CTP TAMU-27/97
LPTENS-97/27
SISSARef-132/97/EP
hep-th/9710119
October 1997

Dualisation of Dualities. I.

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ABSTRACT

We analyse the global (rigid) symmetries that are realised on the bosonic fields of the various supergravity actions obtained from eleven-dimensional supergravity by toroidal compactification followed by the dualisation of some subset of fields. In particular, we show how the global symmetries of the action can be affected by the choice of this subset. This phenomenon occurs even with the global symmetries of the equations of motion. A striking regularity is exhibited by the series of theories obtained respectively without any dualisation, with the dualisation of only the Ramond-Ramond fields of the type IIA theory, with full dualisation to lowest degree forms, and finally for certain inverse dualisations (increasing the degrees of some forms) to give the type IIB series. These theories may be called the GL_A, D, E and GL_B series respectively. It turns out that the scalar Lagrangians of the E series are sigma models on the symmetric spaces $K(E_{11-D})\backslash E_{11-D}$ (where $K(G)$ is the maximal compact subgroup of G) and the other three series lead to models on homogeneous spaces $K(G)\backslash G\ltimes \mathbb{R}^s$. These can be understood from the E series in terms of the deletion of positive roots associated with the dualised scalars, which implies a group contraction. We also propose a constrained Lagrangian version of the even dimensional theories exhibiting the full duality symmetry and begin a systematic analysis of abelian duality subalgebras.

[1] Research supported in part by DOE Grant DE-FG03-95ER40917

[2] Research supported in part by EC under TMR contract ERBFMRX-CT96-0045

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1 Introduction

Eleven-dimensional supergravity [1] occupies the distinguished position of being the highest-dimensional supergravity theory. It provides a window on the elusive M-theory, which would describe the strong coupling limits of ten-dimensional string theories [2]. The fact that M-theory compactified on S^1 gives rise to the type IIA string can be seen both at the level of supergravity [1, 3, 4], and in the sigma-model action [5, 6]. In this paper we shall consider the classical, internal, global symmetry groups of the bosonic sectors of the various maximal supergravities in dimensions $D \le 11$, which can be obtained from eleven-dimensional supergravity by toroidal compactification [7, 8]. These Lie groups, discretised after quantisation, are conjectured to become the duality symmetry groups of the toroidally-compactified type II quantum string theories [9]. As is well known, there exists a formulation of each of these lower-dimensional theories, namely with the canonical (maximal) choice of field dualisations, in which there is a global $E_{(11-D)(11-D)}$ internal symmetry [10, 11].[1] Specifically, these symmetries are realised in the theories that are obtained by performing the toroidal reduction to D dimensions and then dualising any field strength whose degree exceeds $\frac{1}{2}D$. Thus when D is odd, this E_{11-D} symmetry is realised on the gauge potentials and is an invariance of the Lagrangian. In fact the E_{11-D} symmetry for odd dimensions in this canonical choice of fields does not involve any electric/magnetic type of duality at all; the name duality symmetries is nevertheless widely used.

However, the story is different in even dimensions. In what follows we shall use the term "strict duality" to mean a continuous symmetry at the level of the equations of motion, whose Lie algebra generators mix a set of field strengths with their Hodge duals (or with additional field strengths of the dual degrees). By extension, the duality group has come to mean the full global internal symmetry even when there is no strict duality at all. We shall reserve the name dualisation to a discrete Hodge-like duality that exchanges forms of complementary degrees which appear in two dual Lagrangians with locally equivalent equations of motion. Actually we propose the name inverse dualisation for this operation when the degree of the form increases. One of the main questions will be to investigate the effect of dualisations on the strict and the not so strict duality symmetries. In fact when D is even, the field strengths of degree $\frac{1}{2}D$ and their magnetic duals form a single irreducible representation of E_{11-D}, and so for these fields, the strict duality symmetries

[1] In this paper, the exceptional groups E_n will always be in their maximally non-compact form $E_{n(n)}$. For brevity, we shall write them simply as E_n. For $n \le 5$ we have E_0 trivial, $E_1 = \mathbb{R}$, $E_2 = GL(2,\mathbb{R})$, $E_3 = SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$, $E_4 = SL(5,\mathbb{R})$ and $E_5 = O(5,5)$.

can only be implemented locally on the field strengths, rather than on their gauge potentials. Furthermore, only the set of field equations plus Bianchi identities, rather than the conventional Lagrangians, are invariant (recall the example of electric/magnetic duality in $D = 4$). A local implementation of duality on the potentials can only be achieved at the level of equations of motion by the introduction of additional dual potentials. Typically the equations then take the form of a twisted self-duality condition [10]; in this case, the subgroup of symmetries of the Lagrangian is the parity-even subgroup. We may remark that the strict dualities use the metric and thus are not really internal symmetries in the usual sense of commuting with spacetime transformations; there is no absolute Lorentz subgroup of diffeomorphisms in curved space.

It is natural now to ask whether the process of dualising all the field strengths whose degrees exceed $\frac{1}{2}D$ was crucial for obtaining the E_{11-D} global symmetry. It was observed in [12] that the global symmetries can change, depending on whether or not certain dualisations are performed. Indeed when D is odd, and the symmetry is realised at the level of the Lagrangian, it is manifest that the dualisations are necessary for the symmetry to act locally on the gauge potentials, since we cannot assemble two sets of gauge potentials of dual degrees into an irreducible multiplet, unless non-local symmetry transformations are allowed. One might think that this problem could always be circumvented at the level of the equations of motion, since one's experience in simple examples such as electric/magnetic duality in $D = 4$ is that only the field strengths, and not their bare gauge potentials, appear there. Indeed, if only the field strengths appear in the equations of motion and the Bianchi identities, then one could view these field strengths as the fundamental physical quantities on which the true symmetry transformations should be defined. Then, any possible dualisation (or inverse dualisation) that continues to allow the equations of motion and Bianchi identities for a multiplet of field strengths to be written purely in terms of the (now dualised) field strengths will leave the global symmetry of the equations of motion unaffected, since the transformations can be implemented as well on the field strengths of the dualised reformulation. On the other hand, if the result of (inversely) dualising some members of an irreducible multiplet of equal-degree field strengths is to cause the unavoidable appearance of bare potentials (*i.e.* not in the combination of field strengths) for some of the remaining members of this multiplet in the equations of motion and Bianchi identities, then the original global symmetry will be modified. Some examples of this phenomenon are discussed in section 6.3.

In the special case where scalar fields (0-form potentials) are being (inversely) dualised, we shall presently show that the first loss of global symmetry follows the loss of these scalars, because their constant shift symmetries disappear too, or rather, the corresponding group action becomes trivial in that sector: it is no longer faithful. The action of the global symmetry group is a nonlinear realisation on a homogeneous space but after the dualisation the rigid symmetry is partially transmuted to a local gauge type symmetry. As the dualisable scalar potentials appear only through their field strengths and even though these do not mix with bare potentials under the symmetry, the global invariance is already reduced prior to dualisation if one looks only at its action on the (1-form) field strengths.

Note that the mere fact that the equations of motion and Bianchi identities involve a particular field only via its field strength is no guarantee that this field can be dualised. A classic example of this is the 4-form field strength in eleven-dimensional supergravity, which apparently cannot be covariantly dualised to a 7-form.[2] In fact the question of whether or not a particular field can be dualised must be studied at the level of the Lagrangian; the possibilities for dualisation are not enlarged by looking instead at the equations of motion. At each stage of the dualisation process, a sufficient condition for dualisability of a *given* field is that it should appear in the action purely through its field strength. (Here, and generally in these discussions, when we say that a field or rather a collection of fields appear via their field strengths, we mean that this can be achieved after some field redefinitions and/or integrations by parts.) The above considerations lead to the following observations on how the dualisation of fields can affect the global symmetry. If an irreducible multiplet of fields appear in the Lagrangian purely via their field strengths, then the dualisation of any subset (proper or improper) of these fields is possible, and it will not affect the global symmetry of the corresponding equations of motion. The criterion for dualisability of a subset of fields in an irreducible multiplet becomes more complicated if some of the fields in the multiplet require the appearance of bare potentials in the Lagrangian, and we shall not attempt an exhaustive discussion of this issue here. In any case, the general statement about the global symmetry is that if it is to be implemented with a finite number of derivatives on the potentials for the fields of an irreducible multiplet, then all the potentials must have the same degree, in other words dual degrees are forbidden. If instead the symmetry is to be

[2] In terms of the 4-form $F_{(4)} = dA_{(3)}$, the field equation is $d * F_{(4)} = F_{(4)} \wedge F_{(4)}$ and the Bianchi identity is $dF_{(4)} = 0$. To rewrite the field equation as a Bianchi identity we must define $F_{(7)} = *F_{(4)} - A_{(3)} \wedge F_{(4)}$, giving $dF_{(7)} = 0$. However its field equation is $d * F_{(7)} = -d * (A_{(3)} \wedge F_{(4)})$, which cannot be recast into a local equation involving only $F_{(7)}$. A more naive approach would be simply to define $F_{(4)} = *F_{(7)}$, giving $d * F_{(7)} = 0$ and $dF_{(7)} = (*F_{(7)}) \wedge (*F_{(7)})$. This does not work either, since the latter equation cannot be interpreted as a Bianchi identity that is solved in terms of a gauge potential $A_{(6)}$ by writing $F_{(7)} = dA_{(6)} + \cdots$.

implemented only on the field strengths (necessarily in the equations of motion), then all the members of the multiplet must appear in the field equations and Bianchi identities only through their field strengths. If the result of dualisations is to make it that neither of these conditions is satisfied, then the original global symmetry prior to the dualisations will be broken.

The fact that the global symmetry can depend on the choice of dualisation [12], and the fact that not all dualisations are possible, are both consequences of the occurrence of non-linear terms in the D-dimensional Lagrangian. These terms have two origins, namely the $F_{(4)} \wedge F_{(4)} \wedge A_{(3)}$ term in the original eleven-dimensional Lagrangian, and the non-linearity of the eleven-dimensional Einstein-Hilbert action. The latter implies that modifications to the field strengths in lower dimensions will arise in the Kaluza-Klein reduction process. These are sometimes called "Chern-Simons modifications," but the term is really a misnomer since they actually come from the separation between the gauge transformations originating from diffeomorphisms along the compactified directions and the other gauge symmetries. In this paper they will be called Kaluza-Klein modifications. In order to investigate these issues in more detail it is convenient to divide the discussion into two parts, namely for the subsector comprising the scalar fields, and then the remaining sectors involving the higher-degree field strengths.

In any dimension $D \geq 6$, the scalar sector of the D-dimensional theory that is obtained by dimensional reduction from $D = 11$ is unambiguous, since no dualisations of the higher-degree forms can give rise to additional scalars.[3] In these cases, the scalar sector of the Lagrangian has an E_{11-D} symmetry. In $D = 5, 4, 3$, on the other hand, the field content of the scalar manifold depends upon which 4-form, 3-form or 2-form field strengths respectively one chooses to dualise, since these will give additional contributions to the scalar sector. The E_{11-D} global symmetries are achieved in these dimensions if one dualises all such higher-degree fields, so as to maximise the total number of scalars. If the full set of dualisations is not performed, then the global symmetry of the scalar sector is altered. This is because the E_{11-D} symmetry can only be expressed as transformations on the scalars themselves, and not on their "1-form field strengths." (Some of the E_{11-D} transformations would act through non-local functions of higher-degree fields if these were not dualised to scalars, and some would simply disappear together with the axions.[4])

[3]Later though we shall consider the inverse possibility of dualising existing scalars to $(D-2)$-form potentials; this we call an inverse dualisation.

[4]In this context, we are defining an axion to be any scalar field other than the dilatons which come from the dimensional reduction of the diagonal components of the metric. The dilatonic scalars are the moduli

In the Kaluza-Klein reduction of a generic higher-dimensional theory, the global symmetry of the scalar sector may not necessarily extend to the higher-degree sectors of the reduced theory. In fact it is only because of special features of eleven-dimensional supergravity that its dimensional reductions allow the global symmetries of the scalar manifolds to be extended to the full dimensionally-reduced theories including the higher-degree fields. For example, omitting the $F_{(4)} \wedge F_{(4)} \wedge A_{(3)}$ term in $D = 11$ (or even just changing its coefficient) would not affect the E_{11-D} symmetry of the scalar sector in $D \geq 6$, but it would prevent its extension to the higher-degree fields. Even for $D = 11$ supergravity itself, the entire dimensionally-reduced theories may only exhibit the global symmetries of their scalar sectors if appropriate dualisations of higher-degree field strengths are also performed. For example, in $D = 6$ the $E_5 = O(5,5)$ symmetry of the scalar sector only extends to the entire theory if the 3-form gauge potential is dualised to give an additional vector, which, together with the 15 that are already present, can form a 16-dimensional spinor representation of $O(5,5)$. This is an example where bare Kaluza-Klein vector potentials inevitably appear in the original undualised formulation obtained by direct dimensional reduction, even at the level of the equations of motion (see section 6.3). Consequently, only by dualising the 3-form potential can the $O(5,5)$ symmetry be realised in terms of transformations that involve purely local functions of fields, namely on the 16 vector potentials. Thus it should be emphasised that in this example, even at the level of the equations of motion, the $O(5,5)$ symmetry cannot apparently be realised unless the dualisation of the 3-form potential has been performed. We should however point out that the dualisation can be effected in two ways: either by adding a Lagrange multiplier for the Bianchi identity (this indeed requires that no bare potential appears in the Lagrangian) or else by using a first order formalism of the classical type, with field and potential considered as independent variables, and integrating out the potential first. Note that the reverse dualisation exchanges the two types of procedures, and that for instance the Freedman-Townsend dualisation from 2-form potentials to scalars, leading to a sigma model in four dimensions, can be effected despite the presence of bare 2-forms [13].

There are also examples where dualisations of higher-degree fields are not obligatory in order for the global symmetry of the scalar manifold to extend to the entire theory, at least at the level of the equations of motion. For example, the global $E_4 = SL(5, \mathbb{R})$ symmetry of the scalar sector in $D = 7$ can be realised at the level of the equations of motion in the entire theory regardless of whether or not the 3-form potential is dualised to a 2-form

parametrising the size of the compactifying space.

potential. This is because the 3-form potential, together with the four 2-form potentials of the original undualised theory, can all be made to appear in the Lagrangian only via their field strengths. Consequently, there will never be bare potentials in the equations of motion or Bianchi identities, and the set of $1+4$ field strengths will transform as a 5 of $SL(5,\mathbb{R})$, regardless of whether or not the dualisation has been performed.

In this paper we shall analyse the rigid symmetries that are realized on the bosonic fields of the various lower-dimensional maximal supergravities. Each such theory is obtained from eleven-dimensional supergravity by toroidal compactification to D dimensions, with the possible subsequent dualisation of some subset of the fields. A striking regularity is exhibited by the series of theories obtained respectively by making no dualisations; with dualisations of the so-called Ramond-Ramond fields of the type IIA theory; with full dualisation to lower the degrees of all forms; and from there finally by inverse dualisations (raising the degrees of certain forms) to the type IIB series. We shall call these theories the GL_A, D, E and GL_B series respectively. This is only a subset of the large number of classical forms of the theory. It turns out that the scalar Lagrangians of the E series are sigma models on the symmetric spaces $K(E_{11-D})\backslash E_{11-D}$ (where $K(G)$ is the maximal compact subgroup of G), while the other three series lead to models on homogeneous spaces $K(G)\backslash G\ltimes\mathbb{R}^s$, where s is the dimension of a certain linear representation of G. In fact, the E series can be used as a means of generating the other three series by performing appropriate inverse dualisations of some of its axionic scalar fields. The reason for this is that, as will be shown in section 4, the axionic scalars in the fully-dualised supergravities are in one-to-one correspondence with the positive roots of the E_{11-D} algebra [14]. In fact we exploit this to give a simple (triangular or Borel) parameterisation of the $K(E_{11-D})\backslash E_{11-D}$ cosets for the scalar manifolds, in which the axionic scalars are the parameters in the exponentiation of the positive roots, while the dilatonic scalars are the parameters in the exponentiation of the Cartan generators. Let us recall three equivalent formulations of symmetric space sigma models. One possibility is to work in a fixed gauge for the subgroup $K(G)$ and use a triangular representative of each coset (permitted by the Iwasawa decomposition); this amounts to using group elements in a Borel subgroup (morally the upper triangular part of the group). The Borel subgroup itself contains the Cartan subgroup times the strictly upper triangular subgroup called below the group of positive roots. The second possibility is to restore the $K(G)$ local gauge invariance; this form is manifestly invariant under the full global G and not only under its Borel subgroup; the scalar fields (physical and gauge) parameterise, before gauge fixing, the full group G. Finally if we recall the analogy with the moving frames of General Relativity [10], we may use the (local Lorentz invariant) metric instead of the frames and then preserve manifest $GL(4,\mathbb{R})$ invariance without introducing the Lorentz gauge invariance. Analogously, we shall use an internal metric $\mathcal{M}$ instead of an element of the group G; this will be the third formulation of the symmetric space sigma models.

The inverse dualisation of some of the axions appearing in the E series now has the effect of removing the associated positive-root generators from the parameterisation of the coset. For the GL_A and D series, these dualisations involve the subsets of positive roots at the second level according to the grading of the root space of the E_{11-D} Lie algebra along the appropriate simple root (except in $D = 3$, where the scalars associated with both the third and second level positive roots must be dualised for the E to GL_A contraction). For the GL_B series, the construction involves dualising some scalars associated with commuting positive root vectors selected by an appropriate double grading along two of the simple roots. A similar inverse dualisation for a fourth simple root relates E_{11-D} and E_{10-D}. The generators corresponding to the highest level that has not been inversely dualised will consequently now commute. (This means that they could in turn be inversely dualised; we discussed this further in a second paper [36]). The commutativity of a set of generators and an Iwasawa type formula lead to the property that the corresponding scalar fields can be simultaneously covered with derivatives in the sigma model scalar Lagrangian, and hence can be dualised.

