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M-theory Conifolds

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ABSTRACT

Seven-manifolds of G_2 holonomy provide a bridge between M-theory and string theory, via Kaluza-Klein reduction to Calabi-Yau six-manifolds. We find first-order equations for a new family of G_2 metrics $\mathbb{D}_7$, with $S^3 \times S^3$ principal orbits. These are related at weak string coupling to the resolved conifold, paralleling earlier examples $\mathbb{B}_7$ that are related to the deformed conifold, allowing a deeper study of topology change and mirror symmetry in M-theory. The $\mathbb{D}_7$ metrics' non-trivial parameter characterises the squashing of an S^3 bolt, which limits to S^2 at weak coupling. In general the $\mathbb{D}_7$ metrics are asymptotically locally conical, with a nowhere-singular circle action.

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Calabi-Yau manifolds, both compact and non-compact, singular and non-singular, have long been studied because of their significance for string theory, since they provide a way of obtaining $\mathcal{N}=1$ supersymmetry in four dimensions. The principal non-compact example is the singular conifold, and its smoothed-out versions, namely the resolved conifold and the deformed conifold [1]. The singular apex of the cone over $T^{1,1}=(S^3\times S^3)/S^1$ is blown up to a smooth 2-sphere in the former, and to a smooth 3-sphere in the latter. These minimal (calibrated) surfaces are supersymmetric cycles over which D-branes may be wrapped. If one considers a sequence of smooth models in which the cycles shrink to zero, one obtains enhanced gauge symmetry at the conifold point, the resolved and deformed conifolds being related by mirror symmetry [2]. Studying this process has led to an understanding of topology change in quantum gravity [3].

With the advent of M-theory, it has become important to consider the lifts of these 6-manifolds with holonomy $SU(3)$ to seven dimensions and holonomy G_2 [4, 5, 6], in order to set the four-dimensional $\mathcal{N}=1$ theories in an M-theory context. The seven-dimensional and six-dimensional manifolds are related by Kaluza-Klein reduction on a circle, whose variable length R is related to the coupling constant g of type IIA string theory by $R\propto g^{2/3}$. Thus we seek asymptotically locally conical (ALC) G_2 manifolds, for which the size R of the circle tends to a constant at infinity. In the case that R is everywhere constant, and the associated Kaluza-Klein vector field vanishes, the six-dimensional manifold is an exact Calabi-Yau space. If R varies, it will be an approximate Calabi-Yau space. This approximation will be good everywhere if the coupling constant g, or, equivalently, the radius R, never vanishes and is slowly varying. One may show on general grounds that it is never larger than its value at infinity.

Since the principal orbits of the the smoothed-out conifold are $T^{1,1}$, it follows that the principal orbits of the associated seven-dimensional G_2 metrics will be a $U(1)$ bundle over $T^{1,1}$, which is in fact $S^3\times S^3$. Very few examples of cohomogeneity one G_2 metrics can arise [9], and in fact the only explicitly-known examples have principal orbits that are $\mathbb{CP}^3$, the flag manifold $SU(3)/(U(1)\times U(1))$, and $S^3\times S^3$. Asymptotically conical (AC) metrics are known for all three cases [7, 8], but only the $S^3\times S^3$ case has enough freedom to permit ALC metrics of cohomogeneity one to arise.

In previous work [10], we presented complete non-singular G_2 metrics, which we denoted by $\mathbb{C}_7$, for which the coupling constant varied in such a finite positive interval. The associated Calabi-Yau space is the Ricci-flat Kähler metric on a complex line bundle over $S^2\times S^2$ [11, 12]. Other work has provided G_2 metrics $\mathbb{B}_7$ associated with the deformed conifold

Calabi-Yau space [13, 14]. However, in this case the radius R vanishes on an S^3 supersymmetric (calibrated) cycle in the interior. The purpose of this present letter is to extend the picture by providing a new class of complete non-singular G_2 metrics, which we denote by $\mathbb{D}_7$, whose associated Calabi-Yau manifold is the resolved conifold. In this case, as in the $\mathbb{C}_7$ metrics, the coupling constant never vanishes. The new metrics provide a unifying link between the deformed and resolved conifolds, via strong coupling and M-theory.

