}^{\infty} g(k) z_0^{-k} = G(z_0)$, based on the definition of Z-transform. $G(z_0) \neq 0$ provided that $g(k)$ does not change sign, because z_0^k is always positive. However, $G(z_0) = 0$ by the hypothesis; hence, $g(k)$ must change sign at least once. □

Theorem 3.2. *Given $H(z)$ is analytic in $|z| \geq 1$. Then $h(k) \geq 0$ if and only if*

$$H_k(z) \geq 0, \ \forall k \geq 0, \ \forall |z| \geq 1.$$

Proof. The necessity part can be proved as follows. We have

$$H_k(z) = \sum_{l=0}^{\infty} l^k h(l) z^{-l}, \quad \forall |z| \geq 1.$$

The above relationship holds for $k = 0$. Suppose that it holds for $k = 0, 1, 2, \ldots, m$, and consider

$$-z\frac{dH_m(z)}{dz} = -z\frac{d}{dz}[\sum_{l=0}^{\infty} l^m h(l) z^{-l}] = \sum_{l=0}^{\infty} l^{m+1} h(l) z^{-l} = H_{m+1}(z).$$

$H_m(z)$ is analytic for $|z| \geq 1$; hence, its impulse response $\{l^m h(l)\}_{l=0}^{\infty}$ decays asymptotically to zero. From the statement of Lemma 3.1, the impulse response $l^m h(l)$ changes sign provided $H_m(z)$ having at least one real, positive zero for $|z| > 1$ and any m. This implies that $h(l)$, the impulse response of $H(z)$, will also change sign.

The sufficiency part can be proved as follows. For every k and every t, being a natural number, one can define

$$\mathfrak{D}_{k,t}(H(z)) := (\frac{e}{t})^k H_k(e^{\frac{k}{t}}),$$

and also define

$$\mathfrak{D}_t(H(z)) := \lim_{k \to \infty} \mathfrak{D}_{k,t}(H(z)).$$

Clearly,

$$\mathfrak{D}_{k,t}(H(z)) = \sum_{l=0}^{\infty} h(l) l^k e^{-\frac{lk}{t}} (\frac{e}{t})^k = \sum_{l=0}^{\infty} h(l) (\frac{le}{t} e^{-\frac{l}{t}})^k.$$

Let $y = \frac{l}{t}$ and consider the following sequence of functions

$$\phi_k(y) := (ye^{-(y-1)})^k, \quad k = 0, 1, \ldots\ .$$

It can be easily seen that $\phi_k(y) = \phi_0(y)^k$. Also, $\phi_0(y)$ is a monotonically increasing function in the interval $[0, 1]$ and a monotonically decreasing function in the interval $[1, \infty)$ and has exactly one maximum at $y = 1$, which is 1. It can be seen that $\phi_k(y) \to \delta(y-1)$, the Kronecker delta function, as $k \to \infty$, which is 1 for $y = 1$ and is 0 otherwise.

Based on this observation, for every natural number t, we have $\mathfrak{D}_t(H(z)) \to \sum_{l=0}^{\infty} h(l)\delta(\frac{l}{t} - 1) = \sum_{l=0}^{\infty} h(l)\delta(l - t) = h(t)$. Suppose there is a sign change in the impulse response; this implies that there must exist a t_1 and $t_2 > t_1$ such that $h(t_1)h(t_2) < 0$. It is clear that for k being sufficiently large, it must be $H_k(e^{\frac{k}{t_1}})H_k(e^{\frac{k}{t_2}}) < 0$; otherwise, the limit will not hold. Therefore, for all k, sufficiently large, there will be a sign change in $H_k(z)$ for some real positive z which lies between $e^{\frac{k}{t_2}}$ and $e^{\frac{k}{t_1}}$. □

Applying Theorem 3.2 to an appropriate error transfer function yields a sequence of polynomials, with coefficients as polynomial functions of the controller parameters $\mathbf{K} = [k_1, k_2, k_3]^T$, required to be non-negative for all $z \in [1, \infty)$.

3.1.3 A Representation of Non-negative Polynomials

Thus far we showed that the problem of satisfying transient response specifications can be cast a problem of guaranteeing a sequence of polynomials, whose coefficients are polynomial functions of the control design parameters $\mathbf{K}$, to be non-negative on an appropriate interval of the real axis. The Markov-Lukacs theorem [49] provides a sum-of-square representation for non-negative polynomials on any interval of the real axis.

Theorem 3.3. *A polynomial $H(s) = \sum_{n=0}^{N} a_n s^n$ is non-negative on the interval $[0, \infty)$ if and only if there exists polynomials $f(s)$ of degree at most $\frac{N}{2}$ and $g(s)$ of* *degree at most $\frac{N-1}{2}$ such that*

$$H(s) = f^2(s) + sg^2(s). \tag{3.3}$$

The problem of existence of $f(s)$ and $g(s)$ can be checked by a semi-definite program as follows. Consider the vector of monomials

$$M(s) = [1, s, \ldots, s^m],$$

of an appropriate dimension; then one can write the polynomial $H(s)$ as

$$H(s) = M(s)\ F\ M^T(s) + s\ M(s)\ G\ M^T(s), \tag{3.4}$$

where F and G are Hankel matrices of appropriate dimensions. By equating the coefficients of the same powers of s in (3.4) one gets a set of equations that relate the coefficients of $H(s)$ to the entries of matrices F and G. Thus, the problem of non-negativity of polynomial $H(s)$ on the interval $[0, \infty)$ will be equivalent to the feasibility of a semi-definite program defined by $F \succeq 0$, $G \succeq 0$ and the set of equations relating the entries of the matrices F and G to the coefficients of $H(s)$.

The following form of the Markov-Lukacs theorem provides a sum-of-square representation for non-negative polynomials on the interval $[1, \infty)$.

Theorem 3.4. *A polynomial $H(z) = \sum_{n=0}^{N} a_n z^n$ is non-negative on the interval $[1, \infty)$ if and only if there exists polynomials $f(z)$ of degree at most $\frac{N}{2}$ and $g(z)$ of degree at most $\frac{N-1}{2}$ such that*

$$H(z) = f^2(z) + (z-1)g^2(z). \tag{3.5}$$

One may check the existence of the functions $f(z)$ and $g(z)$ by a semi-definite program. Let us write $H(z)$ as

$$H(z) = M(z)\ F\ M^T(z) + (z-1)\ M(z)\ G\ M^T(z), \tag{3.6}$$

where F and G are Hankel matrices and $M(z) = [1, z, \ldots, z^m]^T$ is the vector of monomials, all with appropriate dimensions. One may equate the coefficients of the same powers of z in (3.6) to obtain a set of equations relating the coefficients of $H(z)$ to the entries of matrices F and G. Similar to the previous case, the semi-definite feasibility problem defined by this set of equations and $F \succeq 0$ and $G \succeq 0$ has a solution provide the existence of $f(z)$ and $g(z)$ satisfying (3.5).

3.2 Continuous-time Systems

In this section we present our approach to find an outer approximation of the set of stabilizing PID controllers for continuous-time LTI systems guaranteeing transient response specifications.

3.2.1 An Outer Approximation

Consider the continuous-time unity feedback control system in Fig. 3.1. Let us denote by $E(s, \mathbf{K}) = \frac{N_E(s,\mathbf{K})}{D_E(s,\mathbf{K})}$, the appropriate error transfer function defined with respect to the transient specification. Applying Theorem 3.1, the corresponding error signal $e(t)$ is non-negative for all $t \geq 0$ if and only if

$$E_k(s, \mathbf{K}) = (-1)^k\ \frac{d^k E(s, \mathbf{K})}{ds^k} \geq 0,\ \forall k \geq 0,\ \forall s \geq 0. \tag{3.7}$$

Let us consider the k-th derivative, $E_k(s,\mathbf{K})$, and write it as

$$E_k(s,\mathbf{K}) = \frac{N_{E_k}(s,\mathbf{K})}{D_{E_k}(s,\mathbf{K})} = \frac{\alpha_n(\mathbf{K})s^n + \cdots + \alpha_1(\mathbf{K})s + \alpha_0(\mathbf{K})}{D_{E_k}(s,\mathbf{K})}, \tag{3.8}$$

where $D_{E_k}(s,\mathbf{K})$ is of the form $(D_E(s,\mathbf{K}))^{2k}$, and since $D_E(s,\mathbf{K})$ is Hurwitz stable for $\mathbf{K} \in \mathcal{S}_{stb}$, then $D_{E_k}(s,\mathbf{K}) \geq 0$ for all $s \geq 0$. Therefore, the problem

$$E_k(s,\mathbf{K}) \geq 0,\ \forall s \geq 0, \tag{3.9}$$

is equivalent to

$$N_{E_k}(s,\mathbf{K}) \geq 0,\ \forall s \geq 0, \tag{3.10}$$

where

$$N_{E_k}(s,\mathbf{K}) = \alpha_n(\mathbf{K})s^n + \cdots + \alpha_1(\mathbf{K})s + \alpha_0(\mathbf{K}). \tag{3.11}$$

Using Theorem 3.3, (3.10) is satisfied if and only if there exists polynomials $f(s,\mathbf{K})$ and $g(s,\mathbf{K})$ such that

$$N_{E_k}(s,\mathbf{K}) = f^2(s,\mathbf{K}) + sg^2(s,\mathbf{K}). \tag{3.12}$$

The polynomials $f(s,\mathbf{K})$ and $g(s,\mathbf{K})$ satisfying equation (3.12) exist if and only if there exist positive semi-definite Hankel matrices $F(\mathbf{y}) \succeq 0$ and $G(\mathbf{z}) \succeq 0$ of the form

$$\begin{aligned} F(\mathbf{y}) &= y_1F_1 + y_2F_2 + \cdots\ , \\ G(\mathbf{z}) &= z_1G_1 + z_2G_2 + \cdots\ , \end{aligned} \tag{3.13}$$

where the matrices, $F_1, F_2, \ldots$ and $G_1, G_2, \ldots$ are known and symmetric. The scalar parameters to be determined $y_1, y_2, \ldots$ and $z_1, z_2, \ldots$ will be referred to as Markov-Lukacs variables and y_i, z_i are respectively the i^{th} component of the vectors $\mathbf{y}$ and $\mathbf{z}$. Furthermore,

$$N_{E_k}(s, \mathbf{K}) = M\ F(\mathbf{y})\ M^T + s\ M\ G(\mathbf{z})\ M^T, \tag{3.14}$$

where $M = [1, s, \ldots, s^m]$.

The right hand side of (3.14) is linear in Markov-Lukacs variables; however, the right hand side of (3.11) is linear in $\mathbf{K}$ for the first derivative of $E(s, \mathbf{K})$, is quadratic in $\mathbf{K}$ for the second derivative of $E(s, \mathbf{K})$ and so on. Hence, the polynomial matrix inequalities associated with the first derivative of the error transfer function reduces to Linear Matrix Inequalities (LMIs); for the second derivative of the error transfer function it reduces to Quadratic Matrix Inequalities (QMIs) and so on.

The outer approximation $\mathcal{S}^k_{outer}(k_p^*)$, associated with the k-th derivative of the error transfer function, at a fixed value of $k_p = k_p^*$, can be expressed as the feasible solution of the following feasibility problem.

Feasibility Problem: Find all feasible values of $k_i, k_d, \mathbf{y}, \mathbf{z}$

subject to

$$\begin{aligned} &\mathbb{E}_j(k_p^*, k_i, k_d) = \mathbb{L}_j(\mathbf{y}, \mathbf{z}),\ j = 0, 1, 2, \ldots, n \\ &F(\mathbf{y}) \succeq 0,\ G(\mathbf{z}) \succeq 0, \\ &\mathbb{S}_q(k_p^*, k_i, k_d) \leq 0,\ q = 1, 2, \ldots, m, \end{aligned} \tag{3.15}$$

where $\mathbb{E}_j(k_p^*, k_i, k_d)$ and $\mathbb{L}_j(\mathbf{y}, \mathbf{z})$ are the coefficients of the s^j terms in (3.11) and (3.14), respectively; and the last constraint is the stability constraint determined by

m number of linear inequalities in k_i and k_d.

If we were to represent by $\mathcal{S}^k_{outer}$, the set of all feasible (k_p, k_i, k_d) satisfying the above feasibility problem, for k_p values in a specific range, then, for every k, the set $\mathcal{S}^k_{outer}$ is an outer approximation; and furthermore, the set $\mathcal{S}_{des} := \cap_k \mathcal{S}^k_{outer}$ is also an outer approximation. In fact, this is the desired set of stabilizing PID controllers satisfying the given transient specification, based on the sufficiency part of the Widder's theorem. This approach can also be used to conclude if there exists no stabilizing PID controller satisfying the desired transient specification; if the solution set to any of the outer approximations $\mathcal{S}^k_{outer}$, is empty, it can be concluded that there exists no stabilizing PID controller satisfying the given transient response specification.

The feasibility region of (3.15) in the space of $k_i - k_d$ can be approximated using the following lemma.

Lemma 3.2. *Let k be a given integer. Let (k_p^*, k_i^*, k_d^*) be stabilizing controller gains. If there is no solution corresponding to* (3.15)*, then there exists a valid (nonlinear) inequality in k_i, k_d to the set $\mathcal{S}_{des} \cap \{k_p = k_p^*\}$.*

Proof. Fixing k_p^*, k_i^*, k_d^*, the problem (3.15) is a LMI. If (k_p^*, k_i^*, k_d^*) does not satisfy the constraints of the problem (3.15), then we have the following by the theorem of alternatives: $\exists \lambda, Q_1 \succeq 0, Q_2 \succeq 0$, such that

$$g(\lambda, Q_1, Q_2) < 0, \;\; Q_1 \succeq 0, \;\; Q_2 \succeq 0, \tag{3.16}$$

is feasible, where

$$\begin{aligned} g(\lambda, Q_1, Q_2) &= \inf_{\mathbf{y},\mathbf{z} \in \mathcal{D}} \mathcal{L}(\mathbf{y}, \mathbf{z}, \lambda, Q_1, Q_2) \qquad (3.17) \\ &= \inf_{\mathbf{y},\mathbf{z} \in \mathcal{D}} \{\lambda \cdot [\mathbb{E}(k_p^*, k_i^*, k_d^*) - \mathbb{L}(\mathbf{y}, \mathbf{z})] + (Q_1 \cdot F(\mathbf{y})) + (Q_2 \cdot G(\mathbf{z}))\}, \end{aligned}$$

and $\mathcal{D}$ denotes the domain of Markov-Lukacs variables. Let $\mathcal{F}_{dual}(k_p^*, k_i^*, k_d^*)$ be the set of dual variables (λ, Q_1, Q_2) for which the function $g(\lambda, Q_1, Q_2)$ is well defined. It is clear that $\mathcal{L}(\mathbf{y}, \mathbf{z}, \lambda, Q_1, Q_2)$ is linear in $\mathbf{y}, \mathbf{z}$ and by Ritz representation theorem for linear operators, there exist constants $\alpha_j, \beta_j,\ j = 1, 2, \ldots$ satisfying

$$\mathcal{L}(\mathbf{y}, \mathbf{z}, \lambda, Q_1, Q_2) = \sum_{j=1}^{n} (\alpha_j y_j + \beta_j z_j) + \lambda \cdot \mathbb{E}(k_p^*, k_i^*, k_d^*), \tag{3.18}$$

where the coefficients $\alpha_j,\ \beta_j,\ j = 1, 2, \ldots, n$, are functions of the dual variables $(\lambda, Q_1, Q_2) \in \mathcal{F}_{dual}(k_p^*, k_i^*, k_d^*)$. If α_j's and β_j's are non-zero, the infimum will be $-\infty$, since $\mathbf{y}$ and $\mathbf{z}$ are unconstrained. Hence, α_j's and β_j's must be zero which provide additional linear constraints on λ, Q_1, Q_2. If that is the case, (3.17) simplifies to

$$g(\lambda, Q_1, Q_2) = \lambda \cdot \mathbb{E}(k_p^*, k_i^*, k_d^*). \tag{3.19}$$

The following is a valid (nonlinear) inequality to the set $\mathcal{S}_{des}$:

$$\sum_j \lambda_j \mathbb{E}_j(k_p^*, k_i, k_d) \geq 0. \tag{3.20}$$

□

Remark 3.1. Deepest Cut: *One can even find a deep (nonlinear) cut by solving* *the following problem:*

$$\min \ \sum_j \lambda_j \mathbb{E}_j(k_p^*, k_i^*, k_d^*) \tag{3.21}$$
$$\text{subject to} \ \ \alpha_i(\lambda, Q_1, Q_2) = 0,$$
$$\beta_i(\lambda, Q_1, Q_2) = 0, \ i = 1, \ldots, n$$
$$(\lambda, Q_1, Q_2) \in \mathcal{F}_{dual}.$$

Remark 3.2. Updating the Outer Approximation: *Let $\mathcal{S}_{outer}^b$ be the current best outer approximation of the desired set $\mathcal{S}_{des}$. The idea is to pick a controller $(k_p^*, k_i^*, k_d^*) \in \mathcal{S}_{outer}^b$ and check if it satisfies* (3.15)*. If not, we then find a deep cut using Lemma 3.2 and the cut for the chosen $k_p = k_p^*$ may be plotted in the $k_i - k_d$ plane using the cut inequality:*

$$\sum_j \lambda_j^* \mathbb{E}_j(k_p^*, k_i, k_d) \geq 0. \tag{3.22}$$

If we write

$$\mathcal{S}_{outer}(k_p^*) := \mathcal{S}_{outer}^b \cap \{(k_p, k_i, k_d) : k_p = k_p^*\},$$

then, we may update the current outer approximation of the desired set $\mathcal{S}_{des}$, through

$$\mathcal{S}_{outer}(k_p^*) \leftarrow \mathcal{S}_{outer}(k_p^*) \cap \{(k_p, k_i, k_d) : k_p = k_p^*, \sum_j \lambda_j^* \mathbb{E}_j(k_p^*, k_i, k_d) \geq 0\}.$$

We note that updating $\mathcal{S}_{outer}(k_p^*)$ for possible values of k_p^* is equivalent to updating $\mathcal{S}_{outer}^b$.

3.2.2 First Outer Approximation

The first outer approximation $\mathcal{S}^1_{outer}$, of the set of stabilizing PID controllers satisfying transient response specifications, can be computed by considering the non-negativity of an appropriate error transfer function, defined with respect to the given transient specification, and its first derivative:

$$E(s, \mathbf{K}) \geq 0, \tag{3.23}$$

$$E_1(s, \mathbf{K}) = (-1)\,\frac{dE(s, \mathbf{K})}{ds} \geq 0,\ \forall s \geq 0. \tag{3.24}$$

The polynomial in the numerator of $N_{E_1}(s, \mathbf{K})$, in (3.24), has coefficients which are linear in $\mathbf{K}$. The non-negativity of this polynomial can be stated as the following feasibility problem.

$$\begin{aligned}
&\text{Feasibility Problem: Find all feasible values of } k_i, k_d, \mathbf{y}, \mathbf{z}\\
&\text{subject to}\\
&\quad \mathbb{E}_j(k_p^*, k_i, k_d) = \mathbb{L}_j(\mathbf{y}, \mathbf{z}),\ j = 0, 1, \ldots, n\\
&\quad F(\mathbf{y}) \succeq 0,\ G(\mathbf{z}) \succeq 0,\\
&\quad \mathbb{S}_q(k_p^*, k_i, k_d) \leq 0,\ q = 1, 2, \ldots, m,
\end{aligned} \tag{3.25}$$

where $\mathbb{E}_j(k_p^*, k_i, k_d)$ and $\mathbb{L}_j(\mathbf{y}, \mathbf{z})$ are the coefficients of the s^j terms in (3.11) and (3.14), respectively. Since polynomials $\mathbb{E}_j(k_p^*, k_i, k_d)$ are linear in k_i and k_d, in this case, the cutting hyperplanes become linear inequalities in k_i and k_d. Let $(k_p^*, k_i^*, k_d^*) \in \mathcal{S}_{stb}(k_p^*)$, but (k_p^*, k_i^*, k_d^*) yields the feasibility problem (3.25) infeasible. The corresponding cutting hyperplane can be obtained using Lemma 3.2.

3.2.3 Second Outer Approximation

The second outer approximation $\mathcal{S}^2_{outer}$, can be constructed by adding the non-negativity condition for the second derivative of the appropriate error transfer function, to (3.23) and (3.24), i.e.

$$\begin{aligned} E(s,\mathbf{K}) &\geq 0, \\ E_1(s,\mathbf{K}) = (-1)\ \frac{dE(s,\mathbf{K})}{ds} &\geq 0, \\ E_2(s,\mathbf{K}) = \frac{d^2E(s,\mathbf{K})}{ds^2} &\geq 0,\ \forall s \geq 0. \end{aligned} \tag{3.26}$$

In this case, the numerator of $N_{E_2}(s,\mathbf{K})$ has coefficients which are quadratic functions of $\mathbf{K}$. The non-negativity of this polynomial can be expressed as the following feasibility problem.

Feasibility Problem: Find all feasible values of $k_i, k_d, \mathbf{y}, \mathbf{z}$

subject to

$$\begin{aligned} &\mathbb{E}_j(k_p^*, k_i, k_d) = \mathbb{L}_j(\mathbf{y},\mathbf{z}),\ j = 0,1,\ldots,n \\ &F(\mathbf{y}) \succeq 0,\ G(\mathbf{z}) \succeq 0, \\ &\mathbb{S}_q(k_p^*, k_i, k_d) \leq 0,\ q = 1,2,\ldots,m, \end{aligned} \tag{3.27}$$

where $\mathbb{E}_j(k_p^*, k_i, k_d)$ and $\mathbb{L}_j(\mathbf{y},\mathbf{z})$ are the coefficients of the s^j terms in (3.11) and (3.14), respectively. Since the polynomials $\mathbb{E}_j(k_p^*, k_i, k_d)$ are quadratic functions of k_i and k_d, in this case, the (nonlinear) cuts become quadratic inequalities in k_i and k_d. Let $(k_p^*, k_i^*, k_d^*) \in \mathcal{S}^1_{outer}(k_p^*)$, but (k_p^*, k_i^*, k_d^*) yields the feasibility problem (3.27) infeasible; then, one may find corresponding cutting hyperboloid using Lemma 3.2.

3.2.4 *Estimate of the Minimum Possible Overshoot*

Let $\gamma > 0$ denotes the maximum allowable overshoot to a unit step input using a PID controller. Thus, we want the error transfer function to have a non-negative impulse response:

$$E(s, \mathbf{K}) = \frac{1+\gamma}{s} - \frac{1}{s}\frac{(k_d s^2 + k_p s + k_i)N_p(s)}{sD_p(s) + (k_d s^2 + k_p s + k_i)N_p(s)}.$$

This can be rewritten by $\bar{\gamma} := \frac{1}{\gamma}$ as

$$\bar{E}(s, \mathbf{K}) := \bar{\gamma}E(s, \mathbf{K}) = \frac{(1+\bar{\gamma})D_p(s) + (k_d s^2 + k_p s + k_i)N_p(s)}{s\underbrace{\left(sD_p(s) + (k_d s^2 + k_p s + k_i)N_p(s)\right)}_{\Delta_{cl}(s)}}.$$

The design process requires the impulse response of $\bar{E}(s, \mathbf{K})$ to be non-negative. We can apply the methodology developed here to calculate an outer approximation $\mathcal{S}_{outer}$, for the desired set of controllers satisfying this overshoot specification. In fact, if we have an outer approximation defined by linear constraints, denoted by $\mathcal{S}_{linear}$, in terms of variables $\bar{\gamma}, k_p, k_i$ and k_d, then one may calculate a lower bound on the minimum possible overshoot by solving the following linear optimization problem:

$$\max_{\bar{\gamma}, k_p, k_i, k_d} \bar{\gamma}$$

subject to the constraint $(\bar{\gamma}, k_p, k_i, k_d) \in \mathcal{S}_{linear}$.

3.3 Discrete-time Systems

In this section we propose our approach for calculating an outer approximation of the set of stabilizing digital PID controllers for the class of discrete-time LTI systems guaranteeing transient response specifications.

3.3.1 An Outer Approximation

Consider the discrete-time unity feedback control system depicted in Fig. 3.2. Let us denote by $E(z, \mathbf{K}) = \frac{N_E(z,\mathbf{K})}{D_E(z,\mathbf{K})}$, the appropriate error transfer function defined with respect to the transient specification. Applying Theorem 3.2, the corresponding error signal $e(k)$ is non-negative for all $k \geq 0$ if and only if

$$E_{k+1}(z, \mathbf{K}) = -z \, \frac{dE_k(z, \mathbf{K})}{dz} \geq 0, \ \forall |z| \geq 1, \ \forall k \geq 0, \tag{3.28}$$

where $E_0(z, \mathbf{K}) = E(z, \mathbf{K})$ which is also non-negative for $|z| \geq 1$. Let us consider the k-th derivative, $E_k(z, \mathbf{K})$, and write it as

$$E_k(z, \mathbf{K}) = \frac{N_{E_k}(z, \mathbf{K})}{D_{E_k}(z, \mathbf{K})} = \frac{\alpha_n(\mathbf{K})z^n + \cdots + \alpha_1(\mathbf{K})z + \alpha_0(\mathbf{K})}{D_{E_k}(z, \mathbf{K})}, \tag{3.29}$$

where the denominator is always non-negative for all $|z| \geq 1$, because $D_{E_k}(z, \mathbf{K})$ is of the form $(D_E(z, \mathbf{K}))^{2k}$, and $D_E(z, \mathbf{K})$ is Schur stable for $\mathbf{K} \in \mathcal{S}_{stb}$. Therefore, problem reduces to the non-negativity of $N_{E_k}(z, \mathbf{K})$ on the interval $[1, \infty)$. $N_{E_k}(z, \mathbf{K})$ is a polynomial with coefficients as polynomial functions of the controller parameters $\mathbf{K}$, i.e.

$$N_{E_k}(z, \mathbf{K}) = \ \alpha_n(\mathbf{K})z^n + \cdots + \alpha_1(\mathbf{K})z + \alpha_0(\mathbf{K}). \tag{3.30}$$

By Markov-Lukacs theorem (Theorem 3.4), $N_{E_k}(z, \mathbf{K})$ is non-negative on the interval $[1, \infty)$ provided that there exists polynomials $f(z, \mathbf{K})$ and $g(z, \mathbf{K})$ such that

$$N_{E_k}(z, \mathbf{K}) = \ f^2(z, \mathbf{K}) + (z-1)g^2(z, \mathbf{K}). \tag{3.31}$$

The existence of polynomials $f(z, \mathbf{K})$ and $g(z, \mathbf{K})$ is guaranteed through the existence of positive semi-definite symmetric matrices $F(\mathbf{v}) \succeq 0$, $G(\mathbf{w}) \succeq 0$ satisfying

$$N_{E_k}(z, \mathbf{K}) = \ M(z)\ F(\mathbf{v})\ M^T(z) + (z-1)M(z)\ G(\mathbf{w})\ M^T(z), \tag{3.32}$$

where $M(z) = [1, z, \ldots, z^m]$; $\mathbf{v}$ and $\mathbf{w}$ are vectors of the Markov-Lukacs variables and

$$F(\mathbf{v}) = v_1F_1 + v_2F_2 + \cdots,$$
$$G(\mathbf{w}) = w_1G_1 + w_2G_2 + \cdots .$$

The right hand side of (3.32) is linear in Markov-Lukacs variables $\mathbf{v}, \mathbf{w}$; however, the right hand side of (3.30) is linear in $\mathbf{K}$ for the first derivative of the error transfer function, i.e. $E_1(z, \mathbf{K})$; is quadratic in $\mathbf{K}$ for the second derivative of the error transfer function, i.e. $E_2(z, \mathbf{K})$, and so on.

The outer approximation $\mathcal{S}^k_{outer}(k_3^*)$, associated with the k-th derivative of the error transfer function, at a fixed value of $k_3 = k_3^*$, can be expressed as the feasible solution of the following feasibility problem.

Feasibility Problem: Find all feasible values of $k_1, k_2, \mathbf{v}, \mathbf{w}$

subject to

$$\begin{aligned} &\mathbb{E}_j(k_1, k_2, k_3^*) = \mathbb{L}_j(\mathbf{v}, \mathbf{w}), \\ &F(\mathbf{v}) \succeq 0,\ G(\mathbf{w}) \succeq 0,\ j = 0, 1, 2, \ldots, n, \\ &\mathbb{S}_q(k_1, k_2, k_3^*) \leq 0,\ q = 1, 2, \ldots, m, \end{aligned} \tag{3.33}$$

where $\mathbb{E}_j(k_1, k_2, k_3^*)$ and $\mathbb{L}_j(\mathbf{v}, \mathbf{w})$ are the coefficients of the z^j terms in (3.30) and (3.32), respectively; and the last constraint is the stability constraint determined by m number of linear inequalities in k_1 and k_2.

The feasibility region of (3.33) in the space of $k_1 - k_2$ can be approximated in the same way as with the continuous-time controllers.

3.4 Illustrative Examples

3.4.1 Continuous-time Systems: Non-overshooting Step Response

The following example illustrates how the method developed for the continuous-time systems can be used to construct an outer approximation of the set of stabilizing PID controllers satisfying transient response specifications. Consider the following unstable plant

$$P(s) = \frac{s+1}{s^3 + 2s^2 + s + 3}, \tag{3.34}$$

and a PID controller represented as

$$C(s) = \frac{k_d s^2 + k_p s + k_i}{s}, \tag{3.35}$$

in a unity feedback control system. Assume that the transient specification requires the unit step response of the closed loop system to be non-overshooting.

One may construct the entire stabilizing set as union of interior of convex polygons through an application of the method proposed in [7]. For this example, The stabilizing set is shown in Fig. 3.3, for $-2.5 < k_p < 7$.

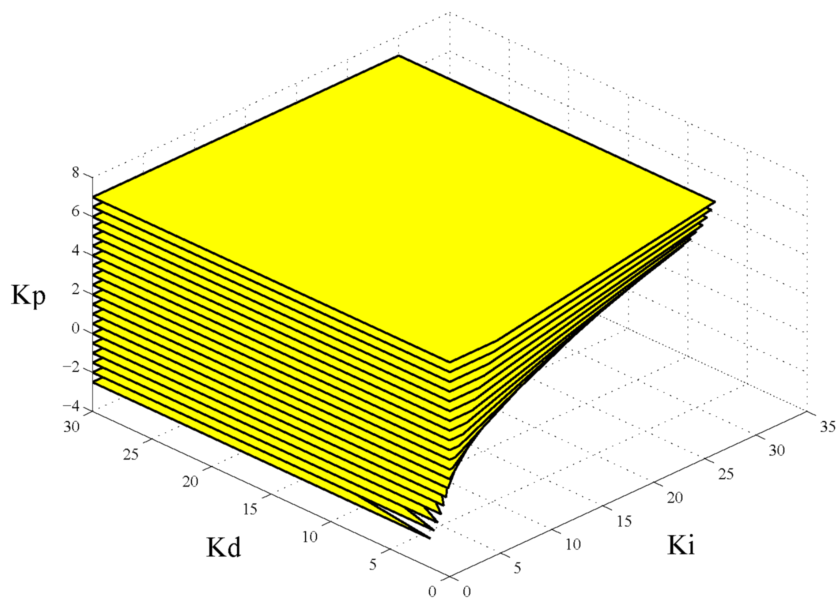

Figure 3.3: Stabilizing set for $-2.5 < k_p < 7$

Consider the stability region in the $k_i - k_d$ plane, at $k_p^* = 5$, defined by

$$\begin{aligned} k_i &> 0, \\ k_d - 0.2k_i + 0.5 &> 0, \end{aligned} \tag{3.36}$$

which is plotted in Fig. 3.4.

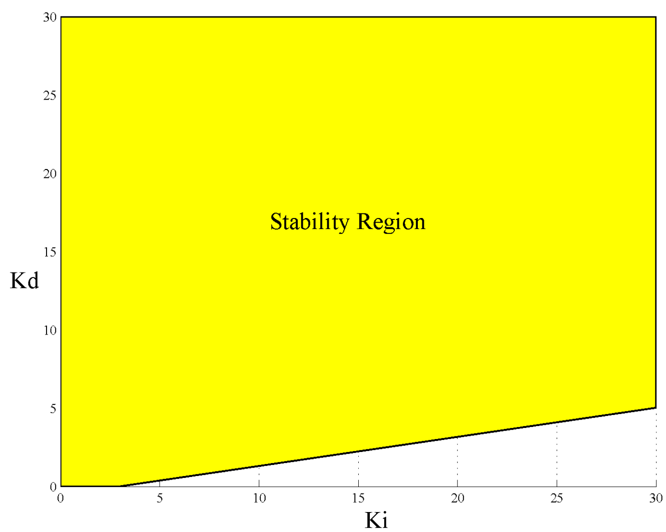

Figure 3.4: Stability region at $k_p = 5$

In this example, the appropriate error function is defined as $e(t) = r(t) - y(t)$, and is non-negative for $t \geq 0$ when response is non-overshooting. The corresponding error transfer function to a unit step input can be written as

$$E(s, \mathbf{K}) = \frac{s^3 + 2s^2 + s + 3}{s^4 + (k_d + 2)\, s^3 + (k_p + k_d + 1)\, s^2 + (k_p + k_i + 3)\, s + k_i}. \tag{3.37}$$

The first outer approximation of the non-overshooting step response set, denoted by $\mathcal{S}_{outer}^{1}$, can be obtained by enforcing

$$E(s, \mathbf{K}) \geq 0, \tag{3.38}$$

$$E_1(s, \mathbf{K}) \geq 0. \tag{3.39}$$

The denominator of (3.37) is Hurwitz stable for $(k_p, k_i, k_d) \in \mathcal{S}_{stb}$; and hence, is non-negative for all $s \geq 0$. The numerator of (3.37) is always non-negative for all $s \geq 0$, because $s^3 + 2s^2 + s + 3 \geq 0$ for all $s \geq 0$. Therefore, (3.38) is automatically satisfied for $\mathbf{K} \in \mathcal{S}_{stb}$, which does not add any further constraint than the stability constraints to the problem. Equation (3.39) is satisfied if $N_{E_1}(s, \mathbf{K}) \geq 0$, which can be calculated as

$$\begin{aligned} N_{E_1}(s, \mathbf{K}) \; = \; & s^6 + 4s^5 + (6 - k_p + k_d)\, s^4 \\ & + (-2k_p + 10 + 2k_d - 2k_i)\, s^3 \\ & + (-k_p + 13 + 10k_d - 5k_i)\, s^2 \\ & + (6 + 6k_d - 4k_i + 6k_p)\, s \\ & + (2k_i + 3k_p + 9) \geq 0, \;\; \forall s \geq 0. \end{aligned} \tag{3.40}$$

The non-negativity condition for $N_{E_1}(s, \mathbf{K})$ is satisfied though the existence of positive semi-definite matrices

$$F(\mathbf{y}) = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \\ y_2 & y_5 & y_6 & y_7 \\ y_3 & y_6 & y_8 & y_9 \\ y_4 & y_7 & y_9 & y_{10} \end{bmatrix} \succeq 0, \tag{3.41}$$

$$G(\mathbf{z}) = \begin{bmatrix} z_1 & z_2 & z_3 & z_4 \\ z_2 & z_5 & z_6 & z_7 \\ z_3 & z_6 & z_8 & z_9 \\ z_4 & z_7 & z_9 & z_{10} \end{bmatrix} \succeq 0, \tag{3.42}$$

where the entries of the matrices $F(\mathbf{y})$ and $G(\mathbf{z})$ are related to the controller parameters by the following set of linear equations

$$\begin{aligned} y_1 &= 2k_i + 3k_p + 9, \\ 2y_2 + z_1 &= 6 + 6k_d - 4k_i + 6k_p, \\ 2z_2 + y_5 + 2y_3 &= -k_p + 13 + 10k_d - 5k_i, \\ 2y_6 + 2y_4 + 2z_3 + z_5 &= -2k_p + 10 + 2k_d - 2k_i, \\ 2z_4 + y_8 + 2y_7 + 2z_6 &= 6 - k_p + k_d, \\ 2y_9 + z_8 + 2z_7 &= 4, \\ y_{10} + 2z_9 &= 1, \\ z_{10} &= 0. \end{aligned} \tag{3.43}$$

The set of all feasible (k_i, k_d), assuming $k_p = 5$, satisfying (3.36), (3.41)-(3.43) forms the first outer approximation of the set of stabilizing PID controllers that guarantee the step response of the closed loop system to be non-overshooting. This outer approximation can be constructed by choosing stabilizing controller gains for which the set of constraints (3.41)-(3.43) is infeasible. Corresponding to such stabilizing controllers, there exist cutting hyperplanes which refine the stabilizing set and yield the outer approximation. Fig. 3.5 shows the first outer approximation of the stable, non-overshooting step response region for this example by considering 10 number of cutting hyperplanes.