As stated earlier, in order to specify the classical theory under consideration one should not restrict oneself to the scalar sector alone but one must also specify the dualisations implemented on higher-degree fields, which exchange degree $p \geq 2$ field strengths with those of degree $D - p$. Once again, the Coxeter-Dynkin diagrams of the E series seem to contain all the information on the four specific nodes corresponding to the simple roots alluded to above (in any dimension for our four series). The general situation is now more difficult to summarise than in the above discussion of abelianisation in the scalar sector. The possibility of dualisation involves the simultaneous existence and use of involutions of the Dynkin diagram of $SL(11-D,\mathbb{R})$ and of that of, for instance, E_{11-D} for the E series; they respectively exchange covectors (*i.e.* 1-forms) and vectors (*i.e.* $(D-1)$-forms) and the corresponding representations of the internal symmetry, and similarly for forms of arbitrary degrees. We shall explain these features in the second paper of this series.

The paper is organised as follows. In section 2, we obtain the bosonic sector of the D-dimensional supergravity following directly from the dimensional reduction of eleven-dimensional supergravity. We show that these non-dualised theories have global $GL(11-D,\mathbb{R})\ltimes\mathbb{R}^q$ global symmetries, where $q=\frac{1}{6}(11-D)(10-D)(9-D)$. In section 3, we study the cases $D=5,4$ and 3 where the full dualisations of all $(D-2)$-form potentials are performed, so as to obtain the maximal numbers of scalars. We show that the symmetry of the scalar Lagrangian is changed by this dualisation. In particular, we show that the symmetry group contains the Borel subgroup of E_{11-D} and that the dimensions of the maximal abelian symmetries are in each case reduced by the dualisation. In section 4, we study the coset structures of the scalar Lagrangians, and their symmetries. We show that the scalar Lagrangians for the fully-dualised E series, where the number of scalars is maximised, have E_{11-D} global symmetries for $3\le D\le 10$. When the dimensional reduction in section 2 is performed by iteratively repeating the $D+1$ to D dimensional reduction, the scalars are precisely the parameters of the generators of the Borel subgroups. In other words, the triangular gauge is always the simplest, and the Borel invariance is the most obvious symmetry.

The symmetric space will be replaced by a double coset in the cases where certain axions associated with some abelian positive roots are undualised or simply have not been manufactured by dualisation. This provides some group theoretical understanding of the dualisation procedure involving scalar fields, and is discussed in section 5. Actually this leads to a situation where the scalar fields take their values in a double coset space on which the normaliser of the suppressed generators still acts transitively.

In section 6, we show that the symmetries of the scalar sectors of the maximal supergravities can be extended to the entire bosonic theories including the higher-degree field strengths, which form linear representations of the symmetry groups. We discuss this in detail in the fully-dualised E series, and show that the toroidally-compactified eleven-dimensional supergravities have E_{11-D} global symmetries after the full dualisation. Our discussion will be simplified, and the full details are postponed to our second paper in this series. We shall also discuss in this section how dualisations of higher-degree field strengths can affect the symmetry group of the full Lagrangian. In section 7, we study a particular case where all the R-R fields are dualised to lower degrees while the NS-NS fields remain intact. In section 8, we study the abelian global symmetries in the various versions of the supergravities. We show how the abelian constant shift symmetries can be grouped into maximal abelian subsets of abelian $\mathbb{R}$ symmetries of the positive-root systems for the theories. In section 9, we study type IIB supergravity and its dimensional reduction. In particular, we are interested in the versions where no dualisations are performed. We conclude our paper in section 10. We also present the full bosonic Lagrangian following from the direct reduction of eleven-dimensional supergravity in Appendix A. In Appendix B, we present a scalar Lagrangian with $SL(2,\mathbb{R})$ global symmetry and study how the symmetry is affected by dualisation. Appendix C contains a discussion of scalar Lagrangians with $O(n,n)$ global symmetries, and their application in supergravity theories.[5]

2 Direct reduction of $D = 11$ supergravity and symmetries

In section 2.1, we shall discuss the toroidal dimensional reduction of eleven-dimensional supergravity to D dimensions. In the cases where none of the D-dimensional fields are dualised, we shall show in section 2.2 that there is a global $GL(11-D,\mathbb{R})\ltimes\mathbb{R}^q$ symmetry, where $q = \frac{1}{6}(11-D)(10-D)(9-D) = \{0,0,0,1,4,10,20,35,56\}$ in $D = \{11,10,9,8,7,6,5,4,3\}$, and the $\ltimes$ symbol denotes a semi-direct product.

2.1 Dimensional reduction of 11-dimensional supergravity

The bosonic sector of eleven-dimensional supergravity contains the metric and a 4-form field strength $F_{(4)} = dA_{(3)}$. The Lagrangian is given by [1]

$$\mathcal{L} = eR - \tfrac{1}{48} eF_{(4)}^2 + \tfrac{1}{6} * (F_{(4)} \wedge F_{(4)} \wedge A_{(3)}) \ . \qquad (2.1)$$

The subscripts on the potential $A_{(3)}$ and its field strength $F_{(4)} = dA_{(3)}$ indicate the degrees of the differential forms, and the normalisation is that of [15]. Note that the relative coefficient $\frac{1}{6}$ of the FFA term is inert under compatible rescalings of the gauge potential and the metric, in the sense that rescalings that preserve the ratio of the coefficients of the Einstein-Hilbert and gauge-field kinetic terms also keep the coefficient of FFA in the same ratio.

This rigid rescaling, which changes the entire action homogeneously, is given by

$$g_{MN} \longrightarrow \lambda^2\, g_{MN} \ , \qquad A_{MNP} \longrightarrow \lambda^3\, A_{MNP} \ . \qquad (2.2)$$

Since it gives a homogeneous rescaling of the action, it is a symmetry of the equations of motion. It can alternatively be viewed as an engineering scale invariance of the eleven-dimensional classical equations, as a consequence of the fact that there is just one overall dimensionful coupling constant, which sits in front of the entire eleven-dimensional action.

[5]This analysis as well as that of sections 6.2.2 and 6.3 was inspired by an ongoing research program of one of us (B.J.) on the important differences between $4k$ and $4k+2$ dimensional spacetime. The next most obvious one beyond the duality properties, namely the change of sign in the Schwinger-Zwanziger quantisation formula, has since then been studied in detail in [37, 38].

We shall reduce the theory to D dimensions in a succession of 1-step compactifications on circles. At each stage in the reduction, say from $(D+1)$ to D dimensions, the metric is reduced according to the standard Kaluza-Klein prescription

$$ds^2_{D+1} = e^{2\alpha\varphi}\, ds^2_D + e^{-2(D-2)\alpha\varphi}\,(dz + \mathcal{A}_{(1)})^2 \ , \tag{2.3}$$

where the D dimensional metric, the Kaluza-Klein vector potential $\mathcal{A}_{(1)} = \mathcal{A}_M\, dx^M$ with $M = 0, 1, ..., D$ and the dilatonic scalar φ are taken to be independent of the ignorable coordinate z on the compactifying circle. The constant α is given by $\alpha^{-2} = 2(D-1)(D-2)$, and the parameterisation of the metric is such that a pure Einstein action is reduced again to a pure Einstein action together with a canonically-normalised kinetic term for φ and a dilated kinetic term for $\mathcal{F}_{(2)} = d\mathcal{A}_{(1)}$:

$$e\, R \longrightarrow e\, R - \tfrac{1}{4} e\, e^{-2(D-1)\alpha\varphi}\, \mathcal{F}^2_{(2)} - \tfrac{1}{2} e\,(\partial\varphi)^2 \ . \tag{2.4}$$

Gauge potentials reduce according to $A_{(n)}(x, z) = A_{(n)}(x) + A_{(n-1)}(x) \wedge dz$, implying that a kinetic term for an n-form field strength $F_{(n)}$ reduces according to the rule:

$$-\frac{1}{2\, n!}\, e\, F^2_{(n)} \longrightarrow -\frac{1}{2\, n!}\, e\, e^{-2(n-1)\alpha\varphi}\, F^2_{(n)} - \frac{1}{2\,(n-1)!}\, e\, e^{2(D-n)\alpha\varphi}\, F^2_{(n-1)} \ . \tag{2.5}$$

There is a subtlety here in the definition of the dimensionally-reduced field strength $F_{(n)}$, which is most easily seen by working with the adapted ("triangular") vielbein of [10] and tangent space (flat) indices, since this facilitates the computation of the inner products in the kinetic terms and in fact uses the bundle principal connection. From the ansatz for the reduction of the gauge potential we have

$$F_{(n)} \longrightarrow dA_{(n-1)} + dA_{(n-2)} \wedge dz = dA_{(n-1)} - dA_{(n-2)} \wedge \mathcal{A}_{(1)} + dA_{(n-2)} \wedge (dz + \mathcal{A}_{(1)}) \ . \tag{2.6}$$

Thus while it is natural to define the dimensionally-reduced field strength $F_{(n-1)}$ by $F_{(n-1)} = dA_{(n-2)}$, we shall define $F_{(n)}$ by $F_{(n)} = dA_{(n-1)} - dA_{(n-2)} \wedge \mathcal{A}_{(1)}$; it is this gauge-invariant field strength that appears on the right-hand side of (2.5). Note that this makes the meaning of the symbol F dimension-dependent. In Appendix A the plain exterior derivative of a potential A is called $\tilde{F}$, tildes are also used here and there for different purposes when there is no ambiguity. Similar non-linear Kaluza-Klein modifications to the lower-dimensional field strengths become progressively more complicated as the descent through the dimensions continues. These definitions are analysed further in Appendix A.

It is not too difficult now to apply the above reduction procedure iteratively [16], to construct the D-dimensional toroidally-compactified theory from the eleven-dimensional starting point. It is easy to see that the original eleven-dimensional fields g_{MN} and A_{MNP} will give rise to the following fields in D dimensions,

$$\begin{array}{rclllll} g_{MN} & \longrightarrow & g_{MN}\ , & \vec{\phi}\ , & \mathcal{A}^i_{(1)}\ , & \mathcal{A}^i_{(0)j}\ , & \\ A_{(3)} & \longrightarrow & A_{(3)}\ , & A_{(2)i}\ , & A_{(1)ij}\ , & A_{(0)ijk}\ , & \end{array} \tag{2.7}$$

where the indices i, j, k run over the $11 - D$ internal toroidally-compactified dimensions, starting from $i = 1$ for the step from $D = 11$ to $D = 10$. The potentials $A_{(1)ij}$ and $A_{(0)ijk}$ are automatically antisymmetric in their internal indices, whereas the 0-form potentials $\mathcal{A}^i_{(0)j}$ that come from the subsequent dimensional reductions of the Kaluza-Klein vector potentials $\mathcal{A}^i_{(1)}$ are defined only for $j > i$. (Note that in the standard notation, the set of potentials $(A_{(2)i}, A_{(1)ij}, A_{(0)ijk})$ correspond to $(A_{\mu\nu i}, A_{\mu ij}, A_{ijk})$. The quantity $\vec{\phi}$ denotes the $(11 - D)$-vector of dilatonic scalar fields coming from the diagonal components of the internal metric.

The detailed expression for the Lagrangian for the bosonic sector of the D-dimensional toroidal compactification of eleven-dimensional supergravity is presented in Appendix A. Note that at this stage the Lagrangian is simply the one obtained directly from dimensional reduction, without performing any dualisations. In the next subsection, we show that this Lagrangian has a $GL(11 - D, \mathbb{R}) \ltimes \mathbb{R}^q$ global symmetry.

2.2 No dualisation and $GL(N, \mathbb{R}) \ltimes \mathbb{R}^q$

The $SL(N, \mathbb{R})$ part of the global symmetry is a completely general consequence of the dimensional reduction to D dimensions of any $(D + N)$-dimensional theory that includes gravity [10]. In order to implement an internal $\mathbb{R}$ symmetry from the last generator of $GL(N, \mathbb{R})$, D must be strictly larger than 2. There is a subtlety here, namely the effect of the Weyl rescalings that are needed in order to go to the so-called Einstein frame. The $\mathbb{R}^q$ part of the symmetry, on the other hand, comes from the local abelian gauge symmetry of an antisymmetric tensor field strength in the original $(D + N)$ dimensions. Specifically, it describes the global shift symmetries of the axionic scalars that are the potentials for 1-form field strengths coming from the dimensional reduction. This $GL(N, \mathbb{R}) \ltimes \mathbb{R}^q$ symmetry can be discussed in any dimension. Let us consider a theory in $(D + N)$ dimensions, containing a metric, a dilaton ϕ and a degree n antisymmetric tensor field strength $F_n = dA_{n-1}$. This theory is invariant under the general coordinate transformations

$$\begin{aligned} \delta x^M &= -\xi^M(x)\ , \qquad \delta\phi = \xi^M \partial_M \phi\ , \\ \delta A_{M_1 \cdots M_{n-1}} &= \xi^M \partial_M A_{M_1 \cdots M_{n-1}} + (n-1)\, \partial_{[M_1} \xi^M A_{|M| M_2 \cdots M_{n-1}]}\ . \end{aligned} \tag{2.8}$$

Now, we compactify the theory to D dimensions, splitting the index $_M$ into a D-dimensional index μ and an N-dimensional internal index i, with coordinates x^μ and y^i respectively, *i.e.* $x^M = (x^\mu, y^i)$. We then impose the toroidal Kaluza-Klein condition that all the D-dimensional fields are independent of the compactifying coordinates y^i, namely $\partial_i \phi = 0 = \partial_i A_{M_1,\ldots,M_{n-1}} = 0$. Note that the Kaluza-Klein ansatz requires that the transformed fields should also be independent of y^i, implying that $\partial_i \xi^\mu = 0$, $\partial_i \partial_\mu \xi^j = 0$ and $\partial_i \partial_j \xi^k = 0$. These equations have the solution

$$\xi^\mu = \xi^\mu(x^\nu) \ , \qquad \xi^i = \Lambda^i{}_j \, y^j + \xi^i(x^\nu) \ , \tag{2.9}$$

where the $\Lambda^i{}_j$ are constants. The resulting transformations imply the following symmetries in D dimensions:

$$\begin{cases} \delta x^\mu = -\xi^\mu(x) \ , & \text{reparameterisation invariance in } D \text{ dimensions} \ , \\ \delta y^i = -\xi^i(x) \ , & \text{local } \mathbb{R}^N \text{ invariance} \ , \\ \delta y^i = -\Lambda^i{}_j \, y^j \ , & \text{global } GL(N, \mathbb{R}) \text{ invariance} \ . \end{cases} \tag{2.10}$$

In the sector with ignorable internal coordinates we cannot distinguish the compactification torus from $\mathbb{R}^N$, but the massive excitations would not transform under $GL(N, \mathbb{R})$. Note that the naive $GL(N, \mathbb{R}) \sim \mathbb{R} \times SL(N, \mathbb{R})$ is to be combined with the rigid rescaling (2.2) to become an internal symmetry (*i.e.* one that leaves the metric invariant). Indeed the plain $\mathbb{R}$ symmetry rescales the volume of the compactifying space, and must be combined with the rescaling (2.2). This defines a new internal scaling symmetry which we call $\mathbb{R}_s$. The $SL(N, \mathbb{R})$ however leaves the volume fixed; this corresponds to the restriction $\sum_i \Lambda^i{}_i = 0$. In particular, from (2.8) we find that $SL(N, \mathbb{R})$ acts on internal world indices on fields according to the rules

$$\delta A_i = \Lambda^j{}_i \, A_j \ , \qquad \delta V^i = -\Lambda^i{}_j \, V^j \ . \tag{2.11}$$

In the above discussion, we showed that the internal part of the $(D+N)$-dimensional reparameterisation invariance describes a $SL(N, \mathbb{R})$ global symmetry from the D-dimensional point of view. There is in addition a local gauge symmetry in $(D+N)$ dimensions, namely

$$\delta A_{M_1 \cdots M_{n-1}} = (n-1) \, \partial_{[M_1} \lambda_{M_2 \cdots M_{n-1}]} \ . \tag{2.12}$$

This gives rise to local gauge symmetries for the dimensionally-reduced gauge potentials $A_{M_1 \cdots M_{n-2}}$ with one or more D-dimensional spacetime indices. In the case of the $N!/((n-

1)!$(N-n+1)!$) 0-form potentials, or axionic scalars, however, only global shift symmetries remain. To see this, we note from (2.12) that the transformation rules for the axions are given by

$$\delta A_{i_1\cdots i_{n-1}} = (n-1)\,\partial_{[i_1}\lambda_{i_2\cdots i_{n-1}]}\ . \tag{2.13}$$

In order for these variations to be nonzero and for (2.12) to be independent of the internal coordinates y^i, we must have $\lambda_{i_2\cdots i_{n-1}} = c_{i_2\cdots i_{n-1}j}\, y^j$, where $c_{i_2\cdots i_{n-1}j}$ is any constant antisymmetric tensor, giving

$$\delta A_{i_1\cdots i_{n-1}} = c_{i_1\cdots i_{n-1}}\ . \tag{2.14}$$

Thus the D-dimensional theory has also an $\mathbb{R}^q$ symmetry, with $q = N!/((n-1)!(N-n+1)!)$. Clearly these $\mathbb{R}$ symmetries commute with each other, since they are derived directly from the abelian gauge symmetry in $(D+N)$ dimensions. However they do not all commute with the $GL(N,\mathbb{R})$ symmetries, since the axions $A_{i_1\cdots i_{n-1}}$ also transform (covariantly) under $GL(N,\mathbb{R})$. This can be seen from the commutation relations

$$[\delta_c, \delta_\Lambda] = \delta_{\tilde c}\ , \qquad \tilde c_{i_1\cdots i_{n-1}} = (n-1)\Lambda^i{}_{[i_1}\, c_{|i|i_2\cdots i_{n-1}]}\ , \tag{2.15}$$

where the $GL(N,\mathbb{R})$ transformations parameterised by $\Lambda^i{}_j$ are given by

$$[\delta_\Lambda, \delta_{\Lambda'}] = \delta_{\tilde\Lambda}\ , \qquad \tilde\Lambda^i{}_j = \Lambda^i{}_k\,\Lambda'^k{}_j - \Lambda'^i{}_k\,\Lambda^k{}_j\ . \tag{2.16}$$

(Examples of fields with upstairs, contravariant world indices have not arisen yet in our discussion, but we shall encounter them later.) As a matter of fact note that, from the general coordinate invariance of the 1-forms $\tilde\gamma^i{}_j\,(dy^j + \hat{\mathcal{A}}^j_{(1)}) \equiv dy^i + \mathcal{A}^i_{(0)j}\, dy^j + \mathcal{A}^i_{(1)}$, we deduce that the axions $\mathcal{A}^i_{(0)j}$ must transform inhomogeneously as

$$\delta\mathcal{A}^i_{(0)j} = \Lambda^i{}_j + \Lambda^k{}_j\,\mathcal{A}^i_{(0)k} \tag{2.17}$$

under $SL(N,\mathbb{R})$, while the 1-forms $\mathcal{A}^i_{(1)}$ are inert. In Appendix A this rule is derived from a careful distinction between tangent and spacetime (internal) indices. $\hat{\mathcal{A}}^j_{(1)}$ and $\gamma^j{}_i$ (the inverse of $\tilde\gamma$) do transform as vectors under $SL(N,\mathbb{R})$.