The metrics are invariant under the action of $SU(2)\times SU(2)$, with left-invariant 1-forms σ_i and Σ_i. The metric ansatz is

$$ds_7^2 = dt^2 + a^2\,((\Sigma_1 + g\,\sigma_1)^2 + (\Sigma_2 + g\,\sigma_2)^2) + b^2\,(\sigma_1^2 + \sigma_2^2) + c^2(\Sigma_3 + g_3\,\sigma_3)^2 + f^2\sigma_3^2\,, \quad (1)$$

where a, b, c, f, g and g_3 are functions only of the radial variable t. If we write σ_i in terms of Euler angles, with $\sigma_1 + \mathrm{i}\,\sigma_2 = e^{-\mathrm{i}\,\psi}\,(d\theta + \mathrm{i}\,\sin\theta\,d\phi)$, $\sigma_3 = d\psi + \cos\theta\,d\phi$, and similar expressions using tilded Euler angles for Σ_i, then the M-theory circle is generated by $\psi \longrightarrow \psi + k$, $\widetilde{\psi} \longrightarrow \widetilde{\psi} + k$, where k is a constant. This $U(1)$ diagonal subgroup of the right translations[1] is generated by the Killing vector $K = \partial/\partial\psi + \partial/\partial\widetilde{\psi}$. The orbits of $SU(2) \times SU(2)$ are generically six-dimensional. In our solutions, the orbits collapse in the interior to a 3-sphere, which in general has a squashed rather than round $SU(2)$-invariant metric. The degenerate orbit is known as a bolt; it is a minimal surface and a supersymmetric (associative) 3-cycle.

The metric will have G_2 holonomy, and thus will also be Ricci flat, if it admits a closed and co-closed associative 3-form

$$\Phi_{(3)} = e^0\,e^3\,e^6 + e^1\,e^2\,e^6 - e^4\,e^5\,e^6 + e^0\,e^1\,e^4 + e^0\,e^2\,e^5 - e^1\,e^3\,e^5 + e^2\,e^3\,e^4\,, \quad (2)$$

where the vielbein is given by $e^0 = dt$, $e^1 = a\,(\Sigma_1 + g\,\sigma_1)$, $e^2 = a\,(\Sigma_2 + g\,\sigma_2)$, $e^3 = c\,(\Sigma_2 + g_3\,\sigma_2)$, $e^4 = b\,\sigma_1$, $e^5 = b\,\sigma_2$ and $e^6 = f\,\sigma_3$. The closure and co-closure implies the algebraic constraints

$$g = -\frac{a\,f}{2b\,c}\,, \qquad g_3 = -1 + 2g^2\,, \quad (3)$$

together with the first-order equations

$$\begin{aligned} \dot a &= -\frac{c}{2a} + \frac{a^5\,f^2}{8b^4\,c^3}\,, & \dot b &= -\frac{c}{2b} - \frac{a^2\,(a^2 - 3c^2)\,f^2}{8b^3\,c^3}\,, \\ \dot c &= -1 + \frac{c^2}{2a^2} + \frac{c^2}{2b^2} - \frac{3a^2\,f^2}{8b^4}\,, & \dot f &= -\frac{a^4\,f^3}{4b^4\,c^3}\,. \end{aligned} \quad (4)$$

[1]The metric ansatz (1) is a specialisation of a nine-function ansatz introduced in [14], in which the metric functions for the $i = 1$ and $i = 2$ directions in the two $SU(2)$ groups are set equal. The diagonal $U(1)$ subgroup of the $SU(2)$ right-translations becomes an isometry, as is needed for Kaluza-Klein reduction, under this specialisation.

Using the closure and co-closure conditions has reduced the Einstein equations, which are of second order and extremely complicated, to a manageable first-order set involving just the four functions a, b, c and f. One can check that the equations are a consistent truncation of the second-order Einstein equations for the more general nine-function ansatz that was given in [14]. It should be emphasised that although the equations here have reduced to a four-function first-order system, the ansatz is inequivalent to the four-function ansatz introduced in [13]. In particular the metric ansatz in [13] admits a Z_2 symmetry under which the σ_i and Σ_i are interchanged and the associative 3-form changes sign, whilst our metric ansatz (1) does not have this symmetry.[2]