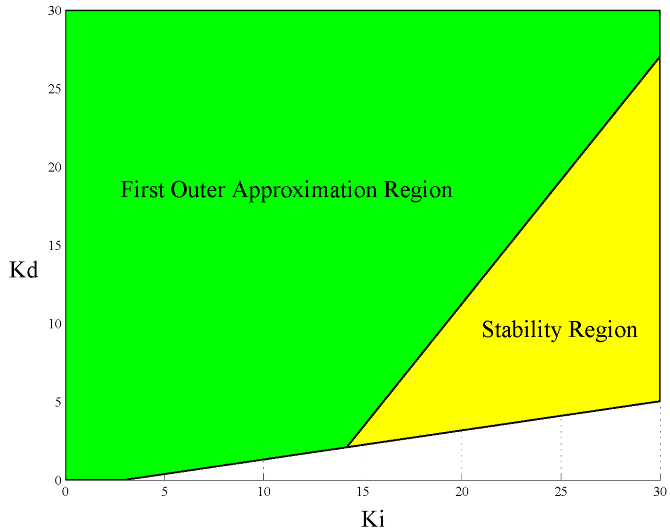

Figure 3.5: First outer approximation at $k_p = 5$

For the calculations of the second outer approximation, the feasibility problem corresponding to $N_{E_2}(s, \mathbf{K}) \geq 0$ will be:

Feasibility Problem: Find all feasible values of $k_i, k_d, \mathbf{y}, \mathbf{z}$

subject to

$$F(\mathbf{y}) = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 & y_5 \\ y_2 & y_6 & y_7 & y_8 & y_9 \\ y_3 & y_7 & y_{10} & y_{11} & y_{12} \\ y_4 & y_8 & y_{11} & y_{13} & y_{14} \\ y_5 & y_9 & y_{12} & y_{14} & y_{15} \end{bmatrix} \succeq 0, \tag{3.44}$$

$$G(\mathbf{z}) = \begin{bmatrix} z_1 & z_2 & z_3 & z_4 & z_5 \\ z_2 & z_6 & z_7 & z_8 & z_9 \\ z_3 & z_7 & z_{10} & z_{11} & z_{12} \\ z_4 & z_8 & z_{11} & z_{13} & z_{14} \\ z_5 & z_9 & z_{12} & z_{14} & z_{15} \end{bmatrix} \succeq 0, \tag{3.45}$$

$$\begin{aligned}
y_1 = {} & 8{k_i}^2 + 4k_ik_p + 6{k_p}^2 + 24k_i \\
& + 36k_p + 54 - 6k_dk_i, \\
2y_2 + z_1 = {} & -6k_dk_i + 18{k_p}^2 + 54k_d \\
& - 24k_i + 72k_p + 54 \\
& + 18k_dk_p + 6{k_i}^2 + 12k_ik_p, \\
2z_2 + y_6 + 2y_3 = {} & 126 + 90k_d - 18k_i + 72k_p \\
& + 18{k_d}^2 - 6k_dk_i + 54k_dk_p \\
& + 6{k_i}^2 - 6k_ik_p + 18{k_p}^2, \\
2y_7 + 2y_4 + 2z_3 + z_6 = {} & 46k_dk_p + 2{k_i}^2 - 2k_ik_p \\
& + 50{k_d}^2 + 130k_d - 62k_i \\
& + 104k_p - 36k_dk_i + 128, \\
y_{10} + 2y_8 + 2y_5 + 2z_4 + 2z_7 = {} & 120 + 36k_p - 6k_dk_p + 180k_d \\
& - 102k_i + 42{k_d}^2 - 24k_dk_i, \\
2y_9 + 2y_{11} + 2z_5 + z_{10} + 2z_8 = {} & -60k_i - 30k_p - 6k_dk_p \\
& + 102k_d + 102 - 6k_dk_i + 6{k_d}^2,
\end{aligned}$$

$$
\begin{aligned}
2z_9 + 2z_{11} + 2y_{12} + y_{13} = \;& 58 - 22k_p + 22k_d - 14k_i \\
& + 2{k_d}^2 - 2k_d k_p, \\
z_{13} + 2z_{12} + 2y_{14} = \;& 6k_d - 6k_p + 30, \\
y_{15} + 2z_{14} = \;& 12, \\
z_{15} = \;& 2. \qquad (3.46)
\end{aligned}
$$

Here, the set of all feasible (k_i, k_d), assuming $k_p = 5$, satisfying (3.36), (3.41)-(3.46) forms the second outer approximation of the stable, non-overshooting step response region for this example. Fig. 3.6 shows the second outer approximation generated by 10 number of cutting hyperplanes and 10 number of cutting hyperboloids.

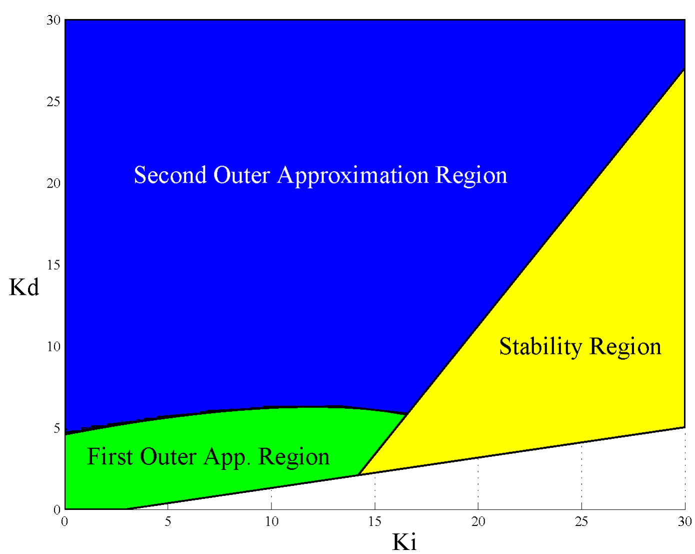

Figure 3.6: Second outer approximation at $k_p = 5$

In order to show that the outer approximations obtained for this example contains the controllers satisfying the stability and non-overshooting step response of the closed loop system, we picked the controller parameters as $k_p = 5, k_i = 1, k_d = 20$, which is inside the second outer approximation shown in Fig. 3.6, and plotted the corresponding unit step response of the closed loop system as depicted in Fig. 3.7. It can be seen from Fig. 3.7 that the unit step response is non-overshooting.

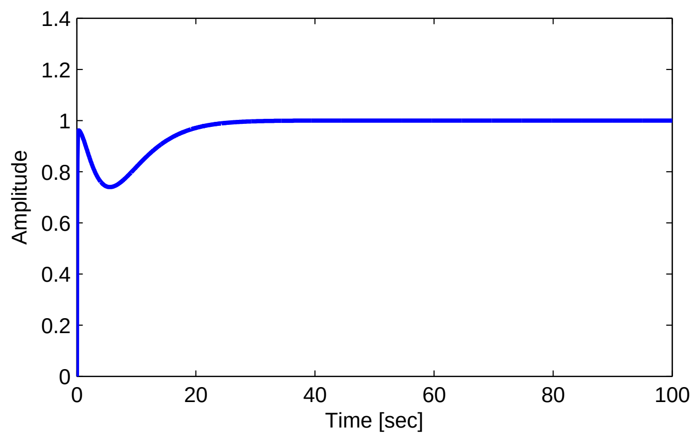

Figure 3.7: Step response of the closed loop system using the controller $k_p = 5, k_i = 1, k_d = 20$

We also picked the controller parameters as $k_p = 5, k_i = 5, k_d = 3$, which is inside the first outer approximation, but outside of the second outer approximations, and plotted the corresponding unit step response of the closed loop system (Fig. 3.8). Fig. 3.8 shows that the response has an overshoot as we expected since the controller chosen here is not inside the second outer approximation.

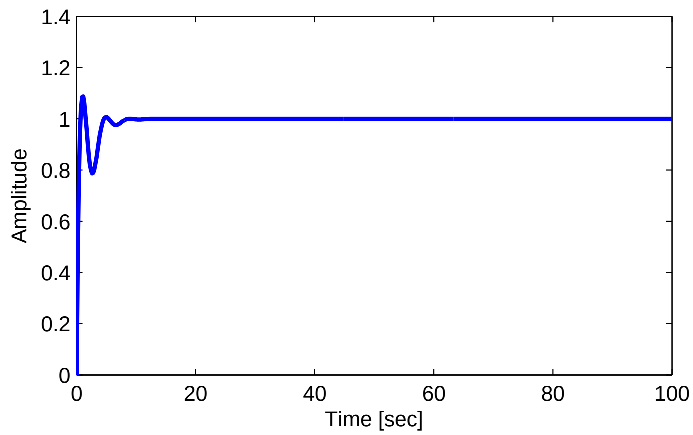

Figure 3.8: Step response of the closed loop system using the controller $k_p = 5, k_i = 5, k_d = 3$

3.4.2 Continuous-time Systems: Maximum Allowable Overshoot

In this example we illustrate how to obtain an outer approximation of the set of PID controllers that guarantee the step response of the closed loop system to have an overshoot less than a maximum allowable value. Recalling the unstable plant from the previous example, and denoting the maximum allowable overshoot by γ, one may define a new error signal as $e(t) = (1+\gamma)r(t) - y(t)$. The corresponding error transfer function can be obtained as

$$E(s, \mathbf{K}) = \frac{1+\gamma}{s} - \frac{1}{s}\frac{(k_d s^2 + k_p s + k_i)(s+1)}{s(s^3+2s^2+s+3) + (k_d s^2 + k_p s + k_i)(s+1)}.$$

Let us assume that the maximum allowable overshoot is $\gamma = 5\%$. Using this error transfer function, and following the same steps presented in the previous example, one may obtain an outer approximation of the set of PID controllers guaranteeing the step response of the close loop system to have an overshoot less than 5%. Fig.

3.9 shows the first and second outer approximations for this example.

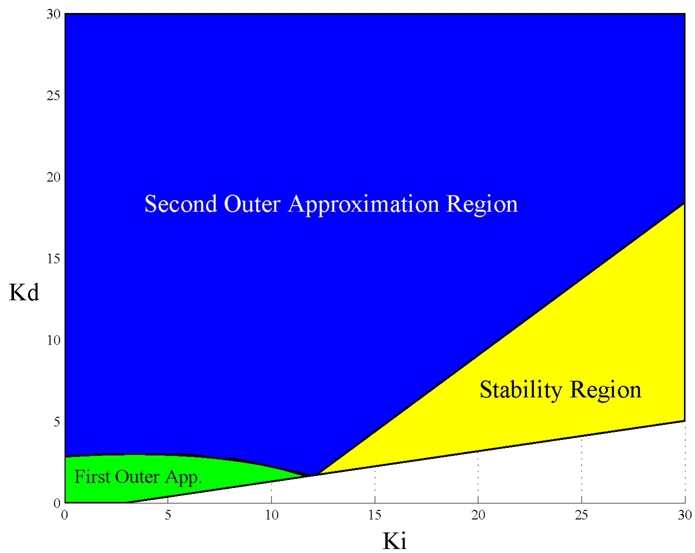

Figure 3.9: Second outer approximation at $k_p = 5$ for the maximum allowable overshoot example

3.4.3 Discrete-time Systems: Non-overshooting Step Response

This example considers a discrete-time system and shows how the approach developed in the previous section can be used to obtain the set of all stabilizing digital PID controllers guaranteeing non-overshooting unit step response. Consider the following discrete-time plant

$$P(z) = \frac{z + 0.5}{z^2 - 0.1z} , \tag{3.47}$$

and the digital PID controller

$$C(z) = \frac{k_2 z^2 + k_1 z + k_0}{z^2 - z}, \tag{3.48}$$

and define $k_3 := k_2 - k_0$. The entire set of stabilizing controller gains (k_1, k_2, k_3) can be constructed upon an application of the method presented in [7]. This set is plotted in Fig. 3.10.

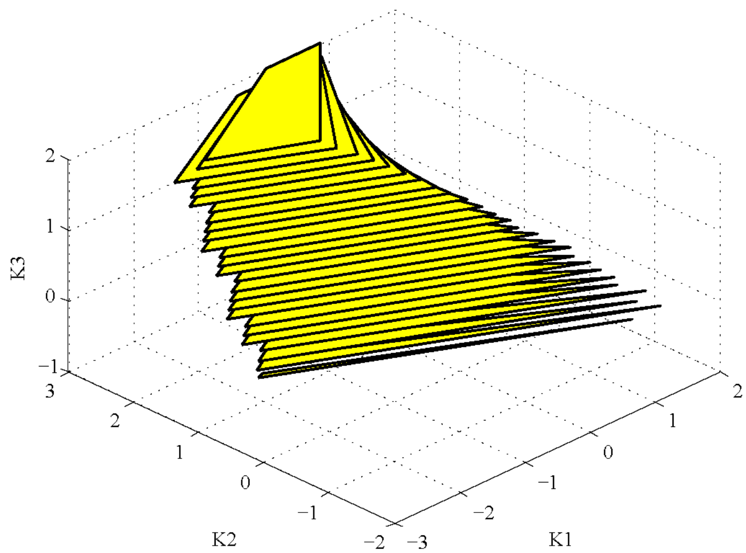

Figure 3.10: Stability set for $-0.5 < k_3 < 2$

Let us fix the value of k_3 to $k_3^* = 1$. At this k_3 value, the stability region can be expressed as the following set of linear inequalities in k_1 and k_2:

$$\begin{aligned} 0.4k_1 + 0.9k_2 - 0.4 &> 0, \\ k_1 + 0.3k_2 - 0.6 &< 0, \\ 0.4k_1 - 0.9k_2 + 2.5 &> 0, \end{aligned} \tag{3.49}$$

which is plotted in Fig. 3.11.

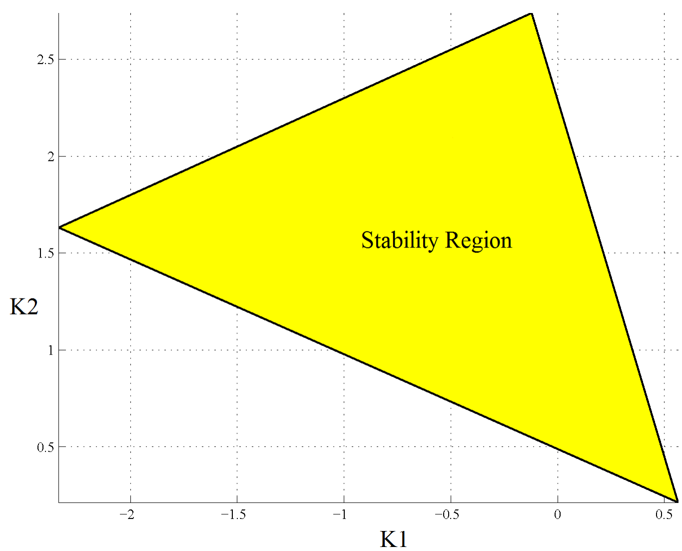

Figure 3.11: Stability region at $k_3 = 1$

In this example, the appropriate error signal is $e(k) = r(k) - y(k)$, which is non-negative for $k \geq 0$ when response is non-overshooting. The corresponding error transfer function to a unit step input signal can be written as

$$E(z, \mathbf{K}) = \frac{N_E(z, \mathbf{K})}{D_E(z, \mathbf{K})}, \tag{3.50}$$

where

$$\begin{aligned} N_E(z, \mathbf{K}) = &\ (10z - 1)z^3, \\ D_E(z, \mathbf{K}) = &\ 10z^4 + (10k_2 - 11)z^3 + (5k_2 + 1 + 10k_1)z^2 \\ &+ (-10k_3 + 10k_2 + 5k_1)z + 5k_2 - 5k_3. \end{aligned} \tag{3.51}$$

The error transfer function $E(z, \mathbf{K})$ in non-negative on the interval $[1, \infty)$. The first outer approximation, $\mathcal{S}^1_{outer}$, of the set of all stabilizing digital PID controllers which render the non-overshooting step response of the closed system can be obtained by enforcing $E_1(z, \mathbf{K})$ to be non-negative on the interval $[1, \infty)$. This means that

$$\begin{aligned} N_{E_1}(z, \mathbf{K}) \ = \ & (100 - 100k_2)\, z^7 + (-200k_1 - 100k_2 - 20)\, z^6 \\ & + (-140k_1 - 295k_2 + 300k_3 + 1.0)\, z^5 \\ & + (10k_1 - 180k_2 + 180k_3)\, z^4 \\ & + (-15k_3 + 15k_2)\, z^3 \geq 0, \ \forall z \in [1, \infty). \end{aligned} \tag{3.52}$$

This requirement can be satisfied through the existence of positive semi-definite matrices:

$$F(\mathbf{v}) = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 \\ v_2 & v_5 & v_6 & v_7 \\ v_3 & v_6 & v_8 & v_9 \\ v_4 & v_7 & v_9 & v_{10} \end{bmatrix} \succeq 0, \tag{3.53}$$

$$G(\mathbf{w}) = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 \\ w_2 & w_5 & w_6 & w_7 \\ w_3 & w_6 & w_8 & w_9 \\ w_4 & w_7 & w_9 & w_{10} \end{bmatrix} \succeq 0, \tag{3.54}$$

where the entries of the matrices $F(\mathbf{v})$ and $G(\mathbf{w})$ are related to the controller parameters by the following set of linear equations:

$$\begin{aligned} v_1 - w_1 &= 0, \\ 2v_2 - 2w_2 + w_1 &= 0, \\ v_5 + 2v_3 - 2w_3 - w_5 + 2w_2 &= 0, \\ 2v_6 + 2v_4 + 2w_3 + w_5 - 2w_4 - 2w_6 &= -15k_3 + 15k_2, \\ 2v_7 + v_8 + 2w_4 + 2w_6 - 2w_7 - w_8 &= 10k_1 - 180k_2 + 180k_3, \\ 2v_9 - 2w_9 + 2w_7 + w_8 &= -140k_1 - 295k_2 + 300k_3 + 1, \\ v_{10} - w_{10} + 2w_9 &= -200k_1 - 100k_2 - 20, \\ w_{10} &= 100 - 100k_2. \end{aligned} \tag{3.55}$$

The set of all feasible (k_1, k_2, k_3) satisfying (3.49), (3.53)-(3.55) forms the first outer approximation $\mathcal{S}^1_{outer}$, of the desired set $\mathcal{S}_{des}$. Fig. 3.12 shows the first outer approximation obtained for this example.

The second outer approximation can be constructed by adding the non-negativity condition for the second derivative of the error transfer function, to the feasibility problem defined for the first outer approximation. The second outer approximation for this example is plotted in Fig. 3.13.

We picked the controller $k_1 = -0.8, k_2 = 0.92, k_3 = 1$, inside the second outer approximation, and another controller $k_1 = 0.38, k_2 = 0.47, k_3 = 1$, inside the first

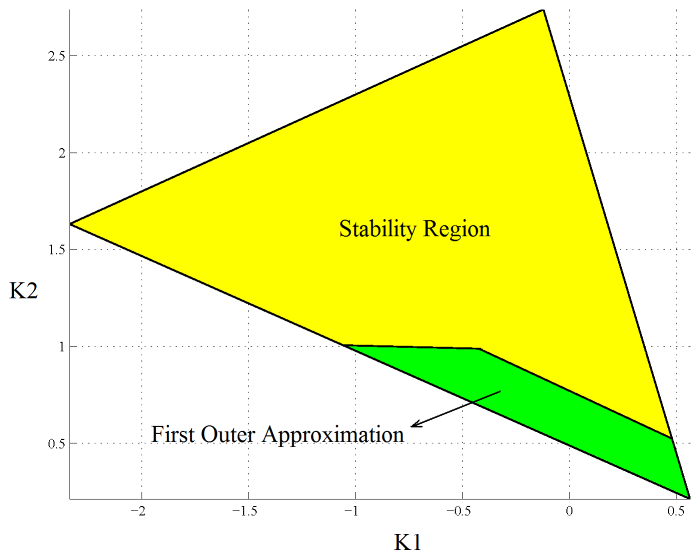

Figure 3.12: First outer approximation at $k_3 = 1$

outer approximation and obtained the unit step response of the closed loop system as plotted in Figs. 3.14, 3.15, respectively.

In the case of discrete-time control systems, one may take an alternative approach to find the non-overshooting transient response region. Let us write the realization of the error transfer function $E(z, \mathbf{K})$, as

$$\begin{aligned} x(k+1) &= A(\mathbf{K})x(k) + Br(k), \\ e(k) &= C(\mathbf{K})x(k) + Dr(k), \end{aligned} \tag{3.56}$$

where $r(k) = 1, \ \forall k \geq 0$ is the unit step input, in this problem, and $\mathbf{K}$ is the vector of the controller parameters. The transient response is non-negative provided that

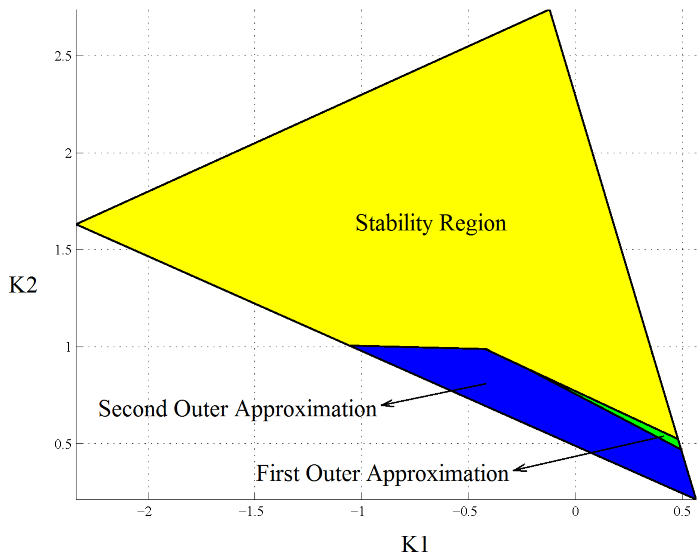

Figure 3.13: Second outer approximation at $k_3 = 1$

$e(k) \geq 0$, $\forall k \geq 0$, which is (assume that $x(0) = 0$)

$$\begin{aligned} e(0) &= D \geq 0, \\ e(1) &= C(\mathbf{K})B + D \geq 0, \\ e(2) &= C(\mathbf{K})(A(\mathbf{K})B + B) + D \geq 0, \\ &\vdots \end{aligned} \tag{3.57}$$

where the inequality corresponding to $e(1) \geq 0$ is linear in $\mathbf{K}$, and the one corresponding to $e(2) \geq 0$ becomes quadratic in $\mathbf{K}$ and so on.

In order to compare the method proposed in this chapter to this alternative approach we obtained $A(\mathbf{K}), B, C(\mathbf{K}), D$ in (3.56) by realizing the error transfer

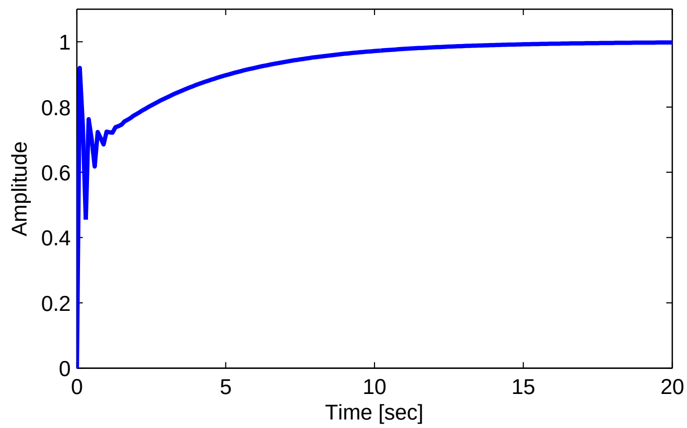

Figure 3.14: Step response of the closed loop system using the controller $k_1 = -0.8, k_2 = 0.92, k_3 = 1$

function $E(z, \mathbf{K})$, for this example, as

$$A(\mathbf{K}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ .5(k_3 - k_2) & k_3 - k_2 - .5k_1 & -k_1 - .5k_2 - .1 & 1.1 - k_2 \end{bmatrix},$$

$$B = [0, 0, 0, 1]^T,$$

$$C(\mathbf{K}) = [.5(k_3 - k_2), k_3 - k_2 - .5k_1, -k_1 - .5k_2, -k_2],$$

$$D = 1. \tag{3.58}$$

Considering the system of inequalities (3.57), it can be easily verified that $e(0) =$

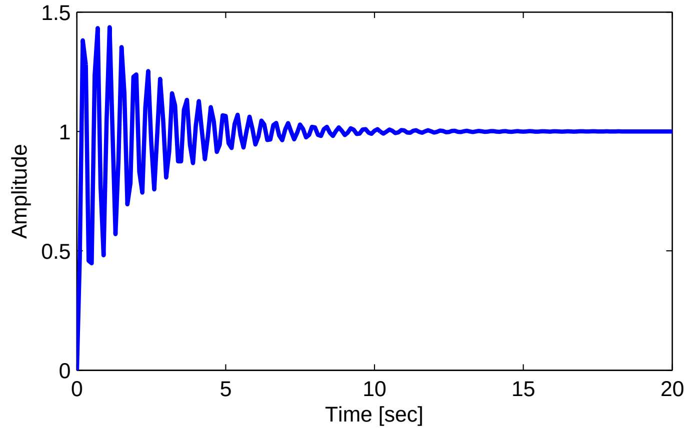

Figure 3.15: Step response of the closed loop system using the controller $k_1 = 0.38, k_2 = 0.47, k_3 = 1$

$D = 1 \geq 0$. The linear inequality corresponding to $e(1) \geq 0$ will be

$$-k_2 + 1 \geq 0, \tag{3.59}$$

which is plotted as the hatched region in Fig. 3.16.

As can be seen in Fig. 3.16, the alternative method cuts off a smaller region, from the stabilizing set, compared to the first outer approximation computed through our proposed approach.

3.4.4 Discrete-time Systems: Response within an Envelope

Recall the previous example and suppose that the transient specification requires that the unit step response of the closed loop system to lie within the envelope shown in Fig. 3.17.

The entire set of stabilizing controller gains (k_1, k_2, k_3) is obtained in the previous

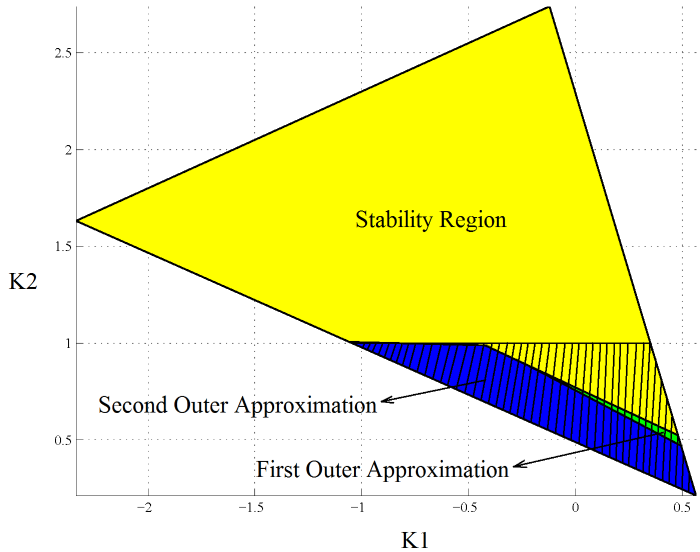

Figure 3.16: Comparing our proposed approach with the alternative approach

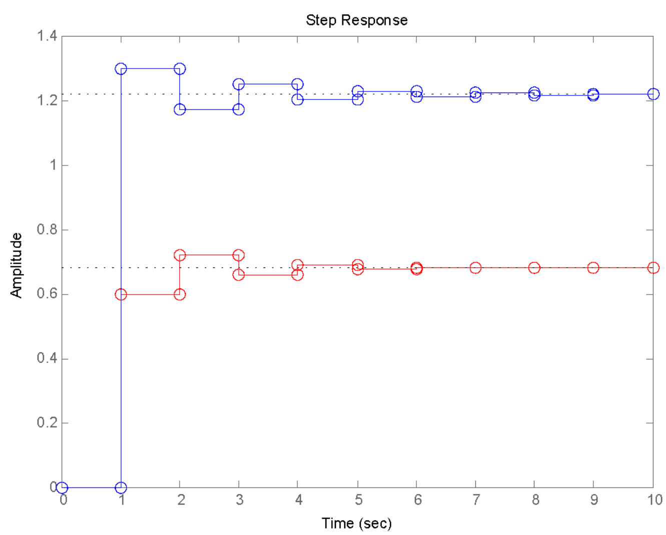

Figure 3.17: The defined envelope

example. Let us fix the value of k_3 to $k_3^* = 1$. At this k_3 value, the stability region can be expressed by the following set of linear inequalities in k_1 and k_2:

$$\begin{aligned} 0.4k_1 + 0.9k_2 - 0.4 &> 0, \\ k_1 + 0.3k_2 - 0.6 &< 0, \\ 0.4k_1 - 0.9k_2 + 2.5 &> 0, \end{aligned} \tag{3.60}$$

which is plotted in Fig. 3.18.

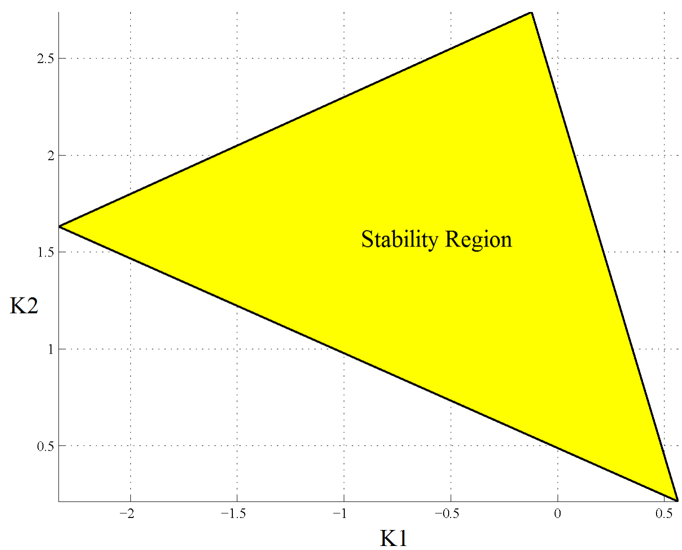

Figure 3.18: Stability region at $k_3 = 1$ (revisited)

In order to compute the set of desired digital PID controllers for which the transient specification is satisfied, appropriate error transfer functions, defined with respect to the bounds of the envelope shown in Fig. 3.17, and their derivatives must be non-negative for $|z| \geq 1$. For instance, the error transfer function to a unit step input signal with respect to the lower bound in Fig. 3.17 can be written as

$$\begin{aligned} E(z,\mathbf{K}) &= \frac{N_E(z,\mathbf{K})}{D_E(z,\mathbf{K})} \\ &= \frac{0.12(10z-1)(z-1)(10z+7)z}{(10z^4+(10k_2-11)z^3+(5k_2+1+10k_1)z^2+(-10k_3+10k_2+5k_1)z+5k_2-5k_3)(2z+1)}, \end{aligned} \tag{3.61}$$

which is non-negative on the interval $[1,\infty)$. The first outer approximation, $\mathcal{S}^1_{outer}$, of the set of all stabilizing digital PID controllers which render the step response of the closed system above the lower bound of the envelope in Fig. 3.17 can be obtained by enforcing $E_1(z,\mathbf{K})$, as defined in (3.28), to be non-negative on the interval $[1,\infty)$. This means that

$$\begin{aligned} N_{E_1}(z,\mathbf{K}) \;=\;& 240z^9 - 192z^8 + (-316.8 - 240k_1 - 336k_2)\, z^7 \\ &+ (236.16 - 480k_1 - 921.6k_2 + 480k_3)\, z^6 \\ &+ (46.92 - 244.8k_1 - 710.4k_2 + 624k_3)\, z^5 \\ &+ (81.6k_1 - 14.4k_2 + 48k_3 - 15.12)\, z^4 \\ &+ (57k_1 + 253.8k_2 - 249.6k_3 + 0.84)\, z^3 \\ &+ (80.4k_2 - 80.4k_3)\, z^2 + (-4.2k_2 + 4.2k_3)\, z \geq 0,\;\; \forall z \in [1,\infty). \end{aligned} \tag{3.62}$$

This requirement can be satisfied through the existence of positive semi-definite matrices:

$$F(\mathbf{v}) = \begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 \\ v_2 & v_6 & v_7 & v_8 & v_9 \\ v_3 & v_7 & v_{10} & v_{11} & v_{12} \\ v_4 & v_8 & v_{11} & v_{13} & v_{14} \\ v_5 & v_9 & v_{12} & v_{14} & v_{15} \end{bmatrix} \succeq 0, \tag{3.63}$$

$$G(\mathbf{w}) = \begin{bmatrix} w_1 & w_2 & w_3 & w_4 & w_5 \\ w_2 & w_6 & w_7 & w_8 & w_9 \\ w_3 & w_7 & w_{10} & w_{11} & w_{12} \\ w_4 & w_8 & w_{11} & w_{13} & w_{14} \\ w_5 & w_9 & w_{12} & w_{14} & w_{15} \end{bmatrix} \succeq 0, \tag{3.64}$$

where the entries of the matrices $F(\mathbf{v})$ and $G(\mathbf{w})$ are related to the controller parameters by the following set of linear equations:

$$\begin{aligned}
v_1 - w_1 &= 0, \\
2v_2 - 2w_2 + w_1 &= -4.2k_2 + 4.2k_3, \\
-2w_3 - w_6 + 2w_2 + v_6 + 2v_3 &= 80.4k_2 - 80.4k_3, \\
2v_7 + 2v_4 - 2w_4 - 2w_7 + 2w_3 + w_6 &= 57k_1 + 253.8k_2 - 249.6k_3 + 0.84, \\
v_{10} + 2v_8 + 2v_5 - 2w_5 & \\
-2w_8 - w_{10} + 2w_4 + 2w_7 &= 81.6k_1 - 14.4k_2 + 48k_3 - 15.12, \\
2v_9 + 2v_{11} - 2w_9 - 2w_{11} + 2w_5 + 2w_8 + w_{10} &= -244.8k_1 - 710.4k_2 + 624k_3 + 46.92, \\
-2w_{12} - w_{13} + 2w_9 + 2w_{11} + 2v_{12} + v_{13} &= -480k_1 - 921.6k_2 + 480k_3 + 236.16, \\
-2w_{14} + 2w_{12} + w_{13} + 2v_{14} &= -240k_1 - 336k_2 - 316.8, \\
v_{15} - w_{15} + 2w_{14} &= -192, \\
w_{15} &= 240.
\end{aligned} \tag{3.65}$$

The set of all feasible (k_1, k_2, k_3) satisfying (3.60), (3.63)-(3.65) forms the first outer approximation $\mathcal{S}^1_{outer}$, of the desired set $\mathcal{S}_{des}$, with respect to the lower bound of the envelope. A corresponding set of constraints, as obtained in (3.63)-(3.65), can be derived with respect to the upper bound of the envelope as well. The second outer approximation can be constructed by adding the non-negativity condition for the second derivative of the appropriate error transfer function, defined with respect to the bounds of the envelope, to the feasibility problem defined for the first outer approximation.

The first and second outer approximations for this example are plotted in Fig. 3.19.

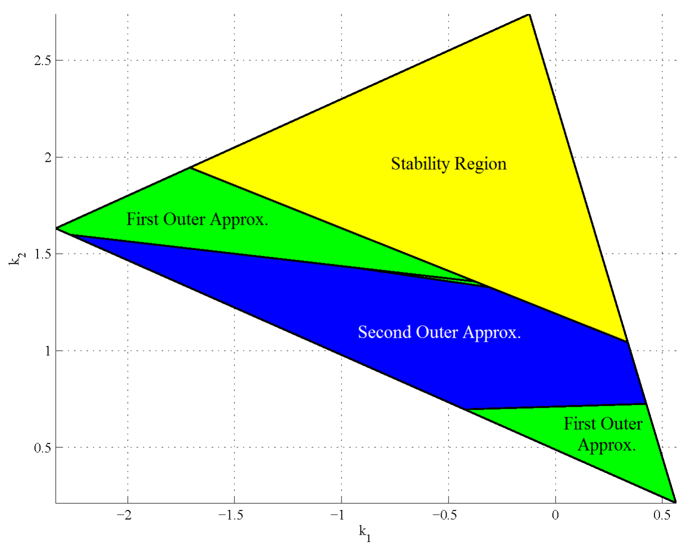

Figure 3.19: First and second outer approximations at $k_3 = 1$

Let us now pick the following 3 stabilizing controllers:

$$\text{Controller 1}: \quad k_1 = -0.5,\ k_2 = 1.05,\ k_3 = 1, \tag{3.66}$$

$$\text{Controller 2}: \quad k_1 = -1,\ k_2 = 1.6,\ k_3 = 1, \tag{3.67}$$

$$\text{Controller 3}: \quad k_1 = -0.5,\ k_2 = 1.7,\ k_3 = 1, \tag{3.68}$$

where Controller 1 is inside the second outer approximation, Controller 2 is inside the first outer approximation but outside of the second approximation, and Controller 3 is outside of the both approximations. We obtained the unit step response of the closed loop system corresponding to (3.66), (3.67) and (3.68) as plotted, by square markers, in Figs. 3.20, 3.21 and 3.22, respectively. In these figures the envelope defined in Fig. 3.17 is also shown by circle markers.

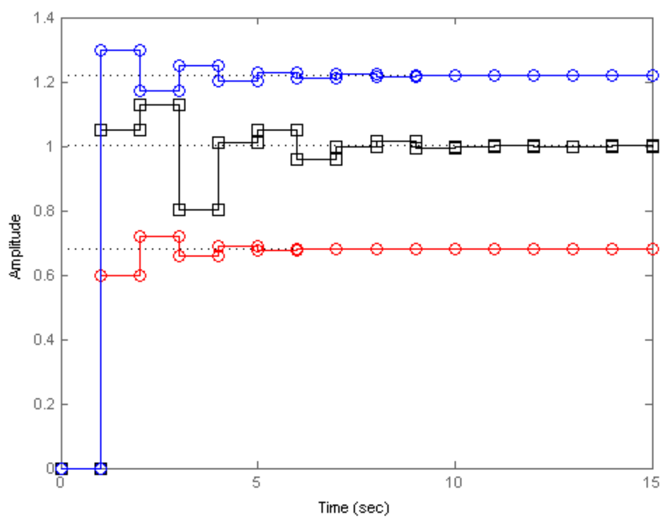

Figure 3.20: Step response of the closed loop system using the controller $k_1 = -0.5, k_2 = 1.05, k_3 = 1$

3.5 Concluding Remarks

In this chapter, we proposed a method to construct an outer approximation of the set of all stabilizing PID controllers for the class of continuous-time and discrete-time LTI control systems guaranteeing transient response specifications. This is

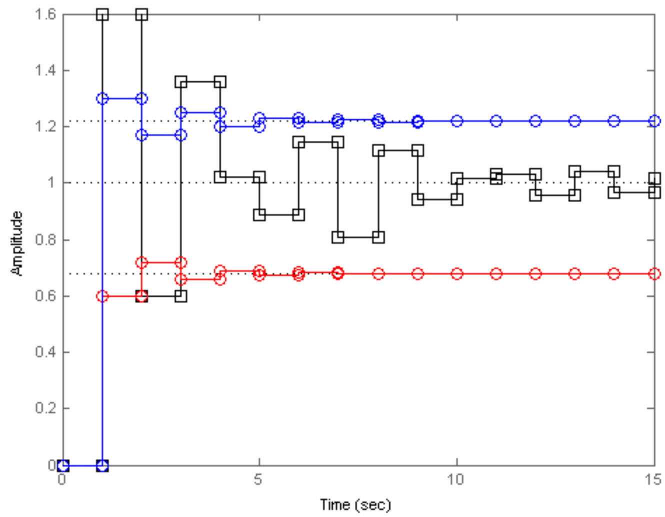

Figure 3.21: Step response of the closed loop system using the controller $k_1 = -1, k_2 = 1.6, k_3 = 1$

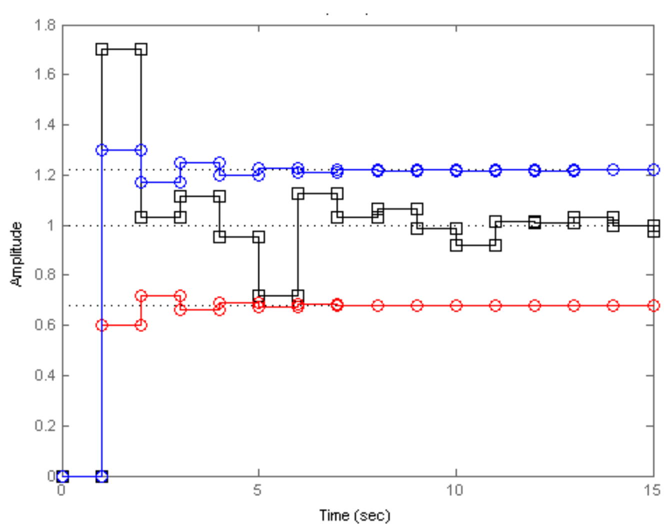

Figure 3.22: Step response of the closed loop system using the controller $k_1 = -0.5, k_2 = 1.7, k_3 = 1$

accomplished by solving a sequence of Semi-Definite Programs (SDPs) developed based on the Widder's theorem, its counterpart for discrete-time transfer functions, and Markov-Lukacs theorem. We also presented a technique to tighten the outer approximation of interest arbitrarily.