Let us now apply the above discussion to the dimensional reduction of eleven-dimensional supergravity, for which we have $n=4$ and $N=11-D$. Thus the D-dimensional theory (without any dualisation) has a global symmetry $GL(11-D,\mathbb{R})\ltimes\mathbb{R}^q$, with $q=\frac{1}{6}(11-D)(10-D)(9-D)$.[6] It should be emphasised that $GL(11-D,\mathbb{R})\ltimes\mathbb{R}^q$ is a symmetry

[6] Note that for all dimensions $D\ge 3$, we could (inversely) dualise all the p axions $A_{(0)ijk}$ to $(D-2)$-form gauge potentials, since these axions may be all simultaneously covered by derivatives everywhere in the Lagrangian. The resulting theory would then have only a $GL(11-D,\mathbb{R})$ global symmetry. In this and the

at the level of the Lagrangian that is derived from direct dimensional reduction without any dualisation. In the process, we also make use of Weyl rescalings so that all the lower dimensional Lagrangians are written in the Einstein frame. This rescaling modifies the $\mathbb{R}$ part of $GL(11-D,\mathbb{R}) = \mathbb{R} \times SL(11-D,\mathbb{R})$, which becomes as a result an internal symmetry ($\mathbb{R}_s$). This is the first instance of a hidden symmetry. It can be traced back to the eleven-dimensional action, which is invariant up to a factor under engineering rescalings, as well as (and equivalently) under Weyl rescalings of the metric coupled to appropriate multiplicative redefinitions of the 3-form. The next hidden symmetry arises in $D = 8$, for which the $\mathbb{R}_s$ factor becomes $SL(2,\mathbb{R})$. In dimension $D = 7$ and below, the obvious and the hidden internal symmetries combine to form a simple group.

In this paper, we obtain lower-dimensional supergravities by iteratively applying the $D+1$ to D dimensional reduction. This has the effect that the manifest $SL(11-D,\mathbb{R})$ symmetry is reduced to the Borel subgroup, generated by the positive-root generators of the group, with infinitesimal transformation parameters $\Lambda^i{}_j$ that are non-zero only for $i < j$, *i.e.* they are upper triangular matrices.

As we shall show in section 4, in the cases $D \ge 6$ the global symmetry of the theory can be extended to E_{11-D}, provided that certain higher-degree fields are dualised appropriately. In fact the scalar Lagrangian in $D \ge 6$ already has the full E_{11-D} global symmetry, and the $GL(11-D,\mathbb{R}) \ltimes \mathbb{R}^q$ symmetry described above is a subgroup of it. The fact that the extra scalars are internal 3-forms is reflected by the fact that the extra root of the E_{11-D} group is above the third root of the $GL(11-D,\mathbb{R})$ subgroup that corresponds to the highest weight of that particular internal $GL(11-D,\mathbb{R})$ representation. Note that in $D \ge 6$ the number $q = \{0, 0, 1, 4, 10\}$ of $\mathbb{R}$ symmetries for $D = \{10, 9, 8, 7, 6\}$ never exceeds the dimension of the maximal abelian subalgebra of the group E_{11-D} corresponding to the fully-dualised theory, namely $\{1, 2, 3, 6, 10\}$ [17].

The situation is different when $D \le 5$. In these lower dimensions, the theory contains $(D-2)$-form gauge potentials which can be dualised to give rise to additional axionic scalars. Before any such dualisation is performed the theory has a $GL(11-D,\mathbb{R}) \ltimes \mathbb{R}^q$ global symmetry, which can be enlarged to E_{11-D} only after performing certain necessary dualisations. At first sight this is rather counter-intuitive, since one might expect that at the level of the equations of motion dualisation should have no effect on the global symmetry of

next sections, we shall concentrate only on those D-dimensional Lagrangians that are direct dimensional reductions of eleven-dimensional supergravity and the ones obtained from these by the dualisation of the 4-form, 3-form or 2-form field strengths to axions in $D = 5$, 4 or 3 respectively. We shall also discuss the dimensional reduction of IIB supergravity in $D = 10$, without performing any dualisation.

the theory. Indeed this is true for the dualisations of fields that appear in the Lagrangian only via their field strengths, since, as we discussed in the Introduction, in such cases the field strengths rather than their potentials can be treated as the fundamental fields in the equations of motion. However, in the case of scalar potentials some transformations act on the scalars themselves and not on their derivatives, and so the global symmetry will be altered if any of the scalars are dualised to $(D-2)$-form potentials. An explicit example of a scalar Lagrangian with $SL(2,\mathbb{R})$ symmetry is discussed in Appendix B.1, to illustrate how the dualisation of the axion alters the symmetry. In the case of supergravities in $3 \le D \le 5$, the global symmetry analysis is more complicated. One way to see that the undualised theories do not have the global E_{11-D} symmetry is to look at the maximal abelian $\mathbb{R}^q$ symmetries in the two versions. In the undualised versions we have certainly at least $\mathbb{R}^{20}$, $\mathbb{R}^{35}$ and $\mathbb{R}^{56}$ abelian symmetries in $D=5$, 4 and 3, whose dimensions exceed that of the maximal abelian subalgebras of E_6, E_7 and E_8, namely 16, 27 and 36 respectively. (The latter dimensions have been determined in [17] by educated inspection.) In the next section we shall show explicitly how the $\mathbb{R}^q$ symmetry changes under the dualisation, in each of the dimensions $D=5$, 4 and 3.

3 Dualisation and the maximal scalar manifolds in $D=5,4,3$

In the previous section, we showed that the D-dimensional supergravities coming from direct dimensional reduction of eleven-dimensional supergravity without any dualisation have $GL(11-D,\mathbb{R})\ltimes\mathbb{R}^q$ global symmetries, where $q=\frac{1}{6}(11-D)(10-D)(9-D)$. The transformations are realised on the gauge potentials, and thus correspond to symmetries of the Lagrangian (hence of the equations of motion). In fact in $D \ge 6$, the Lagrangian for the scalar sector actually has an E_{11-D} global symmetry (see section 4), which contains $GL(11-D,\mathbb{R})\ltimes\mathbb{R}^q$ as a proper subgroup in $6 \le D \le 8$. (In $D \ge 9$, the groups $GL(11-D,\mathbb{R})$ and E_{11-D} coincide.) It is possible to extend the E_{11-D} symmetry of the scalar sector to the full theory, when appropriate dualisations of higher-rank fields are performed. In dimensions 10 and 9 the only subtle symmetry is the modified $\mathbb{R}_s$ generator coming from $GL(N,\mathbb{R})$, but from dimension 8 on we encounter the Ehlers miracle of an extra $SL(2,\mathbb{R})$ factor (the name comes from the analogous phenomenon in ordinary General Relativity reduced to 3 dimensions). In both cases this extra symmetry involves dualisation of gauge forms. We shall return to this in section 6.

In $D=5,4,3$, the scalar Lagrangian coming directly from the dimensional reduction of the eleven-dimensional theory has only the $GL(11-D,\mathbb{R})\ltimes\mathbb{R}^q$ symmetry described above. This is also true at the level of the equations of motion. The extension to E_{11-D} is possible only when all the 3-form, 2-form or 1-form potentials respectively are dualised to give additional scalars. This is because the E_{11-D} symmetry of the dualised formulation must be implemented on the scalars (0-form potentials) themselves, rather than on their derivatives. We shall use the term maximal scalar manifold to refer to this situation where the number of scalar fields is made as large as possible. Recall that in $3 \le D \le 5$ the $GL(11-D,\mathbb{R})\ltimes\mathbb{R}^q$ symmetry is not contained in E_{11-D} as we have seen by observing that the $\mathbb{R}^q = \{\mathbb{R}^{20},\mathbb{R}^{35},\mathbb{R}^{56}\}$ abelian factors are larger than the maximal abelian subalgebras $\{\mathbb{R}^{16},\mathbb{R}^{27},\mathbb{R}^{36}\}$ of $\{E_6,E_7,E_8\}$. Thus the fully-dualised and undualised formulations of the theories have inequivalent global symmetries, neither of which encompasses the other.

We shall now explicitly perform the dualisations of the $(D-2)$-form potentials to give rise to additional scalars in each of the dimensions $D = 5$, 4 and 3, and show how the global symmetries are altered. Note that when we dualise all the $(D-2)$-form potentials, the $GL(11-D,\mathbb{R})$ symmetry is preserved in each case, now becoming a subgroup of the full enlarged E_{11-D} symmetry. There are also other possibilities, in which we may choose to dualise only a subset of the $(D-2)$-forms. For example, we might dualise only those fields which, from the type IIA string point of view, are associated with the Ramond-Ramond sector [12]. In this case, the $GL(11-D,\mathbb{R})$ symmetry is broken. We shall discuss this in detail in section 7.

3.1 Dualisation in $D = 5$ supergravity

In $D = 5$, the 3-form gauge potential $A_{\mu\nu\rho}$, which comes from the dimensional reduction of the 3-form in $D = 11$, can be dualised to give a scalar field. As in the case of the $SL(2,\mathbb{R})$ example in Appendix B, an additional $\mathbb{R}$ symmetry is then created, corresponding to a global shift symmetry of the new scalar field, since it can be covered by a derivative everywhere in the Lagrangian. Naively, one would expect, as in the case of the $SL(2,\mathbb{R})$ example, that the dualisation procedure would always increase the dimension of the commuting $\mathbb{R}$ symmetries, since we have obtained new scalars that can be covered by derivatives everywhere. However, in the present case there is an additional term in the scalar Lagrangian, coming from the dimensional reduction of $\mathcal{L}_{FFA}$ in $D = 11$. Without this additional topological term, the analysis would indeed be analogous to the $SL(2,\mathbb{R})$ example in Appendix B, and we would have a dualised theory in which the $\mathbb{R}^{20}$ symmetry of the original 0-form gauge potentials of the undualised theory was enlarged to $\mathbb{R}^{20+1}$. Let us look in detail at the Lagrangian involving $F_{(4)} = dA_{(3)} + \cdots$ in $D = 5$, given by

$$\mathcal{L}(F_{(4)}) = -\tfrac{1}{48} e e^{\vec{a}\cdot\vec{\phi}}\, F_{(4)}^2 - \tfrac{1}{1728} e\, \epsilon^{ijklmn}\, A_{(0)ijk}\, \partial_\mu A_{(0)\ell mn}\, F_{\nu\rho\sigma\lambda}\, \epsilon^{\mu\nu\rho\sigma\lambda} \ , \tag{3.1}$$

where the second term comes from the $\mathcal{L}_{FFA}$ terms (A.16) and $\vec{a}$ is defined in Appendix A. If no dualisation of $A_{(3)}$ were to be performed, we could add a topological surface term so that all the 20 axions $A_{(0)ijk}$ would be simultaneously covered by derivatives, implying an abelian $\mathbb{R}^{20}$ symmetry. But in order to dualise $A_{(3)}$, we first introduce a Lagrange multiplier χ to impose the Bianchi identity $dF_{(4)} = 0 + \cdots$, by adding the term $\chi dF_{(4)}$ to the Lagrangian (in order to construct the scalar Lagrangian we may neglect the difference $F_{(4)} - dA_{(3)}$). This leads to the first-order Lagrangian

$$\mathcal{L}(F_{(4)}) = -\tfrac{1}{48} e e^{\vec{a}\cdot\vec{\phi}}\, F_{(4)}^2 - \tfrac{1}{1728} e\, \epsilon^{ijklmn}\, A_{(0)ijk} \partial_\mu A_{(0)\ell mn} F_{\nu\rho\sigma\lambda}\, \epsilon^{\mu\nu\rho\sigma\lambda} + \tfrac{1}{24} e\, \chi\, \epsilon^{\mu\nu\rho\sigma\lambda}\, \partial_\mu F_{\nu\rho\sigma\lambda} \ . \tag{3.2}$$

Integrating by parts, we have

$$\mathcal{L}(F_{(4)}) = -\tfrac{1}{48} e e^{\vec{a}\cdot\vec{\phi}}\, F_{(4)}^2 + \tfrac{1}{24} e\, \epsilon^{\mu\nu\rho\sigma\lambda}\, F_{\nu\rho\sigma\lambda} \Big(\partial_\mu \chi - A_{(0)ijk} \partial_\mu A_{(0)\ell mn}\, \tfrac{1}{72} \epsilon^{ijk\ell mn} \Big) \ , \tag{3.3}$$

from which we can solve algebraically for $F_{(4)}$, giving

$$F_{\mu\nu\rho\sigma} = e^{-\vec{a}\cdot\vec{\phi}}\, \epsilon_{\mu\nu\rho\sigma\lambda} \Big(\partial^\lambda \chi - \tfrac{1}{72} A_{(0)ijk} \partial^\lambda A_{(0)\ell mn}\, \epsilon^{ijk\ell mn} \Big) \ . \tag{3.4}$$

Thus after the dualisation of $A_{(3)}$, the Lagrangian (3.1) becomes

$$\mathcal{L} = -\tfrac{1}{2} e e^{-\vec{a}\cdot\vec{\phi}} G_{(1)}^2 \ , \tag{3.5}$$

where

$$G_{(1)} = d\chi - \tfrac{1}{72}\, A_{(0)ijk}\, dA_{(0)\ell mn}\, \epsilon^{ijk\ell mn} \tag{3.6}$$

is the 1-form field strength dual to $F_{(4)}$, *i.e.* $F_{(4)} = e^{-\vec{a}\cdot\vec{\phi}} * G_{(1)}$. Note that the dualisation has the effect of reversing the sign of the dilaton coupling $\vec{a}$. We see that the $\mathbb{R}^{20}$ symmetry $\delta A_{(0)ijk} = c_{ijk}$ of the original Lagrangian (3.1) (under which all the other original axions were inert) becomes

$$\delta A_{(0)ijk} = c_{ijk} \ , \qquad \delta\chi = k + \tfrac{1}{72} c_{ijk}\, A_{(0)\ell mn}\, \epsilon^{ijk\ell mn} \ , \tag{3.7}$$

where k is the constant shift symmetry associated with χ. Under the rescaling symmetry $\mathbb{R}_s$, $\delta\vec{\phi} = \tfrac{1}{2}\mu\, \vec{g}$, where $\vec{g}$ is defined in Appendix A, and $\delta A_{(0)ijk} = -\tfrac{1}{2}\mu\, A_{(0)ijk}$, and $\delta\chi = -\mu\, \chi$. These transformations leave the Lagrangians (3.3) and (3.5) invariant. Thus the original

$\mathbb{R}^{20}$ symmetries no longer all commute once we consider the transformations of the new scalar χ. Indeed we have

$$
\begin{aligned}
&[\delta_c, \delta_{c'}] = \delta_k \ , \qquad && k = \tfrac{1}{36} c'_{ijk}\, c_{\ell mn}\, \epsilon^{ijk\ell mn} \ , \\
&[\delta_\mu, \delta_k] = \delta_{\tilde{k}} \ , \qquad && \tilde{k} = \mu\, k \ , \\
&[\delta_\mu, \delta_c] = \delta_{c'} \ , \qquad && c'_{ijk} = \tfrac{1}{2} \mu\, c_{ijk} \ ,
\end{aligned}
\tag{3.8}
$$

The first line of (3.8) implies that there are now only 10 commuting $\mathbb{R}$ symmetries. Note that the factor $1/36$ is purely combinatorial and would correspond to a one if one were to order the indices of c_{ijk}. Without loss of generality, the $\mathbb{R}^{10}$ symmetry may be taken to correspond to the parameters $c_{\alpha\beta 6}$, where we have split the index i into $i = (\alpha, 6)$, with $\alpha = 1, \ldots 5$. The original abelian global $\mathbb{R}^{20}$ symmetry of the original Lagrangian (3.2) for $F_{(4)}$ therefore becomes non-abelian in general, with an abelian $\mathbb{R}^{11}$ left, after the dualisation that replaces $F_{(4)}$ by $G_{(1)}$. Note that the reduction of the abelian symmetry $\mathbb{R}^{20} \to \mathbb{R}^{11}$ is already seen in (3.3) at the stage when the Lagrange multiplier χ is first introduced, even before the field $F_{(4)}$ is eliminated using its algebraic equation of motion. In this first-order formulation, $F_{\mu\nu\rho\sigma}$ is no longer viewed as a field strength; rather it is an auxiliary field that can be integrated out to give rise to the dualised Lagrangian (3.5), and this route to the dual action forbids bare potentials $A_{\mu\nu\rho}$.