We can find a regular series expansion for the situation where both a and c go to zero at short distance. Substituting the Taylor expansions for the four functions a, b, c and f into (4), we find

$$\begin{aligned} a &= \frac{t}{2} - \frac{(q^2+2)\,t^3}{288} - \frac{(31q^4-29q^2-74)\,t^5}{69120} + \cdots\,, \\ b &= 1 - \frac{(q^2-2)\,t^2}{16} - \frac{(11q^4-21q^2+13)\,t^4}{1152} + \cdots\,, \\ c &= -\frac{t}{2} - \frac{(5q^2-8)\,t^3}{288} - \frac{(157q^4-353q^2+232)\,t^5}{34560} + \cdots\,, \\ f &= q + \frac{q^3\,t^2}{16} + \frac{q^3\,(11q^2-14)\,t^4}{1152} + \cdots\,, \end{aligned} \tag{5}$$

where, without loss of generality, we have set the scale size so that $b = 1$ on the S^3 bolt at $t = 0$. The parameter q is free, and characterises the squashing of the S^3 bolt along its $U(1)$ fibres over the unit S^2. By studying the equations numerically, using the short-distance Taylor expansion to set initial data just outside the bolt, we find that there is a regular asymptotically conical (AC) solution when $q = 1$, and that there are regular ALC solutions for any q in the interval $0 < q < 1$. In fact the AC solution at $q = 1$ is the well-known G_2 metric on the spin bundle of S^3, found in [7, 8]. (One can easily derive this analytically from (4), by noting that it corresponds to the consistent truncation $c = -a$, $f = b$.) The ALC solutions with the non-trivial parameter $0 < q < 1$ are new, and we shall denote them by $\mathbb{D}_7$. They exhibit the unusual phenomenon of admitting a supersymmetric Lagrangian 3-manifold (the bolt) that is not Einstein. The metric function f tends to a constant at infinity, while the remaining functions a, b and c grow linearly with t; in fact a, b and c satisfy the first-order equations governing the Ricci-flat Kähler resolved conifold asymptotically at large distance. One can see from (1) that the $U(1)$ Killing vector $K = \partial/\partial\psi + \partial/\partial\widetilde{\psi}$ has

[2]We understand that S. Gukov, K. Saraikin and N. Volovitch are also considering ansätze that break the Z_2 symmetry [15].

length given by $|K|^2 = f^2 + c^2\,(1+g_3)^2$, and so it follows that its length is nowhere infinite or zero. It ranges from a minimum value $|K| = q$ at short distance to the asymptotic value $|K| = f_\infty$ at infinity.

It may well be that the system (4) is completely integrable,[3] although we have not yet succeeded in finding the general solution to the first-order equations. In a somewhat analogous situation in eight dimensions, we did find the general solution to the first-order equations for an ansatz for ALC metrics of Spin(7) holonomy [17]. In that case, the first-order equations could be reduced to an autonomous third-order equation, whose general solution could be given in term of hypergeometric functions. In the present case, we can again reduce the first-order equations to an autonomous third-order equation for $G \equiv g^2$:

$$
\begin{aligned}
&[(-6\,G^2 + 2G)\,G'^2 - 4(7G^3 - 2G^2)\,G' + 8G^3 - 32G^4]\,G''' \\
&+[(3\,G+1)\,(\,G')^3 + 6(14G^2 - 3G)\,G'^2 - 4(9G^2 - 31G^3)\,G' + 8G^3 - 32G^4]\,G'' \\
&+[6(3G^2 - G)\,G' - 12G^2 + 32G^3]\,G''^2 + 2(3G+1)\,G'^4 - 20(G - 6G^2)\,G'^3 \\
&-8(7G^2 - 29G^3)\,G'^2 - 16(G^3 + 4G^4)\,G' = 0\,, \qquad (6)
\end{aligned}
$$

with $A \equiv c^2/a^2 = 1 + G'/(2G)$, $c^2/b^2 = (A' + 2A^2 - 2A)/(G + 3G\,A - A)$, and $a\,b\,c = e^\rho$. The primes denote derivatives with respect to the new radial variable ρ, defined by $dt = -c\,d\rho$. We have found the following new explicit solution,