4. LINEAR EQUATIONS WITH PARAMETERS*

This chapter describes some basic results on the solution of linear equations containing parameters and the properties of the parametrized solutions.

4.1 Introduction

Consider the system of linear equations

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{4.1}$$

where $\mathbf{A}$ is an $n \times n$ matrix, and $\mathbf{x}$ and $\mathbf{b}$ are $n \times 1$ vectors all with real or complex entries. Let $|.|$ denotes the determinant. Assuming that $|\mathbf{A}| \neq 0$, there exists a unique solution $\mathbf{x}$ and, by Cramer's rule, the i^{th} component x_i of $\mathbf{x}$ is given by

$$x_i = \frac{|\mathbf{A}^i(\mathbf{b})|}{|\mathbf{A}|}, \qquad i = 1, 2, \ldots, n, \tag{4.2}$$

where $\mathbf{A}^i(\mathbf{b})$ is the matrix obtained by replacing the i^{th} column of $\mathbf{A}$ by $\mathbf{b}$.

In many physical problems, $\mathbf{A}$ and $\mathbf{b}$ contain parameters that need to be chosen or designed, as illustrated in the example below.

Example 4.1. Consider the circuit in Fig. 4.1. V is the ideal voltage source, I is the ideal current source, R_1, R_2, R_3 are linear resistors, and R_4 is a gyrator resistance. The gyrator is a linear two port device where the instantaneous currents and the instantaneous voltages are related by $V_2 = R_4 I_2$ and $V_1 = -R_4 I_3$. V_{amp}

*Part of the data reported in this chapter is reprinted with permission from "Linear Circuits: A Measurement Based Approach" by N. Mohsenizadeh, H. Nounou, M. Nounou, A. Datta and S. P. Bhattacharyya, 2013, *Int. J. Circ. Theor. Appl.*, Copyright 2013 John Wiley & Sons, Ltd.

is the dependent voltage of the amplifier where $V_{\text{amp}} = KI_1$, and K represents the amplifier gain. The equations of the system can be written in the following matrix form by applying Kirchhoff's laws,

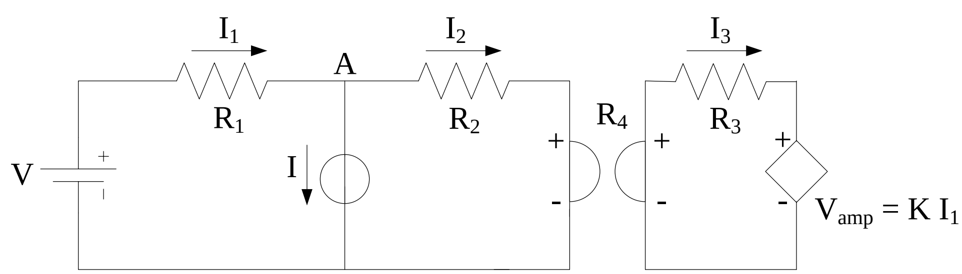

Figure 4.1: A motivational circuit example

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 \\ R_1 & R_2 & -R_4 \\ K & -R_4 & R_3 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} I \\ V \\ 0 \end{bmatrix}}_{\mathbf{b}} . \tag{4.3}$$

To fix notation, we introduce the *parameter* vector $\mathbf{p}$ and the vector of *sources* $\mathbf{q}$:

$$\mathbf{p} := \begin{bmatrix} R_1 \\ R_2 \\ R_3 \\ R_4 \\ K \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{bmatrix} \quad \text{and} \quad \mathbf{q} := \begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}, \tag{4.4}$$

so that (4.1) can be rewritten showing explicitly the dependence on the parameter vector $\mathbf{p}$ and the source vector $\mathbf{q}$ as

$$\mathbf{A}(\mathbf{p})\mathbf{x} = \mathbf{b}(\mathbf{q}). \tag{4.5}$$

Thus, (4.2) can also be rewritten explicitly showing the parametrized solution as

$$x_i(\mathbf{p}, \mathbf{q}) = \frac{|\mathbf{A}^i(\mathbf{p}, \mathbf{b}(\mathbf{q}))|}{|\mathbf{A}(\mathbf{p})|} := \frac{|\mathbf{B}_i(\mathbf{p}, \mathbf{q})|}{|\mathbf{A}(\mathbf{p})|}, \quad i = 1, 2, \ldots, n. \tag{4.6}$$

Furthermore, if $y(\mathbf{p}, \mathbf{q}) = \mathbf{c}^T\mathbf{x}(\mathbf{p}, \mathbf{q}) = c_1 x_1(\mathbf{p}, \mathbf{q}) + \cdots + c_n x_n(\mathbf{p}, \mathbf{q})$ is an output of interest, it follows that

$$y(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{n} c_i \left(\frac{|\mathbf{B}_i(\mathbf{p}, \mathbf{q})|}{|\mathbf{A}(\mathbf{p})|} \right). \tag{4.7}$$

4.2 Parametrized Solutions

Motivated by the above example we consider henceforth the general representation of an arbitrary linear system to be given by (4.5), (4.6) and (4.7). To develop the formula (4.6) in more detail, we note that in (4.3) the parameter $\mathbf{p}$ appears *affinely* in $\mathbf{A}(\mathbf{p})$. Thus, we can write

$$\mathbf{A}(\mathbf{p}) = \mathbf{A}_0 + p_1\mathbf{A}_1 + p_2\mathbf{A}_2 + \cdots + p_l\mathbf{A}_l. \tag{4.8}$$

To proceed, consider the special case of a scalar parameter $\mathbf{p} = p_1$ and

$$\mathbf{A}(\mathbf{p}) = \mathbf{A}_0 + p_1\mathbf{A}_1. \tag{4.9}$$

Lemma 4.1. *With* $\mathbf{A}(\mathbf{p})$ *as in* (4.9), $|\mathbf{A}(\mathbf{p})|$ *is a polynomial of degree at most* r_1 *in* p_1 *where*

$$r_1 = \text{rank}\left[\mathbf{A}_1\right]. \tag{4.10}$$

Proof. The proof follows easily from the properties of determinants. □

Lemma 4.2. *With* $\mathbf{A}(\mathbf{p})$ *as in* (4.8), *let*

$$r_i = \text{rank}\left[\mathbf{A}_i\right], \quad i = 1, 2, \dots, l. \tag{4.11}$$

Then, $|\mathbf{A}(\mathbf{p})|$ *is a multivariate polynomial in* $\mathbf{p}$ *of degree* r_i *or less in* p_i, $i = 1, 2, \dots, l$ *and*

$$|\mathbf{A}(\mathbf{p})| = \sum_{i_l=0}^{r_l} \cdots \sum_{i_2=0}^{r_2} \sum_{i_1=0}^{r_1} \alpha_{i_1 i_2 \cdots i_l} p_1^{i_1} p_2^{i_2} \cdots p_l^{i_l} := \alpha(\mathbf{p}). \tag{4.12}$$

Also, if the parameter $\mathbf{q}$ *is fixed, say* $\mathbf{q} = \mathbf{q}_0$, *then*

$$|\mathbf{B}_i(\mathbf{p}, \mathbf{q}_0)| = \sum_{i_l=0}^{t_l} \cdots \sum_{i_2=0}^{t_2} \sum_{i_1=0}^{t_1} \beta_{i_1 i_2 \cdots i_l} p_1^{i_1} p_2^{i_2} \cdots p_l^{i_l} := \beta_i(\mathbf{p}, \mathbf{q}_0), \tag{4.13}$$

where $\mathbf{B}_i(\mathbf{p}, \mathbf{q}_0)$ *is the matrix obtained by replacing the* i^{th} *column of* $\mathbf{A}(\mathbf{p})$, *in* (4.5), *by the vector* $\mathbf{b}(\mathbf{q}_0)$, *and*

$$t_i = \text{rank}\left[\mathbf{B}_i\right] \leq r_i, \quad i = 1, 2, \dots, l. \tag{4.14}$$

Proof. This follows immediately from Lemma 4.1. □

Remark 4.1. *In the formula* (4.12), *the number of coefficients* $\alpha_{i_1 i_2 \cdots i_l}$ *are*

$$\prod_{i=1}^{l} (r_i + 1). \tag{4.15}$$

Based on the above formula, we have the following characterization of parametrized solutions.

Theorem 4.1. *With* $\mathbf{A}(\mathbf{p})$ *as in* (4.8),

$$x_i(\mathbf{p}, \mathbf{q}_0) = \frac{\beta_i(\mathbf{p}, \mathbf{q}_0)}{\alpha(\mathbf{p})}, \quad i = 1, 2, \ldots, n, \tag{4.16}$$

where $\beta_i(\mathbf{p}, \mathbf{q}_0)$, $i = 1, 2, \ldots, n$, *and* $\alpha(\mathbf{p})$ *are multivariate polynomials in* $\mathbf{p}$.

Proof. The proof follows from (4.6) and Lemma 4.2. □

4.3 Concluding Remarks

In this chapter we explored some properties of the parametrized solutions of sets of linear equations. We expressed these parametrized solutions based on specific type of parameter dependence. These results will be useful in subsequent chapters to develop a measurement based approach to the design and control of linear systems.

5. APPLICATION TO DC CIRCUITS*

In a circuit analysis problem, one can calculate all the currents knowing the circuit model, by applying Kirchhoff's laws [50, 51, 52] and solving a set of linear equations. If the circuit model is unavailable, which is usually the case in practical applications, one can resort to determining the circuit currents by extensive experiments.

In this chapter, we present an alternative new approach which can determine the functional dependency of any circuit variable with respect to any set of design parameters directly from a small set of measurements. The obtained functional dependency can then be used to solve a synthesis problem wherein one or more circuit variables are to be controlled by adjusting the design parameters. We use the results obtained in Chapter 4 to develop this new measurement based approach [35, 36]. Here, we consider current and power level control problems. A similar approach can be used for voltage control problems.

5.1 Current Control

Consider a circuit design problem wherein the current in any branch of an *unknown* linear DC circuit is to be controlled by a set of design elements at arbitrary locations of the circuit. We consider several cases for the set of design parameters, such as a single or multiple resistors, sources and amplifier gains.

The governing equations of a linear DC circuit can be written in the following

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matrix form

$$\mathbf{A}(\mathbf{p})\mathbf{x} = \mathbf{b}(\mathbf{q}), \tag{5.1}$$

where $\mathbf{A}(\mathbf{p})$ is called the circuit characteristic matrix, $\mathbf{p}$ is the vector of circuit *parameters*, including resistors, amplifier gains, gyrators, but excluding independent voltage and current sources, $\mathbf{x}$ represents the vector of unknown currents and $\mathbf{q}$ denotes the vector of independent voltage and current *sources*. The vector $\mathbf{b}(\mathbf{q})$ can be written as

$$\mathbf{b}(\mathbf{q}) = q_1\mathbf{b}_1 + q_2\mathbf{b}_2 + \cdots + q_m\mathbf{b}_m, \tag{5.2}$$

where $q_1, q_2, \ldots, q_m$ are the independent sources. Suppose that we want to control the current I_i, in the i-th branch of the circuit. An application of the Cramer's rule to (5.1) yields

$$x_i = I_i = \frac{|\mathbf{B}_i(\mathbf{p}, \mathbf{q})|}{|\mathbf{A}(\mathbf{p})|}, \tag{5.3}$$

where $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ is the matrix obtained by replacing the i-th column of the characteristic matrix $\mathbf{A}(\mathbf{p})$ by the vector $\mathbf{b}(\mathbf{q})$. We emphasize that in an unknown circuit the matrices $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ and $\mathbf{A}(\mathbf{p})$ are unknown. However, based on Lemma 4.2 and (5.2), if the ranks of the parameters are known, a general rational function for I_i, in terms of the design parameters, can be derived as

$$I_i = \frac{\sum_{j_m=0}^{1} \cdots \sum_{j_1=0}^{1} \sum_{i_l=0}^{t_l} \cdots \sum_{i_1=0}^{t_1} \alpha_{i_1 \cdots i_l j_1 \cdots j_m} \, p_1^{i_1} \cdots p_l^{i_l} q_1^{j_1} \cdots q_m^{j_m}}{\sum_{i_l=0}^{r_l} \cdots \sum_{i_1=0}^{r_1} \beta_{i_1 \cdots i_l} \, p_1^{i_1} \cdots p_l^{i_l}}. \tag{5.4}$$

In the above formula, α's and β's are constants, $t_1, \ldots, t_l$ are the ranks of the coefficient matrices of the parameters $p_1, \ldots, p_l$ in the matrix $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$, and $r_1, \ldots, r_l$ are the ranks of the coefficient matrices of the parameters $p_1, \ldots, p_l$ in the matrix $\mathbf{A}(\mathbf{p})$.

5.1.1 Current Control using a Single Resistor

Consider the unknown linear DC circuit shown in Fig. 5.1. Assume that the objective is to control I_i, the current in the i-th branch, by adjusting the resistor R_j at an arbitrary location. In general, R_j will appear in $\mathbf{A}$, in (5.1), with rank 1 dependency, unless it is a gyrator resistance, in which case the rank dependency is 2.

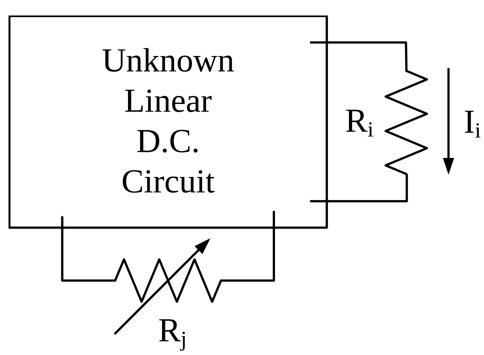

Figure 5.1: An unknown linear DC circuit for Section 5.1.1

Theorem 5.1. *In a linear DC circuit, the functional dependency of any current I_i on any resistance R_j can be determined by at most 3 measurements of the current I_i obtained for 3 different values of R_j.*

Proof. Consider two cases: 1) $i \neq j$, and 2) $i = j$.

Case 1: $i \neq j$

In this case, $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ and $\mathbf{A}(\mathbf{p})$, in (5.3), are both of rank 1 with respect to R_j. According to the statement of Lemma 4.1, the functional dependency of I_i on R_j, i.e. $I_i(R_j)$, can be expressed as

$$I_i(R_j) = \frac{\tilde{\alpha_0} + \tilde{\alpha_1} R_j}{\tilde{\beta_0} + \tilde{\beta_1} R_j}, \tag{5.5}$$

where $\tilde{\alpha_0}, \tilde{\alpha_1}, \tilde{\beta_0}, \tilde{\beta_1}$ are constants. If $\tilde{\beta_0} = \tilde{\beta_1} = 0$, then, $I_i \to \infty$, for any value of the resistance R_j, which is physically impossible. Therefore, we rule out this case. Assuming that $\tilde{\beta_1} \neq 0$, one can divide the numerator and denominator of (5.5) by $\tilde{\beta_1}$ and obtain

$$I_i(R_j) = \frac{\alpha_0 + \alpha_1 R_j}{\beta_0 + R_j}, \tag{5.6}$$

where $\alpha_0, \alpha_1, \beta_0$ are constants. In order to determine $\alpha_0, \alpha_1, \beta_0$ one conducts 3 experiments by setting 3 different values to the resistance R_j, say R_{j1}, R_{j2}, R_{j3}, and measuring the corresponding currents I_i, say I_{i1}, I_{i2}, I_{i3}. Then, the following set of measurement equations can be formed

$$\underbrace{\begin{bmatrix} 1 & R_{j1} & -I_{i1} \\ 1 & R_{j2} & -I_{i2} \\ 1 & R_{j3} & -I_{i3} \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} I_{i1}R_{j1} \\ I_{i2}R_{j2} \\ I_{i3}R_{j3} \end{bmatrix}}_{\mathbf{m}}. \tag{5.7}$$

This set can be uniquely solved for the unknown constants $\alpha_0, \alpha_1, \beta_0$, if and only if $|\mathbf{M}| \neq 0$. If $|\mathbf{M}| = 0$, then the last column of the matrix $\mathbf{M}$ can be written as a linear combination of the first two columns because by assigning different values to the resistance R_j, the first two columns of $\mathbf{M}$ become linearly independent. In this case, $I_i(R_j)$ will be

$$I_i(R_j) = \alpha_0 + \alpha_1 R_j, \tag{5.8}$$

where α_0, α_1 are constants that can be determined from any two of the experiments conducted earlier. The functional dependency in (5.8) corresponds to the case where $\tilde{\beta}_1 = 0$ in (5.5), and the numerator and denominator of (5.5) are divided by $\tilde{\beta}_0$.

Case 2: $i = j$

Here, $\mathbf{A}(\mathbf{p})$ is of rank 1 with respect to R_i; however, $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ is of rank 0 with respect to R_i. Based on Lemma 4.1, $I_i(R_i)$ can be expressed as

$$I_i(R_i) = \frac{\tilde{\alpha_0}}{\tilde{\beta}_0 + \tilde{\beta}_1 R_i}, \tag{5.9}$$

where $\tilde{\alpha_0}, \tilde{\beta}_0, \tilde{\beta}_1$ are constants. Assuming $\tilde{\beta}_1 \neq 0$, and dividing the numerator and denominator of (5.9) by $\tilde{\beta}_1$, gives

$$I_i(R_i) = \frac{\alpha_0}{\beta_0 + R_i}, \tag{5.10}$$

where α_0, β_0 are constants that can be determined by conducting 2 experiments, by setting 2 different values to the resistance R_i, say R_{i1}, R_{i2}, and measuring the corresponding currents I_i, say I_{i1}, I_{i2}. The following set of measurement equations can then be formed

$$\underbrace{\begin{bmatrix} 1 & -I_{i1} \\ 1 & -I_{i2} \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} I_{i1}R_{i1} \\ I_{i2}R_{i2} \end{bmatrix}}_{\mathbf{m}}, \tag{5.11}$$

which has a unique solution for α_0, β_0 if $|\mathbf{M}| \neq 0$. For the case where $|\mathbf{M}| = 0$ in (5.11), it can be easily seen that I_i is a constant,

$$I_i(R_i) = \alpha_0, \tag{5.12}$$

where this constant can be determined from any prior measurements. The functional dependency in (5.12) corresponds to the situation where $\tilde{\beta}_1 = 0$ in (5.9), and the numerator and denominator of (5.9) are divided by $\tilde{\beta}_0$. □

Remark 5.1. *Suppose that $i \neq j$ and $|\mathbf{M}| \neq 0$ in* (5.7)*, then taking the derivative of $I_i(R_j)$ in* (5.6)*, with respect to R_j, yields*

$$\frac{dI_i}{dR_j} = \frac{\alpha_1\beta_0 - \alpha_0}{(\beta_0 + R_j)^2}. \tag{5.13}$$

If $\beta_0 \geq 0$, we have the following:

1. *The function* (5.6) *is monotonic in R_j, i.e. $I_i(R_j)$ monotonically increases or decreases as R_j increases from* 0 *to large values. The upper and lower bounds are: $I_i(0) = \frac{\alpha_0}{\beta_0}$ and $I_i(\infty) = \alpha_1$. If $\frac{\alpha_0}{\beta_0} > \alpha_1$, then* (5.6) *will monotonically decrease, and if $\frac{\alpha_0}{\beta_0} < \alpha_1$, then* (5.6) *will monotonically increase.*

2. *The achievable range for I_i, by varying R_j in the interval* $[0, \infty)$, *is*

$$\min\left\{\frac{\alpha_0}{\beta_0}, \alpha_1\right\} < I_i < \max\left\{\frac{\alpha_0}{\beta_0}, \alpha_1\right\}. \tag{5.14}$$

3. *In a current control problem of this type, this monotonic behavior allows us to uniquely determine a range of values of the design parameter R_j, $R_j^- \leq R_j \leq R_j^+$, for which the current I_i lies within a desired prescribed range, $I_i^- \leq I_i \leq I_i^+$, which of course must be within the achievable range* (5.14).

These observations also are clear from the graph of (5.6). *For instance, if $\beta_0 > 0$, $\alpha_0 < 0$ and $\alpha_1 > 0$, the graph of* (5.6) *has the general shape as depicted below (see Fig. 5.2).*

Thevenin's Theorem (the special case $i = j$): Thevenin's Theorem of circuit theory (see [53, 54, 55]) follows as a special case of the results developed here. To see this, consider the current functional dependency given in (5.10). From this relationship, it is clear that the short circuit current I_{sc} is given by $I_{sc} = \frac{\alpha_0}{\beta_0}$, which is obtained by setting $R_i = 0$. Similarly, the open circuit voltage V_{oc} is obtained by multiplying both sides of (5.10) by R_i and taking the limit as $R_i \to \infty$. This yields $V_{oc} = V_{Th} = \alpha_0$. Thus, the Thevenin resistance is given by $R_{Th} = \frac{V_{oc}}{I_{sc}} = \beta_0$, so that

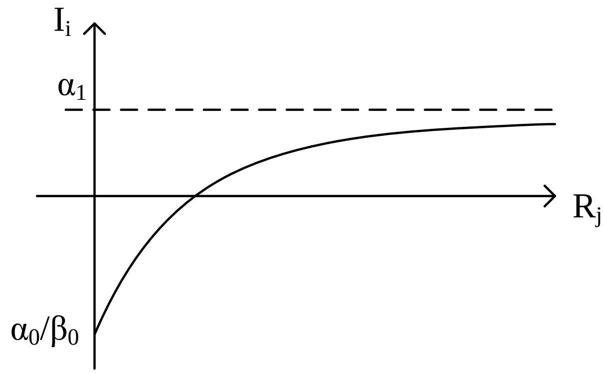

Figure 5.2: Graph of (5.6) for $\beta_0 > 0$, $\alpha_0 < 0$ and $\alpha_1 > 0$

(5.10) becomes

$$I_i(R_i) = \frac{V_{Th}}{R_{Th} + R_i}, \tag{5.15}$$

which is exactly Thevenin's Theorem. We point out that in our approach, it is not necessary to measure short circuit current or open circuit voltage; indeed two *arbitrary* measurements suffice. This has practical and useful implications in circuits where short circuiting and open circuiting may sometimes be impossible. Theorem 5.1 and the subsequent results in this chapter represent generalizations of Thevenin's Theorem.

Current Assignment Problem: After obtaining the desired functional dependency, one of the forms in (5.6), (5.8), (5.10) or (5.12), a synthesis problem can be solved. For example, suppose that it is desirable to assign $I_i = I_i^*$ using R_j, and $i \neq j$. Based on the statement of Theorem 5.1, and the fact that $i \neq j$, one may conduct 3 experiments by setting 3 different values to R_j, and measuring the corresponding I_i.

The matrix $\mathbf{M}$ in (5.7) can then be evaluated from the measurements. If $|\mathbf{M}| \neq 0$, then the functional dependency will be of the form in (5.6), and if $|\mathbf{M}| = 0$, then (5.8) is the functional dependency. Assume that $|\mathbf{M}| \neq 0$ is the case here; therefore, the functional dependency is of the form in (5.6). In order to determine the value of R_j, for $I_i = I_i^*$ is attained, one may solve (5.6) for R_j, with $I_i = I_i^*$,

$$R_j(I_i^*) = \frac{\alpha_0 - I_i^* \beta_0}{I_i^* - \alpha_1}. \tag{5.16}$$

Interval Design Problem: Suppose now that the current I_i is to be controlled to lie within the following range by adjusting R_j, and $i \neq j$:

$$I_i^- \leq I_i \leq I_i^+, \tag{5.17}$$

Assume that the above range is inside the achievable range (5.14) and also after conducting 3 experiments, we got $|\mathbf{M}| \neq 0$ in (5.7) and $\beta_0 \geq 0$. This implies that $I_i(R_j)$ is of the form in (5.6) and thus is monotonic. One can find a unique corresponding interval for R_j values where (5.17) is met. Supposing that I_i, in (5.6), monotonically increases as R_j increases, one gets

$$R_j^- \leq R_j \leq R_j^+, \tag{5.18}$$

where

$$R_j^- = \frac{\alpha_0 - I_i^- \beta_0}{I_i^- - \alpha_1}, \qquad R_j^+ = \frac{\alpha_0 - I_i^+ \beta_0}{I_i^+ - \alpha_1}. \tag{5.19}$$

Following the same strategy, one can solve a design problem for the case $i = j$.

The problem of maintaining several currents in the circuit within prescribed intervals can be tackled in a similar way.

5.1.2 Current Control using Two Resistors

Suppose that the objective is to control the current I_i using any two resistors R_j and R_k at arbitrary locations (see Fig. 5.3). Assume that R_j and R_k are not gyrator resistances. We have the following theorem.

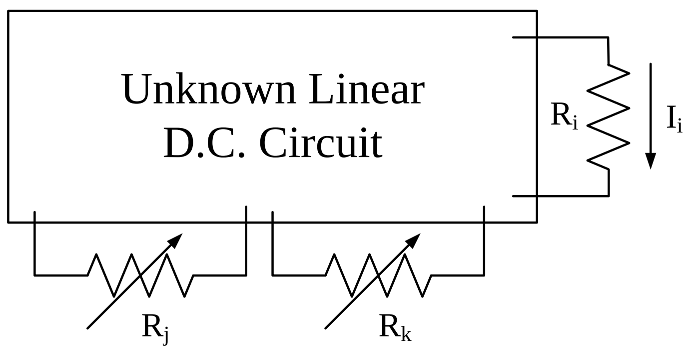

Figure 5.3: An unknown linear DC circuit for Section 5.1.2

Theorem 5.2. *In a linear DC circuit, the functional dependency of any current I_i on any two resistances R_j and R_k can be determined by at most 7 measurements of the current I_i obtained for 7 different sets of values (R_j, R_k).*

Proof. Consider two cases: 1) $i \neq j, k$ and 2) $i = j$ or $i = k$.

Case 1: $i \neq j, k$

In this case, $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ and $\mathbf{A}(\mathbf{p})$, in (5.3), are both of rank 1 with respect to R_j

and R_k. Using Lemma 4.2, the functional dependency of $I_i(R_j, R_k)$ will be

$$I_i(R_j, R_k) = \frac{\tilde{\alpha_0} + \tilde{\alpha_1} R_j + \tilde{\alpha_2} R_k + \tilde{\alpha_3} R_j R_k}{\tilde{\beta_0} + \tilde{\beta_1} R_j + \tilde{\beta_2} R_k + \tilde{\beta_3} R_j R_k}, \tag{5.20}$$

where $\tilde{\alpha_0}$, $\tilde{\alpha_1}$, $\tilde{\alpha_2}$, $\tilde{\alpha_3}$, $\tilde{\beta_0}$, $\tilde{\beta_1}$, $\tilde{\beta_2}$, $\tilde{\beta_3}$ are constants. Assuming that $\tilde{\beta_3} \neq 0$ and dividing the numerator and denominator of (5.20) by $\tilde{\beta_3}$, yields

$$I_i(R_j, R_k) = \frac{\alpha_0 + \alpha_1 R_j + \alpha_2 R_k + \alpha_3 R_j R_k}{\beta_0 + \beta_1 R_j + \beta_2 R_k + R_j R_k}, \tag{5.21}$$

where $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2$ are constants. In order to determine these 7 constants, one needs to conduct 7 experiments by assigning 7 different sets of values to the resistances (R_j, R_k), and measuring the corresponding I_i. The following set of measurement equations will be obtained

$$\underbrace{\begin{bmatrix} 1 & R_{j1} & R_{k1} & R_{j1}R_{k1} & -I_{i1} & -I_{i1}R_{j1} & -I_{i1}R_{k1} \\ 1 & R_{j2} & R_{k2} & R_{j2}R_{k2} & -I_{i2} & -I_{i2}R_{j2} & -I_{i2}R_{k2} \\ 1 & R_{j3} & R_{k3} & R_{j3}R_{k3} & -I_{i3} & -I_{i3}R_{j3} & -I_{i3}R_{k3} \\ 1 & R_{j4} & R_{k4} & R_{j4}R_{k4} & -I_{i4} & -I_{i4}R_{j4} & -I_{i4}R_{k4} \\ 1 & R_{j5} & R_{k5} & R_{j5}R_{k5} & -I_{i5} & -I_{i5}R_{j5} & -I_{i5}R_{k5} \\ 1 & R_{j6} & R_{k6} & R_{j6}R_{k6} & -I_{i6} & -I_{i6}R_{j6} & -I_{i6}R_{k6} \\ 1 & R_{j7} & R_{k7} & R_{j7}R_{k7} & -I_{i7} & -I_{i7}R_{j7} & -I_{i7}R_{k7} \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} I_{i1}R_{j1}R_{k1} \\ I_{i2}R_{j2}R_{k2} \\ I_{i3}R_{j3}R_{k3} \\ I_{i4}R_{j4}R_{k4} \\ I_{i5}R_{j5}R_{k5} \\ I_{i6}R_{j6}R_{k6} \\ I_{i7}R_{j7}R_{k7} \end{bmatrix}}_{\mathbf{m}}. \tag{5.22}$$

This set of equations has a unique solution if $|\mathbf{M}| \neq 0$ in (5.22). In the case where $|\mathbf{M}| = 0$, one can resort to the same procedure used in Section 5.1.1 to obtain the corresponding functional dependency $I_i(R_j, R_k)$. We provided the details of this case in the Appendix.

Case 2: $i = j$ or $i = k$

Suppose that $i = j$ and recall (5.3). Here, $\mathbf{A}(\mathbf{p})$ is of rank 1 with respect to R_i and R_k; however, $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ is of rank 0 with respect to R_i and is of rank 1 with respect to R_k. Using Lemma 4.2 and according to these rank conditions, $I_i(R_i, R_k)$ can be written as

$$I_i(R_i, R_k) = \frac{\tilde{\alpha_0} + \tilde{\alpha_1} R_k}{\tilde{\beta_0} + \tilde{\beta_1} R_i + \tilde{\beta_2} R_k + \tilde{\beta_3} R_i R_k}. \tag{5.23}$$

Assuming that $\tilde{\beta_3} \neq 0$, one can divide the numerator and denominator of (5.23) by $\tilde{\beta_3}$ and obtain

$$I_i(R_i, R_k) = \frac{\alpha_0 + \alpha_1 R_k}{\beta_0 + \beta_1 R_i + \beta_2 R_k + R_i R_k}, \tag{5.24}$$

where $\alpha_0, \alpha_1, \beta_0, \beta_1, \beta_2$ are constants that can be determined by conducting 5 experiments, by assigning 5 different sets of values to the resistances (R_i, R_k), and measuring the corresponding I_i. The following set of measurement equations can then be formed

$$\underbrace{\begin{bmatrix} 1 & R_{k1} & -I_{i1} & -I_{i1}R_{j1} & -I_{i1}R_{k1} \\ 1 & R_{k2} & -I_{i2} & -I_{i2}R_{j2} & -I_{i2}R_{k2} \\ 1 & R_{k3} & -I_{i3} & -I_{i3}R_{j3} & -I_{i3}R_{k3} \\ 1 & R_{k4} & -I_{i4} & -I_{i4}R_{j4} & -I_{i4}R_{k4} \\ 1 & R_{k5} & -I_{i5} & -I_{i5}R_{j5} & -I_{i5}R_{k5} \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} I_{i1}R_{j1}R_{k1} \\ I_{i2}R_{j2}R_{k2} \\ I_{i3}R_{j3}R_{k3} \\ I_{i4}R_{j4}R_{k4} \\ I_{i5}R_{j5}R_{k5} \end{bmatrix}}_{\mathbf{m}}. \tag{5.25}$$

Again, this set has a unique solution for $|\mathbf{M}| \neq 0$ in (5.25). If $|\mathbf{M}| = 0$, following the same strategy used in Section 5.1.1, one can derive the corresponding functional dependency for $I_i(R_i, R_k)$. The details of this case can be found in the Appendix. □

The current $I_i(R_j, R_k)$ can be plotted as a 3D surface. In a synthesis problem, any constraint on I_i results in a corresponding region in the R_j-R_k plane, if the solution set for that constraint is not empty.

5.1.3 Current Control using m Resistors

In this subsection, we want to control I_i by any m resistors R_j, $j = 1, 2, \ldots, m$, that are not gyrator resistances, at arbitrary locations (see Fig. 5.4).

Theorem 5.3. *In a linear DC circuit, the functional dependency of any current I_i on any m resistances R_j, $j = 1, 2, \ldots, m$, can be determined by at most $2^{m+1} - 1$ measurements of the current I_i obtained for $2^{m+1} - 1$ different sets on values of the vector $(R_1, R_2, \ldots, R_m)$.*

Proof. Consider two cases: 1) $i \neq j$ for $j = 1, 2, \ldots, m$, and 2) $i = j$ for some $j = 1, 2, \ldots, m$.

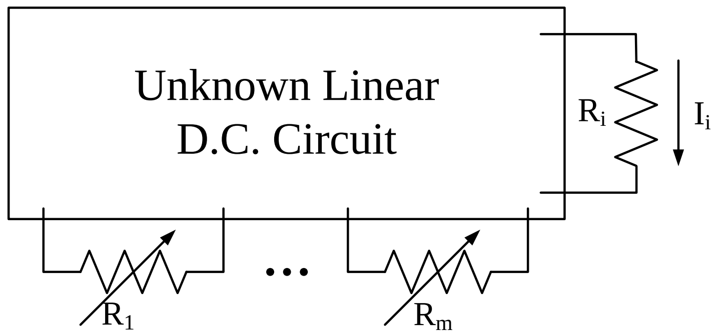

Figure 5.4: An unknown linear DC circuit for Section 5.1.3

Case 1: $i \neq j,\ j = 1, 2, \ldots, m$

In this case, $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ and $\mathbf{A}(\mathbf{p})$, in (5.3), are both of rank 1 with respect to R_j, $j = 1, 2, \ldots, m$. Based on Lemma 4.2, $I_i(R_1, R_2, \ldots, R_m)$ can be written as

$$I_i(R_1, R_2, \ldots, R_m) = \frac{\sum_{i_m=0}^{1} \cdots \sum_{i_2=0}^{1} \sum_{i_1=0}^{1} \tilde{\alpha}_{i_1 i_2 \cdots i_m}\ R_1^{i_1} R_2^{i_2} \cdots R_m^{i_m}}{\sum_{i_m=0}^{1} \cdots \sum_{i_2=0}^{1} \sum_{i_1=0}^{1} \tilde{\beta}_{i_1 i_2 \cdots i_m}\ R_1^{i_1} R_2^{i_2} \cdots R_m^{i_m}}, \tag{5.26}$$

where $\tilde{\alpha}_{i_1 i_2 \cdots i_m}$'s and $\tilde{\beta}_{i_1 i_2 \cdots i_m}$'s are constants. Assuming that $\tilde{\beta}_{11\cdots 1} \neq 0$ and dividing the numerator and denominator of (5.26) by $\tilde{\beta}_{11\cdots 1}$, gives

$$I_i(R_1, R_2, \ldots, R_m) = \frac{\sum_{i_m=0}^{1} \cdots \sum_{i_2=0}^{1} \sum_{i_1=0}^{1} \alpha_{i_1 i_2 \cdots i_m}\ R_1^{i_1} R_2^{i_2} \cdots R_m^{i_m}}{\sum_{i_m=0}^{1} \cdots \sum_{i_2=0}^{1} \sum_{i_1=0}^{1} \beta_{i_1 i_2 \cdots i_m}\ R_1^{i_1} R_2^{i_2} \cdots R_m^{i_m}}, \tag{5.27}$$

where $\beta_{11\cdots 1} = 1$, and $\alpha_{i_1 i_2 \cdots i_m}$'s and $\beta_{i_1 i_2 \cdots i_m}$'s are $2^{m+1} - 1$ constants. In order to determine these constants, one conducts $2^{m+1} - 1$ experiments.