So far we have restricted our discussion only to the Lagrangian for the 4-form field strength. As we discussed earlier, before dualisation the symmetry group of the full scalar Lagrangian is $GL(6, \mathbb{R}) \ltimes \mathbb{R}^{20}$. There is in fact a maximal abelian $\mathbb{R}^9$ subalgebra in $SL(6, \mathbb{R})$, but this symmetry does not commute with the $\mathbb{R}^{20}$. After dualisation, we see that the commuting $\mathbb{R}^{20}$ symmetry is reduced to $\mathbb{R}^{10}$, because of the $\mathcal{L}_{FFA}$ terms. This raises the possibility that some of the $\mathbb{R}^9$ symmetry in $SL(6, \mathbb{R})$ might commute with the remaining $\mathbb{R}^{10}$ symmetry. To see that this indeed occurs, let us denote the $SL(6, \mathbb{R})$ transformation parameters by $\Lambda^i{}_j$. Since the axions $A_{(0)ijk}$ transform covariantly under $SL(6, \mathbb{R})$, we have

$$
\begin{aligned}
[\delta_\Lambda, \delta_{\Lambda'}] &= \delta_{\tilde{\Lambda}} \ , \qquad && \tilde{\Lambda}^i{}_j = \Lambda^i{}_k\, \Lambda'^k{}_j - \Lambda'^i{}_k\, \Lambda^k{}_j \ , \\
[\delta_c, \delta_\Lambda] &= \delta_{\tilde{c}} \ , \qquad && \tilde{c}_{ijk} = 3\, \Lambda^\ell{}_{[i} c_{jk]\ell} \ .
\end{aligned}
\tag{3.9}
$$

It is straightforward to verify that (3.8) and (3.9) generate the complete Borel subalgebra of E_6, (restricting the algebra of $SL(6, R)$ to its Borel subalgebra.) Note that the transformations associated with $\Lambda^\alpha{}_6$ commute with themselves, as well as with those of $c_{\beta\gamma 6}$. In addition, the shift symmetry k commutes with $SL(6, \mathbb{R})$ since the corresponding axion

χ is a singlet under $SL(6,\mathbb{R})$. Thus in the fully dualised $D=5$ supergravity, the theory contains $10+1+5=16$ commuting $\mathbb{R}$ symmetries, corresponding to the parameters

$$c_{\alpha\beta 6}\ , \qquad k\ , \qquad \Lambda^\alpha{}_6\ , \tag{3.10}$$

(These commuting $\mathbb{R}$ symmetries imply that the corresponding axions, namely $A_{(0)\alpha\beta 6}$, χ and $\mathcal{A}^\alpha_{(0)6}$, can be covered by derivatives simultaneously in the Lagrangian.) This result is not an accident. In section 4, we shall show that the scalar Lagrangian of the fully-dualised five-dimensional supergravity has an E_6 global symmetry, and the commuting subset in (3.10) precisely corresponds to the maximal abelian $\mathbb{R}^{16}$ subalgebra of E_6 (recall that here, as in the rest of the paper, we are referring to the maximally non-compact forms (also called split real forms) of our Lie algebras, namely $E_{6(6)}$ in this case). We may insist that maximal abelian subalgebra stands for maximal abelian subalgebra of maximal dimension. Clearly we must stay away from the requirement of ad-semisimplicity of its generators that characterise Cartan tori; in a sense we are looking for maximally nilpotent generators instead.

3.2 Dualisation in $D=4$ supergravity

In $D=4$ dimensions, there are seven 2-form gauge potentials $A_{(2)i}$, which can be dualised to give additional axionic scalar fields. In the undualised form, the theory possesses a $GL(7,\mathbb{R})\ltimes\mathbb{R}^{35}$ global symmetry, with the $\mathbb{R}^{35}$ realised by the transformations $\delta A_{(0)ijk} = c_{ijk}$. We shall now show how, owing to the presence of the $\mathcal{L}_{FFA}$ terms in the Lagrangian, the $\mathbb{R}$ symmetries are modified by the dualisation. The subsector of the Lagrangian involving the 2-form gauge potentials, which eventually turn into axionic scalars in the dualised theory, is given by (see Appendix A)

$$\mathcal{L} = -\tfrac{1}{12} e \sum_{i=1}^{7} e^{\vec{a}_i\cdot\vec{\phi}} (F_{(3)i})^2 - \tfrac{1}{72} A_{(0)ijk}\, dA_{(0)\ell mn}\wedge dA_{(2)p}\, \epsilon^{ijk\ell mnp}\ , \tag{3.11}$$

where the associated field strengths $F_{(3)i}$ have the Kaluza-Klein non-linear modifications

$$F_{(3)i} = \gamma^j{}_i\, dA_{(2)j} + \cdots \tag{3.12}$$

with $\gamma^i{}_j$ given by (A.13).

Multiplying (3.12) by $\tilde\gamma^i{}_k$, the inverse matrix defined in (A.14), and taking an exterior derivative, we see that the Bianchi identities for the field strengths $F_{(3)i}$ are given by

$$d(\tilde\gamma^i{}_k\, F_{(3)i}) = 0 \tag{3.13}$$

modulo non-scalar terms. They can be enforced by introducing seven Lagrange multipliers χ^i, leading to the first-order Lagrangian

$$\begin{aligned} \mathcal{L} &= -\tfrac{1}{12} e \sum_{i=1}^{7} e^{\vec{a}_i\cdot\vec{\phi}} (F_{(3)i})^2 - \tfrac{1}{432} e\, \epsilon^{ijk\ell mnp}\, A_{(0)ijk}\, \partial_\mu A_{(0)\ell mn}\, \tilde\gamma^q{}_p\, F_{\nu\rho\sigma q}\, \epsilon^{\mu\nu\rho\sigma} \\ &\qquad -\tfrac{1}{6} e\, \partial_\mu \chi^i\, \tilde\gamma^j{}_i\, F_{\nu\rho\sigma j}\, \epsilon^{\mu\nu\rho\sigma}\ . \end{aligned} \tag{3.14}$$

It is now easy to solve the algebraic equations of motion for $F_{(3)i} = e^{-\vec{a}_i\cdot\vec{\phi}} * G^i_{(1)}$, and to substitute this back into the Lagrangian, giving

$$\mathcal{L} = -\tfrac{1}{2} e \sum_{i}^{7} e^{-\vec{a}_i\cdot\vec{\phi}} (G^{(i)}_{(1)})^2\ , \tag{3.15}$$

where

$$G^i_{(1)} = \tilde\gamma^i{}_j \Big(d\chi^j + \tfrac{1}{72} A_{(0)k\ell m}\, dA_{(0)npq}\, \epsilon^{jk\ell mnpq}\Big)\ . \tag{3.16}$$

As in the previous five-dimensional case, here the original $\mathbb{R}^{35}$ and the scaling $\mathbb{R}_s$ global symmetry of the undualised Lagrangian (3.11) become

$$\begin{aligned} \delta A_{(0)ijk} &= c_{ijk} - \tfrac{1}{3}\mu\, A_{(0)ijk}\ , \qquad \delta\chi^i = k^i - \tfrac{1}{72} c_{jk\ell}\, A_{(0)mnp}\, \epsilon^{ijk\ell mnp} - \tfrac{2}{3}\mu\, \chi^i\ , \\ \delta\vec{\phi} &= \tfrac{1}{3}\mu\, \vec{g}\ . \end{aligned} \tag{3.17}$$

The transformations associated with the c_{ijk} no longer all commute if one now includes the action on the 7 new scalars, since

$$[\delta_c, \delta_{c'}] = \delta_k\ , \qquad k^i = \tfrac{1}{36} c_{jk\ell}\, c'_{mnp}\, \epsilon^{ijk\ell mnp}\ . \tag{3.18}$$

The maximal commuting subset is the $\mathbb{R}^{15}$ corresponding, for example, to the parameters $c_{\alpha\beta 7}$, where the index $i = (\alpha, 7)$ with $\alpha = 1, 2, \ldots, 6$. Since δ_k commutes with δ_c, the scalar Lagrangian (3.15) now has an $\mathbb{R}^{15+7} = \mathbb{R}^{22}$ global symmetry.

Of course this is not the whole story. The full undualised Lagrangian has also a $SL(7,\mathbb{R})$ global symmetry, which itself contains a maximal abelian subalgebra $\mathbb{R}^{12}$. Prior to dualisation, this $\mathbb{R}^{12}$ symmetry does not commute with the $\mathbb{R}^{35}$, since the axions $A_{(0)ijk}$ transform (covariantly) under $SL(7,\mathbb{R})$, with commutation relations of the form given in (3.9). After dualisation, however, the abelian $\mathbb{R}^{35}$ symmetry becomes non-abelian in general with abelian $\mathbb{R}^{15}$ left. Moreover, the $SL(7,\mathbb{R})$ remains unscathed under this full dualisation, and so it becomes possible that some of the maximal abelian $\mathbb{R}^{12}$ subalgebra of $SL(7,\mathbb{R})$ now commutes with $\mathbb{R}^{15}$. Indeed, following the analogue of the analysis we performed in $D = 5$, we find that six of the $SL(7,\mathbb{R})$ abelian transformations, associated with parameters $\Lambda^\alpha{}_7$, commute with the $\mathbb{R}^{15}$ symmetry described by the parameters $c_{\alpha\beta 7}$: As we remarked above, the $\mathbb{R}^7$ symmetries of the seven new axionic scalars coming from the dualisation commute with $\mathbb{R}^{15}$, but they do not all commute with $\delta_{\Lambda^\alpha{}_7}$; only six of them do, namely the six associated with the parameters k^α. To see this we need, in addition to the previously-given commutators of transformations,

$$
\begin{aligned}
[\delta_c, \delta_\Lambda] &= \delta_{\tilde c}\ , \qquad \tilde c_{ijk} = 3\Lambda_{[i}{}^{\ell}\, c_{jk]\ell}\ ,\\
[\delta_\mu, \delta_c] &= \delta_{c'}\ , \qquad c'_{ijk} = \tfrac13 \mu\, c_{ijk}\ ,\\
[\delta_k, \delta_\Lambda] &= \delta_{\tilde k}\ , \qquad \tilde k^i = -\Lambda^i{}_j\, k^j\ ,\\
[\delta_\mu, \delta_k] &= \delta_{\hat k}\ , \qquad \hat k^i = \tfrac23 \mu\, k^i\ .
\end{aligned}
\tag{3.19}
$$

Thus in the full Lagrangian of the dualised theory, we have abelian symmetries with dimensions $15+6+6=27$, associated with the parameters

$$
c_{\alpha\beta 7}\ , \qquad k^\alpha\ , \qquad \Lambda^\alpha{}_7\ . \tag{3.20}
$$

Note that $\mathbb{R}^{27}$ is precisely the maximal abelian subalgebra of E_7. As in $D=5$, the associated axions, $A_{(0)\alpha\beta 7}$, χ^α and $\mathcal{A}^\alpha_{(0)}{}_7$, can be simultaneously covered by derivatives everywhere in the Lagrangian. As in the previous case, the complete algebra of these transformations is a subalgebra of the Lie algebra E_7 which contains the Borel subalgebra of E_7.

3.3 Dualisation in $D=3$ supergravity

Again, we begin by considering the subsector of the scalar Lagrangian coming from the dualisation of higher-degree field strengths. In this case, it is the 8 2-forms $\mathcal{F}^i_{(2)}$ and 28 2-forms $F_{(2)ij}$ that yield additional scalars after dualisation. From (A.4) and (A.16), we see that these appear in the Lagrangian in the form

$$
\mathcal{L} = -\tfrac14 e \sum_i e^{\vec b_i\cdot\vec\phi}\, (\mathcal{F}^i_{(2)})^2 - \tfrac14 e \sum_{i<j} e^{\vec a_{ij}\cdot\vec\phi}\ (F_{(2)ij})^2 - \tfrac{1}{144} \tilde F_{(1)ijk}\wedge \tilde F_{(1)\ell mn}\wedge A_{(1)pq}\, \epsilon^{ijk\ell mnpq}\ , \tag{3.21}
$$

and we must now use the full non-linear Kaluza-Klein modifications, given by (A.12), because all fields are scalar in this three-dimensional theory;

$$
F_{(2)ij} = \gamma^k{}_i\, \gamma^\ell{}_j \Big(dA_{(1)k\ell} - \gamma^m{}_n\, dA_{(0)k\ell m}\wedge \mathcal{A}^n_{(1)}\Big)\ , \tag{3.22}
$$

$$
\mathcal{F}^i_{(2)} = d\mathcal{A}^i_{(1)} - \gamma^j{}_k\, d\mathcal{A}^i_{(0)j}\wedge \mathcal{A}^k_{(1)}\ . \tag{3.23}
$$

The first task before carrying out the dualisations is to express the Bianchi identities associated with $\mathcal{F}^i_{(2)}$ and $F_{(2)ij}$, and the final cubic interaction term in (3.21), purely in terms of the Kaluza-Klein modified field strengths, so that these may then be eliminated algebraically once the Lagrange multipliers enforcing the Bianchi identities are introduced. Using (A.18-A.19), we see that multiplying (3.23) by $\gamma^j{}_i$ gives

$$\gamma^j{}_i\,\mathcal{F}^i_{(2)} = d(\gamma^j{}_i\,\mathcal{A}^i_{(1)})\ , \tag{3.24}$$

and hence

$$d(\gamma^j{}_i\,\mathcal{F}^i_{(2)}) = 0\ . \tag{3.25}$$

Multiplying (3.22) by $\tilde\gamma^i{}_p\,\tilde\gamma^j{}_q$, and taking the exterior derivative, we obtain, after using (3.24),

$$d(\tilde\gamma^i{}_p\,\tilde\gamma^j{}_q\,F_{(2)ij} - A_{(0)pqm}\,\gamma^m{}_n\,\mathcal{F}^n_{(2)}) = 0\ . \tag{3.26}$$

Equations (3.25) and (3.26) are the relevant Bianchi identities, expressed in terms of the Kaluza-Klein modified field strengths. For the final cubic interaction term $\mathcal{L}_{FFA}$ in (3.21), we first use (3.22), multiplied by two inverse γ matrices, to re-express it as

$$\mathcal{L}_{FFA} = \tfrac{1}{144}\Big(dA_{(0)ijk}\,A_{(0)\ell mn}\,\tilde\gamma^r{}_p\,\tilde\gamma^s{}_q\wedge F_{(2)rs} + dA_{(0)ijk}\,A_{(0)\ell mn}\wedge dA_{(0)pqr}\wedge\gamma^r{}_s\,\mathcal{A}^s_{(1)}\Big)\epsilon^{ijk\ell mnpq}\ . \tag{3.27}$$

For the second term in this expression, we use the Schouten identity that a total antisymmetrisation over the nine indices $ijk\ell mnpqr$ vanishes to show that we may write

$$dA_{(0)ijk}\,A_{(0)\ell mn}\wedge dA_{(0)pqr}\,\epsilon^{ijk\ell mnpq} = -\tfrac13 d\Big(dA_{(0)ijk}\,A_{(0)\ell mn}\,A_{(0)pqr}\,\epsilon^{ijk\ell mnpq}\Big)\ . \tag{3.28}$$

This implies, after an integration by parts and using (3.24), that $\mathcal{L}_{FFA}$ can also be written purely in terms of the modified field strengths, as

$$\mathcal{L}_{FFA} = \tfrac{1}{144}\Big(dA_{(0)ijk}\,A_{(0)\ell mn}\,\tilde\gamma^r{}_p\,\tilde\gamma^s{}_t\wedge F_{(2)rs} + \tfrac13 dA_{(0)ijk}\,A_{(0)\ell mn}\,A_{(0)pqr}\gamma^r{}_s\wedge\mathcal{F}^s_{(2)}\Big)\epsilon^{ijk\ell mnpq}\ . \tag{3.29}$$

It is now a straightforward matter, after introducing Lagrange multipliers λ_i and λ^{ij} to enforce the Bianchi identities (3.25) and (3.26) respectively, to show that the Lagrangian (3.21) becomes, upon elimination of the original 2-form fields by means of their algebraic equation of motion,

$$\mathcal{L} = -\tfrac12 e\sum_i e^{-\vec b_i\cdot\vec\phi}\,(G_{(1)i})^2 - \tfrac12 e\sum_{i<j} e^{-\vec a_{ij}\cdot\vec\phi}\,(G^{ij}_{(1)})^2\ , \tag{3.30}$$

where the dualised field strengths $G_{(1)i}$ and $G^{ij}_{(1)}$ are given by

$$\begin{aligned} G_{(1)i} &= \gamma^j{}_i\,\Big(d\lambda_j - \tfrac12 A_{(0)jk\ell}\,d\lambda^{k\ell} - \tfrac{1}{432}dA_{(0)k\ell m}\,A_{(0)npq}\,A_{(0)rsj}\,\epsilon^{k\ell mnpqrs}\Big)\ , &(3.31)\\ G^{ij}_{(1)} &= \tilde\gamma^i{}_k\,\tilde\gamma^j{}_\ell\,\Big(d\lambda^{k\ell} + \tfrac{1}{72}dA_{(0)mnp}\,A_{(0)qrs}\,\epsilon^{k\ell mnpqrs}\Big)\ . &(3.32)\end{aligned}$$

Thus the original commuting $\mathbb{R}^{56}$ and $\mathbb{R}_s$ symmetry $\delta A_{(0)ijk} = c_{ijk}$ now become

$$\begin{aligned} \delta A_{(0)ijk} &= c_{ijk} - \tfrac{1}{6}\mu\, A_{(0)ijk}\ , \qquad \delta\lambda^{ij} = \kappa^{ij} - \tfrac{1}{72} A_{(0)k\ell m}\, c_{npq}\, \epsilon^{ijk\ell mnpq} - \tfrac{1}{3}\mu\,\lambda^{ij}\ , \\ \delta\lambda_i &= \epsilon_i + \tfrac{1}{2} c_{ijk}\,\lambda^{jk} - \tfrac{1}{432} A_{(0)ijk}\, A_{(0)\ell mn}\, c_{pqr}\, \epsilon^{jk\ell mnpqr} - \tfrac{1}{2}\mu\lambda_i\ , \\ \delta\vec{\phi} &= \tfrac{1}{6}\mu\,\vec{g}\ . \end{aligned} \tag{3.33}$$

These transformations have the following non-trivial commutation relations

$$\begin{aligned} [\delta_c, \delta_{c'}] &= \delta_\kappa\ , \qquad & \kappa^{ij} &= -\tfrac{1}{36} c_{k\ell m}\, c'{}_{npq}\, \epsilon^{ijk\ell mnpq}\ , \\ [\delta_\kappa, \delta_c] &= \delta_\epsilon\ , & \epsilon_i &= \tfrac{1}{2} c_{ijk}\,\kappa^{jk}\ , \\ [\delta_\mu, \delta_\epsilon] &= \delta_{\epsilon'}\ , & \epsilon'_i &= \tfrac{1}{2}\mu\,\epsilon_i\ , \\ [\delta_\mu, \delta_\kappa] &= \delta_{\kappa'}\ , & \kappa'^{ij} &= \tfrac{1}{3}\mu\,\kappa^{ij}\ , \\ [\delta_c, \delta_\Lambda] &= \delta_{\tilde{c}}\ , & \tilde{c}_{ijk} &= 3\Lambda_{[i}{}^{\ell}\, c_{jk]\ell}\ , \\ [\delta_\mu, \delta_c] &= \delta_{c'}\ , & c'_{ijk} &= \tfrac{1}{6}\mu\, c_{ijk}\ , \end{aligned} \tag{3.34}$$

together with the standard $SL(8,\mathbb{R})$ transformations, since $A_{(0)ijk}$, λ^{ij} and λ_i all transform covariantly under $SL(8,\mathbb{R})$. We see that the transformations described by the parameters c_{ijk} no longer all commute, and the maximal set of commuting $\mathbb{R}$ symmetries is associated with the parameters

$$\kappa^{ij}\ , \qquad \epsilon_i\ , \tag{3.35}$$

corresponding to the commuting shift symmetries of all the $36 = 28 + 8$ new axionic scalars coming from the dualisation. Note that this $\mathbb{R}^{36}$ symmetry is precisely the maximal abelian subalgebra of E_8, which is unique up to conjugation. Once more the normalisations have been chosen to include appropriate symmetry factors that correspond to structure constants ± 1.