$$
\begin{aligned}
ds^2 &= h^{-1/3}\,dr^2 + \tfrac{1}{6} r^2\,h^{-1/3}\,[(\Sigma_1 + \frac{k}{r}\sigma_1)^2 + (\Sigma_2 + \frac{k}{r}\sigma_2)^2] \\
&\quad + \tfrac{1}{9} r^2\,h^{-1/3}\,[\Sigma_3 + (-1 + \frac{2k^2}{r^2})\,\sigma_3]^2 + \tfrac{1}{6} r^2\,h^{2/3}\,(\sigma_1^2 + \sigma_2^2) + \tfrac{4}{9} k^2\,h^{2/3}\,\sigma_3^2\,, \qquad (7)
\end{aligned}
$$

where $h \equiv 1 - 9k^2/(2r^2)$. Unlike the smooth $\mathbb{D}_7$ metrics that we have found numerically, (7) has a curvature singularity at $r^2 = 9k^2/2$.

It is useful to summarise some known results for G_2 metrics with $S^3 \times S^3$ principal orbits in the form of a table.

G_2 Metric	Calabi-Yau	Bolt	AC limit	Susy cycle?	$\mathbb{Z}_2$ sym?
$\mathbb{B}_7$	Deformed conifold	S^3_1	$\mathbb{R}^4 \times S^3$	Yes	Yes
$\mathbb{C}_7$	$\mathbb{C} \ltimes (S^2 \times S^2)$	$T^{1,1}_q$	$\sim \mathbb{R}^4 \times S^3$	No	Yes
$\mathbb{D}_7$	Resolved conifold	S^3_q	$\mathbb{R}^4 \times S^3$	Yes	No

Table 1: The three families of G_2 solutions

[3] By contrast, it is expected that the second-order Einstein equations for Ricci flatness are of the type that would give rise to chaotic behaviour [16].

We are including three families of complete non-singular solutions here, each of which has a non-trivial parameter. At one end of the parameter range the metric is asymptotically conical. For the $\mathbb{B}_7$ and $\mathbb{D}_7$ cases, this AC metric is precisely the one found in [7, 8], on the spin bundle of S^3. Since the bundle is trivial, we are denoting this AC metric simply by $\mathbb{R}^4 \times S^3$. In the case of the $\mathbb{C}_7$ metrics [10], the limiting AC member of the family approaches the form of the AC metric of [7, 8] at large distance, but is quite different at short distance, since it instead has the topology of an $\mathbb{R}^2$ bundle over $T^{1,1}$. For the $\mathbb{C}_7$ metrics, and our new $\mathbb{D}_7$ metrics, the non-trivial parameter in the metrics characterises the degree of squashing of the $T^{1,1}$ or S^3 bolt respectively, as denoted by the subscripts q on $T^{1,1}_q$ and S^3_q in the Table. By contrast, for the $\mathbb{B}_7$ metrics the S^3 bolt is always round (denoted by the subscript "1" on S^3_1), and the non-trivial parameter instead characterises "velocities" of the metric functions as one moves outwards from the bolt [14]. (An explicit solution for one specific value of the non-trivial parameter was obtained in [13].)

As the non-trivial parameter in the ALC metric is reduced from its AC limiting value, a circle "splits off" and stabilises its length when one moves out sufficiently far. The geometry is that of a twisted circle bundle over a six-dimensional AC metric. At the lower limit of the parameter range the radius of the circle at infinity becomes vanishingly small. If one performs an appropriate counterbalancing rescaling of the circle coordinate, the Kaluza-Klein vector describing the twist vanishes in the limit and one obtains the Gromov-Hausdorff limit which is just the direct product of S^1 times a Ricci-flat Calabi-Yau six-metric. Thus the Gromov-Hausdorff limit may be identified with the weak coupling limit in this case. These metrics are listed in the second column of the Table. The metric $\mathbb{C} \ltimes (S^2 \times S^2)$ denotes the Ricci-flat Kähler metric on the complex line bundle over $S^2 \times S^2$ that was constructed in [11, 12].

The $\mathbb{B}_7$ and $\mathbb{D}_7$ metrics provide a seven-dimensional link between the six-dimensional deformed and resolved conifolds. This can be seen from the fact that both the $\mathbb{B}_7$ and $\mathbb{D}_7$ families of metrics are encompassed by the ansatz (1). They satisfy two different systems of first-order equations that are each consistent truncations of the same system of six second-order Ricci-flat equations. Each of the $\mathbb{B}_7$ and $\mathbb{D}_7$ families has a continuous non-trivial modulus parameter, with each family having the *same* AC metric at one end of the parameter range, whilst at the other end of the range the $\mathbb{B}_7$ and $\mathbb{D}_7$ metrics approach S^1 times the deformed conifold and the resolved conifold respectively. This implies that the two weakly coupled IIA string theory backgrounds using the deformed and the resolved conifolds are related via strong coupling and eleven dimensions.