Case 2: $i = j$ *for some* $j = 1, 2, \ldots, m$

Without loss of generality, suppose that $i = m$ and recall (5.3). In this case, $\mathbf{A}(\mathbf{p})$ is of rank 1 with respect to R_j, $j = 1, 2, \ldots, m$; however, the matrix $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$

is of rank 0 with respect to R_m and is of rank 1 with respect to R_j, $j = 1, 2, \ldots, m-1$. According to these rank conditions and based on Lemma 4.2, the functional dependency $I_i(R_1, R_2, \ldots, R_m)$ will be

$$I_i(R_1, R_2, \ldots, R_m) = \frac{\sum_{i_{m-1}=0}^{1} \cdots \sum_{i_2=0}^{1} \sum_{i_1=0}^{1} \tilde{\alpha}_{i_1 i_2 \cdots i_{m-1}} \; R_1^{i_1} R_2^{i_2} \cdots R_{m-1}^{i_{m-1}}}{\sum_{i_m=0}^{1} \cdots \sum_{i_2=0}^{1} \sum_{i_1=0}^{1} \tilde{\beta}_{i_1 i_2 \cdots i_m} \; R_1^{i_1} R_2^{i_2} \cdots R_m^{i_m}}, \quad (5.28)$$

where $\tilde{\alpha}_{i_1 i_2 \cdots i_{m-1}}$'s and $\tilde{\beta}_{i_1 i_2 \cdots i_m}$'s are constants. Supposing $\tilde{\beta}_{11\cdots1} \neq 0$, one can divide the numerator and denominator of (5.28) by $\tilde{\beta}_{11\cdots1}$ and get

$$I_i(R_1, R_2, \ldots, R_m) = \frac{\sum_{i_{m-1}=0}^{1} \cdots \sum_{i_2=0}^{1} \sum_{i_1=0}^{1} \alpha_{i_1 i_2 \cdots i_{m-1}} \; R_1^{i_1} R_2^{i_2} \cdots R_{m-1}^{i_{m-1}}}{\sum_{i_m=0}^{1} \cdots \sum_{i_2=0}^{1} \sum_{i_1=0}^{1} \beta_{i_1 i_2 \cdots i_m} \; R_1^{i_1} R_2^{i_2} \cdots R_m^{i_m}}, \quad (5.29)$$

where $\beta_{11\cdots1} = 1$, and there are $3\left(2^{m-1}\right) - 1$ constants. These constants can be determined by conducting $3\left(2^{m-1}\right) - 1$ experiments. □

5.1.4 Current Control using Gyrator Resistance

Now, suppose that the design element is the resistance of a gyrator. The gyrator resistance appears in the matrix $\mathbf{A}(\mathbf{p})$ with rank 2 dependency. Thus, we want to control I_i by a gyrator resistance, denoted by R_g, at an arbitrary location of the circuit.

Theorem 5.4. *In a linear DC circuit, the functional dependency of any current I_i on any gyrator resistance R_g can be determined by at most 5 measurements of the current I_i obtained for 5 different values of R_g.*

Proof. Consider the following two cases:

1) the i-th branch is not connected to either port of the gyrator (Fig. 5.5 left),

2) the i-th branch is connected to one port of the gyrator (Fig. 5.5 right).

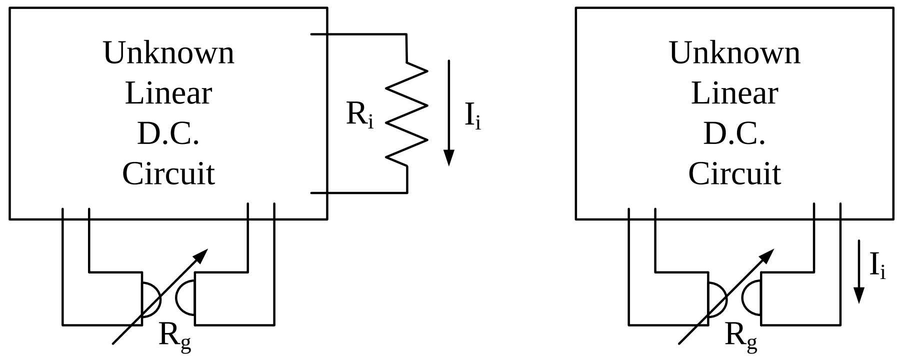

Figure 5.5: An unknown linear DC circuit for Section 5.1.4

Case 1: The i-th branch is not connected to either port of the gyrator

In this case, $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ and $\mathbf{A}(\mathbf{p})$, in (5.3), are both of rank 2 with respect to R_g. Therefore, according to Lemma 4.2, the functional dependency $I_i(R_g)$ can be written as

$$I_i(R_g) = \frac{\tilde{\alpha_0} + \tilde{\alpha_1} R_g + \tilde{\alpha_2} R_g^2}{\tilde{\beta}_0 + \tilde{\beta}_1 R_g + \tilde{\beta}_2 R_g^2}, \tag{5.30}$$

where $\tilde{\alpha_0}$, $\tilde{\alpha_1}$, $\tilde{\alpha_2}$, $\tilde{\beta}_0$, $\tilde{\beta}_1$, $\tilde{\beta}_2$ are constants. Assuming that $\tilde{\beta}_2 \neq 0$, one can divide the numerator and denominator of (5.30) by $\tilde{\beta}_2$ and obtain

$$I_i(R_g) = \frac{\alpha_0 + \alpha_1 R_g + \alpha_2 R_g^2}{\beta_0 + \beta_1 R_g + R_g^2}, \tag{5.31}$$

where α_0, α_1, α_2, β_0, β_1 are constants. In order to determine these constants, one conducts 5 experiments by setting 5 different values to the gyrator resistance R_g, and measuring the corresponding currents I_i. In this case, the set of measurement equations will be

$$\underbrace{\begin{bmatrix} 1 & R_{g1} & R_{g1}^2 & -I_{i1} & -I_{i1}R_{g1} \\ 1 & R_{g2} & R_{g2}^2 & -I_{i2} & -I_{i2}R_{g2} \\ 1 & R_{g3} & R_{g3}^2 & -I_{i3} & -I_{i3}R_{g3} \\ 1 & R_{g4} & R_{g4}^2 & -I_{i4} & -I_{i4}R_{g4} \\ 1 & R_{g5} & R_{g5}^2 & -I_{i5} & -I_{i5}R_{g5} \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \beta_0 \\ \beta_1 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} I_{i1}R_{g1}^2 \\ I_{i2}R_{g2}^2 \\ I_{i3}R_{g3}^2 \\ I_{i4}R_{g4}^2 \\ I_{i5}R_{g5}^2 \end{bmatrix}}_{\mathbf{m}}, \tag{5.32}$$

which has a unique solution for the constants $\alpha_0, \alpha_1, \beta_0, \beta_1, \beta_2$, if and only if $|\mathbf{M}| \neq 0$ in (5.32). If $|\mathbf{M}| = 0$ is the case, one can use the same procedure presented in Section 5.1.1 to derive the corresponding functional dependency of I_i on R_g. The details of this case are provided in the Appendix.

Case 2: The i-th branch is connected to one port of the gyrator

In this case, the matrix $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ is of rank 1 with respect to R_g; however, the matrix $\mathbf{A}(\mathbf{p})$ is of rank 2 with respect to R_g. Therefore, using Lemma 4.2, the functional dependency $I_i(R_g)$ will be written as

$$I_i(R_g) = \frac{\tilde{\alpha_0} + \tilde{\alpha_1} R_g}{\tilde{\beta_0} + \tilde{\beta_1} R_g + \tilde{\beta_2} R_g^2}, \tag{5.33}$$

where $\tilde{\alpha_0}$, $\tilde{\alpha_1}$, $\tilde{\beta_0}$, $\tilde{\beta_1}$, $\tilde{\beta_2}$ are constants. Supposing $\tilde{\beta_2} \neq 0$ and dividing the numerator and denominator of (5.33) by $\tilde{\beta_2}$, one gets

$$I_i(R_g) = \frac{\alpha_0 + \alpha_1 R_g}{\beta_0 + \beta_1 R_g + R_g^2}, \tag{5.34}$$

where α_0, α_1, β_0, β_1 are constants that can be determined by conducting 4 experiments, by assigning 4 different values to the gyrator resistance R_g, and measuring the corresponding currents I_i. Then, the following set of measurement equations can be formed

$$\underbrace{\begin{bmatrix} 1 & R_{g1} & -I_{i1} & -I_{i1}R_{g1} \\ 1 & R_{g2} & -I_{i2} & -I_{i2}R_{g2} \\ 1 & R_{g3} & -I_{i3} & -I_{i3}R_{g3} \\ 1 & R_{g4} & -I_{i4} & -I_{i4}R_{g4} \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \\ \beta_1 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} I_{i1}R_{g1}^2 \\ I_{i2}R_{g2}^2 \\ I_{i3}R_{g3}^2 \\ I_{i4}R_{g4}^2 \end{bmatrix}}_{\mathbf{m}}. \tag{5.35}$$

As before, the system of equations (5.35) can be uniquely solved for the constants α_0, α_1, β_0, β_1, provided $|\mathbf{M}| \neq 0$. For the situations where $|\mathbf{M}| = 0$, one can follow the same procedure used in Section 5.1.1 to find the corresponding functional dependency of I_i on R_g. The details for this case are presented in the Appendix. □

5.1.5 Current Control using m Independent Sources

Here, we consider the problem of controlling the current I_i, by only using the independent current/voltage sources, denoted by $\mathbf{q} = [q_1, q_2, \ldots, q_m]^T$, at arbitrary locations of the circuit (see Fig. 5.6).

Theorem 5.5. *In a linear DC circuit, the functional dependency of any current I_i on the independent sources can be determined by m measurements of the current I_i obtained for m linearly independent sets of values of the source vector $\mathbf{q}$.*

Proof. Recall (5.2),

$$\mathbf{b}(\mathbf{q}) = q_1\mathbf{b}_1 + q_2\mathbf{b}_2 + \cdots + q_m\mathbf{b}_m, \tag{5.36}$$

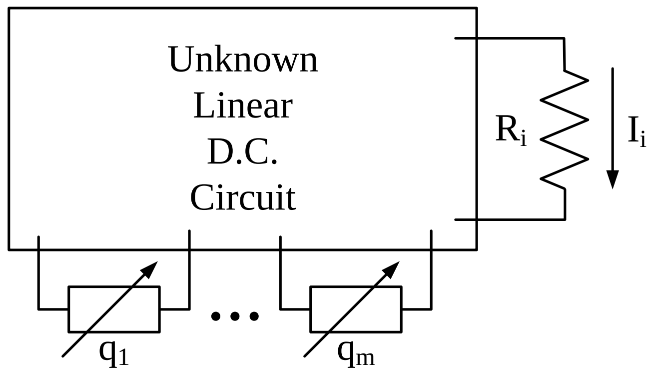

Figure 5.6: An unknown linear DC circuit for Section 5.1.5

where $q_1, q_2, \ldots, q_m$ are the independent sources. $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ in (5.3) can be expanded as

$$\mathbf{B}_i(\mathbf{p}, \mathbf{q}) = [\mathbf{A}_1(\mathbf{p}), \ldots, \mathbf{A}_{i-1}(\mathbf{p}), \mathbf{b}(\mathbf{q}), \mathbf{A}_{i+1}(\mathbf{p}), \ldots, \mathbf{A}_n(\mathbf{p})], \tag{5.37}$$

which can be seen that is of rank 1 with respect to every independent sources $q_1, q_2, \ldots, q_m$. Thus, $|\mathbf{B}_i(\mathbf{p}, \mathbf{q})|$ can be expressed as a linear combination of $q_1, q_2, \ldots, q_m$,

$$|\mathbf{B}_i(\mathbf{p}, \mathbf{q})| = q_1 |\mathbf{B}_{i1}(\mathbf{p})| + q_2 |\mathbf{B}_{i2}(\mathbf{p})| + \cdots + q_m |\mathbf{B}_{im}(\mathbf{p})|, \tag{5.38}$$

where

$$\mathbf{B}_{ij}(\mathbf{p}) = [\mathbf{A}_1(\mathbf{p}), \ldots, \mathbf{A}_{i-1}(\mathbf{p}), \mathbf{b}_j, \mathbf{A}_{i+1}(\mathbf{p}), \ldots, \mathbf{A}_n(\mathbf{p})], \tag{5.39}$$

for $j = 1, 2, \ldots, m$. $\mathbf{A}(\mathbf{p})$ is of rank 0 with respect to $q_1, q_2, \ldots, q_m$, implying that

$|\mathbf{A}(\mathbf{p})|$ is a constant, according to Lemma 4.1. Hence, $I_i(q_1, q_2, \ldots, q_m)$ simplifies to

$$\begin{aligned} I_i(\mathbf{q}) &:= I_i(q_1, q_2, \ldots, q_m) \\ &= \alpha_1 q_1 + \alpha_2 q_2 + \cdots + \alpha_m q_m, \end{aligned} \tag{5.40}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_m$ are constants that can be determined by assigning m sets of linearly independent values to $(q_1, q_2, \ldots, q_m)$, measuring the corresponding I_i, and solving the obtained set of measurement equations. □

Remark 5.2.

1. *Theorem 5.5 is the well-known Superposition Principle of circuit theory.*

2. *If the independent sources vary in the intervals* $q_j^- \leq q_j \leq q_j^+$, $j = 1, 2, \ldots, m$, *then the current* I_i *will vary in an interval whose end values can be computed using the vertices* (q_j^-, q_j^+), $j = 1, 2, \ldots, m$. *For example suppose that* I_i *is given as below,*

$$I_i(\mathbf{q}) = 2q_1 - q_2 + 5q_3 - 3q_4, \tag{5.41}$$

where $q_j^- \leq q_j \leq q_j^+$, $q_j^- \geq 0$, $j = 1, 2, 3, 4$. *One may decompose* I_i *as*

$$I_i(\mathbf{q}) = 2q_1 - q_2 + 5q_3 - 3q_4 = (2q_1 + 5q_3) - (q_2 + 3q_4). \tag{5.42}$$

Then, the maximum and minimum values of I_i, *denoted by* $I_i^{\max}$ *and* $I_i^{\min}$, *respectively, can be obtained from*

$$I_i^{\max} = (2q_1^+ + 5q_3^+) - (q_2^- + 3q_4^-), \tag{5.43}$$

$$I_i^{\min} = (2q_1^- + 5q_3^-) - (q_2^+ + 3q_4^+). \tag{5.44}$$

5.2 Power Level Control

In this section we consider synthesis problems where, in an unknown linear DC circuit, the power in an arbitrary resistor is to be controlled by adjusting the design elements at arbitrary locations. For the sake of simplicity, suppose that the resistor R_i is located in the i-th branch of the circuit and we want to control the power level P_i, in the resistor R_i, by some design elements.

5.2.1 Power Level Control using a Single Resistor

Here, assume that the resistor R_j is not a gyrator resistance and recall the results developed in Section 5.1.1.

Theorem 5.6. *In a linear DC circuit, the functional dependency of the power level P_i, in the resistor R_i, on any resistance R_j can be determined by at most 3 measurements of the current I_i (passing through R_i) obtained for 3 different values of R_j, and 1 measurement of the voltage across the resistor R_i, corresponding to one of the resistance settings.*

Proof. Consider two cases: 1) $i \neq j$, and 2) $i = j$.

Case 1: $i \neq j$

We write the power as $P_i = \frac{V_i}{I_i}\, I_i^2$. The functional dependency $I_i(R_j)$ is of either forms (5.6) or (5.8). Since the ratio $\frac{V_i}{I_i}$ is the same for each experiment, then only one measurement of the voltage V_i, across the resistor R_i, in addition to the 3 measurements of the current I_i, is required to determine $P_i(R_j)$. Assuming one measures V_{i1} from the first experiment, then $P_i(R_j)$ will be of the following forms:

- If $|\mathbf{M}| \neq 0$ in (5.7):

$$P_i(R_j) = \frac{V_{i1}}{I_{i1}} \left(\frac{\alpha_0 + \alpha_1 R_j}{\beta_0 + R_j} \right)^2, \tag{5.45}$$

  where V_{i1} and I_{i1} are the voltage and current signals, at the resistor R_i, measured from the first experiment, and the constants $\alpha_0, \alpha_1, \beta_0$ are obtained by solving (5.7), as explained in Section 5.1.1.

- If $|\mathbf{M}| = 0$ in (5.7):

$$P_i(R_j) = \frac{V_{i1}}{I_{i1}} (\alpha_0 + \alpha_1 R_j)^2, \tag{5.46}$$

  where V_{i1} and I_{i1} are the voltage and current signals, at the resistor R_i, measured from the first experiment, and the constants α_0, α_1 can be determined using any two of the conducted experiments, as discussed in Section 5.1.1.

Case 2: $i = j$

Let us write the power as $P_i = R_i\, I_i^2$. Based on the results of Section 5.1.1, $I_i(R_i)$ will be of either forms in (5.10) or (5.12). Hence, $P_i(R_i)$ will be:

- If $|\mathbf{M}| \neq 0$ in (5.11):

$$P_i(R_i) = R_i \left(\frac{\alpha_0}{\beta_0 + R_i} \right)^2, \tag{5.47}$$

where the constants α_0, β_0 can be obtained as explained in Section 5.1.1.

- If $|\mathbf{M}| = 0$ in (5.11):

$$P_i(R_i) = \alpha_0^2 R_i, \tag{5.48}$$

where α_0 is a constant that can be determined as discussed in Section 5.1.1.

□

Remark 5.3. Maximum Power Transfer Theorem. *Suppose that $i = j$ and $|\mathbf{M}| \neq 0$ in (5.11), then the derivative of $P_i(R_i)$, in (5.47), with respect to R_i, is*

$$\frac{dP_i}{dR_i} = \frac{\alpha_0^2(\beta_0 - R_i)}{(\beta_0 + R_i)^3}. \tag{5.49}$$

We have the following statements:

1. *The functional dependency $P_i(R_i)$, in (5.47), in this case, is not monotonic. For $R_i \to 0$, $P_i \to 0$ and for $R_i \to \infty$, $P_i \to 0$. Therefore, as R_i varies from 0 to ∞, P_i increases from 0 to the maximum achievable value of $\frac{\alpha_0^2}{4\beta_0}$, and then decreases to 0 at very large values of R_i. The maximum occurs at $R_i = \beta_0$.*

2. *The achievable range for the power level P_i, by varying R_i in $[0, \infty)$, is*

$$0 \leq P_i < \frac{\alpha_0^2}{4\beta_0}. \tag{5.50}$$

3. In a power level control problem of this type, for any desired prescribed interval of power P_i, which is within the achievable range (5.50)*, one may find two ranges of values for the design resistance R_i.*

5.2.2 Power Level Control using Two Resistors

For this case, assuming that R_j and R_k are not gyrator resistances and recall the results of Section 5.1.2.

Theorem 5.7. *In a linear DC circuit, the functional dependency of the power level P_i, in any resistor R_i, on any two resistances R_j and R_k can be determined by at most 7 measurements of the currents I_i (passing through R_i) obtained for 7 different sets of values (R_j, R_k), and 1 measurement of the voltage across the resistor R_i, corresponding to one of the resistance settings.*

Proof. The proof is similar to the previous case and thus omitted here. □

The power level $P_i(R_j, R_k)$ can be depicted as a 3D surface. In a design problem, any constraint on P_i yields in a corresponding region in the R_j-R_k plane, if the solution set to that constraint is not empty.

Remark 5.4. *For the case of m resistors and gyrator resistance, corresponding functional dependencies can be derived using the results of Sections 5.1.3 and 5.1.4, respectively.*

5.3 Illustrative Examples

Example 5.1. In this illustrative example we show how the method proposed in Sections 5.1.1 and 5.2.1 can be used toward synthesis problems in unknown linear DC circuits. Consider the unknown circuit in Fig. 5.7.

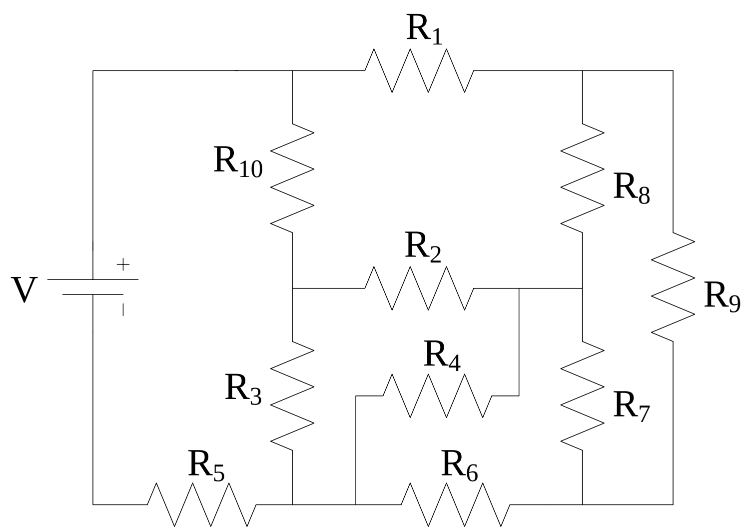

Figure 5.7: An unknown resistive circuit example

In this example, it is desired to find the functional dependency of the current I_1 on the resistance R_9. Based on the results given in Section 5.1.1, one conducts 3 experiments, by setting 3 different values to R_9, and measuring the corresponding currents I_1. Suppose that experiments are done and let Table 5.1 summarize the numerical values, for this example, obtained from the 3 experiments.

Table 5.1: Measurements for the DC circuit example 5.1

Exp. No.	R_9 (Ω)	I_1 (A)
1	1	0.054
2	5	0.056
3	10	0.058

Substituting the numerical values obtained from the experiments into the matrix $\mathbf{M}$ in (5.7) resulted in $|\mathbf{M}| \neq 0$. Therefore, (5.7) can be uniquely solved for the constants and yield the following functional dependency which is plotted in Fig. 5.8.

$$I_1(R_9) = \frac{78.4 + 0.66R_9}{181.3 + R_9}. \tag{5.51}$$

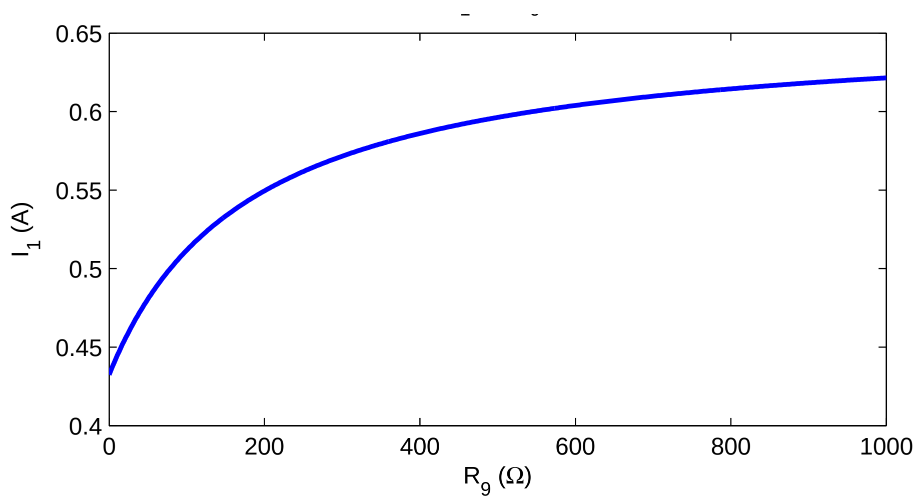

Figure 5.8: I_1 vs. R_9

Remark 5.5.

1. *The current I_1 monotonically increases as R_9 increases.*

2. *By varying R_9 in the range $[0, \infty)$, the achievable range for I_1 becomes $[\frac{\alpha_0}{\beta_0}, \alpha_1] = [0.43, 0.66]$.*

3. *In a synthesis problem where the current I_1 is to be controlled to stay within an acceptable interval, since I_1 is monotonic in R_9, one can find a corresponding*

interval for R_9 values for which the current I_1 stays within the acceptable range.

Suppose that we wish to design R_9 such that I_1 lies within the following achievable range

$$0.5 \leq I_1 \leq 0.6 \ (A). \tag{5.52}$$

Using (5.51), or Fig. 5.8, the corresponding range for the design resistor R_9 can be obtained as

$$79 \leq R_9 \leq 550 \ (\Omega).$$

Example 5.2. For this example we constructed a resistive DC circuit. Our objective was to find the functional dependency of the voltage, V, across a specific resistor, in terms of a design resistor, R. We performed 3 experiments and obtained the numerical values given in Table 5.2.

Table 5.2: Measurements for the DC circuit example 5.2

Exp. No.	R (Ω)	V (V)
1	10.3	0.651
2	98.8	0.613
3	984	0.425

These numerical values resulted in the following function for $V(R)$:

$$V(R) = \frac{618.2962 + 0.2038R}{942.6883 + R}. \tag{5.53}$$

We set the resistor R to some other values and measured the corresponding voltage V as shown in Table 5.3. The evaluation of function (5.53) for these values of R is also provided in Table 5.3.

Table 5.3: Additional measurements for the DC circuit example 5.2

R (Ω)	V (V) (practice)	V (V) (evaluating (5.53))
51.5	0.632	0.633
501.5	0.499	0.499
741	0.456	0.457
2023	0.348	0.348

As we expected, there is a very good agreement between the practical measurements and the evaluation of the obtained functional dependency.

Example 5.3. Consider the same circuit as in the Example 5.1 (Fig. 5.7). Suppose now that the power levels within R_1, R_3 and R_9, denoted by P_1, P_3 and P_9, respectively, must remain in the following ranges:

$$6\ (W) \leq P_1 \leq 7\ (W), \tag{5.54}$$

$$7\ (W) \leq P_3 \leq 8\ (W), \tag{5.55}$$

$$3\ (W) \leq P_9 \leq 3.5\ (W). \tag{5.56}$$

Assume that the design resistor is R_9. Based on the results of Section 5.2.1, one conducts 3 experiments by assigning 3 different values to R_9, and measuring the corresponding currents I_1, I_3 and I_9, passing through the resistors R_1, R_3 and R_9, respectively. In this problem, one also needs to measure the voltage across R_1 and R_3

from one of the experiments. Suppose that the experiments are done and let Table 5.4 summarize the numerical values for this example, obtained from the experiments.

Table 5.4: Measurements for the DC circuit example 5.3

Exp. No.	R_9 (Ω)	I_1 (A)	I_3 (A)	I_9 (A)
1	1	0.437	0.964	0.301
2	5	0.438	0.972	0.295
3	10	0.444	0.982	0.287

Exp. No.	R_9 (Ω)	V_1 (V)	V_3 (V)
1	1	8.67	4.82

Substituting the numerical values from Table 5.4 into the matrix $\mathbf{M}$ in (5.7), for the currents I_1 and I_3, and into the matrix $\mathbf{M}$ in (5.11), for the current I_9, yields $|\mathbf{M}| \neq 0$, for all cases. Therefore, the functional dependencies of P_1, P_3 and P_9 on R_9 will be

$$P_1(R_9) = \frac{8.67}{0.437}\left(\frac{78.4 + 0.66R_9}{181.3 + R_9}\right)^2, \tag{5.57}$$

$$P_3(R_9) = \frac{4.82}{0.964}\left(\frac{174.4 + 1.34R_9}{181.3 + R_9}\right)^2, \tag{5.58}$$

$$P_9(R_9) = R_9\left(\frac{54.9}{181.3 + R_9}\right)^2. \tag{5.59}$$

Fig. 5.9 shows the plots of the power levels P_1, P_3 and P_9 obtained above.

Using (5.57)-(5.59), shown graphically in Fig. 5.9, one imposes the power level constraints (5.54)-(5.56) to find the corresponding ranges of R_9 values. A necessary

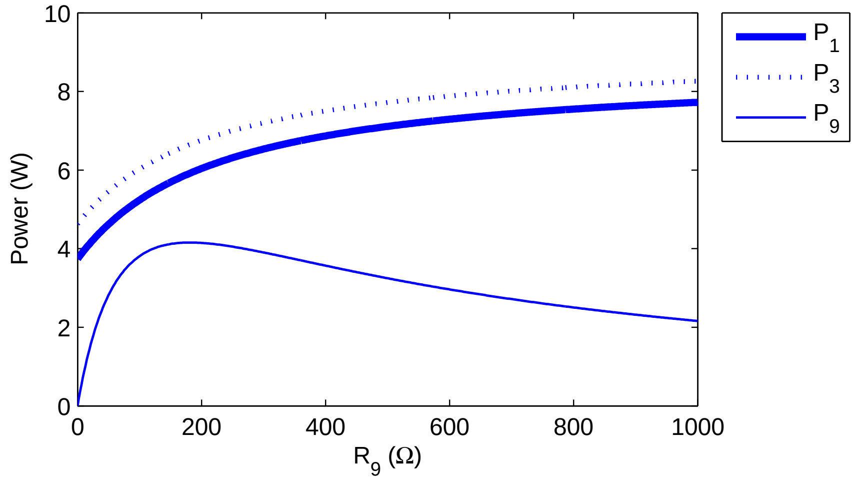

Figure 5.9: P_1, P_3, P_9 vs. R_9

condition for the existence of a solution is that the constraints (5.54)-(5.56) must be within their corresponding achievable ranges. For this example, the power level constraints are within the achievable ranges; hence, we can find the following ranges for R_9 values:

$$190\ (\Omega) \leq R_9 \leq 450\ (\Omega), \tag{5.60}$$

$$250\ (\Omega) \leq R_9 \leq 690\ (\Omega), \tag{5.61}$$

$$60\ (\Omega) \leq R_9 \leq 80\ \cup\ 420\ (\Omega) \leq R_9 \leq 580\ (\Omega), \tag{5.62}$$

corresponding to the power level constraints (5.54), (5.55) and (5.56), respectively. Therefore, the range for R_9 values where (5.54), (5.55) and (5.56) are achieved simultaneously is the intersection of the ranges calculated above, that is

$$420\ (\Omega) \leq R_9 \leq 450\ (\Omega). \tag{5.63}$$

Example 5.4. Consider the unknown linear DC circuit depicted in Fig. 5.10.

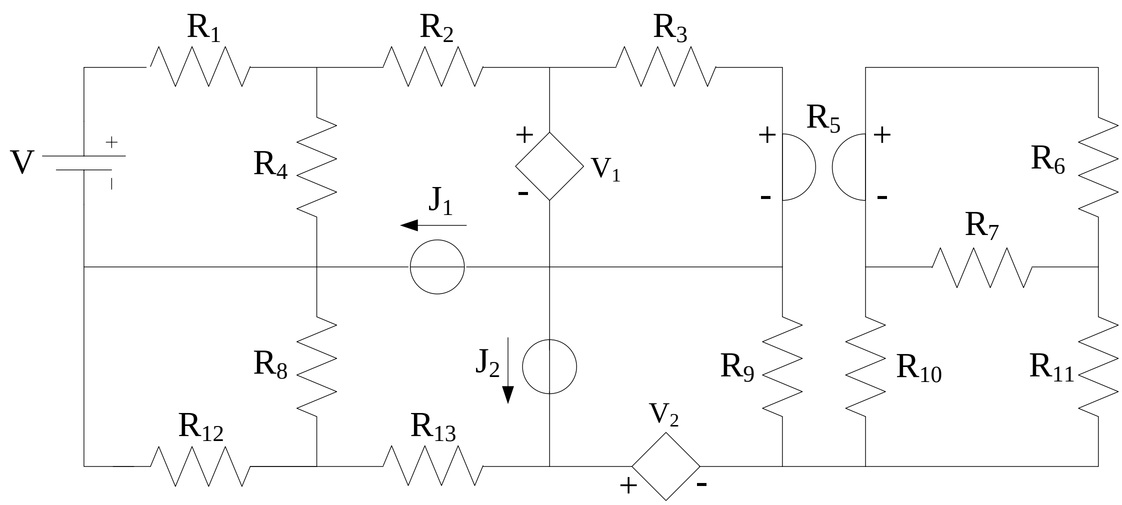

Figure 5.10: An unknown linear DC circuit example

In this example, $R_i,\ i = 1, 2, \ldots, 13,\ i \neq 5$ are resistors, R_5 is a gyrator resistance, V, J_1, J_2 are independent sources and V_1, V_2 are dependent sources. Suppose that the design objective is to control the power levels in R_3, R_6 and R_{11}, denoted by P_3, P_6

and P_{11}, respectively, to lie within the ranges below:

$$40\ (W) \leq P_3 \leq 60\ (W), \tag{5.64}$$

$$1\ (W) \leq P_6 \leq 8\ (W), \tag{5.65}$$

$$0.5\ (W) \leq P_{11} \leq 5\ (W). \tag{5.66}$$

Assume that the design parameters are R_1 and R_6. Thus, we need to find the region in the R_1-R_6 plane where (5.64), (5.65) and (5.66) are met. Based on the approach presented in Section 5.2.2, in order to find the functional dependency of any power level in terms of any two arbitrary resistances, one has to do at most 7 measurements of current and one measurement of voltage. Let us treat each power level problem separately as follows:

- P_3 vs. R_1 and R_6:

  Based on the results in Section 5.2.2, $P_3(R_1, R_6)$ can be determined by conducting 7 experiments by setting 7 different sets of values to (R_1, R_6), and measuring the corresponding I_3. In addition, one measurement of the voltage, across the resistor R_3, is needed. Suppose that this measurement is taken from the first experiment and denote it by V_{31}. Let Table 5.5 summarize the numerical values assigned to the resistances R_1 and R_6 along with the corresponding measurements of I_3 and V_{31}. Substituting the numerical values of Table 5.5 into the matrix $\mathbf{M}$, in (5.22), it can be checked that $|\mathbf{M}| \neq 0$. Thus

$$P_3(R_1, R_6) = \frac{V_{31}}{I_{31}} \underbrace{\left(\frac{\alpha_0 + \alpha_1 R_1 + \alpha_2 R_6 + \alpha_3 R_1 R_6}{\beta_0 + \beta_1 R_1 + \beta_2 R_6 + R_1 R_6} \right)^2}_{I_3^2(R_1, R_6)}, \tag{5.67}$$

  where the constants $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2$ can be determined by solving (5.22),

Table 5.5: Measurements for the DC circuit example 5.4

Exp. No.	$R_1(\Omega)$	$R_6(\Omega)$	$I_3(A)$
1	7	1	3.33
2	13	8	2.71
3	21	19	2.47
4	35	26	2.57
5	40	32	2.52
6	52	45	2.47
7	59	56	2.44

Exp. No.	$R_1(\Omega)$	$R_6(\Omega)$	$V_{31}(V)$
1	7	1	33.3

using the numerical values of Table 5.5. For this example, the constants are obtained as: $\alpha_0 = 98.4$, $\alpha_1 = 36$, $\alpha_2 = 6.6$, $\alpha_3 = 2.4$, $\beta_0 = 58.5$, $\beta_1 = 5$, $\beta_2 = 11.7$. Hence, $P_3(R_1, R_6)$ will be

$$P_3(R_1, R_6) = \frac{33.3}{3.33}\left(\frac{98.4 + 36R_1 + 6.6R_6 + 2.4R_1R_6}{58.5 + 5R_1 + 11.7R_6 + R_1R_6}\right)^2. \tag{5.68}$$

Fig. 5.11 shows the plot of P_3 as a function of R_1 and R_6, obtained in (5.68). Applying constraint (5.64) on P_3, one may obtain the region in the R_1-R_6 plane, shown in black color in Fig. 5.12, where this constraint is satisfied.

- P_6 vs. R_1 and R_6:

The functional dependency $P_6(R_1, R_6)$ can be determined by at most 5 measurements of current and one measurement of voltage as discussed in Section 5.2.2 (Case 2). The plot of $P_6(R_1, R_6)$ is shown in Fig. 5.13. Applying constraint (5.65) on P_6, one gets the region in the R_1-R_6 plane, shown in black color in Fig. 5.14, where this constraint is valid.

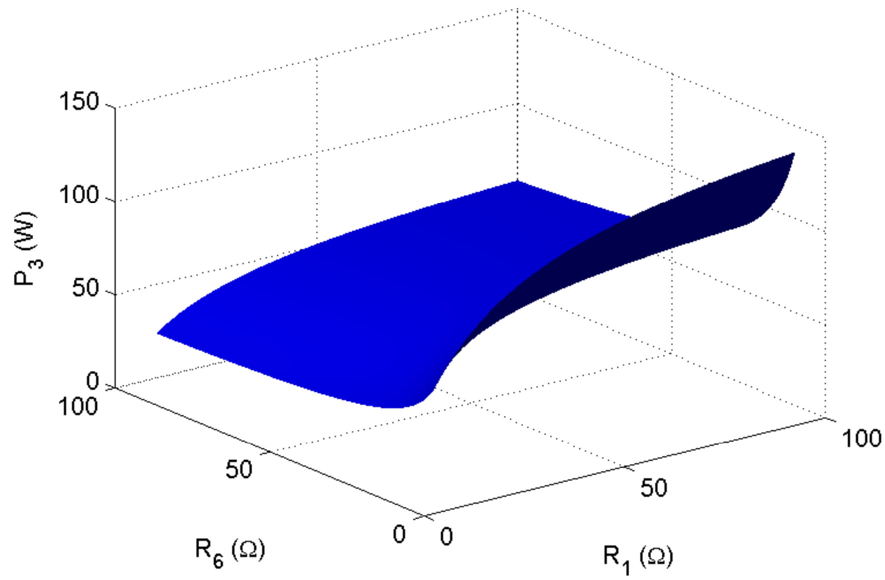

Figure 5.11: P_3 vs. R_1 and R_6

- P_{11} vs. R_1 and R_6:

  Following the same procedure used to determine $P_3(R_1, R_6)$, one may find $P_{11}(R_1, R_6)$. The plot of $P_{11}(R_1, R_6)$ is depicted in Fig. 5.15. Applying constraint (5.66) on P_{11}, one gets the region in the R_1-R_6 plane, shown in black color in Fig. 5.16, where this constraint is satisfied.

In order to satisfy the constraints in (5.64), (5.65) and (5.66), simultaneously, one needs to intersect the regions shown in Figs. 5.12, 5.14 and 5.16. Fig. 5.17 shows the region (in black color) in the R_1-R_6 plane where the constraints (5.64), (5.65) and (5.66) are satisfied, simultaneously.

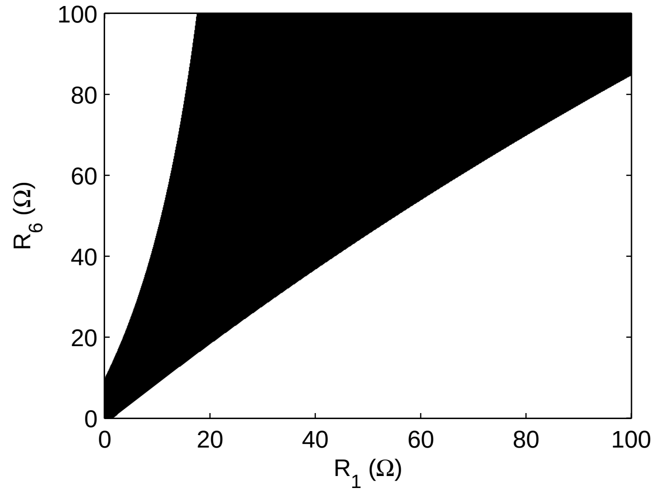

Figure 5.12: Region (in black color) where (5.64) is satisfied.