4 Coset structure of scalar Lagrangians and their symmetries

In this section, we examine the complete scalar Lagrangians for all maximal supergravities in $D \le 11$ dimensions. These scalars include the $(11-D)$ dilatons $\vec{\phi} = (\phi_1, \phi_2, \ldots, \phi_{11-D})$, the $\frac{1}{2}(11-D)(10-D)$ axions $\mathcal{A}^i_{(0)j}$ and the $\frac{1}{6}(11-D)(10-D)(9-D)$ axions $A_{(0)ijk}$. In addition, in dimensions $D \le 5$ there is an option, as we saw in the previous section, to include further axions obtained as the potentials for the duals of $(D-1)$-form field strengths.

The construction and the analysis of the symmetries of the scalar Lagrangians turns out to be very simple in the field variables that we are using here, which arise from the step-by-step Kaluza-Klein reduction from $D = 11$. Let us first consider the dimensions $D \ge 6$, where there are no complications coming from the possibility of further axions arising from the dualisation of higher-degree field strengths. The key observation, which is shown by elementary computation using the definitions given in Appendix A, is that in each dimension $D \ge 6$ the set of dilaton vectors $\vec{b}_{ij}$ and $\vec{a}_{ijk}$ for the corresponding axions $\mathcal{A}^i_{(0)j}$ and $A_{(0)ijk}$ are in one-to-one correspondence with the positive-root vectors of the E_{11-D} algebra. In fact it is easy to see from (A.5) and (A.8) that one can take $\vec{b}_{i,i+1}$ and $\vec{a}_{123}$ (for $D \le 8$) to be the simple roots, with all others generated as sums of these with non-negative integer coefficients. Furthermore, in the dimensions $D = 5$, $D = 4$ or $D = 3$, one can also easily see from (A.5) and (A.8) that if the dilaton vectors $-\vec{a}$, $-\vec{a}_i$, or $(-\vec{b}_i, -\vec{a}_{ij})$ respectively, corresponding to the axions coming from the dualisations discussed in the previous section, are included then the entire sets of dilaton vectors for all the axions coincide with the positive roots of E_6, E_7 and E_8 respectively. Again, the simple roots can be taken to be $\vec{b}_{i,i+1}$ and $\vec{a}_{123}$. The former are the simple roots of $SL(11-D, \mathbb{R})$, and the latter, which arises after the appropriate Weyl rescaling, is the seed of E_{11-D}. Thus in all dimensions we may summarise the information about the dot products of the dilaton vectors for the full sets of axions by the Dynkin diagram:

$$
\begin{array}{ccccccccccccc}
\vec{b}_{12} & & \vec{b}_{23} & & \vec{b}_{34} & & \vec{b}_{45} & & \vec{b}_{56} & & \vec{b}_{67} & & \vec{b}_{78} \\
\mathrm{o} & — & \mathrm{o} & — & \mathrm{o} & — & \mathrm{o} & — & \mathrm{o} & — & \mathrm{o} & — & \mathrm{o} \\
 & & & & | & & & & & & & & \\
 & & & & \mathrm{o} & & & & & & & & \\
 & & & & \vec{a}_{123} & & & & & & & &
\end{array}
$$

Diagram 1: The dilaton vectors $\vec{b}_{i,i+1}$ with $i \le n-1$ and $\vec{a}_{123}$ generate the E_n Dynkin diagram

In each dimension D, the diagram is truncated to the part that survives when only the simple roots with indices $i \le 11-D$ are retained. This defines the simple root that has to to be removed in order to disintegrate E_{11-D} to E_{10-D}. Note that the undualised versions of supergravities discussed in section 2 have symmetry $SL(11-D, \mathbb{R})$, corresponding to removing the root $\vec{a}_{123}$. In section 7, we shall discuss the case of R-R dualisation, where all R-R fields are dualised when this results in fields of lower degrees. In this case the symmetry group contains $O(10-D, 10-D)$ as a subgroup, corresponding to removing the simple root

$\vec{b}_{12}$. In section 9, we shall discuss the direct dimensional reduction of the type IIB theory, where no dualisations are performed. These cases correspond to removing both $\vec{b}_{23}$ and $\vec{a}_{123}$, and hence they contain $SL(2,R)\times SL(9-D,R)$ as subgroups. For future convenience, we shall denote the simple roots $(\vec{a}_{123},\vec{b}_{12},\vec{b}_{23},\ldots,\vec{b}_{78})$ by $\vec{r}_i$ with $i=(1,2\ldots,8)$.

In the next two subsections, we shall prove that the scalar Lagrangians of toroidally-compactified eleven-dimensional supergravities have E_{11-D} global symmetries, after all the $(D-2)$-form potentials are dualised to give rise to the maximal number of scalar fields, and after the canonical Weyl rescaling. Our strategy will be to give $K(E_{11-D})\backslash E_{11-D}$ coset constructions of scalar Lagrangians, and then to show that these coincide precisely with the scalar Lagrangians of the fully-dualised supergravities that we obtained in section 3. The occurrence of the global E_{11-D} symmetries was conjectured in [10, 11], based on the structure of the scalar fields in each dimension. In fact, maximal supergravities with these symmetries have been obtained in all dimensions, but in general by direct construction, rather than by dimensional reduction from $D=11$ (see, for example, [18]). However, a complete proof that they can be obtained from eleven-dimensional supergravity has been given only for the cases of $D=9$ [19], $D=4$ [10] and $D=3$ [20, 21].

4.1 Scalar manifolds in $D\geq 6$

First let us consider the dimensions $D\geq 6$, where we just have the axions $\mathcal{A}^i_{(0)j}$ and $A_{(0)ijk}$ associated with the dilaton vectors $\vec{b}_{ij}$ and $\vec{a}_{ijk}$ respectively. Since these are given by (A.5)

$$\vec{b}_{ij}=-\vec{f}_i+\vec{f}_j\ ;\qquad \vec{a}_{ijk}=\vec{f}_i+\vec{f}_j+\vec{f}_k-\vec{g}\ ,\tag{4.1}$$

it follows immediately that

$$\vec{b}_{ij}+\vec{b}_{jk}=\vec{b}_{ik}\ ,\qquad \vec{a}_{ijk}+\vec{b}_{i\ell}=\vec{a}_{\ell jk}\ .\tag{4.2}$$

Defining the generators associated with the positive roots $\vec{b}_{ij}$ and $\vec{a}_{ijk}$ as $E_i{}^j$ and E^{ijk} respectively, we see from (4.2) that they will obey commutation relations of the form

$$\begin{aligned}
[E_i{}^j,E_k{}^\ell] &= \delta^j_k\,E_i{}^\ell-\delta^\ell_i\,E_k{}^j\ , &(4.3)\\
[E_\ell{}^m,E^{ijk},] &= -3\delta^{[i}_\ell\,E^{|m|jk]}\ , &(4.4)\\
[E^{ijk},E^{\ell mn}] &= 0\ , &(4.5)
\end{aligned}$$

where it is understood that $E_i{}^j$ is defined only for $i<j$, while E^{ijk} is totally antisymmetric and is defined for any ordering of its indices. Note that the commutators (4.3) and (4.4) arise in all dimensions: (4.3) describes the positive-root (nilpotent) subalgebra of $SL(11-D,\mathbb{R})$,

while (4.4) reflects the fact that E^{ijk} transforms covariantly under $SL(11-D,\mathbb{R})$. The third commutator (4.5) is dimension dependent. For $D\geq 6$, which we are currently considering, the generators E^{ijk} commute. For $D\leq 5$, whether they commute or not depends on whether we perform dualisations or not. We shall discuss this later, case by case. Note also that if we include the $\mathbb{R}_s$ factor of $GL(11-D,\mathbb{R})$ and write the set of Cartan generators as a vector $\vec{H}$, then we will have

$$[\vec{H}, E_i{}^j] = \vec{b}_{ij}\, E_i{}^j\ , \qquad [\vec{H}, E^{ijk}] = \vec{a}_{ijk}\, E^{ijk} \qquad \text{no sum}\ . \tag{4.6}$$

In fact having defined the positive-root subalgebra, the summation rules (4.2) for the dilaton vectors come as no surprise, since they are then a direct consequence of the Jacobi identities of this subalgebra.

Now define $\mathcal{V}=\mathcal{V}_1\,\mathcal{V}_2\,\mathcal{V}_3$, with

$$\begin{aligned}\mathcal{V}_1 &= e^{\frac12\vec{\phi}\cdot\vec{H}}\ ,\\ \mathcal{V}_2 &= \prod_{i<j} U_{ij} = \cdots U_{24}\,U_{23}\cdots U_{14}\,U_{13}\,U_{12}\ ,\\ \mathcal{V}_3 &= e^{\sum_{i<j<k} A_{(0)ijk}\,E^{ijk}}\ ,\end{aligned} \tag{4.7}$$

where

$$U_{ij}\equiv e^{\mathcal{A}^i_{(0)j}\,E_i{}^j} \qquad (\text{no sum})\ , \tag{4.8}$$

and the right-hand-side of $\mathcal{V}_2$ is defined with anti-lexical ordering, as indicated in (4.7). It follows that

$$d\mathcal{V}\,\mathcal{V}^{-1} = d\mathcal{V}_1\,\mathcal{V}_1^{-1} + \mathcal{V}_1\,(d\mathcal{V}_2\,\mathcal{V}_2^{-1})\,\mathcal{V}_1^{-1} + \mathcal{V}_1\,\mathcal{V}_2\,(d\mathcal{V}_3\,\mathcal{V}_3^{-1})\,\mathcal{V}_2^{-1}\,\mathcal{V}_1^{-1}\ . \tag{4.9}$$

One can now see from the commutation relations above that

$$\begin{aligned}d\mathcal{V}_1\,\mathcal{V}_1^{-1} &= \tfrac12 d\vec{\phi}\cdot\vec{H}\\ d\mathcal{V}_2\,\mathcal{V}_2^{-1} &= \sum_{i<j}\mathcal{F}^i_{(1)j}\,E_i{}^j\ ,\\ d\mathcal{V}_3\,\mathcal{V}_3^{-1} &= \sum_{i<j<k} dA_{(0)ijk}\,E^{ijk}\ ,\end{aligned} \tag{4.10}$$

where $\mathcal{F}^i_{(1)j}$ are the fully Kaluza-Klein modified field strengths given by

$$\mathcal{F}^i_{(1)j} = \gamma^k{}_j\, d\mathcal{A}^i_{(0)k}\ , \tag{4.11}$$

and $\gamma^i{}_j$ is given in (A.13). (We have made use of the relation $de^X\,e^{-X} = dX + \frac12[X,dX] + \frac16[X,[X,dX]]+\cdots$, where, because of the nature of the parameterisation of $\mathcal{V}$, we need only the first term in this series. Also, we need $e^X\,Y\,e^{-X} = Y + [X,Y] + \frac{1}{2}[X,[X,Y]] + \cdots$.) Note that in terms of the fundamental representation of $E_i{}^j$, we have $(\mathcal{V}_2)^i{}_j = \tilde\gamma^i{}_j$ and $(\mathcal{V}_2^{-1})^i{}_j = \gamma^i{}_j$. The anti-lexical ordering in $\mathcal{V}_2$ is crucial to the above derivation. This in particular implies that $(\mathcal{V}_2)^i{}_j$ becomes linear in $\mathcal{A}^i_{(0)j}$ and makes the proof of (4.10) straightforward.

Conjugating the expression for $d\mathcal{V}_2\,\mathcal{V}_2^{-1}$ with $\mathcal{V}_1$, we find

$$\mathcal{V}_1\,(d\mathcal{V}_2\,\mathcal{V}_2^{-1})\,\mathcal{V}_1^{-1} = \sum_{i<j} e^{\frac{1}{2}\vec b_{ij}\cdot\vec\phi}\,\mathcal{F}^i_{(1)j}\,E_i{}^j\ . \tag{4.12}$$

Also, we can see that

$$\mathcal{V}_2\,(d\mathcal{V}_3\,\mathcal{V}_3^{-1})\,\mathcal{V}_2^{-1} = \sum_{i<j<k} F_{(1)ijk}\,E^{ijk}\ , \tag{4.13}$$

where $F_{(1)ijk}$ are the fully modified field strengths given by

$$F_{(1)ijk} = \gamma^\ell{}_i\,\gamma^m{}_j\,\gamma^n{}_k\,dA_{(0)\ell mn}\ . \tag{4.14}$$

After conjugating (4.13) with $\mathcal{V}_1$, and adding together all the terms in (4.9), we obtain the result

$$d\mathcal{V}\,\mathcal{V}^{-1} = \tfrac{1}{2}d\vec\phi\cdot\vec H + \sum_{i<j} e^{\frac{1}{2}\vec b_{ij}\cdot\vec\phi}\,\mathcal{F}^i_{(1)j}\,E_i{}^j + \sum_{i<j<k} e^{\frac{1}{2}\vec a_{ijk}\cdot\vec\phi}\,F_{(1)ijk}\,E^{ijk}\ . \tag{4.15}$$

It follows from this that the entire scalar Lagrangian in $D\ge 6$ is expressible as

$$\begin{aligned}\mathcal{L} &= \tfrac14 e\,\mathrm{tr}\Big(\partial\mathcal{M}^{-1}\,\partial\mathcal{M})\Big)\ , &(4.16)\\ &= -\tfrac12 e\,\mathrm{tr}\Big(\partial\mathcal{V}\,\mathcal{V}^{-1}\,(\partial\mathcal{V}\,\mathcal{V}^{-1})^{\mathrm{T}}\Big) - \tfrac12 e\,\mathrm{tr}\Big(\partial\mathcal{V}\,\mathcal{V}^{-1}\,(\partial\mathcal{V}\,\mathcal{V}^{-1})\Big)\ . &(4.17)\end{aligned}$$

where we have defined the internal metric

$$\mathcal{M} := \mathcal{V}^{\mathrm{T}}\,\mathcal{V}\ , \tag{4.18}$$

and the superscript "T" denotes the transpose. The first two normalisations below are imposed by (4.16):

$$\mathrm{tr}(H_i\,H_j) = 2\delta_{ij}\ ,\qquad \mathrm{tr}(E_i{}^j\,E_k^{T\ell}) = \delta_{ik}\,\delta^{j\ell}\ ,\qquad \mathrm{tr}(E^{ijk}\,E^{T\ell mn}) = 6\delta^i_{[\ell}\,\delta^j_m\,\delta^k_{n]}\ , \tag{4.19}$$

where in the second expression it is understood that $i<j$ and $k<\ell$. The last normalisation is at our disposal provided we respect the $SL(11-D,\mathbb{R})$ covariance but the relative normalisation of the first two terms (which form the Casimir invariant in the defining representation of $SL(11-D,\mathbb{R})$) is dictated by the obvious invariance under that group. The present choice of normalisation is canonical for simply laced groups and leads to the most symmetric expressions. Also, we have

$$\mathrm{tr}(E_i{}^j\, E_k{}^\ell) = 0 \qquad \mathrm{tr}(E^{ijk}\, E^{\ell mn}) = 0\ , \tag{4.20}$$

again for $i < j$, $k < \ell$. Comparison shows that the Lagrangian obtained by substituting (4.15) into (4.17) is identical to the scalar sector of the D-dimensional supergravity Lagrangian given in (A.4), when $D \geq 6$. Using the manifest $SL(11-D, \mathbb{R})$ invariance one only has to adjust the coefficient of the exponent in $\mathcal{V}_3$ to establish the invariance. The commutativity of the extra generators (4.5) allows for this arbitrary rescaling. Of course one still makes use of the presence of the antisymmetric third order tensor representation and the special Weyl rescaling that modifies the $\mathbb{R}_s$ subgroup of $GL(11-D, \mathbb{R})$ to exhibit the E_{11-D} symmetry in the scalar sector.

Having written the scalar Lagrangian in the form (4.17), its global symmetry is now made manifest. If U is a constant matrix in the global symmetry group, we can send

$$\mathcal{V} \longrightarrow \mathcal{V}'' = \mathcal{V}\, U\ , \tag{4.21}$$

which leaves $d\mathcal{V}\, \mathcal{V}^{-1}$ invariant. This takes us out of the "positive-root" gauge, in that $\mathcal{V}''$ is no longer expressible in the form (4.7). We now define $\mathcal{V}'$ by

$$\mathcal{V}' = \mathcal{O}\, \mathcal{V}\, U\ , \tag{4.22}$$

where $\mathcal{V}'$ *is* in the positive-root gauge (4.7), and $\mathcal{O}$ is some appropriate compensating element of the maximal compact subgroup (the denominator group), for which $\mathcal{O}^{\mathrm{T}}\, \mathcal{O} = 1$. Then under the compensated transformation, we have

$$\mathcal{M} \longrightarrow \mathcal{M}' = U^{\mathrm{T}}\, \mathcal{V}^{\mathrm{T}}\, \mathcal{O}^{\mathrm{T}}\, \mathcal{O}\, \mathcal{V}\, U = U^{\mathrm{T}}\, \mathcal{V}^{\mathrm{T}}\, \mathcal{V}\, U = U^{\mathrm{T}}\, \mathcal{M}\, U\ , \tag{4.23}$$

which is easily seen to leave (4.16) invariant. Since in $D \geq 6$ the dilaton vectors have been established to correspond to the positive roots of the E_{11-D} algebra, it follows that this is the global symmetry group of the scalar Lagrangian. We may restore the $K(E_{11-D})$ local gauge invariance to streamline some formulas because the Lagrangian is built out of $K(E_{11-D})$ invariants.