An important issue for future work is the phenomenologically central question of chiral fermions localised at isolated singularities [18, 19, 20]. Physically, these can arise in M-theory from massless states associated to membranes wrapped around vanishing cycles. Mathematically, they correspond to solutions of the massless Dirac equation in the M-theory background. The process of localisation is as yet imperfectly understood. What is needed is explicit metrics permitting explicit calculations. Our metrics are certainly sufficiently simple for this purpose. What requires further investigation is whether one can model the appropriate co-dimension seven singularities using them.

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References

[1] P. Candelas and X.C. de la Ossa, *Comments on conifolds*, Nucl. Phys. **B342**, 246 (1990).

[2] Many articles can be found in "Essays on Mirror Manifolds," S.-T. Yau, ed., International Press, 1992.

[3] A. Strominger, *Massless black holes and conifolds in string theory*, Nucl. Phys. **b451**, 96 (1995), hep-th/9504090.

[4] B. S. Acharya and C. Vafa, *On domain walls of $N = 1$ supersymmetric Yang-Mills in four dimensions*, hep-th/0103011.

[5] M. Atiyah, J. Maldacena and C. Vafa, *An M-theory flop as a large n duality*, J. Math. Phys. **42**, 3209 (2001), hep-th/0011256.

[6] M. Atiyah and E. Witten, *M-theory dynamics on a manifold of G_2 holonomy*, hep-th/0107177.

[7] R.L. Bryant and S. Salamon, *On the construction of some complete metrics with exceptional holonomy*, Duke Math. J. **58**, 829 (1989).

[8] G.W. Gibbons, D.N. Page and C.N. Pope, *Einstein metrics on S^3, $\mathbb{R}^3$ and $\mathbb{R}^4$ bundles*, Commun. Math. Phys. **127**, 529 (1990).

[9] R. Cleyton and A. Swann, *Cohomogeneity-one G_2 structures*, math.DG/0111056.

[10] M. Cvetič, G. W. Gibbons, H. Lü and C. N. Pope, *Orientifolds and slumps in G_2 and Spin(7) metrics* hep-th/0111096.

[11] L. Berard-Bergery, *Quelques exemples de varietes riemanniennes completes non compactes a courbure de Ricci positive*, C.R. Acad. Sci., Paris, Ser. I302, 159 (1986).

[12] D.N. Page and C.N. Pope, *Inhomogeneous Einstein metrics on complex line bundles*, Class. Quantum Grav. **4**, 213 (1987).

[13] A. Brandhuber, J. Gomis, S.S. Gubser and S. Gukov, *Gauge theory at large N and new G_2 holonomy metrics*, hep-th/0106034.

[14] M. Cvetič, G.W. Gibbons, H. Lü and C.N. Pope, *Cohomogeneity one manifolds of Spin(7) and G_2 holonomy,* hep-th/0108245.

[15] S. Gukov, UPenn seminar, December 2001, and communications with S. Gukov.

[16] J. Demaret, Y. De Rop and M. Henneaux, *Chaos in nondiagonal spatially homogeneous cosmological models in space-time dimensions* ≤ 10, Phys. Lett. **B211**, 37 (1988).

[17] M. Cvetič, G. W. Gibbons, H. Lü and C. N. Pope, *New complete non-compact Spin(7) manifolds,* hep-th/0103155; *New cohomogeneity one metrics with Spin(7) holonomy,* math.DG/0105119.

[18] M. Cvetič, G. Shiu and A.M. Uranga, *Chiral four-dimensional $N = 1$ supersymmetric type 2A orientifolds from intersecting D6-branes*, Nucl. Phys. **B615**, 3 (2000), hep-th/0107166.

[19] E. Witten, *Anomaly cancellation on G_2 manifolds*, hep-th/0108165.

[20] B. Acharya and E. Witten, *Chiral fermions from manifolds of G_2 holonomy*, hep-th/0109152.