5.4 Concluding Remarks

This chapter showed that the analysis and design problems in linear DC circuits can be carried out without knowledge of the circuit model, provided a few measurements can be made. These measurements, processed appropriately, can yield the complete information regarding the functional dependency of circuit variables on the design elements.

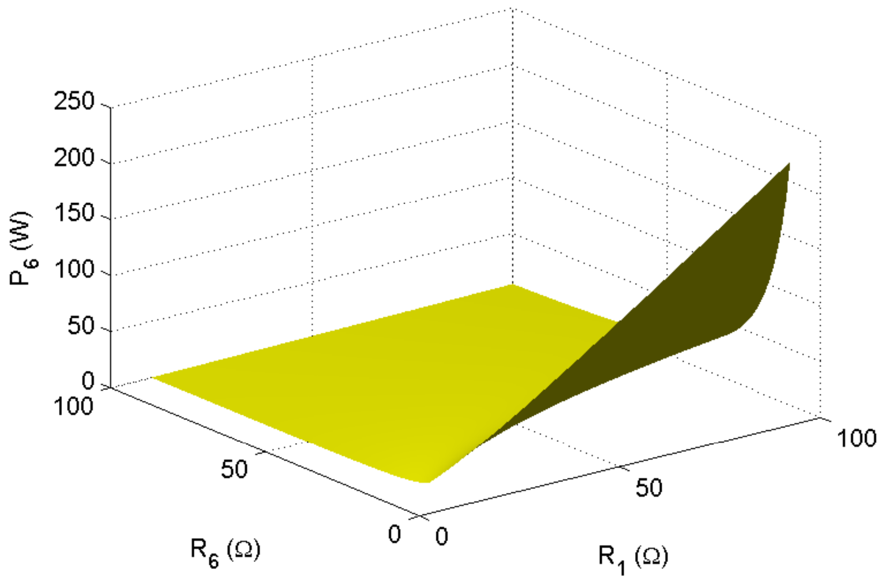

Figure 5.13: P_6 vs. R_1 and R_6

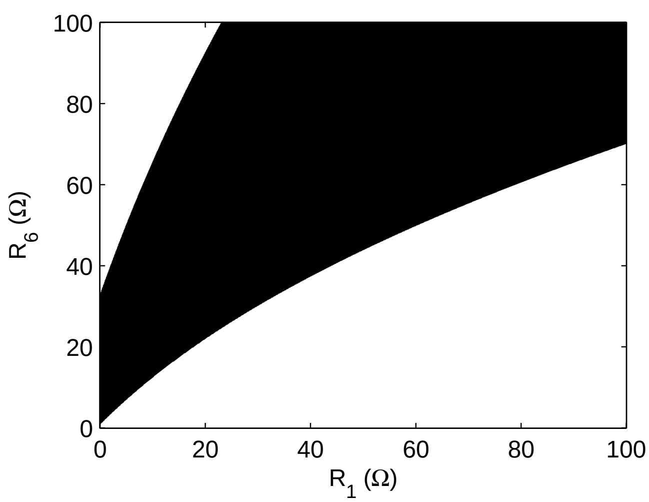

Figure 5.14: Region (in black color) where (5.65) is satisfied.

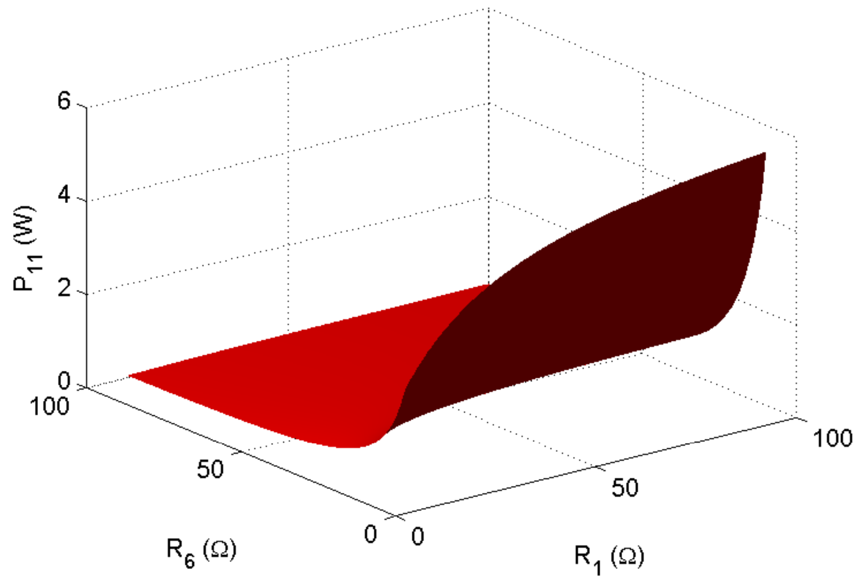

Figure 5.15: P_{11} vs. R_1 and R_6

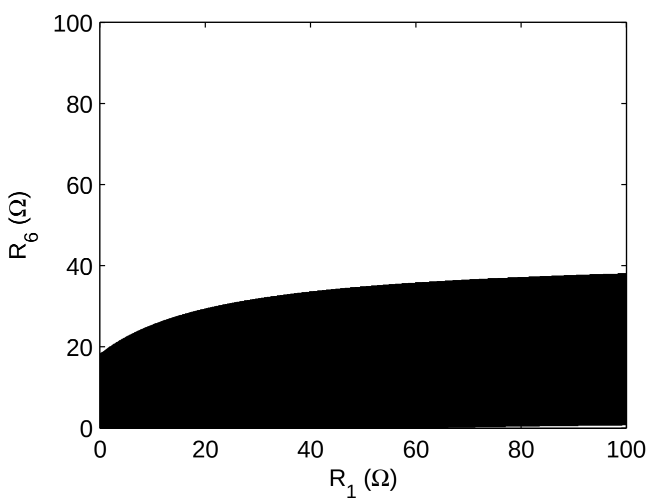

Figure 5.16: Region (in black color) where (5.66) is satisfied.

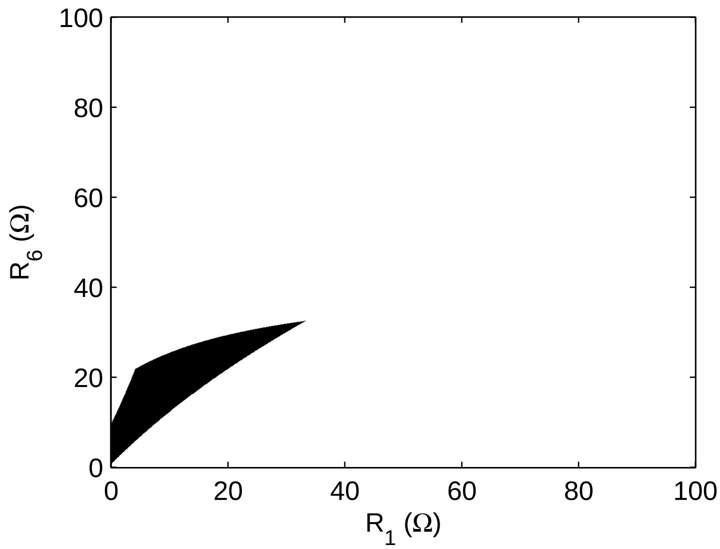

Figure 5.17: Region (in black color) where (5.64), (5.65) and (5.66) are simultaneously satisfied.

6. APPLICATION TO AC CIRCUITS*

In this chapter we extend the proposed measurement based approach to the domain of AC circuits operating in steady state at a fixed frequency. The main difference between application of this method to AC circuits and its DC circuit counterpart (Chapter 5) is that in an AC circuit the variables are complex quantities which are usually called phasors and impedances, rather than real quantities.

6.1 Current Control

Suppose that a linear AC circuit is operating at a fixed frequency ω, Let us write its governing steady state equations in the following matrix form

$$\mathbf{A}(\mathbf{p}(j\omega))\mathbf{x}(j\omega) = \mathbf{b}(\mathbf{q}(j\omega)), \tag{6.1}$$

where $\mathbf{A}(\mathbf{p}(j\omega))$ is called the circuit characteristic matrix which contains the circuit impedances, $\mathbf{x}(j\omega)$ represents the vector of unknown current phasors and $\mathbf{q}(j\omega)$ is the vector of independent voltage and current sources. Suppose that the current phasor in the i-th branch of the circuit, denoted by $I_i(j\omega)$, is of interest. Applying Cramer's rule to (6.1), $I_i(j\omega)$ can be calculated from

$$x_i(j\omega) = I_i(j\omega) = \frac{|\mathbf{B}_i(\mathbf{p}(j\omega), \mathbf{q}(j\omega))|}{|\mathbf{A}(\mathbf{p}(j\omega))|}, \tag{6.2}$$

where $\mathbf{B}_i(\mathbf{p}(j\omega), \mathbf{q}(j\omega))$ may be obtained by replacing the i-th column of $\mathbf{A}(\mathbf{p}(j\omega))$ by $\mathbf{b}(\mathbf{q}(j\omega))$. We emphasize that if the circuit is unknown, then the matrices

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$\mathbf{B}_i(\mathbf{p}(j\omega), \mathbf{q}(j\omega))$ and $\mathbf{A}(\mathbf{p}(j\omega))$ are unknown. In the following subsections, for each case of the design elements, a general rational form for the current phasor $I_i(\omega)$, as a function of the design elements, will be derived. For the sake of simplicity, we drop the argument $(j\omega)$ in writing the equations from now on.

6.1.1 Current Control using a Single Impedance

Suppose that in an unknown linear AC circuit we want to control the current phasor in the i-th branch, I_i, using any impedance Z_j at an arbitrary location (see Fig. 6.1). Assume that Z_j is not a gyrator resistance. We have the following theorem.

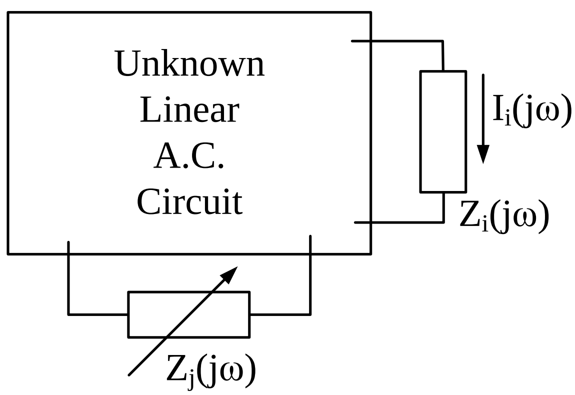

Figure 6.1: An unknown linear AC circuit for Section 6.1.1

Theorem 6.1. *In a linear AC circuit, the functional dependency of any current phasor I_i on any impedance Z_j can be determined by at most 3 measurements of the current phasor I_i obtained for 3 different complex values of Z_j.*

Proof. The proof is similar to its DC circuit counterpart presented in Section 5.1.1.

The main difference is that the circuit variables and the constants appearing in the functional dependencies will be complex quantities rather than real numbers. Hence, we provide the results and leave the details to the reader.

Case 1: $i \neq j$

The function $I_i(Z_j)$ will be

$$I_i(Z_j) = \frac{\alpha_0 + \alpha_1 Z_j}{\beta_0 + Z_j}, \tag{6.3}$$

where $\alpha_0, \alpha_1, \beta_0$ are complex quantities that can be uniquely determined by solving the following set of measurement equations,

$$\underbrace{\begin{bmatrix} 1 & Z_{j1} & -I_{i1} \\ 1 & Z_{j2} & -I_{i2} \\ 1 & Z_{j3} & -I_{i3} \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} Z_{j1}I_{i1} \\ Z_{j2}I_{i2} \\ Z_{j3}I_{i3} \end{bmatrix}}_{\mathbf{m}}, \tag{6.4}$$

provided $|\mathbf{M}| \neq 0$. These complex quantities can be written as

$$\begin{aligned} \alpha_0(j\omega) &= \alpha_{0r}(\omega) + j\alpha_{0i}(\omega), \\ \alpha_1(j\omega) &= \alpha_{1r}(\omega) + j\alpha_{1i}(\omega), \\ \beta_0(j\omega) &= \beta_{0r}(\omega) + j\beta_{0i}(\omega). \end{aligned}$$

If $|\mathbf{M}| = 0$ in (6.4), then $I_i(Z_j)$ can be written as

$$I_i(Z_j) = \alpha_0 + \alpha_1 Z_j, \tag{6.5}$$

where α_0, α_1 can be determined from any 2 of the experiments conducted earlier.

Case 2: $i = j$

In this case, $I_i(Z_i)$ can be written as

$$I_i(Z_i) = \frac{\alpha_0}{\beta_0 + Z_i}, \tag{6.6}$$

where α_0, β_0 can be calculated by solving the following set of measurement equations,

$$\underbrace{\begin{bmatrix} 1 & -I_{i1} \\ 1 & -I_{i2} \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} I_{i1} Z_{i1} \\ I_{i2} Z_{i2} \end{bmatrix}}_{\mathbf{m}}. \tag{6.7}$$

if and only if $|\mathbf{M}| \neq 0$. In a situation where $|\mathbf{M}| = 0$ in (6.7), I_i will be a constant,

$$I_i(Z_i) = \alpha_0, \tag{6.8}$$

where α_0 may be obtain from one of the experiments conducted earlier. □

As noted earlier, since the main difference between the results of this chapter and their D.C. circuit counterparts is that in AC circuits the variables are complex quantities, rather than real numbers, the proofs of the following theorems are omitted.

6.1.2 Current Control using Two Impedances

In this subsection we want to control I_i using any two impedances Z_j and Z_k, that are not gyrator resistances, at arbitrary locations (see Fig. 6.2).

Theorem 6.2. *In a linear AC circuit, the functional dependency of any current*

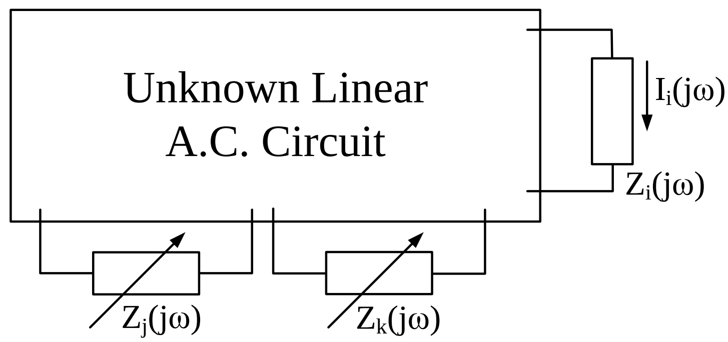

Figure 6.2: An unknown linear AC circuit for Section 6.1.2

phasor I_i on any two impedances Z_j and Z_k can be determined by at most 7 measurements of the current phasor I_i obtained for 7 different sets of complex values (Z_j, Z_k).

Remark 6.1. *For the case of m impedances, one may resort to the results in Section 5.1.3 to derive the corresponding functional dependencies.*

6.1.3 Current Control using Gyrator Resistance

Here, the design parameter is the resistance of a gyrator, R_g, located at an arbitrary location of the circuit. We can state the following theorem.

Theorem 6.3. *In a linear AC circuit, the functional dependency of any current phasor I_i on any gyrator resistance R_g can be determined by at most 5 measurements of the current phasor I_i obtained for 5 different values of R_g.*

6.1.4 Current Control using m Independent Sources

Consider the problem of controlling the current phasor I_i using the independent current and voltage sources, denoted by $q_1, q_2, \ldots, q_m$, at arbitrary locations of the circuit (Fig. 6.3).

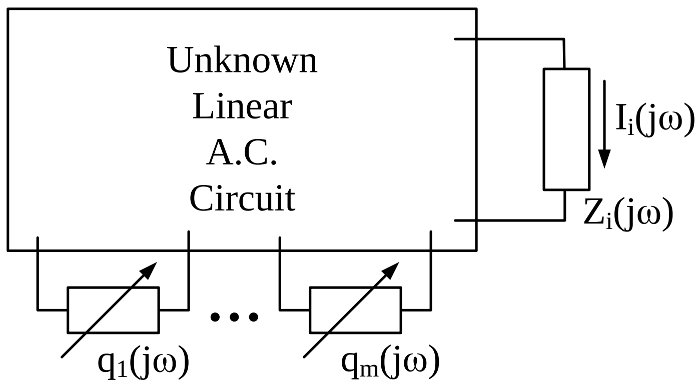

Figure 6.3: An unknown linear AC circuit for Section 6.1.4

Theorem 6.4. *In a linear AC circuit, the functional dependency of any current phasor I_i on the independent sources can be determined by m measurements of the current phasor I_i obtained for m linearly independent sets of values of the source vector* $\mathbf{q} = [q_1, q_2, \ldots, q_m]^T$.

6.2 Power Control

6.2.1 Power Control using a Single Impedance

In this problem, the objective is to control the complex power P_i, in the impedance Z_i, located in the i-th branch of an unknown linear AC circuit, by adjusting any impedance Z_j at an arbitrary location of the circuit. Assuming that Z_j is not a gyrator resistance and recalling the results presented in Section 6.1.1, we can state the following theorem.

Theorem 6.5. *In a linear AC circuit, the functional dependency of the complex power P_i on any impedance Z_j can be determined by at most 3 measurements of the current phasor I_i (passing through Z_i) obtained for 3 different complex values of Z_j, and 1 measurement of the voltage across the impedance Z_i, for one such setting of the impedance.*

6.2.2 Power Control using Two Impedances

Suppose that the power P_i is to be controlled by any two impedances Z_j and Z_k, that are not gyrator resistances, at arbitrary locations. Using the results of Section 6.1.2, we have the following theorem.

Theorem 6.6. *In a linear AC circuit, the functional dependency of the power level P_i, in any impedance Z_i, on any two impedances Z_j and Z_k can be determined by at most 7 measurements of the current phasor I_i (passing through Z_i) obtained for 7 different sets of complex values (Z_j, Z_k), and 1 measurement of the voltage across the impedance Z_i, for one of the impedance settings.*

Remark 6.2. *For the case of m impedances, the corresponding functional dependencies can be derived using the results presented in Section 5.1.3.*

6.2.3 Power Control using Gyrator Resistance

In this case, the power P_i is to be controlled by a gyrator resistance R_g, at an arbitrary location. We use the results obtained in Section 6.1.3.

Theorem 6.7. *In a linear AC circuit, the functional dependency of the complex power P_i on any gyrator resistance R_g can be determined by at most 5 measurements* *of the current phasor I_i obtained for 5 different values of R_g, and 1 measurement of the voltage across the impedance Z_i, corresponding to one of the impedance settings.*

6.3 Illustrative Example

Example 6.1. Consider the unknown linear AC circuit depicted in Fig. 6.4 which is operation at the frequency $f = 60\ (Hz)$.

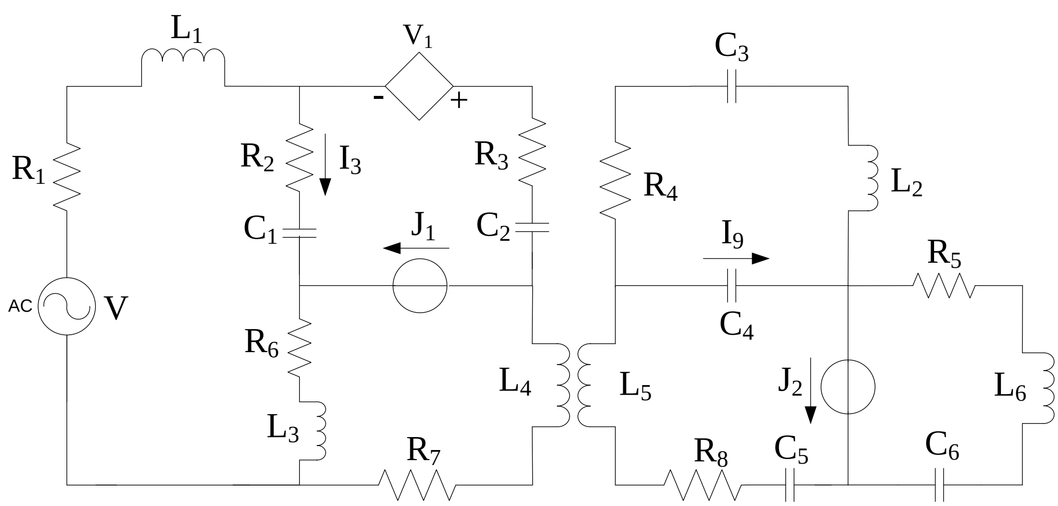

Figure 6.4: An unknown linear AC circuit example

We want to control the current phasors I_3 and I_9 to lie within the following ranges:

$$0\ (A) \leq |I_3| \leq 4\ (A), \tag{6.9}$$

$$10\ (deg) \leq \angle I_3 \leq 30\ (deg), \tag{6.10}$$

$$0\ (A) \leq |I_9| \leq 2.5\ (A), \tag{6.11}$$

$$-30\ (deg) \leq \angle I_9 \leq -10\ (deg). \tag{6.12}$$

Assume that the design elements are the inductor L_1 and the capacitor C_2. Thus, we need to calculate the region in the L_1-C_2 plane where (6.9)-(6.12) are met. Based on the results developed in Section 6.1.2, one can determine the functional dependency of any current phasor in terms of any two impedances by taking at most 7 measurements of the current phasor. Let us treat each current phasor problem separately as follows:

- I_3 vs. L_1 and C_2:

  To determine $I_3(L_1, C_2)$, one has to do 7 measurements of current phasor I_3 for 7 different sets of values (L_1, C_2). Let Table 6.1 summarize the numerical values assigned to L_1 and C_2 and the corresponding measurements of I_3. For this case, the general functional dependency can be written as

$$I_3(L_1, C_2) = \frac{\alpha_0 + \alpha_1 L_1 j\omega_0 + \alpha_2/(C_2 j\omega_0) + \alpha_3 L_1/C_2}{\beta_0 + \beta_1 L_1 j\omega_0 + \beta_2/(C_2 j\omega_0) + L_1/C_2}, \tag{6.13}$$

  where the complex constants $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2$ can be determined by solving the set of 7 measurement equations.

  Substituting the numerical values given in Table 6.1 into the measurement

Table 6.1: Measurements for the AC circuit example

Exp. No.	$L_1(mH)$	$C_2(\mu F)$	$I_3(A)$
1	13	10	3.3-2.9i
2	25	20	2.7-3.2i
3	32	23	2.3-3.4i
4	45	29	1.4-3.6i
5	54	33	.7-3.5i
6	68	40	-.5-2.9i
7	90	47	-1.4-1.3i

equations and solving for the unknown complex constants gives

$$\begin{aligned} &\alpha_0 = -1502 - 2772j, && \alpha_1 = 173 + 74j, \\ &\alpha_2 = 106 + 151j, && \alpha_3 = 0, \\ &\beta_0 = -481 - 316j, && \beta_1 = 13 + 13j, \\ &\beta_2 = 30 + 15j. && \end{aligned} \tag{6.14}$$

The function $I_3(L_1, C_2)$ will then be

$$I_3(L_1, C_2) = \frac{(-1502 - 2772j) + (173 + 74j)L_1 j\omega_0 + (106 + 151j)/(C_2 j\omega_0)}{(-481 - 316j) + (13 + 13j)L_1 j\omega_0 + (30 + 15j)/(C_2 j\omega_0) + L_1/C_2}. \tag{6.15}$$

Figs. 6.5 and 6.6 depict the magnitude and the phase of I_3 as a function of the design elements L_1 and C_2, as obtained in (6.15). Applying constraints (6.9) and (6.10) on I_3, one can find the region in the L_1-C_2 plane, shown in black color in Fig. 6.7, where these constraints are satisfied.

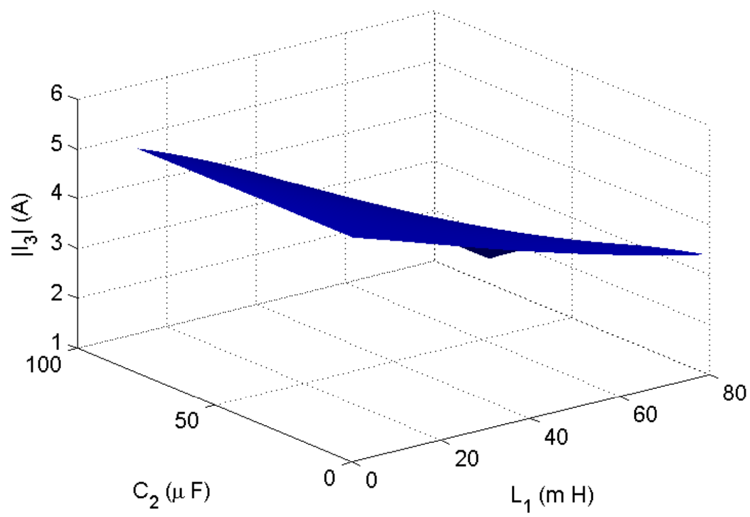

Figure 6.5: $|I_3(j\omega_0)|$ vs. L_1 and C_2

- I_9 vs. L_1 and C_2:

  Following the same procedure, one may obtain $I_9(L_1, C_2)$. Plots of the magnitude and the phase of I_9 as a function of L_1 and C_2 are shown in Figs. 6.8 and 6.9, respectively. Applying constraints (6.11) and (6.12) on I_9, one may calculate the region in the L_1-C_2 plane, shown in black color in Fig. 6.10, where these constraints are valid.

Intersecting the regions in Figs. 6.7 and 6.10, one finds the region where the constraints in (6.9)-(6.12) are satisfies simultaneously; this region is shown in Fig. 6.11.

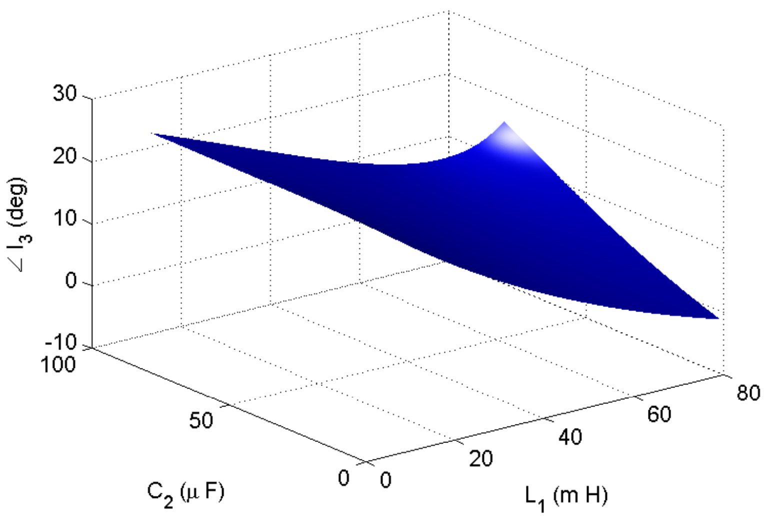

Figure 6.6: $\angle I_3(j\omega_0)$ vs. L_1 and C_2

6.4 Concluding Remarks

In this chapter we extended the measurement based approach, developed in Chapter 5 for DC circuits, to linear AC circuits. The main difference here is that the linear equations describing the system contain complex quantities. All the results of Chapter 5 carry over to the analysis and design of unknown linear AC circuits.

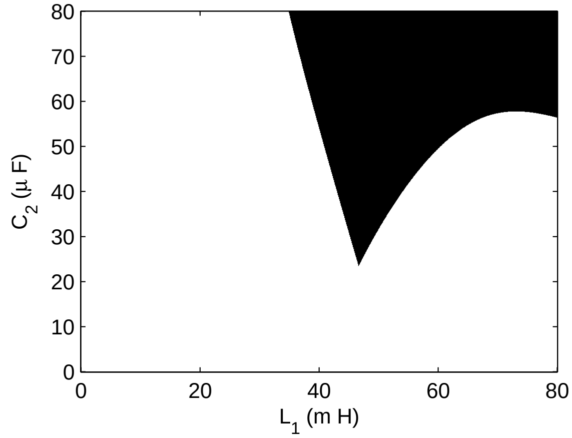

Figure 6.7: Region (in black color) where (6.9) and (6.10) are satisfied.

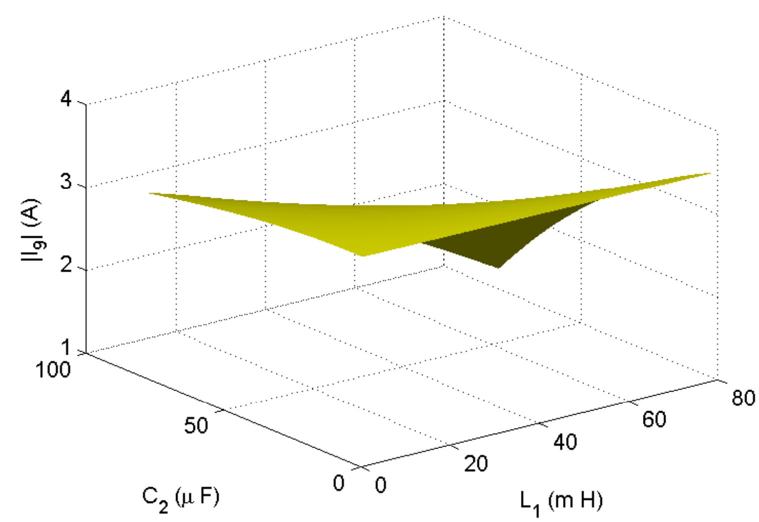

Figure 6.8: $|I_9(j\omega_0)|$ vs. L_1 and C_2

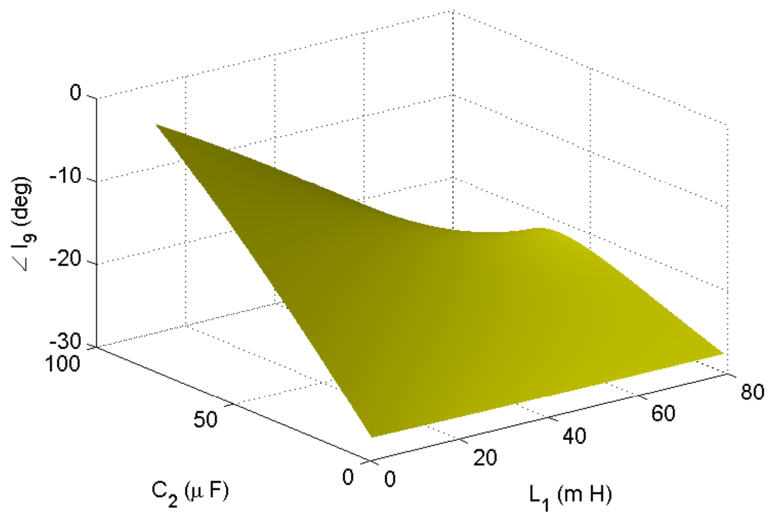

Figure 6.9: $\angle I_9(j\omega_0)$ vs. L_1 and C_2

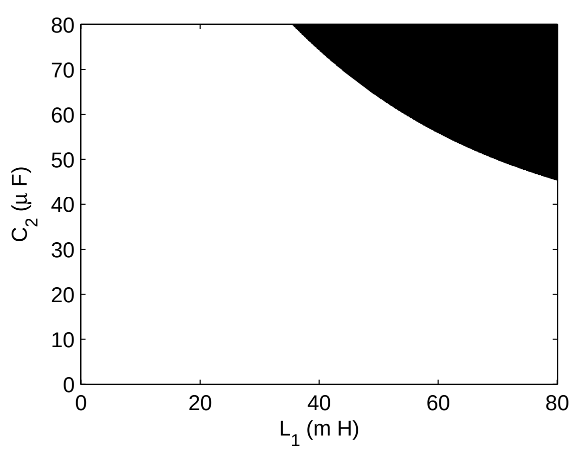

Figure 6.10: Region (in black color) where (6.11) and (6.12) are satisfied.

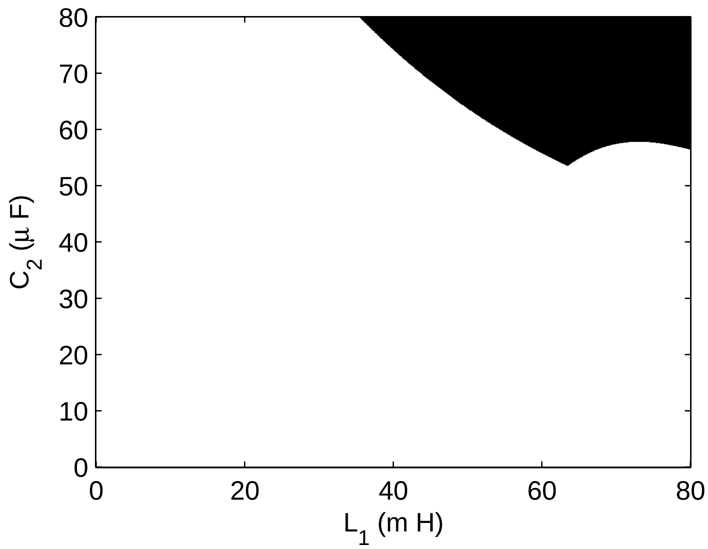

Figure 6.11: Region (in black color) where (6.9)-(6.12) are satisfied.

7. APPLICATION TO MECHANICAL SYSTEMS

This chapter deals with the application of the measurement based approach to linear mechanical systems, truss structures and linear hydraulic networks (see [37]).

7.1 Mass-Spring Systems

In this section we consider design problems where in an unknown mass-spring system the displacements of the masses are to be controlled by adjusting the spring stiffness constants at arbitrary locations. Consider the unknown linear mass-spring system shown in Fig. 7.1.

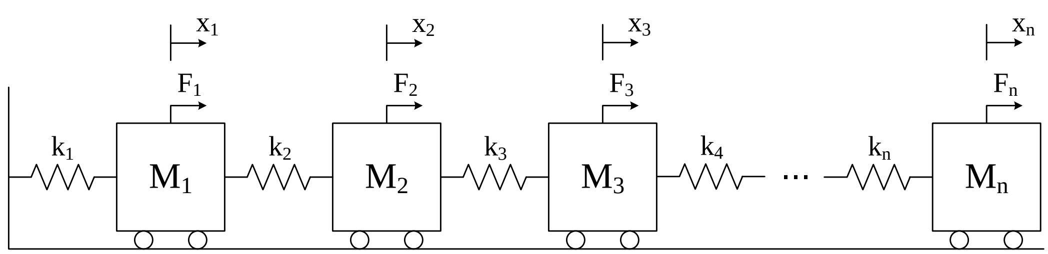

Figure 7.1: An unknown general mass-spring system

Suppose that we want to control the displacement of the i-th mass, denoted by x_i, by adjusting the spring stiffness k_j at an arbitrary location. Assume that the spring k_j is composed of piezoelectric materials (see [56]) such that its stiffness can be controlled by applying an electrical field. The displacements can be measured using a variety of sensors such as potentiometers or Linear Variable Differential Transformers

(LVDTs) (see [57]). In this problem the system of governing linear equations can be constructed in the form:

$$\underbrace{\begin{bmatrix} k_1+k_2 & -k_2 & 0 & \cdots & 0 & 0 \\ -k_2 & k_2+k_3 & -k_3 & \cdots & 0 & 0 \\ 0 & -k_3 & k_3+k_4 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -k_{n-1} & k_n \end{bmatrix}}_{\mathbf{A}(\mathbf{p})} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ \vdots \\ F_n \end{bmatrix}}_{\mathbf{b}(\mathbf{q})}, \quad (7.1)$$

where $\mathbf{p} = [k_1, k_2, \ldots, k_n]^T$, $\mathbf{x}$ represents the vector of unknown displacements and $\mathbf{q} = [F_1, F_2, \ldots, F_n]^T$ is the vector of external forces. Applying the Cramer's rule to (7.1) to calculate x_i gives

$$x_i(\mathbf{p}, \mathbf{q}) = \frac{|\mathbf{B}_i(\mathbf{p}, \mathbf{q})|}{|\mathbf{A}(\mathbf{p})|}, \quad i = 1, 2, \ldots, n, \quad (7.2)$$

where $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ is obtained by replacing the i^{th} column of $\mathbf{A}(\mathbf{p})$ by $\mathbf{b}(\mathbf{q})$. We have the following theorem.