So far, we have obtained cosets $K(E_{11-D})\backslash E_{11-D}$ for $D \geq 6$. We constructed them using the Borel subgroups of E_{11-D}, which are generated by the positive-root and Cartan generators. The scalars are the parameters for these generators, with the axions associated with positive-root generators and dilatons associated with the Cartan generators. (The Iwasawa decomposition and the Borel subgroups of the global supergravity symmetry groups

E_{11-D} were extensively studied in [22, 23, 24].) We showed that these scalar cosets are precisely the same as the scalar manifolds obtained from eleven-dimensional supergravity by dimensional reduction to D dimensions. The cosets have manifest $SL(11-D)$ symmetries, since all the generators carry $SL(11-D)$ indices. In Appendix C, we give an explicit construction of these scalar cosets with manifest E_{11-D} invariance, for $D \geq 6$.

Before we go to the next subsection to study the coset structure of the scalar manifolds for $D \leq 5$, there are some group theoretical points that need to be addressed. The structure of the matrix $\mathcal{M}$ defined in (4.18) is correct for $D \geq 6$, owing to the nature of the E_{11-D} groups in these cases, but it is not as it stands precisely applicable in lower dimensions. For a generic non-compact Lie group G, one has $\mathcal{M} = \tau(\mathcal{V}^{-1})\,\mathcal{V}$, where τ is the Cartan involution whose fixed point set is the maximal compact subgroup of G. For the groups $SL(N, \mathbb{R})$ or $SO(N,N)$, where the maximal compact subgroups are $SO(N)$ and $SO(N) \times SO(N)$ respectively, we simply have $\tau(\mathcal{V}^{-1}) = \mathcal{V}^{\rm T}$. That is why we could use ordinary transpose "T" to discuss the coset structure for $D \geq 6$. For E_6 in $D = 5$, we have in the fundamental representation: $\mathcal{M} = (\Omega \otimes \Omega)\,\mathcal{V}^\dagger\,\mathcal{V} = \mathcal{V}^T(\Omega \otimes \Omega)\,\mathcal{V}$, where following the second reference of [11] Ω is the constant invariant symplectic matrix in $USp(8)$, which is the maximal compact subgroup of E_6. Another example is E_7, with $SU(8)$ as its maximal compact subgroup. In this case we have $\tau(\mathcal{V}^{-1}) = \mathcal{V}^\dagger$. For convenience, we shall introduce a "generalised transpose" #, defined as $\mathcal{V}^\# = \tau(\mathcal{V}^{-1})$. Then the formulae obtained in this section become applicable to general dimensions simply by replacing the transpose "T" by the generalised transpose "#" everywhere, and in particular the scalar Lagrangian (4.17) becomes

$$\mathcal{L} = -\tfrac{1}{2} e\, {\rm tr}\Big(\partial\mathcal{V}\,\mathcal{V}^{-1}\,(\partial\mathcal{V}\,\mathcal{V}^{-1})^\#\Big) - \tfrac{1}{2} e\, {\rm tr}\Big(\partial\mathcal{V}\,\mathcal{V}^{-1}\,(\partial\mathcal{V}\,\mathcal{V}^{-1})\Big)\,. \tag{4.24}$$

Acting on the generators $E_i{}^j$ and E^{ijk} corresponding to the positive roots, the Cartan involution has the effect of turning them into minus their conjugate negative-root generators, $E^{\#}_i{}^j$ and $E^{\#ijk}$, and changes the sign of the Cartan generators. (In other words, τ sends $(E^+, E^-, \vec H) \to (-E^-, -E^+, -\vec H)$, where E^+ and E^- denote the sets of positive and negative root generators. This implies that all the compact generators, $(E^+ - E^-)$, are left unchanged, while all the non-compact generators, $(E^+ + E^-)$ and $\vec H$, are reversed in sign.) Note that in this positive-root system, $E_i{}^j$, and hence also their conjugate generators $E^{\#}_i{}^j$, are defined only for $i < j$.

It turns out that the maximal duality symmetries of supergravities are real Lie groups in split form (maximally noncompact real form with Cartan subalgebra that may be chosen along the noncompact directions). For all these real forms the Cartan involutions follow exactly the pattern we have just discussed.

4.2 Maximal scalar manifolds in $D = 5, 4, 3$

The above discussion was for the cases $D \geq 6$, where there are no extra axions coming from dualisations of higher-rank field strengths. In $D \leq 5$, there can be additional scalars coming from the dualisations of $(D-2)$-form potentials. In this subsection, we shall consider the cases where the maximal numbers of such dualisations are performed, leading to scalar manifolds with global E_6, E_7 and E_8 symmetries in $D = 5, 4$ and 3. The essential point is that again all the axions, including those obtained by dualisation, have dilaton vectors corresponding to the positive roots of the E_{11-D} algebra, and all the dilatonic scalars are associated with the Cartan generators. In fact the manifestly $SL(11-D, \mathrm{I\!R})$-covariant expressions that initially arise in the theories are nothing but the decompositions of E_{11-D} covariant expressions with respect to the $SL(11-D, \mathrm{I\!R})$ subgroup. Thus we again have a very simple way of expressing the coset representatives in a straightforward generalisation of the previous parameterisation $\mathcal{V} = \mathcal{V}_1\, \mathcal{V}_2\, \mathcal{V}_3$ used above, leading eventually to an expression for the scalar Lagrangian in the form (4.17), implying that it has a manifest E_{11-D} global symmetry.

First, consider $D = 5$. We have, in addition to the commutation relations implied by (4.2) that

$$\vec{a}_{ijk} + \vec{a}_{\ell mn} = -\vec{a}\ , \tag{4.25}$$

when $ijk\ell mn$ are all different. As we saw in the previous section, $-\vec{a}$ is indeed the dilaton vector associated with the axion χ obtained from the dualisation of $F_{(4)}$. Introducing a new generator D, associated with this extra axion, we see that it will satisfy the commutation relations

$$[E^{ijk}, E^{\ell mn}] = -\epsilon^{ijk\ell mn}\, D\ , \qquad [\vec{H}, D] = -\vec{a}\, D\ , \qquad [E_i{}^j, D] = 0\ , \qquad [E^{ijk}, D] = 0\ , \tag{4.26}$$

while the remaining commutators are unchanged from their previous form. The normalisation of the extra generator D is chosen to be canonical, as in the case of the other positive-root generators, namely $\mathrm{tr}(D^{\#}\, D) = 1$. The coefficient on the right-hand side of the first commutator in (4.26) is the one that arises in E_6 with the same canonical normalisations of the generators. This can easily be seen by using the Weyl group of E_6 to relate this commutator to a commutator of generators with already-known normalisation. For example, we may begin from the commutator for $[E_1{}^2, E_2{}^3] = E_1{}^3$, which involves only the generators of $SL(6, \mathrm{I\!R})$, and is given in (4.3). In terms of the associated dilaton vectors, this corresponds to the sum rule $\vec{b}_{12} + \vec{b}_{23} = \vec{b}_{13}$. Since our positive roots all have length 2, the

Weyl reflection of a weight $\vec{\gamma}$ in the root $\vec{\alpha}$ is given by $\vec{\gamma} \to \vec{\gamma} - (\vec{\alpha}\cdot\vec{\gamma})\vec{\alpha}$. Elementary computations using the results in Appendix A show that under a Weyl reflection in the root $\vec{a}_{145}$, the previous dilaton sum is mapped to $\vec{a}_{245} + \vec{b}_{23} = \vec{a}_{345}$, and that a second Weyl reflection in the root $\vec{a}_{126}$ then maps this into $\vec{a}_{245} + \vec{a}_{136} = -\vec{a}$. Since the positive-root generators X_a have all been canonically normalised so that $\mathrm{tr}(X_a^{\#}\, X_b) = \delta_{ab}$, which is invariant under Weyl reflections, this shows that the commutator $[E_1{}^2, E_2{}^3] = E_1{}^3$, after the two Weyl reflections, becomes $[E^{245}, E^{136}] = \pm D$, in agreement with the scale factor in the first commutator in (4.26).[7] Note that the result of the first Weyl reflection provides a consistency check on the normalisation of the commutator (4.4), since it implies that $[E^{245}, E_2{}^3] = \pm E^{345}$. Note also that in fact the form of the first commutation relation in (4.26) can be seen already in (3.8); it is dictated by $SL(6,\mathbb{R})$ covariance, and the summation rule (4.25) is the direct consequence of the Jacobi identity for $\vec{H}$, E^{ijk} and $E^{\ell mn}$. The internal dilation covariance dictates the dimensions of the generators for the $\mathbb{R}$ factor of $GL(6,\mathbb{R})$.

We now extend the parameterisation of the previous $D \geq 6$ cosets, by introducing an extra factor $\mathcal{V}_4$, so that $\mathcal{V} = \mathcal{V}_1\, \mathcal{V}_2\, \mathcal{V}_3\, \mathcal{V}_4$, where

$$\mathcal{V}_4 = e^{\chi\, D}\ , \tag{4.27}$$

and $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$ are given by (4.7) as before. There will then be the following changes when we calculate $d\mathcal{V}\, \mathcal{V}^{-1}$. Firstly, we pick up an extra term $d\chi\, D$, from $d\mathcal{V}_4\, \mathcal{V}_4^{-1}$. In addition, the computation of $d\mathcal{V}_3\, \mathcal{V}_3^{-1}$ will be modified, because of the non-zero commutators of the form $[E^{ijk}, E^{\ell mn}]$. This will cause non-linear Kaluza-Klein modifications to $d\chi$ to be generated, producing precisely the field strength $G_{(1)}$ obtained in (3.6) from the dualisation of $F_{(4)}$. After conjugating this with $\mathcal{V}_1$, and including the other contributions, we find that

$$d\mathcal{V}\, \mathcal{V}^{-1} = \tfrac{1}{2} d\vec{\phi}\cdot\vec{H} + \sum_{i<j} e^{\frac{1}{2}\vec{b}_{ij}\cdot\vec{\phi}}\, \mathcal{F}^i_{(1)j}\, E_i{}^j + \sum_{i<j<k} e^{\frac{1}{2}\vec{a}_{ijk}\cdot\vec{\phi}}\, F_{(1)ijk}\, E^{ijk} + e^{-\frac{1}{2}\vec{a}\cdot\vec{\phi}}\, G_{(1)}\, D\ . \tag{4.28}$$

Substituting into (4.24), we get precisely the $D = 5$ scalar Lagrangian obtained in the previous section by dualising the 3-form potential $A_{(3)}$ to give an additional axionic scalar. It is now manifest from (4.28) that it will have an E_6 global symmetry. A crucial feature here is that the normalisation of the first commmutation relation in (4.26), which is dictated now by the nonabelian structure of the E_6 algebra, is identical to the normalisation that was needed above for the coset construction of the $D = 5$ Lagrangian.

[7] This argument fixes the magnitudes, but not the signs, of the right-hand sides of the commutators. The signs are in general determinable by more subtle arguments [25]. For our present purposes, it suffices to note that any choice of signs that is consistent with the $SL(11-D,\mathbb{R})$ Jacobi identities represents a valid reduction of E_{11-D} to $SL(11-D,\mathbb{R})$, with the different possible such choices being related by trivial redefinitions of generators.

Now consider $D = 4$. In this case, we have 3-form field strengths $F_{(3)i}$ that are dualised to 1-forms, and hence give the further seven axions χ^i, $i = 1, \ldots, 7$ that we discussed in section 3.2. The associated change in the commutation relations is reflected in the relation

$$\vec{a}_{ijk} + \vec{a}_{\ell mn} = -\vec{a}_p \ , \tag{4.29}$$

where $ijk\ell mnp$ are all different. This follows from (4.1), together with the fact that the dilaton vector for $F_{(3)i}$ is $\vec{a}_i = \vec{f}_i - \vec{g}$, and that the dualisation reverses the sign of the dilaton vector, as we saw previously. Introducing generators D_i associated with the extra axions χ^i, we see that they will obey the commutation relations

$$\begin{aligned} [E^{ijk}, E^{\ell mn}] &= \epsilon^{ijk\ell mnp}\, D_p \ , \\ [E_j{}^k, D_i] &= \delta_i{}^k\, D_j \ , \qquad\qquad [D_i, E^{jk\ell}] = 0 \ , \\ [\vec{H}, D_i] &= -\vec{a}_i\, D_i \ , \qquad\qquad \text{no sum} \ . \end{aligned} \tag{4.30}$$

Note that the non-trivial commutator in $D = 4$, namely the one for E^{ijk} with $E^{\ell mn}$, corresponds to (3.18), and that (4.29) can be derived from the Jacobi identity for $\vec{H}$, E^{ijk} and $E^{\ell mn}$. As usual, we choose the canonical normalisation for the extra generators, so that $\mathrm{tr}(D_i^{\#}\, D_j) = \delta_{ij}$. The normalisation of the first commutator in (4.30) is the one dictated by decomposing E_7 commutation relations under $SL(7, \mathbb{R})$, and can be established by the same technique that we used in $D = 5$, namely by making E_7 Weyl reflections to relate it to an already-known commutator. In fact successive Weyl reflections in the roots $\vec{a}_{145}$ and $\vec{a}_{126}$ map the commutator $[E_1{}^2, E_2{}^3] = E_1{}^3$ into $[E^{245}, E^{136}] = \pm D_7$.

We now write the coset representative as $\mathcal{V}_1\, \mathcal{V}_2\, \mathcal{V}_3\, \mathcal{V}_4$, where $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$ are given by (4.7) as before, and here

$$\mathcal{V}_4 = e^{\sum_i \chi^i\, D_i} \ . \tag{4.31}$$

From this, and the commutation relations, we see that

$$\begin{aligned} d\mathcal{V}\, \mathcal{V}^{-1} &= d\mathcal{V}_1\, \mathcal{V}_1^{-1} + \mathcal{V}_1\, d\mathcal{V}_2\, \mathcal{V}_2^{-1}\, \mathcal{V}_1 + \mathcal{V}_1\, \mathcal{V}_2\, d\mathcal{V}_3\, \mathcal{V}_3^{-1}\, \mathcal{V}_2^{-1}\, \mathcal{V}_1^{-1} + \mathcal{V}_1\, \mathcal{V}_2\, d\mathcal{V}_4\, \mathcal{V}_4^{-1}\, \mathcal{V}_2^{-1}\, \mathcal{V}_1^{-1} \\ &= \tfrac{1}{2} d\vec{\phi} \cdot \vec{H} + \sum_{i<j<k} e^{\frac{1}{2}\vec{a}_{ijk}\cdot\vec{\phi}}\, F_{(1)ijk}\, E^{ijk} \\ &\quad + \sum_{i<j} e^{\frac{1}{2}\vec{b}_{ij}\cdot\vec{\phi}}\, \mathcal{F}^i_{(1)j}\, E_i{}^j + \sum_i e^{-\frac{1}{2}\vec{a}_i\cdot\vec{\phi}}\, G^i_{(1)}\, D_i \ , \end{aligned} \tag{4.32}$$

where $G^i_{(1)}$ are the 1-forms dual to $F_{(3)i}$, defined in (3.16). Substituting this into (4.24), we obtain precisely the $D = 4$ scalar Lagrangian of section 3.2, with its E_7 global symmetry made manifest. Again, it should be emphasised that the normalisations of the commutators in (4.30), which are dictated by the structure of the E_7 algebra, are exactly such as to give the correct expressions for the kinetic terms for the new axions arising from dualisation. The E_7 symmetry of the maximal 4-dimensional supergravity coming from compactification of eleven-dimensional supergravity with full dualisation was obtained in [10].

Finally, we turn to the coset construction for the scalar manifold of 3-dimensional maximal supergravity. In this case, the dualisation of the 2-form field strengths $\mathcal{F}^i_{(2)}$ and $F_{(2)ij}$ gives rise to $8+28=36$ additional axions λ_i and λ^{ij}, with the associated dilaton vectors $-\vec{b}_i$ and $-\vec{a}_{ij}$, as we showed in the previous section. From (A.5), one easily sees that the following relations hold,

$$\vec{a}_{ijk} + \vec{a}_{\ell mn} = -\vec{a}_{pq} \ , \qquad \vec{a}_{ijk} + (-\vec{a}_{jk}) = -\vec{b}_i \ , \tag{4.33}$$

where in the first equation, $ijk\ell mnpq$ are all different. These relations amongst the positive roots of E_8 result from the following commutation relations for the extra generators D^i and D_{ij} associated with $\vec{b}_i$ and $\vec{a}_{ij}$:

$$\begin{aligned}
[E^{ijk}, E^{\ell mn}] &= -\sum_{p<q} \epsilon^{ijk\ell mnpq} D_{pq} \ , \\
[E^{ijk}, D_{\ell m}] &= -6\delta^{[i}{}_{[\ell}\delta^{j}{}_{m]}\, D^{k]} \ , \qquad [E^{ijk}, D^\ell] = 0 \ , \\
[E_i{}^j, D_{k\ell}] &= 2\delta^j_{[k}\, D_{|i|\ell]} \ , \qquad\qquad [E_i{}^j, D^k] = -\delta^k_i\, D^j \ , \\
[\vec{H}, D^i] &= -\vec{b}_i\, D^i \ , \qquad\qquad\quad [\vec{H}, D_{ij}] = -\vec{a}_{ij}\, D_{ij} \ .
\end{aligned} \tag{4.34}$$

The first three commutators, characteristic of the $D=3$ case, were encountered in section 3.3, and from these we can use the Jacobi identities to derive the dilaton-vector summation rules (4.33). As in the previous cases, we are taking the extra generators to be canonically normalised, namely $\mathrm{tr}(D^{\#i}\, D^j) = \delta^{ij}$ and $\mathrm{tr}(D^{\#}_{ij}\, D_{k\ell}) = 2\delta^i_{[k}\,\delta^j_{\ell]}$. The normalisations of the commutators that produce D^i and D_{ij} in (4.34) are those dictated by the E_8 algebra, and can be established, as in the previous cases, by relating them to already-known commutators by means of Weyl reflections. For example, starting again from $[E_1{}^2, E_2{}^3] = E_1{}^3$, and applying successive Weyl reflections in the roots $\vec{a}_{245}$ and then $\vec{a}_{126}$ gives $[E^{245}, E^{136}] = \pm D_{78}$. A third Weyl reflection in the root $\vec{a}_{278}$ then gives $[E^{245}, D_{45}] = \pm D^2$.