Theorem 7.1. *In a linear mass-spring system, the functional dependency of any mass displacement x_i on any spring stiffness k_j can be determined by 3 measurements of the displacement x_i obtained for 3 different values of k_j.*

Proof. Note that the matrices $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ and $\mathbf{A}(\mathbf{p})$, in (7.2), are both of rank 1 with respect to k_j. Based on Lemma 4.1, $x_i(k_j)$ can be expressed as

$$x_i(k_j) = \frac{\tilde{\alpha}_0 + \tilde{\alpha}_1 k_j}{\tilde{\beta}_0 + \tilde{\beta}_1 k_j}, \quad (7.3)$$

where $\tilde{\alpha}_0, \tilde{\alpha}_1, \tilde{\beta}_0, \tilde{\beta}_1$ are constants. We rule out the case where $\tilde{\beta}_0 = \tilde{\beta}_1 = 0$, because if that is the case then $x_i \to \infty$, for any value of k_j, which is physically impossible. Assuming that $\tilde{\beta}_1 \neq 0$, one can simplify (7.3) to

$$x_i(k_j) = \frac{\alpha_0 + \alpha_1 k_j}{\beta_0 + k_j}, \tag{7.4}$$

where $\alpha_0, \alpha_1, \beta_0$ are constants. In order to determine $\alpha_0, \alpha_1, \beta_0$ one conducts 3 experiments by setting 3 different values to the spring stiffness k_j, say k_{j1}, k_{j2}, k_{j3} and measuring the corresponding displacements x_i, say x_{i1}, x_{i2}, x_{i3}. The following set of measurement equations can then be formed:

$$\underbrace{\begin{bmatrix} 1 & k_{j1} & -x_{i1} \\ 1 & k_{j2} & -x_{i2} \\ 1 & k_{j3} & -x_{i3} \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} x_{i1}k_{j1} \\ x_{i2}k_{j2} \\ x_{i3}k_{j3} \end{bmatrix}}_{\mathbf{m}}. \tag{7.5}$$

This set has a unique solution provided that $|\mathbf{M}| \neq 0$. If $|\mathbf{M}| = 0$, then the last column of $\mathbf{M}$ can be expressed as a linear combination of the first two columns because by assigning different values to the spring stiffness k_j, the first two columns of $\mathbf{M}$ become linearly independent. In this case, $x_i(k_j)$ will be

$$x_i(k_j) = \alpha_0 + \alpha_1 k_j, \tag{7.6}$$

where α_0, α_1 are constants that can be determined from any two of the experiments conducted earlier. The functional dependency (7.6) corresponds to the case where $\tilde{\beta}_1 = 0$ in (7.3), and the numerator and denominator of (7.3) are divided by $\tilde{\beta}_0$. □

Remark 7.1. *If the design parameters are the external forces then x_i can be expressed* *as*

$$x_i(F_1, F_2, \ldots, F_n) = \beta_1 F_1 + \beta_2 F_2 + \cdots + \beta_n F_n, \tag{7.7}$$

and the constants $\beta_1, \beta_2, \ldots, \beta_n$ *can be determined by applying* n *different sets of linearly independent vectors* $(F_1, F_2, \ldots, F_n)$ *to the system and measuring the corresponding displacements* x_i*. This is the well-known Superposition Principle in mechanical systems. In addition, if the external forces vary in the intervals* $F_j^- \leq F_j \leq F_j^+$*,* $j = 1, 2, \ldots, n$*, then the displacement* x_i *will vary in a convex hull whose vertices can be computed using the vertices* (F_j^-, F_j^+)*,* $j = 1, 2, \ldots, n$*.*

7.2 Truss Structures

Here, we consider truss structures and want to control the displacements of some set of truss joints using the stiffness of some set of design elements. Fig. 7.2 represents an unknown general truss structure.

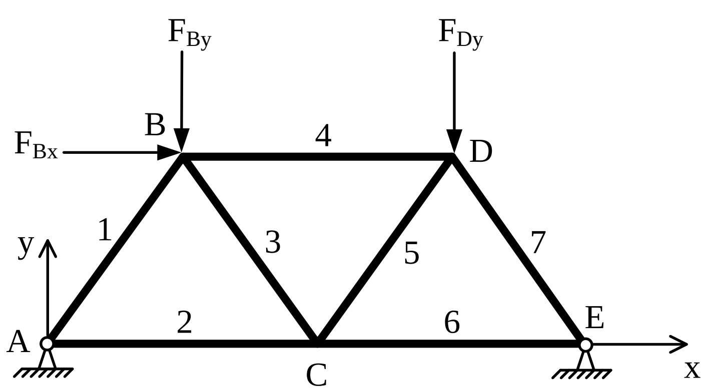

Figure 7.2: An unknown general truss structure

The element-wise stiffness matrix K_k, associated with the element k, can be constructed as

$$K_k = \frac{E_k A_k}{L_k} \begin{bmatrix} \cos^2\theta_k & \frac{1}{2}\sin 2\theta_k & -\cos^2\theta_k & -\frac{1}{2}\sin 2\theta_k \\ \frac{1}{2}\sin 2\theta_k & \sin^2\theta_k & -\frac{1}{2}\sin 2\theta_k & -\sin^2\theta_k \\ -\cos^2\theta_k & -\frac{1}{2}\sin 2\theta_k & \cos^2\theta_k & \frac{1}{2}\sin 2\theta_k \\ -\frac{1}{2}\sin 2\theta_k & -\sin^2\theta_k & \frac{1}{2}\sin 2\theta_k & \sin^2\theta_k \end{bmatrix}, \tag{7.8}$$

where E_k denotes the modulus of elasticity, A_k is the cross section area, L_k is the length of the element and θ_k is the angle of the element. For the sake of simplicity let us define $R_k := E_k A_k / L_k$. The global stiffness matrix $\mathbf{A}(\mathbf{p})$, in (7.9), can then formed from element-wise stiffness matrices [58]. Let s and c denote sin(.) and cos(.) functions, respectively. Then, the governing linear equations, for the truss structure depicted in Fig. 7.2, can be written in the following matrix form:

$$\underbrace{\begin{bmatrix} \mathbf{A}_{11}(\mathbf{p}) & \mathbf{A}_{12}(\mathbf{p}) \\ \mathbf{A}_{12}^T(\mathbf{p}) & \mathbf{A}_{22}(\mathbf{p}) \end{bmatrix}}_{\mathbf{A}(\mathbf{p})} \underbrace{\begin{bmatrix} \delta_{Ax} \\ \delta_{Ay} \\ \vdots \\ \delta_{Ex} \\ \delta_{Ey} \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} F_{Ax} \\ F_{Ay} \\ \vdots \\ F_{Ex} \\ F_{Ey} \end{bmatrix}}_{\mathbf{b}(\mathbf{q})}, \tag{7.9}$$

where $\mathbf{A}(\mathbf{p})$ is the global stiffness matrix, with

$$\mathbf{A}_{11}(\mathbf{p}) = \begin{bmatrix} R_1c^2\theta_1 + R_2c^2\theta_2 & R_1s\theta_1 + R_2s\theta_2 & -R_1c^2\theta_1 & -R_1s\theta_1 & -R_2c^2\theta_2 \\ & R_1s^2\theta_1 + R_2s^2\theta_2 & -R_1s\theta_1 & -R_1s^2\theta_1 & -R_2s\theta_2 \\ & & R_1c^2\theta_1 + R_3c^2\theta_3 + R_4c^2\theta_4 & R_1s\theta_1 + R_3s\theta_3 + R_4s\theta_4 & -2R_3c^2\theta_3 \\ & & & R_1s^2\theta_1 + R_3s^2\theta_3 + R_4s^2\theta_4 & -2R_3s\theta_3 \\ & & & & R_2c^2\theta_2 + R_3c^2\theta_3 + R_5c^2\theta_5 + R_6c^2\theta_6 \end{bmatrix},$$

$$\mathbf{A}_{12}(\mathbf{p}) = \begin{bmatrix} -R_2s\theta_2 & 0 & 0 & 0 & 0 \\ -R_2s^2\theta_2 & 0 & 0 & 0 & 0 \\ -2R_3s\theta_3 & -R_4c^2\theta_4 & -R_4s\theta_4 & 0 & 0 \\ -2R_3s^2\theta_3 & -R_4s\theta_4 & -R_4s^2\theta_4 & 0 & 0 \\ R_2s\theta_2 + R_3s\theta_3 + R_5s\theta_5 + R_6s\theta_6 & -R_5c^2\theta_5 & -R_5s\theta_5 & -R_6c^2\theta_6 & -R_6s\theta_6 \end{bmatrix},$$

$$\mathbf{A}_{22}(\mathbf{p}) = \begin{bmatrix} R_2s^2\theta_2 + R_3s^2\theta_3 + R_5s^2\theta_5 + R_6s^2\theta_6 & -R_5s\theta_5 & -R_5s^2\theta_5 & -R_6s\theta_6 & -R_6s^2\theta_6 \\ & R_4c^2\theta_4 + R_5c^2\theta_5 + R_7c^2\theta_7 & R_4s\theta_4 + R_5s\theta_5 + R_7s\theta_7 & -R_7c^2\theta_7 & -R_7s\theta_7 \\ & & R_4s^2\theta_4 + R_5s^2\theta_5 + R_7s^2\theta_7 & -R_7s\theta_7 & -R_7s^2\theta_7 \\ & & & R_6c^2\theta_6 + R_7c^2\theta_7 & R_6s\theta_6 + R_7s\theta_7 \\ & & & & R_6s^2\theta_6 + R_7s^2\theta_7 \end{bmatrix},$$

and $\mathbf{p}$ denotes the vector of elements parameters, $\mathbf{x}$ is the vector of unknown joints displacements (in x and y directions) and $\mathbf{q}$ represents the vector of external forces applied to the truss structure. Similar to the previous section, if one applies the Cramer's rule to (7.9) to calculate the i^{th} component of x, denoted by x_i, then

$$x_i(\mathbf{p}, \mathbf{q}) = \frac{|\mathbf{B}_i(\mathbf{p}, \mathbf{q})|}{|\mathbf{A}(\mathbf{p})|}, \quad i = 1, 2, \ldots, n, \tag{7.10}$$

where $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ is the matrix $\mathbf{A}(\mathbf{p})$ with the i^{th} column replaced by $\mathbf{b}(\mathbf{q})$.

Assuming that the design elements are composed of piezoelectric materials, one can control their cross section areas, and thus their stiffness constants, by applying an electrical field. Suppose that in this problem, the design parameters are the cross section areas of some set of design elements.

Theorem 7.2. *In a linear truss structure, the functional dependency of a given joint displacement δ_i, at a given direction, on A_j can be determined by 3 measurements of the joint displacement δ_i, in the respective direction, obtained for 3 different values of A_j.*

Proof. The proof is similar to the results in Section 7.1. Based the procedure of assembling element-wise stiffness matrices into the global stiffness matrix $\mathbf{A}(\mathbf{p})$, it can be concluded that the matrices $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ and $\mathbf{A}(\mathbf{p})$, in (7.10), are both of rank 1 with respect to A_j. Based on Lemma 4.1, $\delta_i(A_j)$ can be written as

$$\delta_i(A_j) = \frac{\alpha_0 + \alpha_1 A_j}{\beta_0 + A_j}, \tag{7.11}$$

where $\alpha_0, \alpha_1, \beta_0$ are constants that can be determined by conducting 3 experiments.

□

7.3 Hydraulic Networks

In this section we consider linear hydraulic networks. Suppose that, in a hydraulic network, all the flows are in the laminar state resulting in the governing steady state equations to be linear. The objective is to control the flow rates passing through some set of pipes.

In a laminar flow, the pressure drop occurring in a pipe can be obtained from,

$$\Delta P = \frac{8\mu LQ}{\pi r^4}, \tag{7.12}$$

where μ is the dynamic viscosity of the fluid, L is the length of the pipe, Q is the volume flow rate and r is the inner radius of the pipe. Let us rewrite (7.12) as

$$\Delta P = RQ, \tag{7.13}$$

where

$$R = \frac{8\mu L}{\pi r^4}, \tag{7.14}$$

is called the pipe resistance constant which is a function of the mechanical properties (length L and radius r) of the pipe.

To illustrate the approach, let us consider an unknown general hydraulic network as depicted in Fig. 7.3.

Similar to linear circuits, applying Kirchhoff's laws to a linear hydraulic network (Fig. 7.3) yields the set of governing linear equations as shown below:

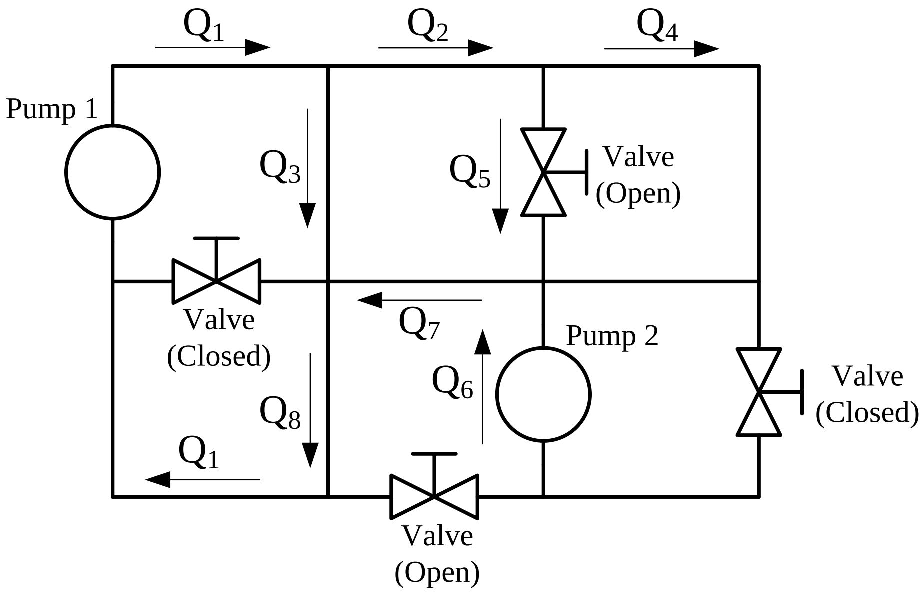

Figure 7.3: An unknown general hydraulic network

$$\underbrace{\begin{bmatrix} 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ R_1 & 0 & R_3 & 0 & 0 & 0 & 0 & R_8 \\ 0 & -R_2 & R_3 & 0 & -R_5 & R_6 & 0 & R_8 \\ 0 & 0 & 0 & -R_4 & R_5 & 0 & 0 & 0 \\ 0 & -R_2 & R_3 & -R_4 & 0 & 0 & -R_7 & 0 \end{bmatrix}}_{\mathbf{A}(\mathbf{p})} \underbrace{\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \\ Q_7 \\ Q_8 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ P_1 \\ P_2 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{b}(\mathbf{q})}, \tag{7.15}$$

where $\mathbf{p} = [R_1, R_2, \ldots, R_8]$ is the vector of the pipe resistances ($R_i,\ i = 1, 2, \ldots, 8$ is the resistance of the set of pipes through which $Q_i,\ i = 1, 2, \ldots, 8$ flows), $\mathbf{x}$ is the vector of unknown flow rates and $\mathbf{q}$ represents the vector of input parameters including the pump pressures. The flow rate Q_i can be calculated from (7.15) using the Cramer's rule,

$$Q_i = x_i(\mathbf{p}, \mathbf{q}) = \frac{|\mathbf{B}_i(\mathbf{p}, \mathbf{q})|}{|\mathbf{A}(\mathbf{p})|}, \quad i = 1, 2, \ldots, n, \tag{7.16}$$

where $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ is the matrix $\mathbf{A}(\mathbf{p})$ with the i^{th} column replaced by $\mathbf{b}(\mathbf{q})$.

Observation 7.1. *Upon an application of Kirchhoff's laws, each pipe resistance R_j appears in only one column of the characteristic matrix $\mathbf{A}(\mathbf{p})$.*

In the following subsections we consider different sets of design parameters.

7.3.1 Flow Rate Control using a Single Pipe Resistance

Assume that the design element is the resistance of one pipe, denoted by R_j, at an arbitrary location of the network. Therefore, we want to control the flow rate at some location of the network, denoted by Q_i, by adjusting the pipe resistance R_j.

Theorem 7.3. *In a linear hydraulic network, the functional dependency of any flow rate Q_i on the pipe resistance R_j can be determined by at most 3 measurements of the flow rate Q_i obtained for 3 different values of R_j.*

Proof. Consider two cases: 1) $i \neq j$ and 2) $i = j$.

Case 1: $i \neq j$

Based on the Observation 7.1, $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ and $\mathbf{A}(\mathbf{p})$, in (7.16), are both of rank 1 with respect to R_j. Therefore, based on Lemma 4.1, the function $Q_i(R_j)$ can be written as

$$Q_i(R_j) = \frac{\tilde{\alpha_0} + \tilde{\alpha_1} R_j}{\tilde{\beta_0} + \tilde{\beta_1} R_j}, \tag{7.17}$$

where $\tilde{\alpha_0}, \tilde{\alpha_1}, \tilde{\beta_0}, \tilde{\beta_1}$ are constants. Assuming that $\tilde{\beta_1} \neq 0$, one can divide the numerator and denominator of (7.17) by $\tilde{\beta_1}$ which gives

$$Q_i(R_j) = \frac{\alpha_0 + \alpha_1 R_j}{\beta_0 + R_j}, \tag{7.18}$$

where $\alpha_0, \alpha_1, \beta_0$ are constants that can be determined by conducting 3 experiments. The measurement equations can be written as

$$\underbrace{\begin{bmatrix} 1 & R_{j1} & -Q_{i1} \\ 1 & R_{j2} & -Q_{i2} \\ 1 & R_{j3} & -Q_{i3} \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} Q_{i1}R_{j1} \\ Q_{i2}R_{j2} \\ Q_{i3}R_{j3} \end{bmatrix}}_{\mathbf{m}}, \tag{7.19}$$

which can be uniquely solved provided that $|\mathbf{M}| \neq 0$. For the situations where $|\mathbf{M}| = 0$, which corresponds to the case $\tilde{\beta_1} = 0$ in (7.17), one may use a similar strategy presented in Section 5.1.1 to derive the corresponding functional dependency. Hence,

$$Q_i(R_j) = \alpha_0 + \alpha_1 R_j. \tag{7.20}$$

Case 2: $i = j$

Recalling (7.16) and based on the Observation 7.1, the matrix $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ is of rank 0 with respect to R_i; however, the matrix $\mathbf{A}(\mathbf{p})$ is of rank 1 with respect to R_j.

According to Lemma 4.1, $Q_i(R_i)$ will be

$$Q_i(R_i) = \frac{\tilde{\alpha_0}}{\tilde{\beta_0} + \tilde{\beta_1} R_i}, \tag{7.21}$$

where $\tilde{\alpha_0}, \tilde{\beta_0}, \tilde{\beta_1}$ are constants. Assuming that $\tilde{\beta_1} \neq 0$, and dividing the numerator and denominator of (7.21) by $\tilde{\beta_1}$ results in

$$Q_i(R_i) = \frac{\alpha_0}{\beta_0 + R_i}, \tag{7.22}$$

where α_0, β_0 are constants that can be determined by conducting 2 experiments. The following set of measurement equations can then be formed

$$\underbrace{\begin{bmatrix} 1 & -Q_{i1} \\ 1 & -Q_{i2} \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \alpha_0 \\ \beta_0 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} Q_{i1} R_{i1} \\ Q_{i2} R_{i2} \end{bmatrix}}_{\mathbf{m}}, \tag{7.23}$$

and uniquely solved if and only if $|\mathbf{M}| \neq 0$. If $|\mathbf{M}| = 0$ in (7.23), it can be concluded that Q_i is a constant,

$$Q_i(R_i) = \alpha_0. \tag{7.24}$$

This case corresponds to $\tilde{\beta_1} = 0$ in (7.21). $\square$

7.3.2 Flow Rate Control using Two Pipe Resistances

Suppose that the design parameters are any two pipe resistances, denoted by R_j and R_k, at arbitrary locations of the network, and the flow rate Q_i, at some location of the network, is to be controlled by adjusting these two pipe resistances.

Theorem 7.4. *In a linear hydraulic network, the functional dependency of any flow rate Q_i on the pipe resistances R_j and R_k can be determined by at most 7 measurements of the flow rate Q_i obtained for 7 different sets of values of (R_j, R_k).*

Proof. Again, let us consider two cases: 1) $i \neq j, k$ and 2) $i = j$ or $i = k$.

Case 1: $i \neq j, k$

In this case, based on the Observation 7.1, $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ and $\mathbf{A}(\mathbf{p})$, in (7.16), are both of rank 1 with respect to R_j and R_k. According to Lemma 4.2, $Q_i(R_j, R_k)$ will be

$$Q_i(R_j, R_k) = \frac{\tilde{\alpha_0} + \tilde{\alpha_1} R_j + \tilde{\alpha_2} R_k + \tilde{\alpha_3} R_j R_k}{\tilde{\beta_0} + \tilde{\beta_1} R_j + \tilde{\beta_2} R_k + \tilde{\beta_3} R_j R_k}, \tag{7.25}$$

where $\tilde{\alpha_0}$, $\tilde{\alpha_1}$, $\tilde{\alpha_2}$, $\tilde{\alpha_3}$, $\tilde{\beta_0}$, $\tilde{\beta_1}$, $\tilde{\beta_2}$, $\tilde{\beta_3}$ are constants. Assuming that $\tilde{\beta_3} \neq 0$, dividing the numerator and denominator of (7.25) by $\tilde{\beta_3}$ yields

$$Q_i(R_j, R_k) = \frac{\alpha_0 + \alpha_1 R_j + \alpha_2 R_k + \alpha_3 R_j R_k}{\beta_0 + \beta_1 R_j + \beta_2 R_k + R_j R_k}, \tag{7.26}$$

where $\alpha_0, \alpha_1, \alpha_2, \alpha_3, \beta_0, \beta_1, \beta_2$ are constants. In order to determine these constants, one conducts 7 experiments by setting 7 different sets of values to the pipe resistances (R_j, R_k), and measuring the corresponding flow rates Q_i. The following set of measurement equations can then be obtained

$$\underbrace{\begin{bmatrix} 1 & R_{j1} & R_{k1} & R_{j1}R_{k1} & -Q_{i1} & -Q_{i1}R_{j1} & -Q_{i1}R_{k1} \\ 1 & R_{j2} & R_{k2} & R_{j2}R_{k2} & -Q_{i2} & -Q_{i2}R_{j2} & -Q_{i2}R_{k2} \\ 1 & R_{j3} & R_{k3} & R_{j3}R_{k3} & -Q_{i3} & -Q_{i3}R_{j3} & -Q_{i3}R_{k3} \\ 1 & R_{j4} & R_{k4} & R_{j4}R_{k4} & -Q_{i4} & -Q_{i4}R_{j4} & -Q_{i4}R_{k4} \\ 1 & R_{j5} & R_{k5} & R_{j5}R_{k5} & -Q_{i5} & -Q_{i5}R_{j5} & -Q_{i5}R_{k5} \\ 1 & R_{j6} & R_{k6} & R_{j6}R_{k6} & -Q_{i6} & -Q_{i6}R_{j6} & -Q_{i6}R_{k6} \\ 1 & R_{j7} & R_{k7} & R_{j7}R_{k7} & -Q_{i7} & -Q_{i7}R_{j7} & -Q_{i7}R_{k7} \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} Q_{i1}R_{j1}R_{k1} \\ Q_{i2}R_{j2}R_{k2} \\ Q_{i3}R_{j3}R_{k3} \\ Q_{i4}R_{j4}R_{k4} \\ Q_{i5}R_{j5}R_{k5} \\ Q_{i6}R_{j6}R_{k6} \\ Q_{i7}R_{j7}R_{k7} \end{bmatrix}}_{\mathbf{m}}. \tag{7.27}$$

This set of equations can be uniquely solved provided $|\mathbf{M}| \neq 0$. If $|\mathbf{M}| = 0$, one can follow a similar procedure presented in Section 5.1.2 to develop the corresponding function $Q_i(R_j, R_k)$.

Case 2: $i = j$ or $i = k$

Suppose that $i = j$ and recall (7.16). Based on the Observation 7.1, in this case, $\mathbf{A}(\mathbf{p})$ is of rank 1 with respect to R_i and R_k; however, $\mathbf{B}_i(\mathbf{p}, \mathbf{q})$ is of rank 0 with respect to R_i and is of rank 1 with respect to R_k. According to Lemma 4.2 and these rank conditions, $Q_i(R_i, R_k)$ becomes

$$Q_i(R_i, R_k) = \frac{\tilde{\alpha_0} + \tilde{\alpha_1} R_k}{\tilde{\beta_0} + \tilde{\beta_1} R_i + \tilde{\beta_2} R_k + \tilde{\beta_3} R_i R_k}, \tag{7.28}$$

where $\tilde{\alpha_0}$, $\tilde{\alpha_1}$, $\tilde{\beta_0}$, $\tilde{\beta_1}$, $\tilde{\beta_2}$, $\tilde{\beta_3}$ are constants. Assuming that $\tilde{\beta_3} \neq 0$, one can divide the numerator and denominator of (7.28) by $\tilde{\beta}_3$ and obtain

$$Q_i(R_i, R_k) = \frac{\alpha_0 + \alpha_1 R_k}{\beta_0 + \beta_1 R_i + \beta_2 R_k + R_i R_k}, \tag{7.29}$$

where $\alpha_0, \alpha_1, \beta_0, \beta_1, \beta_2$ are constants that can be determined by conducting 5 experiments, by assigning 5 different sets of values to the pipe resistances (R_i, R_k), measuring the corresponding flow rates Q_i, and solving the obtained set of measurement equations. □

7.4 Illustrative Examples

Example 7.1. Mass-Spring Systems. Consider a three-story building frame as shown in Fig. 7.4 (A similar two-story building frame example can be found in [59], pg. 362, exercise 5.24). Suppose that the mechanical properties of the building components are unknown and the building is modeled as a mass-spring system shown is Fig. 7.5 with unknown parameters.

Suppose that we want to control the displacement of the second floor (mass M_2), denoted by x_2, by adjusting the stiffness constants of the links connecting the first and the second floors (spring constant k_2), to be within the range

$$-0.05 \leq x_2 \leq -0.03 \ (m). \tag{7.30}$$

Hence, we need to find an interval of k_2 values for which (7.30) is satisfied. Based on Theorem 7.1 the function $x_2(k_2)$ can be written as

$$x_2 = \frac{\alpha_0 + \alpha_1 k_2}{\beta_0 + k_2}, \tag{7.31}$$

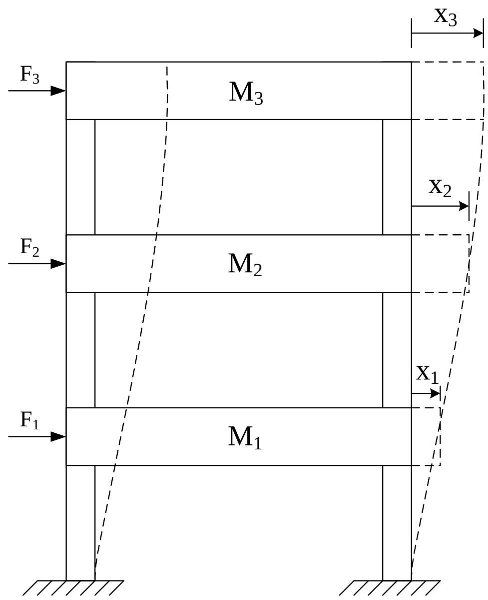

Figure 7.4: An unknown 3-story building example

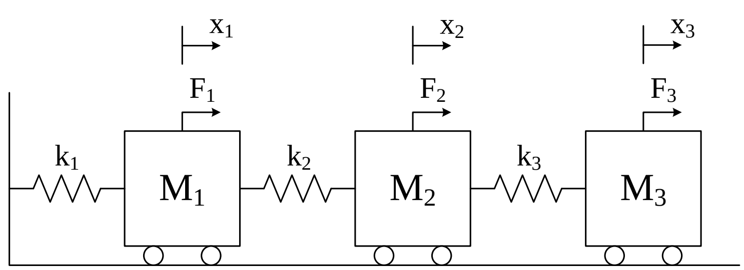

Figure 7.5: A mass-spring model of the 3-story building

where $\alpha_0, \alpha_1, \beta_0$ are constants that can be determined by conducting 3 experiments, by setting 3 different values to the spring constant k_2, say k_{21}, k_{22}, k_{23}, measuring the corresponding displacements x_2, say x_{21}, x_{22}, x_{23}, and then solving the following system of measurement equations

$$\begin{bmatrix} 1 & k_{21} & -x_{21} \\ 1 & k_{22} & -x_{22} \\ 1 & k_{23} & -x_{23} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} k_{21}x_{21} \\ k_{22}x_{22} \\ k_{23}x_{23} \end{bmatrix}. \tag{7.32}$$

Let Table 7.1 show the numerical values for the experiments performed for this example.

Table 7.1: Measurements for the mass-spring example

Exp. No.	k_2 (N/m)	x_2 (m)
1	2×10^5	-0.035
2	3×10^5	-0.030
3	5×10^5	-0.026

Substituting these numerical values into (7.32) and solving for the constants yields

$$x_2 = \frac{-3000 - 0.02k_2}{k_2}, \tag{7.33}$$

which is plotted in Fig. 7.6. Applying the design constraint given in (7.30) yields the following range of k_2 values:

$$10^5 \leq k_2 \leq 3 \times 10^5 \ (N/m).$$

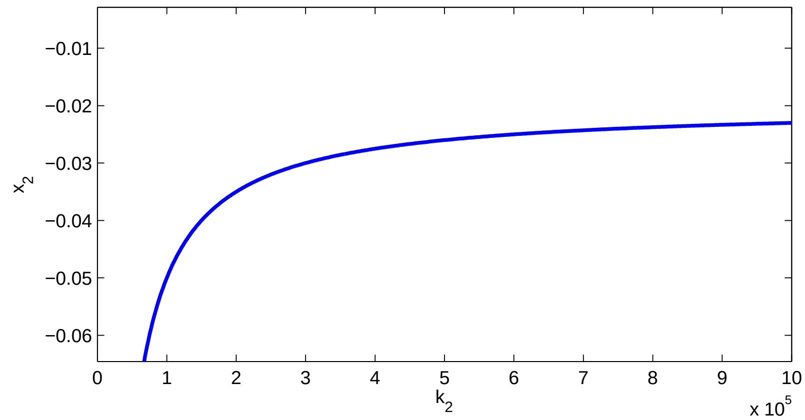

Figure 7.6: x_2 vs. k_2 for the mass-spring example

Example 7.2. A Network of Springs. For this example we constructed a network of springs as depicted in Fig. 7.7. Our objective was to find the functional dependency of the displacement of point "P", x_p, in terms of the spring stiffness constant k_2.

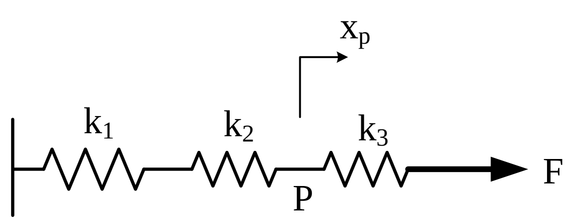

Figure 7.7: An unknown network of springs example

For this example, we performed 3 experiments and obtained the numerical values summarized in Table 7.2.

Table 7.2: Measurements for the network of springs example

Exp. No.	k_2 (N/m)	x_p (mm)
1	135.41	28.99
2	180.02	25.14
3	318.47	19.50

These numerical values resulted in the following function for $x_p(k_2)$:

$$x_p(k_2) = \frac{3479.86 + 10.53k_2}{33.83 + k_2}. \tag{7.34}$$

We set k_2 to some other values and measured the corresponding displacement x_p as shown in Table 7.3. The evaluation of function (7.34) for these values of k_2 is also provided in Table 7.3.

Table 7.3: Additional measurements for the network of springs example

k_2 (N/m)	x_p (mm) (practice)	x_p (mm) (evaluating (7.34))
176.99	25.62	25.35
285.31	20.96	20.32
310.08	19.56	19.62

As we expected, there is a very good agreement between the practical measurements and the evaluation of the obtained functional dependency.

Example 7.3. Truss Structures. Consider the truss structure shown in Fig. 7.8 (see [58], pg. 196, example 4.6.1) with unknown parameters.

Assuming that the deign element is the cross section area of link "AC", denoted by

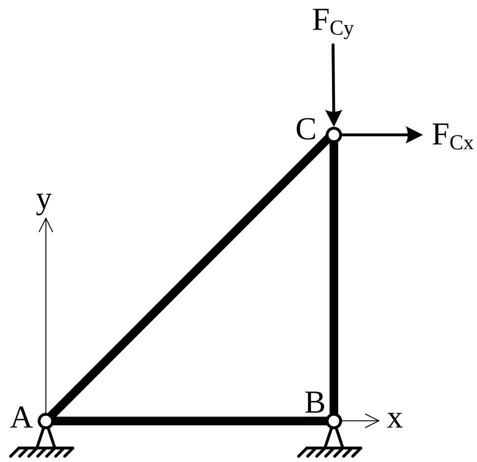

Figure 7.8: An unknown truss structure example

A_{AC}, and it is of interest to control the deflection of the joint "C" in the x-direction, denoted by δ_{Cx}, to be within the range

$$0 \leq \delta_{Cx} \leq 0.02\ (m). \tag{7.35}$$

Based on the statement of Theorem 7.2, $\delta_{Cx}(A_{AC})$ can be written as

$$\delta_{Cx} = \frac{\alpha_0 + \alpha_1 A_{AC}}{\beta_0 + A_{AC}}, \tag{7.36}$$

where $\alpha_0, \alpha_1, \beta_0$ are constants that can be determined by conducting 3 experiments and solving the following system of equations

$$\begin{bmatrix} 1 & A_{AC1} & -\delta_{Cx1} \\ 1 & A_{AC2} & -\delta_{Cx2} \\ 1 & A_{AC3} & -\delta_{Cx3} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \beta_0 \end{bmatrix} = \begin{bmatrix} A_{AC1}\delta_{Cx1} \\ A_{AC2}\delta_{Cx2} \\ A_{AC3}\delta_{Cx3} \end{bmatrix}. \tag{7.37}$$

Let Table 7.4 summarize the numerical values for the measurements taken for this example. Substituting these numerical values into (7.37) and solving for the unknown constants yields

$$\delta_{Cx} = \frac{1.1 \times 10^{-6} + 6.67 \times 10^{-3} A_{AC}}{A_{AC}}, \tag{7.38}$$

which is plotted in Fig. 7.9.

Table 7.4: Measurements for the truss structure example

Exp. No.	A_{AC} (m^2)	δ_{Cx} (m)
1	100×10^{-6}	0.018
2	150×10^{-6}	0.014
3	200×10^{-6}	0.012

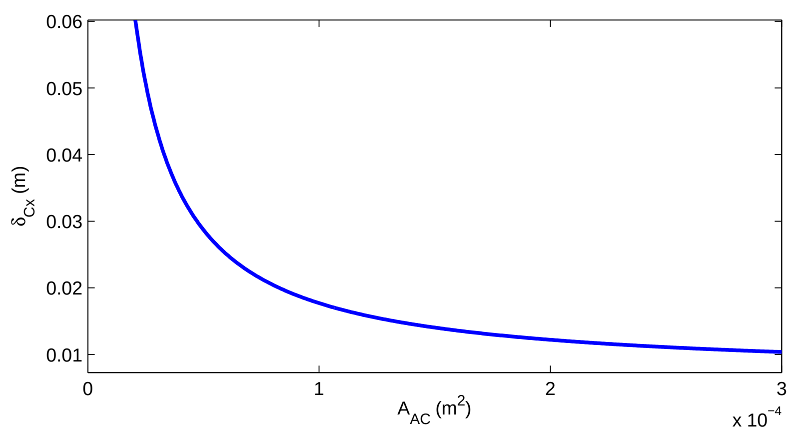

Figure 7.9: δ_{Cx} vs. A_{AC} for the truss structure example

Applying the design constraint, given in (7.35), on δ_{Cx} yields

$$A_{AC} \geq 0.83 \times 10^{-4}\ (m^2).$$

Example 7.4. Hydraulic Networks. Consider the unknown hydraulic network shown in Fig. 7.10 and suppose that the flow is laminar, which results in the governing steady state equations to be linear.

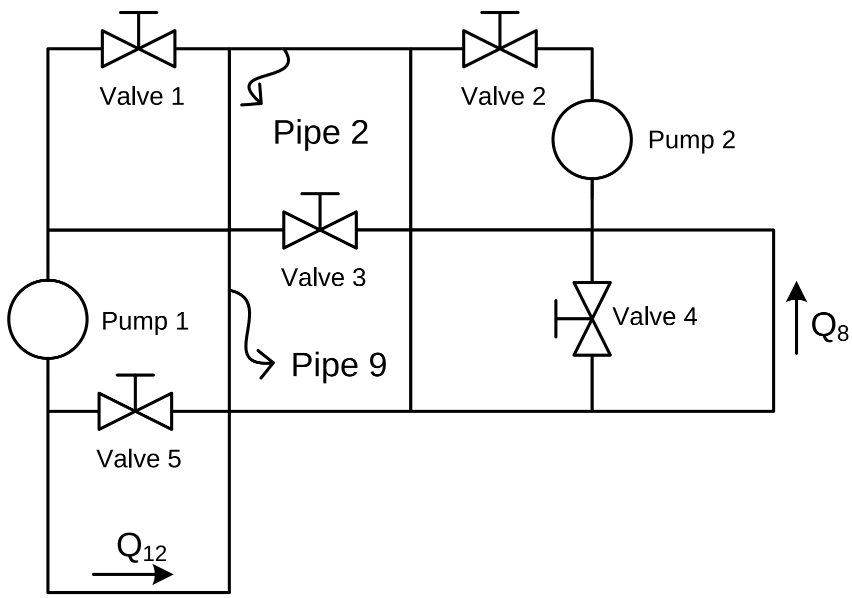

Figure 7.10: An unknown hydraulic network example

Assume that the design objective is to control the flow rates Q_8 and Q_{12} (as

Table 7.5: Measurements for the hydraulic network example

Exp. No.	r_2 (m)	R_2 $(Pa.s/m^3)$	r_9 (m)	R_9 $(Pa.s/m^3)$	Q_8 (m^3/s)
1	0.05	408	0.05	408	0.038
2	0.07	107	0.08	62	0.043
3	0.09	39	0.11	17	0.049
4	0.1	26	0.13	9	0.051
5	0.12	12	0.15	5	0.054
6	0.14	6	0.17	3	0.055
7	0.17	3	0.2	1.6	0.056

shown in Fig. 7.10) to stay within the following ranges:

$$0.045 \leq Q_8 \leq 0.055 \ (m^3/s), \tag{7.39}$$

$$0.01 \leq Q_{12} \leq 0.03 \ (m^3/s), \tag{7.40}$$

by adjusting the radii of the pipes numbered 2 and 9, denoted by r_2 and r_9, respectively. Therefore, the design objective is to find regions in the r_2-r_9 plane for which the desired flow rates in (7.39) and (7.40) are met.

Based on the results in Section 7.3.2, one can determine the functional dependency of any flow rate on any two pipe resistances by at most 7 measurements. Let Table 7.5 show the numerical values of the measurements taken to find $Q_8(r_2, r_9)$. Substituting these values into (7.27) and solving for the unknown constants yields (also recall (7.14))

$$Q_8(r_2, r_9) = \frac{8.7 \times 10^7 + \frac{1600}{r_2^4} + \frac{3500}{r_9^4} + \frac{0.034}{r_2^4 r_9^4}}{1.5 \times 10^9 + \frac{48000}{r_2^4} + \frac{75000}{r_9^4} + \frac{1}{r_2^4 r_9^4}}, \tag{7.41}$$

which is plotted in Fig. 7.11. Applying the constraint (7.39) to (7.41) gives the region shown (in black) in Fig. 7.12 in the r_2-r_9 plane.

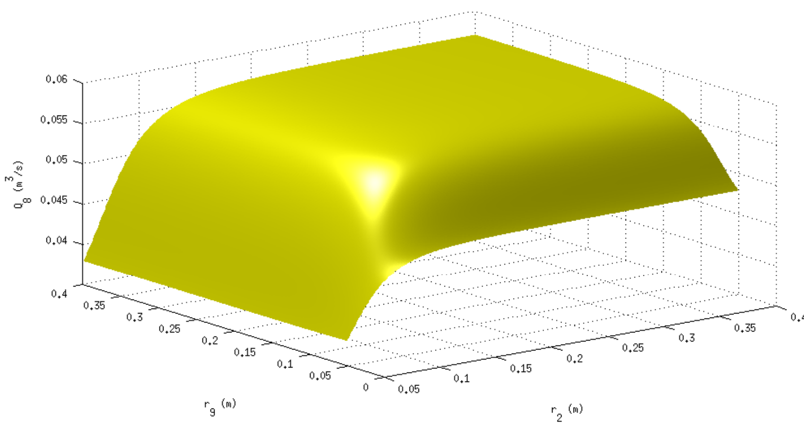

Figure 7.11: Q_8 vs. r_2 and r_9

Similarly one can find $Q_{12}(r_2, r_9)$, as plotted in Fig. 7.13. Fig. 7.14 shows the region (in black) in the r_2-r_9 plane where the constraint (7.40) is satisfied.

Intersecting the regions in Figs. 7.12 and 7.14, one finds the region where both the constraints (7.39) and (7.40) are met simultaneously (see Fig. 7.15).

7.5 Concluding Remarks

In this chapter we extended our measurement based approach to the domain of linear mechanical systems, such as mass-spring networks, truss structures and hydraulic systems. We showed that measurements processed appropriately can be directly used to solve analysis and design problem in linear mechanical systems.

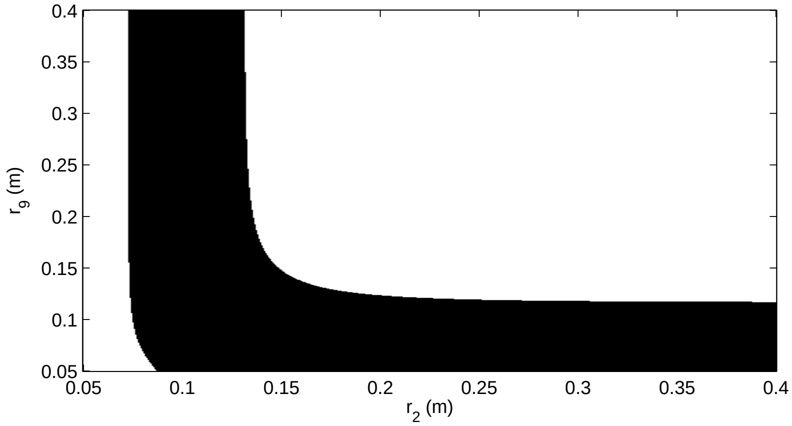

Figure 7.12: Region where (7.39) is satisfied.

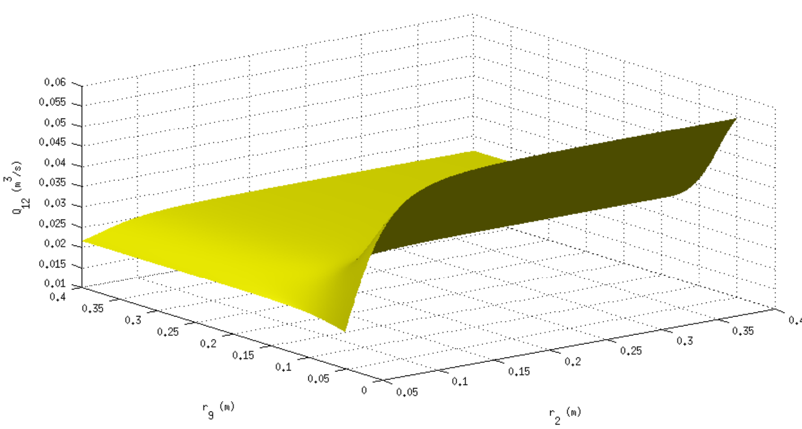

Figure 7.13: Q_{12} vs. r_2 and r_9

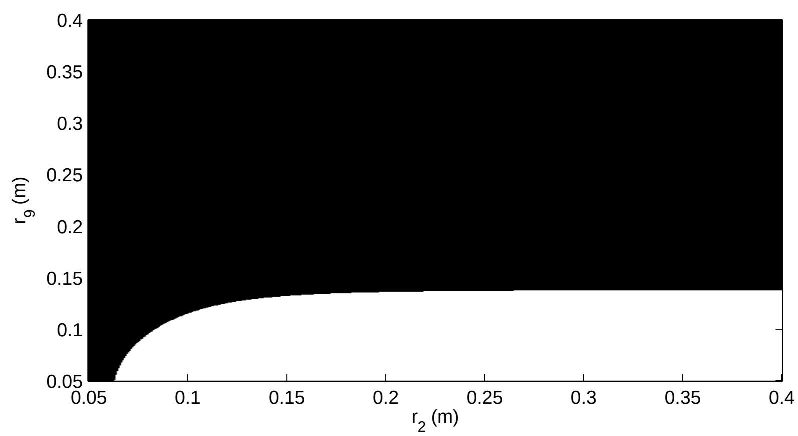

Figure 7.14: Region where (7.40) is satisfied.

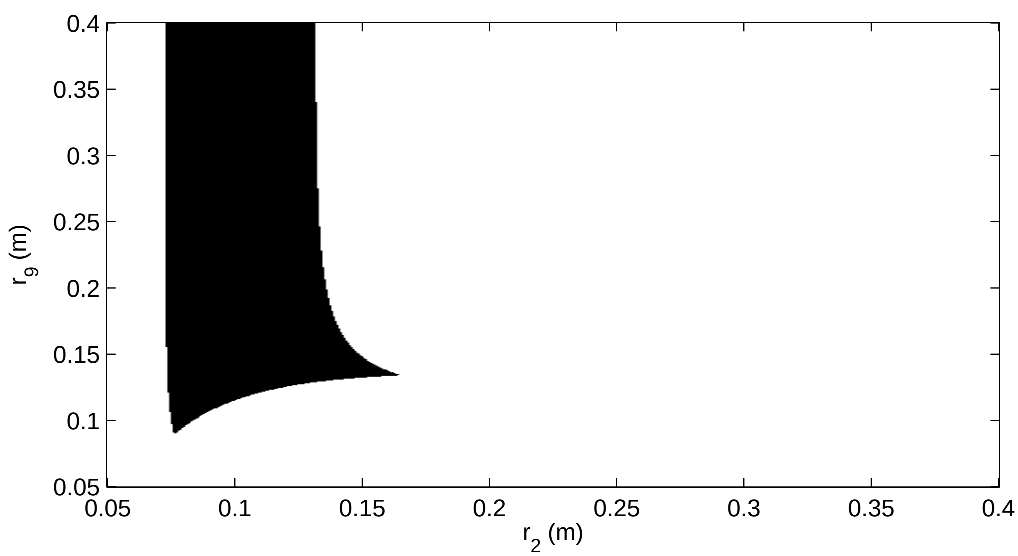

Figure 7.15: Region where (7.39) and (7.40) are satisfied

8. APPLICATION TO CONTROL SYSTEMS

The problem of designing controllers satisfying stability and performance requirements has many practical important applications. Most of the classical control design techniques require a mathematical model of the plant, such as a transfer function representation or state space equations, a priori. In practice one usually deals with very complex systems where modeling is not an easy task. In general, if a model is to be proposed for a complex system, it will be of higher order, which makes the design process difficult. This observation motivates the search for a new approach which can determine the controller parameters directly from measurements and without requiring a mathematical model of the system.

This chapter presents a new measurement based approach to the problem of controller design for Linear Time-Invariant (LTI) control systems. The objective is to guarantee stability and a prescribed desired closed loop frequency response, meeting the design specifications in the frequency domain.

8.1 Block Diagrams

Let us begin by considering a general block diagram as shown in Fig. 8.1.

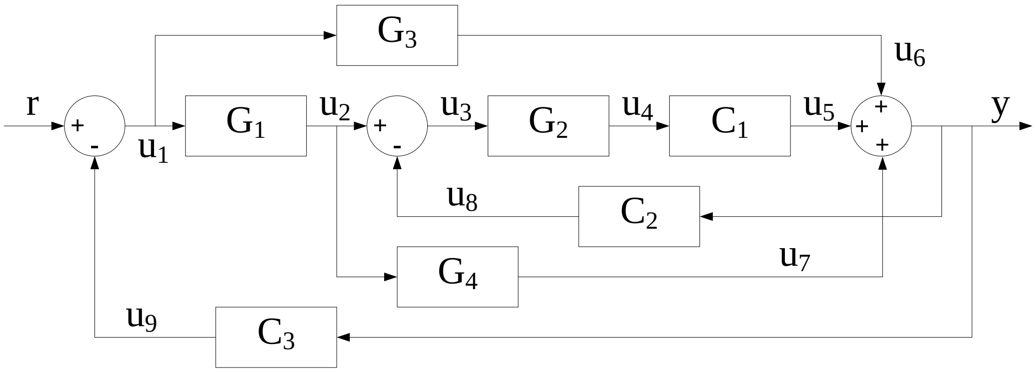

Figure 8.1: Block diagram of an unknown multivariable system

Writing the system equations in the matrix form gives

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ G_1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & G_2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ G_3 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & G_4 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & C_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & C_3 \end{bmatrix}}_{\mathbf{A}(\mathbf{p})} \underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ y \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} r \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{b}(\mathbf{q})}, \tag{8.1}$$

where $\mathbf{A}(\mathbf{p})$ is called the system characteristic matrix, $\mathbf{x}$ is the vector of unknown signals and $\mathbf{q} = q_1 = r$ is the input to the system. It can be easily observed that in (8.1), the characteristic matrix $\mathbf{A}(\mathbf{p})$ is of rank 1 with respect to each of the elements, $G_i,\ i = 1, 2, 3, 4$ and $C_j,\ j = 1, 2, 3$ of the block diagram. Therefore, considering any of these elements as the design parameter, the transfer function between any two points of the control system shown in Fig. 8.1 can be expressed as a linear rational function of the design parameter. For instance, suppose that the design parameter is C_1, which appears in $\mathbf{A}(\mathbf{p})$ with rank 1 dependency. Then, the closed loop transfer function between r and y, denoted by $H(s)$, can be represented as

$$H(s) = \frac{\alpha_0(s) + \alpha_1(s)C_1(s)}{\beta_0(s) + C_1(s)}, \tag{8.2}$$

where $\alpha_0(s), \alpha_1(s)$ and $\beta_0(s)$ are unknown rational functions and are to be determined by performing experiments, as explained in the following section.

8.2 SISO Control Systems

8.2.1 Functional Dependency on a Single Controller

Consider the unknown Single-Input Single-Output (SISO) control system shown in Fig. 8.2 and assume that the controller to be designed is denoted by $C(s)$. Let us write the governing equations as

$$\mathbf{A}(\mathbf{p})\mathbf{x} = \mathbf{b}(\mathbf{q}). \tag{8.3}$$

Recalling the rank dependency observation presented in the previous section, the closed loop transfer function can be expressed as

$$H(s) = \frac{\alpha_0(s) + \alpha_1(s)C(s)}{\beta_0(s) + C(s)}, \tag{8.4}$$

where $\alpha_0(s), \alpha_1(s)$ and $\beta_0(s)$ are unknown and $H(s)$ is the transfer function connect-

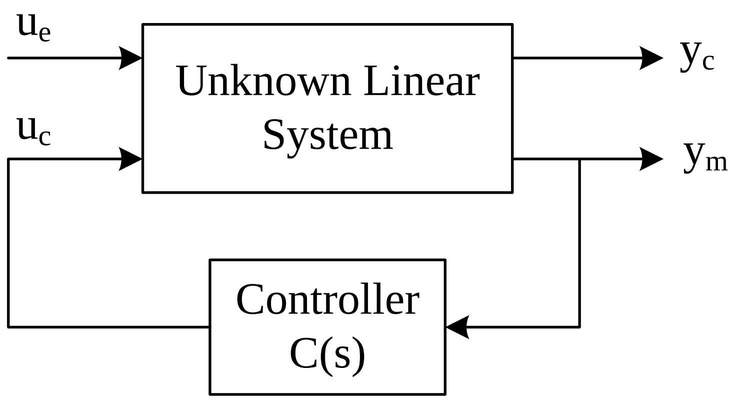

Figure 8.2: Unknown linear system with controller

ing u_e to y_c. Let us rewrite (8.4) in the frequency domain as

$$H(j\omega) = \frac{\alpha_0(j\omega) + \alpha_1(j\omega)C(j\omega)}{\beta_0(j\omega) + C(j\omega)}, \tag{8.5}$$

where the unknown complex coefficients can be determined by embedding 3 stabilizing controllers, C_1, C_2 and C_3, into the closed loop system and measuring the corresponding closed loop frequency responses, $H_1(j\omega), H_2(j\omega)$ and $H_3(j\omega)$, at a finite set of frequencies ω_k, $k = 1, 2, \ldots, N$. The following system of linear measurement equations can be formed at each frequency ω_k, $k = 1, 2, \ldots, N$:

$$\underbrace{\begin{bmatrix} 1 & C_1(j\omega_k) & -H_1(j\omega_k) \\ 1 & C_2(j\omega_k) & -H_2(j\omega_k) \\ 1 & C_3(j\omega_k) & -H_3(j\omega_k) \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \alpha_0(j\omega_k) \\ \alpha_1(j\omega_k) \\ \beta_0(j\omega_k) \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} H_1(j\omega_k)C_1(j\omega_k) \\ H_2(j\omega_k)C_2(j\omega_k) \\ H_3(j\omega_k)C_3(j\omega_k) \end{bmatrix}}_{\mathbf{m}}, \tag{8.6}$$

which can be solved for the unknown complex quantities $\alpha_0(j\omega_k), \alpha_1(j\omega_k)$ and $\beta_0(j\omega_k)$.

8.2.2 Determining a Desired Response

Suppose that in a control design problem, the design specifications are given in the frequency domain. For example, the specifications may include a desired gain margin, phase margin and bandwidth. Based on these specifications, one may consider a desired closed loop frequency response, denoted by $H^*(j\omega)$. Equation (8.5) can then be solved for the controller $C^*(j\omega)$ as

$$C^*(j\omega) = \frac{H^*(j\omega)\beta_0(j\omega) - \alpha_0(j\omega)}{\alpha_1(j\omega) - H^*(j\omega)}, \tag{8.7}$$

which guarantees that the desired closed loop frequency response, $H^*(j\omega)$, is attained. Next subsection summarizes the design steps to find frequency response $C^*(j\omega)$ and finally solve a controller design problem.

8.2.3 Steps to Controller Design

In this subsection, we summarize the design steps toward solving a general control design problem directly from frequency domain data and based on the approach provided in this chapter. Suppose that the frequency response data of plant P, denoted by $P(j\omega)$, is available, and the design objective is to find a controller which guarantees the stability and a set of frequency domain specifications for the closed loop system, such as gain margin, phase margin and bandwidth. One may take the following steps to design such controller.

1. Connect 3 stabilizing controllers, C_1, C_2 and C_3, to the control system and measure the corresponding closed loop frequency responses, $H_1(j\omega)$, $H_2(j\omega)$ and $H_3(j\omega)$.

2. Solve (8.6), at a finite set of frequencies, for the unknown complex quantities, $\alpha_0(j\omega)$, $\alpha_1(j\omega)$ and $\beta_0(j\omega)$.

3. Define a desired closed loop frequency response, $H^*(j\omega)$, based on the desired frequency domain design specifications.

4. Calculate the corresponding frequency response $C^*(j\omega)$ using (8.7).

5. Realize $C^*(j\omega)$ using system identification methods. An alternative approach is to consider a fixed structure controller, such as a PID controller, and solve a least-square minimization problem to determine the controller gains in the stabilizing set. Denote the realized controller by $C_r(s)$.

6. Check $C_r(s)$ for stability. If $P(j\omega)$ and $C_r(j\omega)$ satisfy certain conditions at specific frequencies, then the closed loop stability is guaranteed (see [33]).

7. If $C_r(s)$ is not a stabilizing controller go to step 3, define a new $H^*(j\omega)$, and repeat steps 4 through 6 until the realized controller is a stabilizing one.

8.3 Illustrative Example

Example 8.1. *Problem.* Suppose that the frequency response data of an unknown linear plant is depicted in Fig. 8.3, and the controller to be designed has a PID structure,

$$C(s) = \frac{k_d s^2 + k_p s + k_i}{s}. \tag{8.8}$$

Also, suppose that the required closed loop frequency domain specifications are:

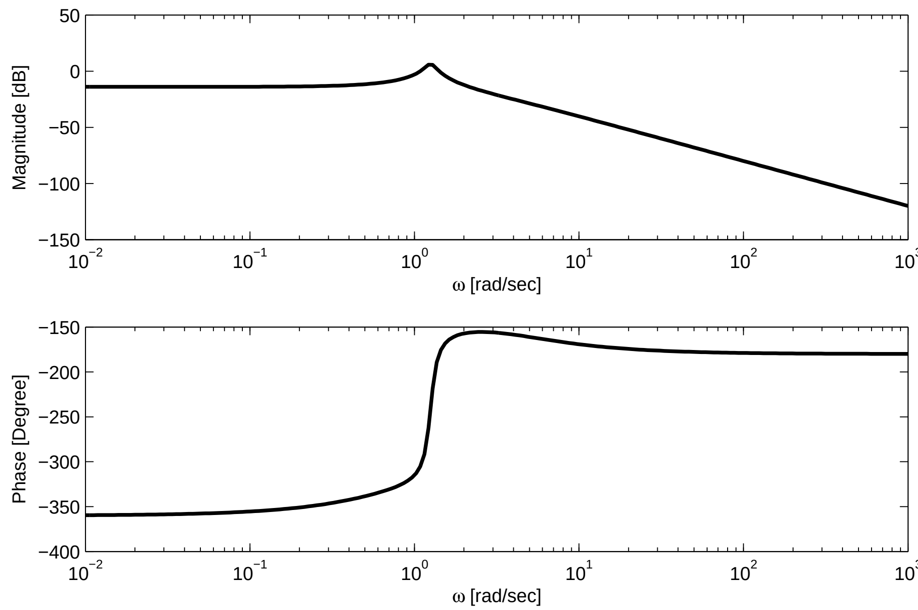

Figure 8.3: Frequency response of an unknown linear plant

1. Bandwidth $\approx$ 10 rad/sec,
2. PM > 100 deg.

The complete set of stabilizing PID controllers for the plant with the frequency response as shown in Fig. 8.3 can be constructed as shown in Fig. 8.4 (see [7]).

Approach. The design objective is to find the "best" PID controller inside the stabilizing set (Fig. 8.4) which guarantees the specifications. One can follow the steps given in the previous section.

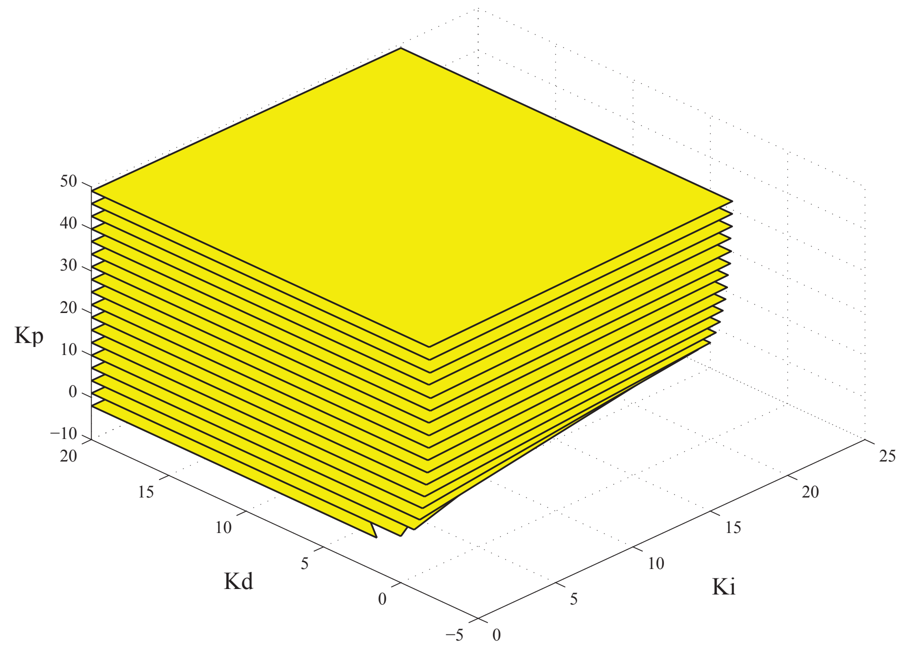

Figure 8.4: The complete set of stabilizing PID controllers for this example

1. Select 3 arbitrary controllers from the stabilizing set (Fig. 8.4), for example,

$$\begin{aligned} C_1(s) =& \frac{s^2 + 2s + 0.5}{s}, \\ C_2(s) =& \frac{2s^2 + 2s + 1}{s}, \\ C_3(s) =& \frac{3s^2 + 2s + 2}{s}, \end{aligned} \tag{8.9}$$

   and place them in the closed loop system. For each case, measure the closed loop frequency response, $H(j\omega)$, as shown in Fig. 8.5 (solid lines).

2. Solve (8.6) for $\alpha_0(j\omega), \alpha_1(j\omega)$ and $\beta(j\omega)$, at a finite set of frequencies ω_k, $k = 1, 2, \ldots, N$.

3. Based on the given specifications, we defined a desired closed loop frequency

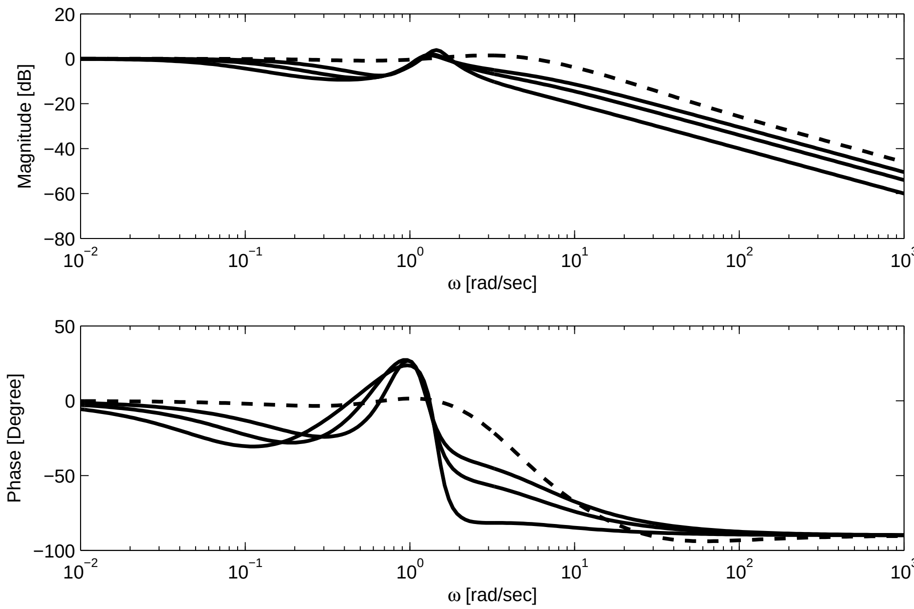

Figure 8.5: Frequency response of the closed loop system $H_1(j\omega)$, $H_2(j\omega)$ and $H_3(j\omega)$ (solid lines) after embedding the controllers in (8.9) and the desired response $H^*(j\omega)$ (dashed line)

response, $H^*(j\omega)$, as plotted with the dashed line in Fig. 8.5.

4. Calculate $C^*(j\omega)$ using (8.7), which is depicted in Fig. 8.6.

5. The frequency response $C^*(j\omega)$ is realized by a PID controller transfer function as

$$C_r(s) = \frac{5s^2 + 39.8s + 13.3}{s}. \tag{8.10}$$

Fig. 8.7 shows the frequency response of $C^*(j\omega)$ (solid line) and $C_r(s)$ (dashed line).

6. Referring to the complete stabilizing set (Fig. 8.4), it can be easily verified

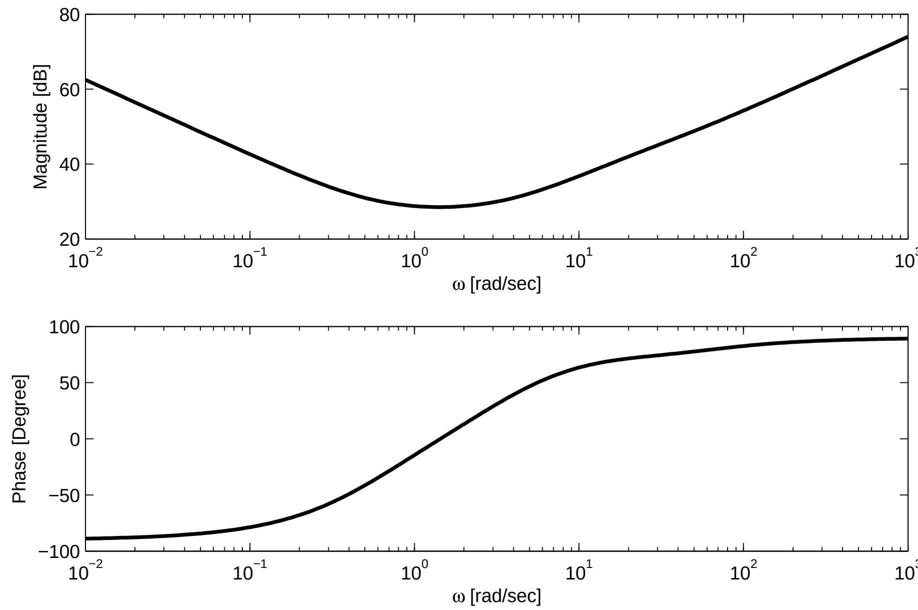

Figure 8.6: Frequency response $C^*(j\omega)$

that $C_r(s)$ is a stabilizing controller; hence, it is a solution to our control design problem.

The PID controller obtained in (8.10) is connected to the system and the closed loop frequency response is measured as shown (with solid line) in Fig. 8.8 (the dashed line represents $H^*(j\omega)$). The closed loop system obtained by connecting $C_r(s)$ in (8.10) has the following bandwidth and phase margin (see Fig. 8.8):

1. Bandwidth = 10.4 rad/sec,

2. PM = 104 deg.

Therefore, the controller $C_r(s)$ in (8.10), designed through this new measurement based approach, guarantees the stability and the required design specifications of the closed loop system.

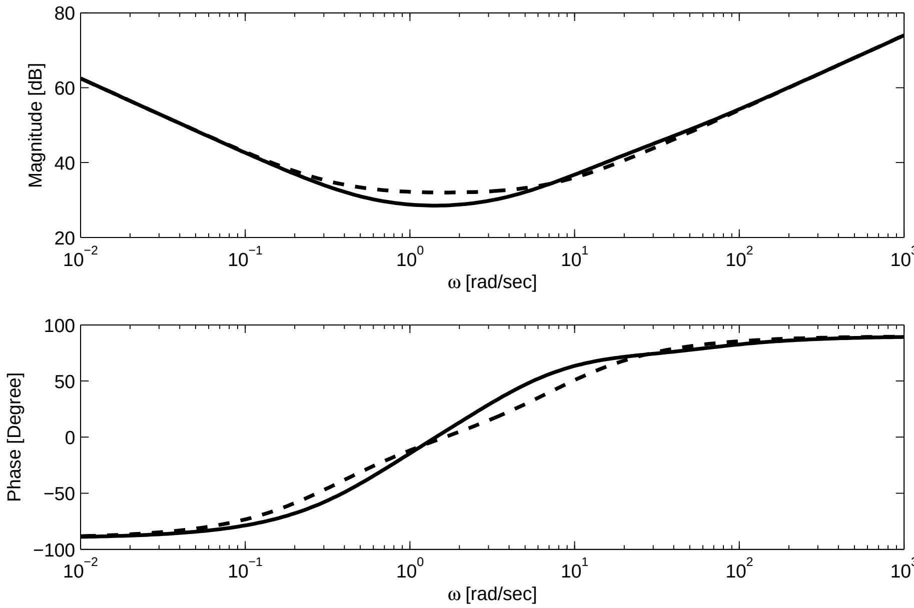

Figure 8.7: Realization of $C^*(j\omega)$ (solid line) by $C_r(s)$ (dashed line)

8.4 Concluding Remarks

We have shown here that a few strategic measurements can solve the controller design problem for general unknown linear systems even without the knowledge of the mathematical model of the system. The resulting controllers guarantee the stability and performance of the closed loop system. These results can apply broadly to any system described by linear equations.

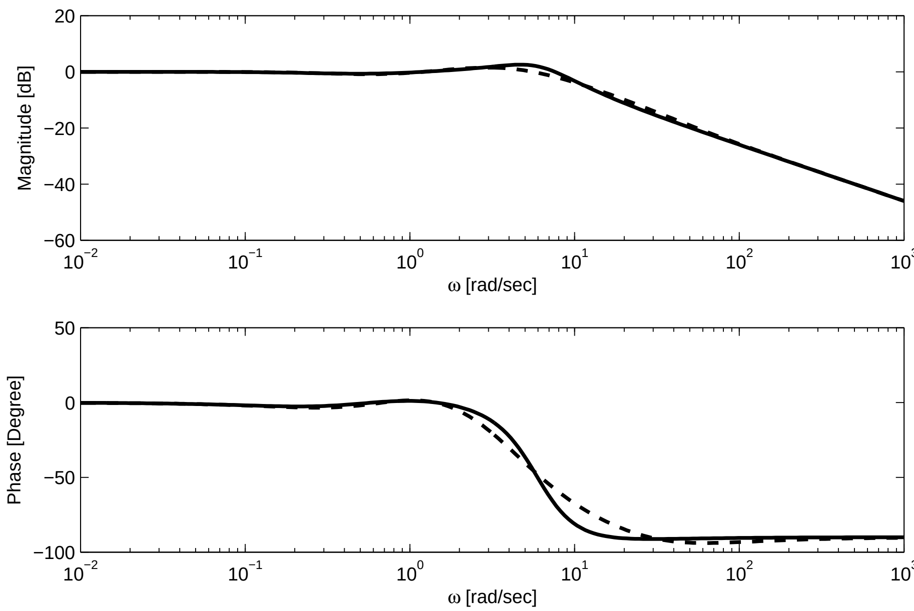

Figure 8.8: The desired frequency response $H^*(j\omega)$ (dashed line) and the closed loop response after connecting $C_r(s)$ (solid line)

9. AN EXTREMAL RESULT FOR UNKNOWN INTERVAL LINEAR SYSTEMS

This chapter explores some important characteristics of a system of linear equations containing parameters. As studied in the earlier chapters, such a system of equations arises in many branches of engineering. A parametrized solution of a set of linear equations can be obtained by applying Cramer's rule. In many practically important cases the parameters appear with rank one dependency, resulting in parametrized solutions to be of a rational multilinear form, which will be monotonic in each parameter. This monotonic characteristic has practical importance in the analysis and design of linear systems with parameters having interval uncertainties. In particular, extremal values of system variables occur at the vertices of the parameter boxes [47].

This chapter is organized as follows. Section 9.1 presents our extremal result for unknown linear systems with parameters appearing with rank one dependency. Some illustrative examples of current, power level and flow rate control problems are given in Section 9.2. Finally, we summarize with our concluding remarks in Section 9.3.

9.1 Main Results

Suppose that a physical system can be described by the following set of linear equations

$$\mathbf{A}(\mathbf{p})\mathbf{x} = \mathbf{b}(\mathbf{q}), \tag{9.1}$$

where $\mathbf{A}(\mathbf{p})$ is referred to as the system characteristic matrix, $\mathbf{p}$ and $\mathbf{q}$ are vectors of system parameters and inputs, respectively, and $\mathbf{x}$ is the vector of unknown system variables. Assuming that $|\mathbf{A}(\mathbf{p})| \neq 0$, there exists a unique solution $\mathbf{x}$ and, by Cramer's rule, the i^{th} element x_i of $\mathbf{x}$ is given by

$$x_i(\mathbf{p}, \mathbf{q}) = \frac{|\mathbf{B}(\mathbf{p}, \mathbf{q})|}{|\mathbf{A}(\mathbf{p})|}, \qquad i = 1, 2, \ldots, n. \tag{9.2}$$

We make the following crucial assumption regarding the set of equations (9.1).

Assumption 9.1. *There exists no* $\mathbf{p}$ *such that* $\mathbf{A}(\mathbf{p})$ *is a singular matrix.*

This assumption is usually true for physical systems, because if there exists a vector $\mathbf{p}_0$ so that $\mathbf{A}(\mathbf{p}_0)$ becomes a singular matrix, then the corresponding vector of system variables, $\mathbf{x}$ in (9.1), will not have a unique value which is not the case for physical systems.

Suppose that the parameter vector $\mathbf{p}$ appears *affinely* in $\mathbf{A}(\mathbf{p})$. Thus, we can write

$$\mathbf{A}(\mathbf{p}) = \mathbf{A}_0 + p_1\mathbf{A}_1 + p_2\mathbf{A}_2 + \cdots + p_l\mathbf{A}_l. \tag{9.3}$$

Recalling Lemma 4.2, $x_i(\mathbf{p}, \mathbf{q})$ in (9.2) can be expressed as

$$x_i(\mathbf{p}, \mathbf{q}) = \frac{|\mathbf{B}(\mathbf{p}, \mathbf{q})|}{|\mathbf{A}(\mathbf{p})|} := \frac{\beta(\mathbf{p}, \mathbf{q})}{\alpha(\mathbf{p})}, \qquad i = 1, 2, \ldots, n, \tag{9.4}$$

where $\beta(\mathbf{p}, \mathbf{q})$ and $\alpha(\mathbf{p})$ are multivariate polynomials in $(\mathbf{p}, \mathbf{q})$ and $\mathbf{p}$, respectively.

We define the following sets:

$$\mathcal{P} := \{\mathbf{p}, \mathbf{q}\} = \{p_1, p_2, \ldots, p_l, q_1, q_2, \ldots, q_m\}, \tag{9.5}$$

$$\mathcal{X} := \{x_1, x_2, \ldots, x_n\}. \tag{9.6}$$

Let us consider the i^{th} element of $\mathcal{X}$, x_i, whose value over a box in the parameter space $\mathcal{D}$, where $\mathcal{D} \subset \mathcal{P}$, is to be evaluated. In the following subsections we summarize our results for 3 cases:

1. $\mathcal{D} = \{p_1\}$,
2. $\mathcal{D} = \{p_1, p_2\}$,
3. $\mathcal{D} = \mathcal{P}$.

9.1.1 Case 1: $\mathcal{D} = \{p_1\}$

In this case there is only one parameter, p_1. The matrix $\mathbf{A}(\mathbf{p})$ in (9.1) can be decomposed as

$$\mathbf{A}(\mathbf{p}) = \mathbf{A}_0 + p_1\mathbf{A}_1. \tag{9.7}$$

Recalling Lemma 4.1, we state the following theorem.

Theorem 9.1. *Supposing that* $\text{rank}[\mathbf{A}_1] = 1$ *in* (9.7)*, the function* $x_i(p_1)$ *in* (9.4) *can be determined by setting* p_1 *to 3 different values and measuring the corresponding* x_i *values.*

Proof. Since $\text{rank}[\mathbf{A}_1] = 1$, and based on Lemma 4.1, then $x_i(p_1)$ can be expressed as

$$x_i(p_1) = \frac{\tilde{\beta}_0 + \tilde{\beta}_1 p_1}{\tilde{\alpha}_0 + \tilde{\alpha}_1 p_1}. \tag{9.8}$$

We note that for $\tilde{\alpha}_0 = \tilde{\alpha}_1 = 0$, $x_i \to \infty, \forall p_1$, which is not physically possible. Hence, we rule out this case. If $\tilde{\alpha}_1 \neq 0$, then the numerator and denominator of (9.8) can be divided by $\tilde{\alpha}_1$:

$$x_i(p_1) = \frac{\beta_0 + \beta_1 p_1}{\alpha_0 + p_1}. \tag{9.9}$$

The function $x_i(p_1)$ in (9.9) can be determined by setting p_1 to 3 different values, measuring the corresponding x_i values and solving the following set of *measurement* equations:

$$\underbrace{\begin{bmatrix} 1 & p_1^1 & -x_i^1 \\ 1 & p_1^2 & -x_i^2 \\ 1 & p_1^3 & -x_i^3 \end{bmatrix}}_{\mathbf{M}} \underbrace{\begin{bmatrix} \beta_0 \\ \beta_1 \\ \alpha_0 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} x_i^1 p_1^1 \\ x_i^2 p_1^2 \\ x_i^3 p_1^3 \end{bmatrix}}_{\mathbf{m}}. \tag{9.10}$$

The set of equations (9.10) has a unique solution for β_0, β_1 and α_0 if and only if $|\mathbf{M}| \neq 0$. If $|\mathbf{M}| = 0$, then as the first two columns of $\mathbf{M}$ are linearly independent, x_i will be

$$x_i(p_1) = \beta_0 + \beta_1 p_1, \tag{9.11}$$

where β_0 and β_1 can be obtained from any 2 experiments conducted earlier. Equation (9.11) corresponds to the case where $\tilde{\alpha}_1 = 0$ in (9.8) and the numerator and denominator of (9.8) are divided by $\tilde{\alpha}_0$. □

The linear fractional form in (9.9) has some important practical aspects which is explained below.