The parameterisation of the coset in this case is taken to be

$$g = \mathcal{V}_1\, \mathcal{V}_2\, \mathcal{V}_3\, \mathcal{V}_4\, \mathcal{V}_5, \tag{4.35}$$

where $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{V}_3$ are given by (4.7) as usual, and

$$\mathcal{V}_4 = e^{\sum_i \lambda_i\, D^i} \qquad \mathcal{V}_5 = e^{\sum_{i<j} \lambda^{ij}\, D_{ij}} \ . \tag{4.36}$$

A mechanical calculation gives the result that

$$
\begin{aligned}
d\mathcal{V}\,\mathcal{V}^{-1} \;=\;& \tfrac{1}{2}d\vec{\phi}\cdot\vec{H} + \sum_{i<j} e^{\frac{1}{2}\vec{b}_{ij}\cdot\vec{\phi}}\,\mathcal{F}^i_{(1)j}\,E_i{}^j + \sum_{i<j<k} e^{\frac{1}{2}\vec{a}_{ijk}\cdot\vec{\phi}}\,F_{(1)ijk}\,E^{ijk}\\
&+\sum_i e^{-\frac{1}{2}\vec{b}_i\cdot\vec{\phi}}\,G_{(1)i}\,D^i + \sum_{i<j} e^{-\frac{1}{2}\vec{a}_{ij}\cdot\vec{\phi}}\,G^{ij}_{(1)}\,D_{ij}\ ,
\end{aligned}
\tag{4.37}
$$

where in addition to the usual field strengths $\mathcal{F}^i_{(1)}$ and $F_{(1)ijk}$ for the axions $\mathcal{A}^i_{(0)j}$ and $A_{(0)ijk}$, the field strengths $G_{(1)i}$ and $G^{ij}_{(1)}$ for the additional axions λ_i and λ^{ij} are precisely the ones given in (3.31) and (3.32). Substituting (4.37) into (4.24) gives a manifestly E_8 invariant formulation for the $D=3$ scalar Lagrangian obtained in section 2.1. Again, the commutator structures that are dictated by E_8 covariance are precisely the ones needed to reproduce the kinetic terms of the additional axions arising from dualisation. Note that in $D=3$, this scalar Lagrangian describes the entire bosonic sector of the theory. The E_8 symmetry of the bosonic sector of $D=3$ maximal supergravity, obtained from dimensional reduction of $D=4$ maximal supergravity, was proved in [20]. A rather different proof of the E_8 symmetry in $D=3$ was given recently in [21].

5 Dualisation and double coset structure

In the previous section, we constructed the cosets that give the scalar Lagrangians for the maximal supergravities (obtained from dimensional reduction of eleven-dimensional supergravity) in all dimensions $D\ge 3$, in the formulations where the numbers of scalars are maximised by dualising all $(D-2)$-form potentials (in $D\le 5$). We also showed that they have global E_{11-D} symmetries. If no such dualisations were performed, the original scalar Lagrangians in dimensions $3\le D\le 5$ would instead have $GL(11-D,\mathrm{I\!R})\ltimes\mathrm{I\!R}^q$ global symmetries. The easiest way to understand the relation between the global symmetries of these two formulations, and indeed to understand the symmetries for any other choice of dualisations, is to take the fully-dualised versions with the E_{11-D} symmetries as the starting point.

Let us illustrate this by examining the relation between the global E_{11-D} symmetries of the fully-dualised formulations and the $GL(11-D,\mathrm{I\!R})\ltimes\mathrm{I\!R}^q$ symmetries of the undualised formulations. $SL(11-D,\mathrm{I\!R})$ has positive-root generators $E_i{}^j$, whose algebra is given by (4.3). The E^{ijk} form a q-dimensional linear representation under $SL(11-D,\mathrm{I\!R})$. Curiously enough, in $D\ge 6$ the positive-root generators $E_i{}^j$ of $SL(11-D,\mathrm{I\!R})$, together with E^{ijk}, are precisely the positive-root generators for E_{11-D}. In dimensions $3\le D\le 5$, on the other hand, the situation is different. In these cases, the positive-root algebra for $SL(11-D,\mathrm{I\!R})$

and its $\mathbb{R}^q$ representation can instead be obtained as a subalgebra of a contraction of E_{11-D} positive-root algebra. For example in $D=5$, E_6 has an additional generator in its positive-root algebra, namely D, which is a singlet under $SL(6,\mathbb{R})$. It is generated by the commutation of E^{ijk} with $E^{\ell mn}$:

$$[E^{ijk}, E^{\ell mn}] = -\epsilon^{ijk\ell mn}\, D\ . \tag{5.1}$$

In a Wigner-Inönu contraction, we would rescale $D \to \lambda\, D$ and send $\lambda \to 0$, then all the generators E_{ijk} would become commuting, giving rise to an $\mathbb{R}^{20}$ symmetry. Furthermore, we can then consistently truncate the generator D, to obtain a subalgebra. Here both steps are done simultaneously by setting $D=0$. The remaining positive-root generators are precisely those of $SL(6,\mathbb{R})$ and its $\mathbb{R}^{20}$ linear representation, which together generate exactly the symmetry of the undualised scalar manifold, namely $GL(6,\mathbb{R})\ltimes\mathbb{R}^{20}$. The necessity of the truncation of the generator D from the algebra is a reflection of the fact that the corresponding scalar χ is now replaced by its dualised version, namely a $(D-2)$-form gauge potential, and thus disappears from the scalar Lagrangian. It should be emphasised that although we can legitimately set the generator D to zero in the positive-root algebra, the corresponding scalar field χ cannot be consistently set to zero if the remaining scalars are non-vanishing. (In other words, setting χ to zero would in general be inconsistent with its equation of motion.) This can be seen from the non-linear Kaluza-Klein structure given in (3.6). Similarly, if we dualise χ to $A_{(3)}$ then the field strength $F_{(4)}$, unlike other higher-rank fields, cannot be consistently set to zero either. However in this case, the terms involving $F_{(4)}$ are no longer part of the scalar Lagrangian.

From the group theoretical point of view, the above truncation of the generator D corresponds to a double coset $USp(8)\backslash E_6/N_2^{\vec r_1}$, since the generator D corresponds to the unique level-2 positive root with respect to the simple root $\vec r_1$, it generates the Lie algebra of $N_2^{\vec r_1}$. $N_2^{\vec r_1}$ is by definition the algebra of the set of positive root vectorss of level 2 with respect to the simple root $\vec r_1$.[8] The single coset construction, which was given in section 4, has the effect of removing all the negative roots of the group. In the double coset, additional higher-level positive roots are removed as well. (Note that our notation for simple roots $\vec r_i$ of E_{11-D} is defined below Diagram 1, and in particular $\vec r_1$ denotes the simple root $\vec a_{123}$.) There are in total 36 positive roots in the E_6 algebra, with only one that is at level 2 with respect to the simple root $\vec r_1$ in E_6. The remainder comprise 20 level 1 and 15 level 0 positive

[8] Recall that any positive root can be written as a sum $\sum_i \ell_i\, \vec r_i$ over the simple roots, where the coefficients ℓ_i are non-negative integers which define the level of the positive root with respect to each of the simple roots $\vec r_i$. The total level of the root is $\ell = \sum_i \ell_i$.

roots. Since the levels of roots are additive under commutation of the associated generators, it follows that the 20 level 1 generators become commuting, as indicated by (5.1), if the would-be level 2 generator is contracted and truncated out. As a result, the double coset described above can then be re-expressed as the single coset $SO(6)\backslash GL(6,\mathbb{R})\ltimes\mathbb{R}^{20}$, and the associated theory has a $GL(6,R)\ltimes\mathbb{R}^{20}$ global symmetry. By general arguments, the double coset remains a coset for the normalizer of the $N_2^{\vec{r}_1}$ subgroup acting on the right. Clearly this normalizer is generated by the level 0 and 1 roots, *i.e.* it is precisely the group $GL(6,\mathbb{R})\ltimes\mathbb{R}^{20}$.

A similar analysis applies also in $D=4$ and $D=3$, where the extra generators coming from the dualisations can again be rescaled, allowing the positive-root algebra to be contracted. We can then extract a subalgebra, namely the positive-root algebra of $GL(11-D,\mathbb{R})$ and its $\mathbb{R}^q$ linear representation and verify the symmetry $GL(11-D,\mathbb{R})\ltimes\mathbb{R}^q$ of the undualised theory. For $D=4$ and $D=3$, these dualisations correspond to the double cosets $SU(8)\backslash E_7/N_2^{\vec{r}_1}$ and $SO(16)\backslash E_8/N_{2,3}^{\vec{r}_1}$ respectively. To be more specific, in $D=4$ the global E_7 symmetry algebra has 63 positive roots. With respect to the simple root $\vec{r}_1$, their gradings are: 7 level 2 roots, 35 level 1 roots and 21 level 0 roots. The 7 level 2 roots correspond precisely to the 7 axions that are generated by dualisation of the 7 2-form potentials. When these 7 axions are inversely dualised back to 2-form potentials, the 7 level 2 roots are contracted and truncated out, and as a consequence the 35 level 1 generators become commuting. The resulting theory then has a global $GL(7,\mathbb{R})\ltimes\mathbb{R}^{35}$ symmetry. In both $D=5$ and $D=4$ the directly-reduced theories, where no dualisations to give additional scalars are performed, correspond to truncating out the level 2 roots of the E_6 or E_7 algebras respectively. In $D=3$ the story is slightly different. The E_8 group has 120 positive roots, and with respect to the simple root $\vec{r}_1$ there are 8 at level 3, 28 at level 2, 56 at level 1 and 28 at level 0. The level 3 and level 2 positive roots are commuting, corresponding to the axions that are generated by dualising all the 36 vectors of the theory. If these axions were absent because we left undualised the corresponding vectors, so that the corresponding level 3 and level 2 roots were omitted, then the 56 level 1 generators would become commuting on the remaining fields. Thus the non-dualised theory in $D=3$ possesses a $GL(8,\mathbb{R})\ltimes\mathbb{R}^{56}$ global symmetry.

There are many other possible choices of dualisations that could be performed on the scalar sectors of the theories. Some of these will correspond to partial dualisations of a subset of the $(D-2)$-form potentials of the original undualised theories, while others can involve "inverse" dualisations of fields that were originally scalars. In all cases, the global symmetries can be deduced by taking the fully-dualised theory as the starting point, and studying the algebra of the subset of E_{11-D} positive roots corresponding to the axions that remain after the chosen dualisations. For example in $D=3$, we can truncate out 28 level 2 or 8 level 3 generators, since they are commuting and thus are associated with axions that can be dualised. In fact, if we truncate out all of them, then we get the non-dualised case with $GL(8,\mathbb{R})\ltimes\mathbb{R}^{56}$, which we discussed earlier. If we dualise only the level 3 roots we again end up with a coset, but if however we dualise the 28 level 2 roots without dualising the higher 8, then the normaliser is only the level 0 $GL(8,\mathbb{R})$ which does not act transitively and this is not a coset situation. In section 7 we shall study one particular class of examples, in which only those original $(D-2)$-form potentials that lie in the Ramond-Ramond sector (from the ten-dimensional type IIA viewpoint) are dualised to give additional scalars. This corresponds to truncating out the highest-level positive roots with respect to the simple root $\vec{r}_2 = \vec{b}_{12}$, rather than the truncations with respect to $\vec{r}_1 = \vec{a}_{123}$ that we discussed above. In section 9, we discuss the symmetry group of the direct dimensional reductions of type IIB supergravity in $D=10$, where no dualisations are performed. These correspond to truncating out sets of commuting positive roots under a double grading with respect to the two simple roots $\vec{r}_1 = \vec{a}_{123}$ and $\vec{r}_3 = \vec{b}_{23}$. For completeness we may recall that the disintegration of E_{11-D} to E_{10-D} amounts to omitting the roots of level 1 with respect to the last simple root that appeared during the dimensional reduction, except for the case $D=3$ where both level 1 and 2 roots must be omitted.

In all these cases, the symmetry can be understood from the point of view of the double coset of the E_{11-D} group. In section 2, the dimensional reduction of eleven-dimensional supergravity was performed by iteratively repeating the $D+1$ to D dimensional reduction. It followed that the scalars are then precisely the parameters of the generators of the Borel group modulo the maximal compact subgroup. Thus the fully-dualised Lagrangian is already naturally written with the coset $K(E_{11-D})\backslash E_{11-D}$ structure, where $K(G)$ denotes the maximal compact subgroup of G. To understand the dualisation, we recall that in section 4 we saw that the axions are in one-to-one correspondence with the positive roots, while the dilatons are associated with the Cartan generators. We shall show in section 8 that the maximal abelian subalgebras of the positive-root (nilpotent) algebras precisely describe the commuting $\mathbb{R}$ symmetries of the sets of axions that can be simultaneously covered by a derivative. Thus inverse dualisation of scalars to higher-forms can be understood as removals of commuting generators from the coset $K(G)\backslash G$, giving rise to a double coset.

6 Global symmetries of higher-rank fields

In the previous sections, we studied the global symmetries of the scalar sectors of the toroidally compactified supergravity Lagrangians, showing in particular that they have global E_{11-D} symmetries when the numbers of scalars are maximised by dualising all $(D-2)$-form potentials. We also showed that alternative choices of dualisation change the scalar Lagrangian, and its symmetries. In this section we shall study how the global symmetries of the scalar sector can also be realised on the higher-degree fields. We shall show that in general it is necessary to perform appropriate dualisations in the higher-degree sectors in order to preserve the global symmetry of the scalar sector. In particular, when the dualisation of $(D-2)$-form potentials to scalars has been performed, the E_{11-D} symmetry is preserved for the higher-degree sectors of the Lagrangian provided that all field strengths with degrees greater than $\frac{1}{2}D$ are dualised. In some cases, these dualisations are necessary in order to preserve the E_{11-D} symmetry, while in others, the symmetry may still be realised (on the higher-rank field strengths rather than their potentials) at the level of the equations of motion even in the absence of the dualisations. An example of the former is in $D=6$, where the $E_5 = O(5,5)$ global symmetry of the scalar Lagrangian is broken (even at the level of the equations of motion) unless the 4-form field strength is dualised to give an additional 2-form. An example where the dualisation is optional is in $D=7$, where the $E_4 = SL(5,\mathbb{R})$ symmetry of the scalar Lagrangian is preserved in the full Lagrangian if the 4-form is dualised to give another 3-form field strength, and it is also preserved at the level of the equations of motion if the 4-form is left undualised. We shall discuss these, and other examples, below.

To begin, let us consider the cases where all field strengths of ranks greater than $\frac{1}{2}D$ are dualised, and the theories have the E_{11-D} global symmetries. The discussion divides into two, depending on whether D is odd or even. In odd dimensions, the symmetry acts on the potentials, and is realised at the level of the Lagrangian. In an even dimension, on the other hand, the field strengths of rank $\frac{1}{2}D$ and their duals form a single irreducible multiplet under the E_{11-D} group, and thus the symmetry can be implemented only at the level of the equations of motion, realised on the field strengths rather than the potentials. (The symmetry transformations still act on the potentials for the lower-degree field strengths.)

6.1 Odd dimensions

In $D=9$, no dualisations are necessary, and the global $GL(2,\mathbb{R})$ symmetry of the scalar manifold extends to the entire Lagrangian, as we described in section 2.2. The bosonic

Lagrangian and its $GL(2,\mathbb{R})$ symmetry was proved in [19, 14]. The situation is more complicated in $D = 7$ and $D = 5$; we shall describe the former in complete detail, and postpone a similar discussion of the latter to a subsequent publication. (In $D = 3$, the fully-dualised theory contains only scalar fields, and its E_8 symmetry was fully discussed already in section 4.2.)

We shall first consider $D = 7$. The scalar Lagrangian, which has a manifest global $GL(4,\mathbb{R})$ symmetry, in fact has a larger $SL(5,\mathbb{R})$ global symmetry. This can be made manifest by defining generators $E_\alpha{}^\beta$, where $\alpha = (i, 5)$ is an $SL(5,\mathbb{R})$ index, and $E_i{}^5 = \frac{1}{6}\epsilon_{ijk\ell}\, E^{jk\ell}$. The commutation relations for $E_i{}^j$ and E^{ijk} now become

$$[E_\alpha{}^\beta, E_\gamma{}^\delta] = \delta_\gamma^\beta\, E_\alpha{}^\delta - \delta_\alpha^\delta\, E_\gamma{}^\beta\ . \tag{6.1}$$

The manifest $O(5)\backslash SL(5,R)$ coset is presented in Appendix C.

At the level of the Lagrangian, where this symmetry acts on the gauge potentials, it extends to the higher-degree sectors of the theory only if we dualise $A_{(3)}$ to give a further 2-form potential $A_{(2)}$ which, together with the four potentials $A_{(2)i}$, (and after some field redefinitions, which we shall discuss in detail below) form an irreducible 5 under $SL(5,\mathbb{R})$. The grouping of $SL(4,\mathbb{R})$ representations into $SL(5,\mathbb{R})$ representations can already be seen in the structure of the dilaton vectors. Consider first the 3-form field strengths, including the dualisation of the 4-form. The associated dilaton vectors are $(\vec{a}_i, -\vec{a})$. As we saw in section 4, the simple roots of the $SL(5,\mathbb{R})$ algebra in $D = 7$ are given by $\vec{b}_{12}$, $\vec{b}_{23}$, $\vec{b}_{34}$ and $\vec{a}_{123}$. It is a simple matter to see, from the definitions in Appendix A, that $-\vec{a}$ is the highest-weight vector of the 5-dimensional representation of $SL(5,\mathbb{R})$, with the rest of the multiplet filled out by acting with the negatives of the simple roots, according to the scheme

$$\vec{a}_4 = -\vec{a} - \vec{a}_{123}\ , \qquad \vec{a}_3 = \vec{a}_4 - \vec{b}_{34}\ , \qquad \vec{a}_2 = \vec{a}_3 - \vec{b}_{23}\ , \qquad \vec{a}_1 = \vec{a}_2 - \vec{b}_{12}\ . \tag{6.2}$$

Similarly, the dilaton vectors for the 2-form field strengths, namely $(\vec{a}_{ij}, \vec{b}_i)$, are the weight vectors of the $\bar{10}$ representation of $SL(5,\mathbb{R})$, with $\vec{a}_{34}$ as the highest-weight vector.