Remark 9.1. *Taking the derivative of* (9.9) *with respect to* p_1 *yields*

$$\frac{dx_i}{dp_1} = \frac{\beta_1\alpha_0 - \beta_0}{(\alpha_0 + p_1)^2}. \tag{9.12}$$

Therefore, we can state the followings:

1. *The function in* (9.9) *is monotonic in* p_1*. For example, if* $\beta_1\alpha_0 - \beta_0 > 0$ *(see Fig. 9.1), then* x_i *will monotonically increase as* p_1 *increases. The upper and lower bounds of* x_i *for this case are:*

$$\frac{\beta_0}{\alpha_0} \leq x_i \leq \beta_1. \tag{9.13}$$

The range in (9.13) *is called the achievable range.*

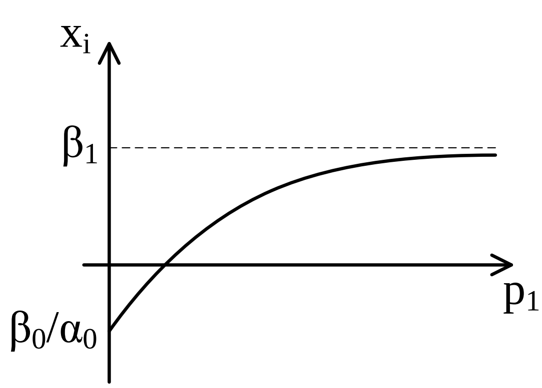

Figure 9.1: $x_i(p_1)$ for the case where $\beta_1\alpha_0 - \beta_0 > 0$

2. *This monotonic characteristic is beneficial in solving design problems. For instance, suppose that the system variable x_i is to lie within the range $x_i^- \leq x_i \leq x_i^+$ by adjusting p_1. If $[x_i^-, x_i^+]$ is inside the achievable range, then there exists a unique interval of values for p_1, $p_1^- \leq p_1 \leq p_1^+$, such that the constraint on x_i is satisfied.*

The parameter p_1 can be viewed as an uncertain parameter varying in an interval $\mathcal{I} = [p_1^-, p_1^+]$. We now state our first extremal result.

Theorem 9.2. *Assuming that* $\text{rank}[\mathbf{A}_1] = 1$ *in* (9.7)*, and* p_1 *is varying in an interval,* $\mathcal{I} = [p_1^-, p_1^+]$*, then the extremal values of* x_i *can be obtained from:*

$$\min_{p_1 \in \mathcal{I}} x_i(p_1) = \min\{x_i(p_1^-), x_i(p_1^+)\},$$

$$\max_{p_1 \in \mathcal{I}} x_i(p_1) = \max\{x_i(p_1^-), x_i(p_1^+)\}.$$

Proof. The proof follows from Theorem 9.1 and Remark 9.1. □

9.1.2 Case 2: $\mathcal{D} = \{p_1, p_2\}$

Here there are two parameters, p_1 and p_2, and therefore the characteristic matrix $A(p)$ can written as

$$\mathbf{A}(\mathbf{p}) = \mathbf{A}_0 + p_1\mathbf{A}_1 + p_2\mathbf{A}_2. \tag{9.14}$$

We state the following theorem.

Theorem 9.3. *Supposing that* $\text{rank}[\mathbf{A}_1] = \text{rank}[\mathbf{A}_2] = 1$ *in* (9.14)*, the function* $x_i(p_1, p_2)$ *in* (9.4) *can be determined by assigning 7 different sets of values to* (p_1, p_2) *and measuring the corresponding* x_i *values.*

Proof. According to Lemma 4.2, since rank$[\mathbf{A}_1]$ = rank$[\mathbf{A}_2]$ = 1, by following the same strategy described in the proof on Theorem 9.1, $x_i(p_1, p_2)$ will be

$$x_i(p_1, p_2) = \frac{\beta_0 + \beta_1 p_1 + \beta_2 p_2 + \beta_3 p_1 p_2}{\alpha_0 + \alpha_1 p_1 + \alpha_2 p_2 + p_1 p_2}. \tag{9.15}$$

A corresponding function for $x_i(p_1, p_2)$ can be obtained if $|\mathbf{M}| = 0$ in this case (see proof of Theorem 9.1). $\square$

Remark 9.2. *Taking the derivative of x_i in* (9.15) *with respect to p_1 and fixing $p_2 = p_2^*$ yields*

$$\left[\frac{dx_i}{dp_1}\right]_{p_2 = p_2^*} = \frac{a + bp_2^* + cp_2^{*2}}{(\alpha_0 + \alpha_2 p_2^* + (\alpha_1 + p_2^*)p_1)^2}, \tag{9.16}$$

where

$$a = \alpha_0 \beta_1 - \alpha_1 \beta_0, \tag{9.17}$$

$$b = \alpha_0 \beta_3 + \alpha_2 \beta_1 - \alpha_1 \beta_2 - \beta_0, \tag{9.18}$$

$$c = \alpha_2 \beta_3 - \beta_2, \tag{9.19}$$

which is of the form in (9.12) *and is monotonic in p_1. A similar relationship for $[(dx_i/dp_2)]_{p_1 = p_1^*}$ can be derived. Therefore, the function $x_i(p_1, p_2)$ in* (9.15) *is monotonic in each parameter p_1 and p_2.*

Theorem 9.2 can be generalized for this case as below.

Theorem 9.4. *If* rank$[\mathbf{A}_1]$ = rank$[\mathbf{A}_2]$ = 1 *in* (9.14)*, and p_1 and p_2 are varying in a rectangle, $\mathcal{R}$ (see Fig. 9.2),*

$$\mathcal{R} = \{(p_1, p_2) \mid p_1^- \le p_1 \le p_1^+,\ p_2^- \le p_2 \le p_2^+\}, \tag{9.20}$$

with vertices:

$$A = (p_1^-, p_2^-),\ B = (p_1^-, p_2^+),$$

$$C = (p_1^+, p_2^+),\ D = (p_1^+, p_2^-),$$

then the extremal values of x_i *happen at the vertices of* $\mathcal{R}$*:*

$$\min_{p_1, p_2 \in \mathcal{R}} x_i(p_1, p_2) = \min\{x_i(A), x_i(B), x_i(C), x_i(D)\},$$

$$\max_{p_1, p_2 \in \mathcal{R}} x_i(p_1, p_2) = \max\{x_i(A), x_i(B), x_i(C), x_i(D)\}.$$

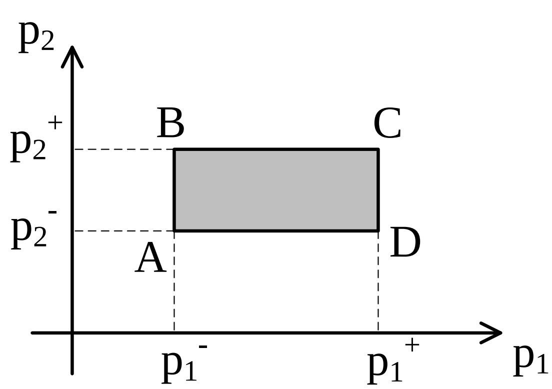

Figure 9.2: Rectangle of (p_1, p_2)

Proof. The proof follows immediately from Remark 9.2. □

9.1.3 Case 3: $\mathcal{D} = \mathcal{P}$

The results developed in the previous subsections can be generalized to the case where all system parameters $(\mathbf{p}, \mathbf{q})$ are considered. In this case $\mathbf{A}(\mathbf{p})$ can be decomposed as the form given in (9.3),

$$\mathbf{A}(\mathbf{p}) = \mathbf{A}_0 + p_1\mathbf{A}_1 + p_2\mathbf{A}_2 + \cdots + p_l\mathbf{A}_l, \tag{9.21}$$

and $\mathbf{B}(\mathbf{p}, \mathbf{q})$ will be

$$\begin{aligned}\mathbf{B}(\mathbf{p}, \mathbf{q}) = \mathbf{B}_0 + p_1\mathbf{B}_1 + \cdots + p_l\mathbf{B}_l \\ + q_1\mathbf{B}_{l+1} + \cdots + q_m\mathbf{B}_{l+m}.\end{aligned} \tag{9.22}$$

We now state the following general theorems. The proofs follow from the results provided in the previous subsections and are thus omitted here.

Theorem 9.5. *If* $\operatorname{rank}[\mathbf{A}_i] = 1,\ i = 1, 2, \ldots, l,$ *in* (9.21)*, the function* $x_i(\mathbf{p}, \mathbf{q})$ *in* (9.4) *can be determined by assigning* $2^l(2^m + 1) - 1$ *linearly independent sets of values to* $(\mathbf{p}, \mathbf{q})$*, measuring the corresponding values of* x_i *and solving a system of measurement equations.*

Theorem 9.6. *If* $\operatorname{rank}[\mathbf{A}_i] = 1,\ i = 1, 2, \ldots, l,$ *in* (9.21)*, and* $(\mathbf{p}, \mathbf{q})$ *are varying in a box,* $\mathcal{B}$*,*

$$\begin{aligned}\mathcal{B} = \{(\mathbf{p}, \mathbf{q}) \mid\ & p_i^- \le p_i \le p_i^+,\ i = 1, 2, \ldots, l, \\ & q_j^- \le q_j \le q_j^+,\ j = 1, 2, \ldots, m\},\end{aligned} \tag{9.23}$$

with $v := 2^{l+m}$ *vertices, labeled* $V_1, V_2, \ldots, V_v$*, then the extremal values of* x_i *occur at* *the vertices of $\mathcal{B}$:*

$$\min_{\mathbf{p},\mathbf{q}\in\mathcal{B}} x_i(\mathbf{p},\mathbf{q}) = \min\{x_i(V_1), x_i(V_2), \ldots, x_i(V_v)\},$$

$$\max_{\mathbf{p},\mathbf{q}\in\mathcal{B}} x_i(\mathbf{p},\mathbf{q}) = \max\{x_i(V_1), x_i(V_2), \ldots, x_i(V_v)\}.$$

Before ending this section, we mention that the evaluation of extremal values of x_i can be accomplished by either of the following ways:

1. Directly assign values corresponding to the vertices of $\mathcal{B}$, to the vector of parameters and measure x_i, or

2. First, find the functional dependency for x_i, as states in Theorem 9.5 by conducting a small number of measurements, and then evaluate that function at the vertices of $\mathcal{B}$.

9.2 Illustrative Examples

In this section three illustrative examples are presented to explain the results developed in Section 9.1.

Example 9.1. Consider the linear DC circuit shown in Fig. 9.3. This system can be described mathematically by the following set of linear equations

$$\mathbf{A}(\mathbf{p})\mathbf{x} = \mathbf{b}(\mathbf{q}), \tag{9.24}$$

where $\mathbf{p} = [R_1, R_2, \ldots, R_{13}, K_1, K_2]^T$, $\mathbf{q} = [V, J_1, J_2]^T$, and $\mathbf{x}$ is the vector of unknown currents. In this example $R_i,\ i = 1, 2, \ldots, 13,\ i \neq 5$ are resistors, R_5 is a gyrator resistance, V, J_1, J_2 are independent sources and V_1, V_2 are dependent sources with amplifier gains K_1 and K_2, respectively. We assume that the system is unknown, implying that $\mathbf{p}$ and $\mathbf{q}$ are unknown.

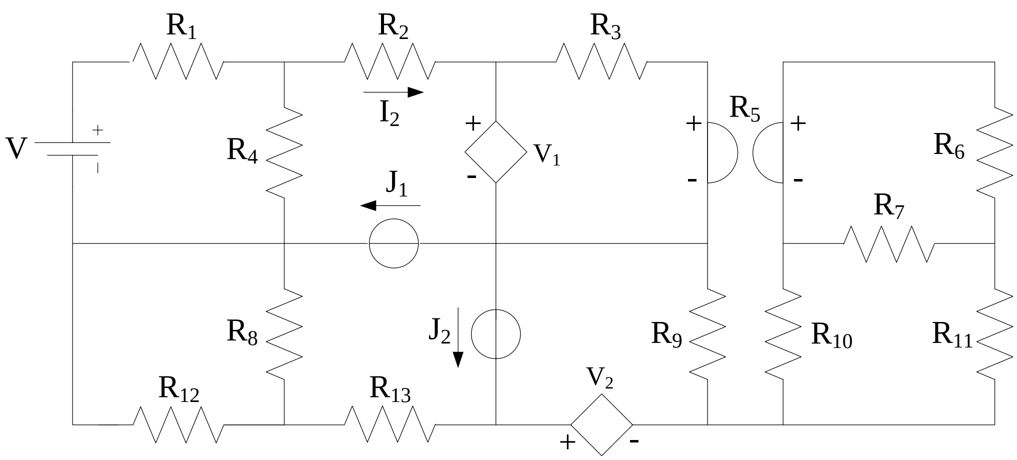

Figure 9.3: An unknown DC circuit

Suppose that the objective is to find the extremal values of I_2, if R_1 is varying in the interval $\mathcal{I} = [R_1^-, R_1^+] = [10, 30]$ (Ω). Since the circuit is unknown, $\mathbf{A}(\mathbf{p})$ and $\mathbf{b}(\mathbf{q})$ in (9.24) are unknown; but, in fact, one can write

$$\mathbf{A}(R_1) = \mathbf{A}_0 + R_1\mathbf{A}_1, \tag{9.25}$$

with rank$[\mathbf{A}_1] = 1$. This infers that R_1 appears in $\mathbf{A}(\mathbf{p})$ with rank one dependency, and accordingly the results of Section 9.1.1 can be applied. Based on Theorem 9.2, the extremal values of I_2 occur at $R_1^- = 10$ (Ω) and $R_1^+ = 30$ (Ω). Assigning these values to R_1 and measuring I_2 gives:

$$I_{2,\min} = 4.7\ (A),$$
$$I_{2,\max} = 6.3\ (A). \tag{9.26}$$

An alternative approach to evaluate the extremal values of I_2 is to firstly find the function $I_2(R_1)$. Based on Theorem 9.1, one can find the function $I_2(R_1)$ by assigning 3 different values to R_1, measuring the corresponding current I_2, and solving the measurement equations (9.10) for β_0, β_1 and α_0. Table 9.1 shows the numerical values of the measurements for this example. Solving (9.10) for β_0, β_1 and α_0 and substituting these constants into (9.9) yields

$$I_2(R_1) = \frac{21.9 + 8R_1}{11.7 + R_1}, \tag{9.27}$$

which is plotted in Fig. 9.4. It can be verified from Fig. 9.4 that the extremal values of I_2 are as the ones obtained in (9.26).

Table 9.1: Measurements for example 9.1

Exp. No.	R_1 (Ω)	I_2 (A)
1	7	4.2
2	18	5.6
3	32	6.4

Example 9.2. In this example we consider the same circuit as in the Example 9.1.

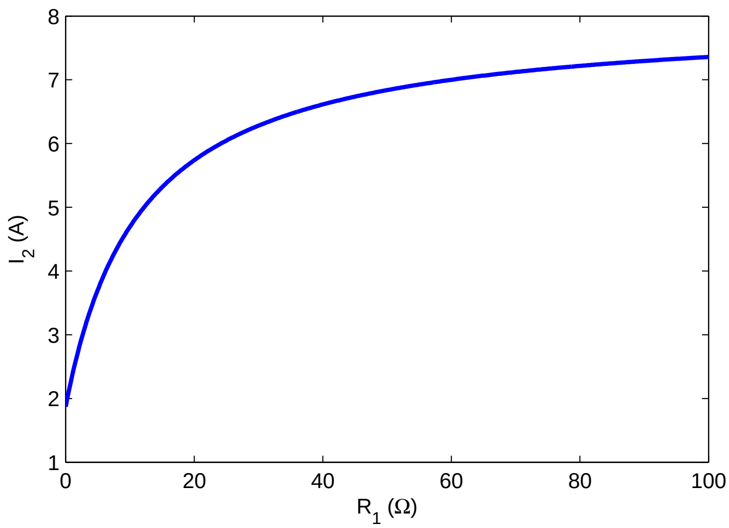

Figure 9.4: $I_2(R_1)$ for example 9.1

Suppose that the uncertain parameters R_1 and R_6 are varying in the rectangle,

$$\mathcal{R} = \{(R_1, R_6) \mid 5 \leq R_1 \leq 15,\ 2 \leq R_6 \leq 5\ (\Omega)\}, \tag{9.28}$$

with vertices:

$$A = (5, 2),\ B = (5, 5),$$
$$C = (15, 5),\ D = (15, 2),$$

and one is interested to evaluate the extremal values of the power level P_3, in the resistor $R_3 = 10\ (\Omega)$, over the rectangle $\mathcal{R}$ in (9.28). The power level P_3 can be expressed in the terms of the uncertain parameters as

$$P_3(R_1, R_6) = R_3\ I_3^2(R_1, R_6), \tag{9.29}$$

but since, according to Remark 9.2, $I_3(R_1, R_6)$ is monotonic in R_1 and R_6, Theorem 9.4 is valid to evaluate the extremal values of P_3 at the vertices. Setting (R_1, R_6) to the values corresponding to vertices A, B, C, D, one gets:

$$P_{3,\min} = 49.4 \ (W) \text{ at vertex B,}$$
$$P_{3,\max} = 150 \ (W) \text{ at vertex D.} \tag{9.30}$$

Also, one can plot the function $P_3(R_1, R_6)$ (see Fig. 9.5) following Theorem 9.3 and by conducting 7 experiments. The rectangle $\mathcal{R}$, defined in (9.28), is also shown in Fig. 9.5. It can be seen that the extremal values of P_3 are the same as those obtained in (9.30).

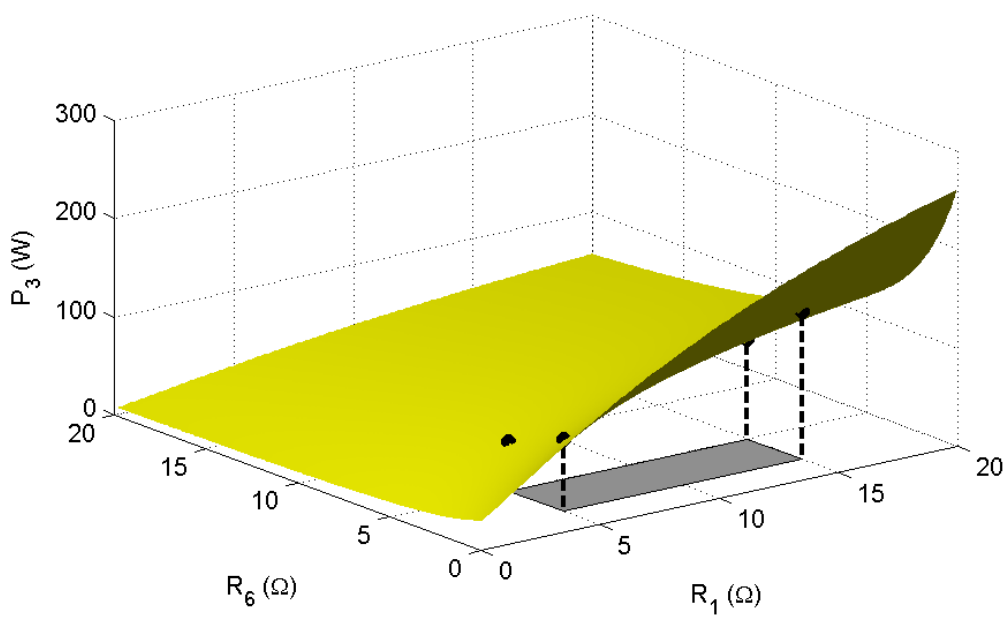

Figure 9.5: $P_3(R_1, R_6)$ for example 9.2

Example 9.3. Consider the unknown hydraulic network shown in Fig. 9.6. Assuming

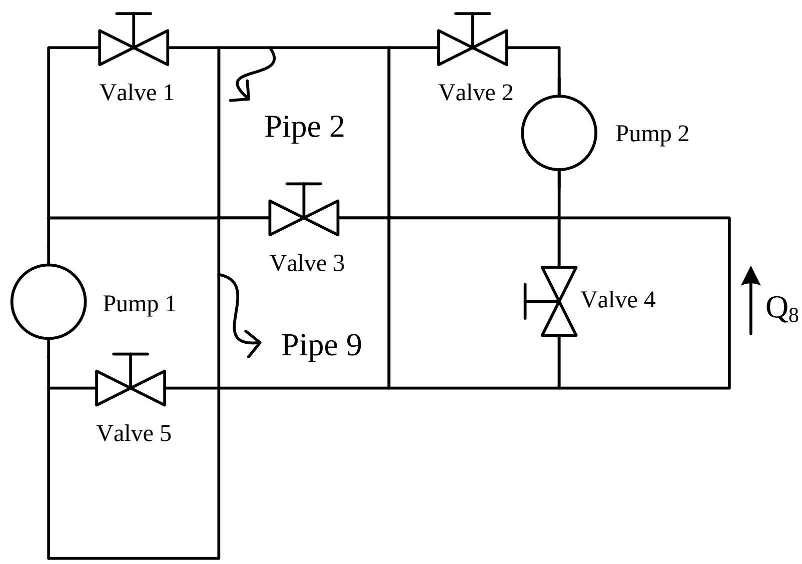

Figure 9.6: An unknown hydraulic network

that the flows are in the laminar state, the system can be described, by applying Kirchhoff's laws, as a set of linear equations

$$\mathbf{A}(\mathbf{p})\mathbf{x} = \mathbf{b}(\mathbf{q}), \tag{9.31}$$

where $\mathbf{A}(\mathbf{p})$ is the system characteristic matrix, $\mathbf{p}$ denotes the vector of pipe resistances, $\mathbf{q}$ is the vector of inputs such as pump pressures, and $\mathbf{x}$ is the vector of unknown flow rates. A pipe resistance is related to the properties of the fluid flowing through it and its geometric dimensions by

$$R = \frac{8\mu L}{\pi r^4}, \tag{9.32}$$

where μ is the dynamic viscosity of the fluid, and L and r represent the length and radius of the pipe. It can be observed that each pipe resistance appears with a rank one dependency in the characteristic matrix $\mathbf{A}(\mathbf{p})$ of the system. Suppose that the radii of pipes numbered 2 and 9 are varying in intervals described by the following rectangle,

$$\mathcal{R} = \{(r_2, r_9) \mid 0.08 \leq r_2 \leq 0.14,\ 0.07 \leq r_9 \leq 0.10\ (m)\}, \tag{9.33}$$

where the vertices are labelled as

$$A = (0.08, 0.07),\ B = (0.08, 0.10),$$
$$C = (0.14, 0.10),\ D = (0.14, 0.07).$$

It is of interest to evaluate the extremal values of the flow rate Q_8 over the rectangle $\mathcal{R}$ in (9.33). Similar to the previous example, since the assumptions in Theorem 9.4 hold, the extremal values of Q_8 occur at the vertices of the rectangle $\mathcal{R}$:

$$Q_{8,\min} = 0.045\ (m^3/s) \text{ at vertex A,}$$
$$Q_{8,\max} = 0.053\ (m^3/s) \text{ at vertex C.} \tag{9.34}$$

The function $Q_8(r_2, r_9)$ can be found by taking 7 measurements as explained in Theorem 9.3, and is depicted in Fig. 9.7. The rectangle $\mathcal{R}$, defined in (9.33), is also shown.

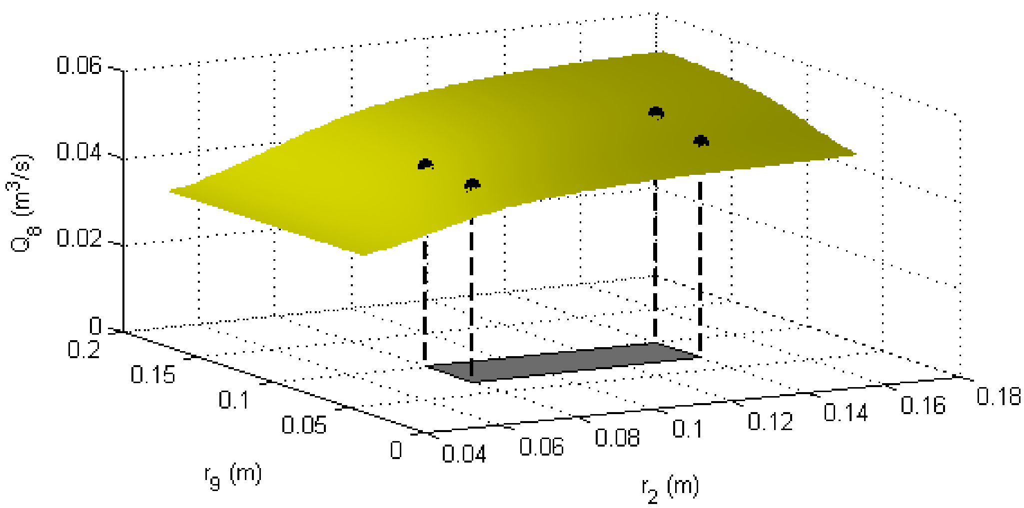

Figure 9.7: $Q_8(r_2, r_9)$ for example 9.3

9.3 Concluding Remarks

In this chapter we described some important characteristics of parametrized solutions of a system of linear equations containing interval parameters. If the interval parameters (uncertain parameters) appear in the characteristic matrix of the system with rank one dependency, which is usually the case in practical applications, then the parametrized solutions, which are system variables, will be monotonic in these parameters. This fact is used to show that the extremal values of the parametrized solutions over a box in the parameter space occur at the vertices of the box. This result is explained through illustrative examples.

10. CONCLUSIONS

This dissertation presented several new methods in control theory and its applications. Chapter 2 employed computational algebraic geometry techniques, such as Groebner bases and elimination theory in polynomial rings, to construct the set of stabilizing controllers of fixed structure for Linear Time Invariant (LTI) control systems. Chapter 3 proposed a new method to compute the set of all stabilizing PID controllers for the class of continuous-time and discrete-time LTI control systems guaranteeing transient response specifications. A desired transient response can be defined as an envelope and the calculation of the desirable set of PID controllers, for which the transient response of the system lies within that envelope, can be carried out by solving a sequence of Semi-Definite Programs (SDPs) developed based on Widder's theorem, its discrete-time counterpart and Markov-Lukacs representation of non-negative polynomials. Chapter 4 explored some important characteristics of parametrized solutions of sets of linear equations containing parameters. A general rational polynomial form, in terms of the system parameters, can be derived for the parametrized solutions. This mathematical result is used to develop a new measurement based approach to linear systems. Chapters 5 and 6 showed how this new measurement based approach can be applied to the analysis and design of linear DC and AC circuits, respectively. An application of this approach to the domain of linear mechanical systems is studied in Chapter 7. Chapter 8 presented a new method to synthesize stabilizing controllers for LTI control systems satisfying a set of prescribed desired frequency domain specifications. This method makes use of frequency response measurements directly to extract the design controller and does not require a mathematical model of the system a priori. Finally, Chapter 9 provided an extremal result on linear interval systems. If in a set of linear equations with interval parameters, the parameters appear with rank one dependency in the characteristic matrix, then the extremal values of the solution set over a box in the parameter space occur at the vertices of that box.

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APPENDIX A

PROOFS

A.1 Current Control using Two Resistors

Recall the proof of Theorem 5.2. We considered two cases: 1) $i \neq j, k$ and 2) $i = j$ or $i = k$.

Case 1: $i \neq j, k$

Suppose that $|\mathbf{M}| = 0$ in (5.22),

$$\begin{vmatrix} 1 & R_{j1} & R_{k1} & R_{j1}R_{k1} & -I_{i1} & -I_{i1}R_{j1} & -I_{i1}R_{k1} \\ 1 & R_{j2} & R_{k2} & R_{j2}R_{k2} & -I_{i2} & -I_{i2}R_{j2} & -I_{i2}R_{k2} \\ 1 & R_{j3} & R_{k3} & R_{j3}R_{k3} & -I_{i3} & -I_{i3}R_{j3} & -I_{i3}R_{k3} \\ 1 & R_{j4} & R_{k4} & R_{j4}R_{k4} & -I_{i4} & -I_{i4}R_{j4} & -I_{i4}R_{k4} \\ 1 & R_{j5} & R_{k5} & R_{j5}R_{k5} & -I_{i5} & -I_{i5}R_{j5} & -I_{i5}R_{k5} \\ 1 & R_{j6} & R_{k6} & R_{j6}R_{k6} & -I_{i6} & -I_{i6}R_{j6} & -I_{i6}R_{k6} \\ 1 & R_{j7} & R_{k7} & R_{j7}R_{k7} & -I_{i7} & -I_{i7}R_{j7} & -I_{i7}R_{k7} \end{vmatrix} = 0, \qquad \text{(A.1)}$$

then it can be concluded that $\tilde{\beta}_3 = 0$ in (5.20). In this case, $\tilde{\beta}_2 = 0$ in (5.20), if

$$|\mathbf{M}'| = \begin{vmatrix} 1 & R_{j1} & R_{k1} & R_{j1}R_{k1} & -I_{i1} & -I_{i1}R_{j1} \\ 1 & R_{j2} & R_{k2} & R_{j2}R_{k2} & -I_{i2} & -I_{i2}R_{j2} \\ 1 & R_{j3} & R_{k3} & R_{j3}R_{k3} & -I_{i3} & -I_{i3}R_{j3} \\ 1 & R_{j4} & R_{k4} & R_{j4}R_{k4} & -I_{i4} & -I_{i4}R_{j4} \\ 1 & R_{j5} & R_{k5} & R_{j5}R_{k5} & -I_{i5} & -I_{i5}R_{j5} \\ 1 & R_{j6} & R_{k6} & R_{j6}R_{k6} & -I_{i6} & -I_{i6}R_{j6} \end{vmatrix} = 0, \qquad \text{(A.2)}$$

and $\tilde{\beta}_1 = 0$ in (5.20), if

$$|\mathbf{M}''| = \begin{vmatrix} 1 & R_{j1} & R_{k1} & R_{j1}R_{k1} & -I_{i1} & -I_{i1}R_{k1} \\ 1 & R_{j2} & R_{k2} & R_{j2}R_{k2} & -I_{i2} & -I_{i2}R_{k2} \\ 1 & R_{j3} & R_{k3} & R_{j3}R_{k3} & -I_{i3} & -I_{i3}R_{k3} \\ 1 & R_{j4} & R_{k4} & R_{j4}R_{k4} & -I_{i4} & -I_{i4}R_{k4} \\ 1 & R_{j5} & R_{k5} & R_{j5}R_{k5} & -I_{i5} & -I_{i5}R_{k5} \\ 1 & R_{j6} & R_{k6} & R_{j6}R_{k6} & -I_{i6} & -I_{i6}R_{k6} \end{vmatrix} = 0. \tag{A.3}$$

Therefore, the results for this case can be summarized as follows:

- if $|\mathbf{M}| = 0$, $|\mathbf{M}'|, |\mathbf{M}''| \neq 0$:

$$I_i(R_j, R_k) = \frac{\alpha_0 + \alpha_1 R_j + \alpha_2 R_k + \alpha_3 R_j R_k}{\beta_0 + \beta_1 R_j + R_k}. \tag{A.4}$$

- if $|\mathbf{M}| = |\mathbf{M}'| = 0$, $|\mathbf{M}''| \neq 0$:

$$I_i(R_j, R_k) = \frac{\alpha_0 + \alpha_1 R_j + \alpha_2 R_k + \alpha_3 R_j R_k}{\beta_0 + R_j}. \tag{A.5}$$

- if $|\mathbf{M}| = |\mathbf{M}''| = 0$, $|\mathbf{M}'| \neq 0$:

$$I_i(R_j, R_k) = \frac{\alpha_0 + \alpha_1 R_j + \alpha_2 R_k + \alpha_3 R_j R_k}{\beta_0 + R_k}. \tag{A.6}$$

- if $|\mathbf{M}| = |\mathbf{M}'| = |\mathbf{M}''| = 0$:

$$I_i(R_j, R_k) = \alpha_0 + \alpha_1 R_j + \alpha_2 R_k + \alpha_3 R_j R_k. \quad \text{(A.7)}$$

For each case above the constants can be determined using measurements.

Case 2: $i = j$ *or* $i = k$

If $|\mathbf{M}| = 0$ in (5.25),

$$|\mathbf{M}| = \begin{vmatrix} 1 & R_{k1} & -I_{i1} & -I_{i1}R_{j1} & -I_{i1}R_{k1} \\ 1 & R_{k2} & -I_{i2} & -I_{i2}R_{j2} & -I_{i2}R_{k2} \\ 1 & R_{k3} & -I_{i3} & -I_{i3}R_{j3} & -I_{i3}R_{k3} \\ 1 & R_{k4} & -I_{i4} & -I_{i4}R_{j4} & -I_{i4}R_{k4} \\ 1 & R_{k5} & -I_{i5} & -I_{i5}R_{j5} & -I_{i5}R_{k5} \end{vmatrix} = 0, \quad \text{(A.8)}$$

then $\tilde{\beta}_3 = 0$ in (5.23). The following cases are possible: $\tilde{\beta}_2 = 0$ in (5.23), which happens if

$$|\mathbf{M}'| = \begin{vmatrix} 1 & R_{k1} & -I_{i1} & -I_{i1}R_{j1} \\ 1 & R_{k2} & -I_{i2} & -I_{i2}R_{j2} \\ 1 & R_{k3} & -I_{i3} & -I_{i3}R_{j3} \\ 1 & R_{k4} & -I_{i4} & -I_{i4}R_{j4} \end{vmatrix} = 0, \quad \text{(A.9)}$$

and $\tilde{\beta}_1 = 0$ in (5.23), if

$$|\mathbf{M}''| = \begin{vmatrix} 1 & R_{k1} & -I_{i1} & -I_{i1}R_{k1} \\ 1 & R_{k2} & -I_{i2} & -I_{i2}R_{k2} \\ 1 & R_{k3} & -I_{i3} & -I_{i3}R_{k3} \\ 1 & R_{k4} & -I_{i4} & -I_{i4}R_{k4} \end{vmatrix} = 0. \quad \text{(A.10)}$$

For this case we can state the results as follows:

- if $|\mathbf{M}| = 0$, $|\mathbf{M}'|, |\mathbf{M}''| \neq 0$:

$$I_i(R_j, R_k) = \frac{\alpha_0 + \alpha_1 R_k}{\beta_0 + \beta_1 R_j + R_k}. \tag{A.11}$$

- if $|\mathbf{M}| = |\mathbf{M}'| = 0$, $|\mathbf{M}''| \neq 0$:

$$I_i(R_j, R_k) = \frac{\alpha_0 + \alpha_1 R_k}{\beta_0 + R_j}. \tag{A.12}$$

- if $|\mathbf{M}| = |\mathbf{M}''| = 0$, $|\mathbf{M}'| \neq 0$:

$$I_i(R_j, R_k) = \frac{\alpha_0 + \alpha_1 R_k}{\beta_0 + R_k}. \tag{A.13}$$

- if $|\mathbf{M}| = |\mathbf{M}'| = |\mathbf{M}''| = 0$:

$$I_i(R_j, R_k) = \alpha_0 + \alpha_1 R_k. \tag{A.14}$$

A.2 Current Control using Gyrator Resistance

Recalling the proof of Theorem 5.4, we considered two cases: 1) The i-th branch is not connected to either port of the gyrator, and 2) The i-th branch is connected to one port of the gyrator.

Case 1: The i-th branch is not connected to either port of the gyrator

Here, we consider the case where $|\mathbf{M}| = 0$ in (5.32),

$$|\mathbf{M}| = \begin{vmatrix} 1 & R_{g1} & R_{g1}^2 & -I_{i1} & -I_{i1}R_{g1} \\ 1 & R_{g2} & R_{g2}^2 & -I_{i2} & -I_{i2}R_{g2} \\ 1 & R_{g3} & R_{g3}^2 & -I_{i3} & -I_{i3}R_{g3} \\ 1 & R_{g4} & R_{g4}^2 & -I_{i4} & -I_{i4}R_{g4} \\ 1 & R_{g5} & R_{g5}^2 & -I_{i5} & -I_{i5}R_{g5} \end{vmatrix} = 0. \tag{A.15}$$

This implies that $\tilde{\beta}_2 = 0$ in (5.30). Also, $\tilde{\beta}_1 = 0$ in (5.30), if

$$|\mathbf{M}'| = \begin{vmatrix} 1 & R_{g1} & R_{g1}^2 & -I_{i1} \\ 1 & R_{g2} & R_{g2}^2 & -I_{i2} \\ 1 & R_{g3} & R_{g3}^2 & -I_{i3} \\ 1 & R_{g4} & R_{g4}^2 & -I_{i4} \end{vmatrix} = 0. \tag{A.16}$$

The results for this case can be summarized as:

- if $|\mathbf{M}| = 0$, $|\mathbf{M}'| \neq 0$:

$$I_i(R_g) = \frac{\alpha_0 + \alpha_1 R_g + \alpha_2 R_g^2}{\beta_0 + R_g}. \tag{A.17}$$

- if $|\mathbf{M}| = |\mathbf{M}'| = 0$:

$$I_i(R_g) = \alpha_0 + \alpha_1 R_g + \alpha_2 R_g^2. \tag{A.18}$$

Case 2: The i-th branch is connected to one port of the gyrator

Here, suppose that $|\mathbf{M}| = 0$ in (5.35),

$$|\mathbf{M}| = \begin{vmatrix} 1 & R_{g1} & -I_{i1} & -I_{i1}R_{g1} \\ 1 & R_{g2} & -I_{i2} & -I_{i2}R_{g2} \\ 1 & R_{g3} & -I_{i3} & -I_{i3}R_{g3} \\ 1 & R_{g4} & -I_{i4} & -I_{i4}R_{g4} \end{vmatrix} = 0. \tag{A.19}$$

Therefore $\tilde{\beta}_2 = 0$ in (5.33). In this case, $\tilde{\beta}_1 = 0$ in (5.33), if

$$|\mathbf{M}'| = \begin{vmatrix} 1 & R_{g1} & -I_{i1} \\ 1 & R_{g2} & -I_{i2} \\ 1 & R_{g3} & -I_{i3} \end{vmatrix} = 0. \tag{A.20}$$

We summarize the results for this case as follows:

- if $|\mathbf{M}| = 0$, $|\mathbf{M}'| \neq 0$:

$$I_i(R_g) = \frac{\alpha_0 + \alpha_1 R_g}{\beta_0 + R_g}. \tag{A.21}$$

- if $|\mathbf{M}| = |\mathbf{M}'| = 0$:

$$I_i(R_g) = \alpha_0 + \alpha_1 R_g. \tag{A.22}$$