To see the $SL(5,\mathbb{R})$ symmetry explicitly at the level of the Lagrangian, we begin by dualising the 3-form potential $A_{(3)}$. To do this, we introduce a 2-form Lagrange multiplier $A_{(2)}$, to enforce the Bianchi identity for $F_{(4)}$, which will now be treated as an auxiliary field, adding an extra term $dA_{(2)} \wedge \tilde{F}_{(4)}$ to the Lagrangian. As usual, it is advantageous to replace $\tilde{F}_{(4)}$ immediately by its Kaluza-Klein modified field strength $F_{(4)} = \hat{F}_4$ as given in (A.29), and likewise to replace all occurrences of $\tilde{F}_{(4)}$ in the Wess-Zumino terms given in (A.16) by $F_{(4)}$. This makes the algebraic equation for the auxiliary field $F_{(4)}$ very easy to solve, giving

$F_{(4)} = e^{-\vec{a}\cdot\vec{\phi}} * G_{(3)}$, where

$$G_{(3)} = dA_{(2)} + \Big(\tfrac{1}{6} dA_{(2)i}\, A_{(0)jk\ell} + \tfrac{1}{8} A_{(1)ij} \wedge dA_{(1)k\ell}\Big)\, \epsilon^{ijk\ell} \ , \tag{6.3}$$

At this point, it is convenient to introduce some redefined potentials, along the lines of those described in Appendix A. In fact for the 1-forms $A_{(1)ij}$, we make exactly the redefinition given in (A.18). For the 2-forms $A_{(2)i}$, however, it turns out that the most appropriate redefinition here is different from the hatted potentials given in (A.18), and is instead given by

$$\bar{A}_{(2)i} = A_{(2)i} + \tfrac{1}{2} A_{(1)ij} \wedge \hat{\mathcal{A}}^j_{(1)} \ . \tag{6.4}$$

At the same time, we make the following redefinition for the dualised potential $A_{(2)}$:

$$\bar{A}_{(2)} = A_{(2)} + \tfrac{1}{12} \epsilon^{ijk\ell}\, A_{(0)ijk}\, A_{(1)\ell m} \wedge \hat{\mathcal{A}}^m_{(1)} \ . \tag{6.5}$$

Note that $\bar{A}_{(2)i}$ and $\bar{A}_{(2)}$ have the transformations

$$\delta \bar{A}_{(2)i} = 0 \ , \qquad \delta \bar{A}_{(2)} = \Lambda^i{}_5\, \bar{A}_{(2)i} \ , \tag{6.6}$$

under the variations $\delta A_{(0)ijk} = c_{ijk}$, where $\Lambda^i{}_5 = \frac{1}{6} \epsilon^{ijk\ell}\, c_{jk\ell}$.

We are now in a position to define $SL(5,\mathrm{I\!R})$-covariant potentials, $B^{\alpha\beta}_{(1)}$ and $B_{(2)\alpha}$, related to those described above by

$$\begin{aligned} B^{ij}_{(1)} &= \tfrac{1}{2} \epsilon^{ijk\ell}\, \hat{A}_{(1)k\ell} \ , \qquad && B^{i5}_{(1)} = \hat{\mathcal{A}}^i_{(1)} \ , \\ B_{(2)i} &= \bar{A}_{(2)i} \ , && B_{(2)5} = \bar{A}_{(2)} \ . \end{aligned} \tag{6.7}$$

We also define their field strengths $H^{\alpha\beta}_{(2)}$ and $H_{(3)\alpha}$, by

$$H^{\alpha\beta}_{(2)} = dB^{\alpha\beta}_{(1)} \ , \qquad H_{(3)\alpha} = dB_{(2)\alpha} + \tfrac{1}{8} \epsilon_{\alpha\beta\gamma\delta\sigma}\, B^{\beta\gamma}_{(1)} \wedge dB^{\delta\sigma}_{(1)} \ . \tag{6.8}$$

Note that while the gauge transformation for $B_{(2)\alpha}$ is simply $\delta B_{(2)\alpha} = d\Lambda_{(1)\alpha}$, the one for $B^{\alpha\beta}_{(1)}$ must be accompanied by a compensating transformation for $B_{(2)\alpha}$, in order to ensure the gauge invariance of $H_{(3)\alpha}$:

$$\delta B^{\alpha\beta}_{(1)} = d\Lambda^{\alpha\beta}_{(0)} \ , \qquad \delta B_{(2)\alpha} = -\tfrac{1}{8} \epsilon_{\alpha\beta\gamma\delta\sigma}\, B^{\beta\gamma}_{(1)} \wedge d\Lambda^{\delta\sigma}_{(0)} \ . \tag{6.9}$$

We now find that all the remaining Wess-Zumino terms given in (A.16), together with the additional terms acquired from the introduction of the Lagrange multiplier, turn out, after calculations of not inconsiderable complexity, to be expressible in the simple $SL(5,\mathrm{I\!R})$-covariant form

$$\mathcal{L}_{\rm WZ} = -\tfrac{1}{2} \Big(H_{(3)\alpha} \wedge H_{(3)\beta} \wedge B^{\alpha\beta}_{(1)} - 4 H_{(3)\alpha} \wedge B_{(2)\beta} \wedge dB^{\alpha\beta}_{(1)} \Big) \ . \tag{6.10}$$

Note that this is invariant, up to a total derivative, under the gauge transformations not only for $B_{(2)\alpha}$ but also for $B_{(1)}^{\alpha\beta}$, given in (6.9). In terms of $H_{(2)}^{\alpha\beta}$ and $H_{(3)\alpha}$, the field strengths appearing in the Lagrangian given in Appendix A take the form

$$
\begin{aligned}
\hat{\mathcal{F}}_{(2)}^{i} &= H_{(2)}^{i5}\ , \qquad \hat{F}_{(2)ij} = \tfrac{1}{2}\epsilon_{ijk\ell}\, H_{(2)}^{k\ell} + A_{(0)ijk}\, H_{(2)}^{k5}\ ,\\
\hat{F}_{(3)i} &= H_{(3)i}\ , \qquad G_{(3)} = H_{(3)5} - \tfrac{1}{6}\epsilon^{ijk\ell}\, A_{(0)ijk}\, H_{(3)\ell}\ ,
\end{aligned}
\tag{6.11}
$$

where $G_{(3)}$ is defined by (6.3).

The scalar Lagrangian for coset $O(5)\backslash SL(5,\mathbb{R})$ is discussed in Appendix C. Putting this together with the results for the higher-rank fields obtained above, we can write down the manifestly $SL(5,\mathbb{R})$ invariant bosonic Lagrangian for $D=7$ maximal supergravity:

$$
\begin{aligned}
\mathcal{L} &= e\,R + \tfrac{1}{4}e\,\mathrm{tr}(\partial_\mu G^{-1}\,\partial^\mu G) - \tfrac{1}{12}e\,H_{(3)\alpha}G^{\alpha\beta}H_{(3)\beta} - \tfrac{1}{8}e\,H_{(2)}^{\alpha\beta}G_{\alpha\gamma}G_{\beta\delta}H_{(2)}^{\gamma\delta}\\
&\quad -\tfrac{1}{2}\Big(H_{(3)\alpha}\wedge H_{(3)\beta}\wedge B_{(1)}^{\alpha\beta} - 4H_{(3)\alpha}\wedge B_{(2)\beta}\wedge dB_{(1)}^{\alpha\beta}\Big)\ .
\end{aligned}
\tag{6.12}
$$

where $G^{\alpha\beta}$ and $G_{\alpha\beta}$ are given by (C.8).

The discussion for five dimensions proceeds in a similar way. The supergravity Lagrangian in $D=5$ with manifest E_6 symmetry was constructed in the second reference in [11]. In order to have E_6 as the global symmetry, we must first of all dualise the 3-form potential $A_{(3)}$ to give another scalar χ, as discussed in section 4. For the higher-form gauge potentials, it is necessary to dualise the six 2-form potentials $A_{(2)i}$ to give additional vectors, which together with the six Kaluza Klein vectors, and fifteen vectors from anti-symmetric tensor in $D=11$ form the 27-dimensional representation of E_6. The detailed discussion of this example will be presented in a forthcoming paper.

6.2 Even dimensions

The discussion becomes more complicated in even dimensions only because of the occurrence of field strengths whose degree is equal to $\frac{1}{2}D$. For all the other higher-degree fields, the global E_{11-D} symmetry of the fully-dualised scalar manifold acts on the potentials in the same manner as in the case of odd dimensions. However, because the field strengths of degree $\frac{1}{2}D$ together with their duals form a single irreducible representation of E_{11-D}, the symmetry can only be realised on these field strengths themselves, rather than on their potentials. Consequently, the E_{11-D} symmetry of the full theory can only be realised at the level of the equations of motion in even dimensions.

There is, however, a convenient way to reformulate the theory so that the global E_{11-D} symmetry can in fact be implemented at the level of an auxiliary Lagrangian. We shall call it the doubled Lagrangian. This can be done by introducing an auxiliary set of field strengths of degree $\frac{1}{2}D$, equal in number to the original set, together with their associated potentials. One can then construct a Lagrangian whose full set of field equations can be consistently truncated to give the equations of motion of the original theory. By this means, the global E_{11-D} symmetry can be implemented on the doubled set of potentials, whose total number is now equal to the dimension of the relevant irreducible representation of E_{11-D}. (A similar trick was recently used in order to construct a Lagrangian for type IIB supergravity, by adding the anti-self-dual half of the 4-form potential, whose equations of motion could be consistently truncated to those of type IIB supergravity [28].) We shall now use this technique to discuss the global E_{11-D} symmetries of the fully-dualised theories in $D = 8$, 6 and 4. We shall give the complete construction of the E_7-invariant bosonic theory in $D = 4$, while, in the more complicated cases of $D = 6$ and $D = 8$, we shall just focus our attention on the field strengths of degrees 3 and 4 respectively.

$D = 8$:

The scalar Lagrangian in this case has a global $SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$ symmetry. The 3-form potential $A_{(3)}$ is a singlet under $SL(3,\mathbb{R})$, since it carries no internal indices and the $SL(3,\mathbb{R})$ is contained in the standard global $GL(3,\mathbb{R})$ symmetry that results from compactification on a 3-torus. (The $SL(3,\mathbb{R})$ has simple roots $\vec{b}_{12}$ and $\vec{b}_{23}$, while the $SL(2,\mathbb{R})$ has the simple root $\vec{a}_{123}$.) However, the field strength of $A_{(3)}$ is actually a doublet under $SL(2,\mathbb{R})$, which can be seen from the fact that $(\vec{a}, -\vec{a})$ form the weight vectors of the 2 of $SL(2,\mathbb{R})$, with $-\vec{a}$ as highest weight. These two dilaton vectors are associated with the 4-form field strength and its dual. The 2-form potentials $A_{(2)i}$ form a singlet under $SL(2,\mathbb{R})$, and a triplet under $SL(3,\mathbb{R})$. This can be seen from their dilaton vectors $(\vec{a}_1, \vec{a}_2, \vec{a}_3)$, which are the weight vectors of the 3 of $SL(3,\mathbb{R})$, with $\vec{a}_3$ as highest weight. Finally, the vector potentials $A_{(1)ij}$ and $\mathcal{A}^i_{(1)}$ have dilaton vectors $(\vec{a}_{ij}, \vec{b}_i)$ that form the weights of the $(3, 2)$ representation of $SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$, with $\vec{a}_{23}$ as highest weight.

The extension to the higher-degree fields of the $SL(3,\mathbb{R})$ factor in the global symmetry of the scalar manifold is straightforward, since it is contained in the $GL(3,\mathbb{R})$ that was described in section 2. We shall instead concentrate on the $SL(2,\mathbb{R})$ factor, since this involves new issues associated with the occurrence of the 4-form field strength, whose degree is half the spacetime dimension. In fact, we can consistently truncate out all the fields that are non-singlets under $SL(3,\mathbb{R})$, namely the 2-form and 3-form field strengths and the scalars of the $SL(3,\mathbb{R})$ factor in the scalar manifold, since the remaining $SL(2,\mathbb{R})$ scalars and the 4-form field strength, which is an $SL(3,\mathbb{R})$ singlet, cannot act as a source for them. We may therefore simplify the discussion of the $SL(2,\mathbb{R})$ structure of the theory by performing this truncation. We then introduce a second 3-form potential $\tilde{A}_{(3)}$, and define the 4-form field strengths

$$H_{(4)} = \begin{pmatrix} dA_{(3)} \\ d\tilde{A}_{(3)} \end{pmatrix} , \tag{6.13}$$

which form a doublet under $SL(2,\mathbb{R})$. The $SL(2,\mathbb{R})$-invariant Lagrangian can then be written in the form

$$\begin{aligned} \mathcal{L} &= eR + \tfrac{1}{4} e\, \mathrm{tr}(\partial_\mu \mathcal{M}^{-1}\, \partial^\mu \mathcal{M}) - \tfrac{1}{48} e\, H^{\mathrm{T}}_{(4)}\, \mathcal{M}\, H_{(4)} \ , \\ &= eR - \tfrac{1}{2} e\, (\partial\phi)^2 - \tfrac{1}{2} e\, e^{2\phi}\, (\partial\chi)^2 - \tfrac{1}{96} e\, e^{-\phi}\, F^2_{(4)} - \tfrac{1}{96} e\, e^{\phi}\, \tilde{F}^2_{(4)} \ , \end{aligned} \tag{6.14}$$

where $F_{(4)} = dA_{(3)}$, $\tilde{F}_{(4)} = d\tilde{A}_{(3)} - \chi\, dA_{(3)}$, and $\phi = -\vec{a}\cdot\vec{\phi}$, $\chi = A_{(0)123}$. The Bianchi identities and equations of motion for $F_{(4)}$ and $\tilde{F}_{(4)}$ are therefore

$$\begin{aligned} dF_{(4)} &= 0 \ , \qquad\qquad d(\tilde{F}_{(4)} + \chi\, F_{(4)}) = 0 \ , \\ d*(e^{\phi}\, \tilde{F}_{(4)}) &= 0 \ , \qquad d*(e^{-\phi}\, F_{(4)} - \chi\, e^{\phi}\, \tilde{F}_{(4)}) = 0 \ . \end{aligned} \tag{6.15}$$

We see from these equations that it is consistent to impose the relation

$$\widetilde{F}_{(4)} = e^{-\phi} * F_{(4)} \ , \tag{6.16}$$

leading to the equations

$$dF_{(4)} = 0 \ , \qquad\qquad d(e^{-\phi} * F_{(4)} + \chi\, F_{(4)}) = 0 \ , \tag{6.17}$$

which are precisely the Bianchi identity and equation of motion for the 4-form field strength $F_{(4)}$ that follow from the $D=8$ Lagrangian in Appendix A. In fact the latter, after performing the truncation to the $SL(3,\mathbb{R})$ singlets described above, takes the form

$$\mathcal{L} = eR - \tfrac{1}{2} e(\partial\phi)^2 - \tfrac{1}{2} e\, e^{2\phi}\, (\partial\chi)^2 - \tfrac{1}{48} e\, e^{-\phi}\, F_4^2 + \tfrac{1}{48} e\, \chi\, F_4 \cdot {}^*F_4 \ , \tag{6.18}$$

which can easily be verified to give the same equations of motion as the ones coming from (6.14) together with the constraint (6.16). Note that the truncation (6.16) is $SL(2,\mathbb{R})$ covariant, and thus the Bianchi identity and equation of motion (6.17) indeed inherit the global $SL(2,\mathbb{R})$ symmetry that was manifest in the Lagrangian (6.14) prior to the truncation. Before the truncation, $A_{(3)}$ and $\tilde{A}_{(3)}$ form an $SL(2,\mathbb{R})$ doublet and the symmetry is realised in the Lagrangian; after the truncation, $F_{(4)}$ and $e^{-\phi} * F_{(4)}$ form an $SL(2,\mathbb{R})$ doublet and the symmetry is realised only in the equations of motion. Note that prior to truncation, the Bianchi identities and equations of motion can be written in the manifestly $SL(2,\mathbb{R})$-covariant forms

$$dH_{(4)} = 0 \ , \qquad\qquad d * (\mathcal{M} H_{(4)}) = 0 \ . \tag{6.19}$$

The truncation (6.16) then takes the manifestly $SL(2,\mathbb{R})$-covariant form

$$H_{(4)} = \Omega \mathcal{M} * H_{(4)} \ , \tag{6.20}$$

where Ω is the $SL(2,\mathbb{R})$-invariant antisymmetric rank-two tensor, which appears here in a (noncovariant) canonical form. One can raise one index of this *a fortiori* $SO(2)$ invariant tensor and obtain the relation $\Omega^2 = -1$.

$D = 6$:

The global symmetry of the scalar manifold is $E_5 = O(5,5)$ in six dimensions. After dualising the 3-form potential to give an additional vector potential $A_{(1)}$, the vectors $A_{(1)ij}$, $\mathcal{A}^i_{(1)}$ and $A_{(1)}$ (after appropriate field redefinitions) form a 16-dimensional irreducible multiplet under $O(5,5)$, corresponding to the weight vectors $(\vec{a}_{ij}, \vec{b}_i, -\vec{a})$. The highest-weight vector is $-\vec{a}$. The five 2-form potentials $A_{(2)i}$ cannot themselves form an $O(5,5)$ multiplet, but their field strengths, together with the duals, form an irreducible 10-dimensional representation. The associated dilaton vectors $(\vec{a}_i, -\vec{a}_i)$ are the weight vectors of the 10, with $-\vec{a}_1$ as the highest weight. We may give an analogous discussion to the one in $D = 8$, and focus just on the sectors comprising the scalars and the 3-form field strengths. (The vectors can be truncated consistently from the theory, thus simplifying the discussion.) We may then introduce a second set of five 2-form potentials $\tilde{A}^i_{(2)}$, in terms of which we define

$$H_{(3)} = \begin{pmatrix} dA_{(2)i} \\ d\tilde{A}^i_{(2)} \end{pmatrix} \ . \tag{6.21}$$

This set of field strengths transform as the 10-dimensional vector representation under $O(5,5)$. The Lagrangian for the kinetic terms for the scalars and 2-form potentials can then be written in the manifestly $O(5,5)$-invariant form

$$\mathcal{L} = eR + \tfrac{1}{4} e \, \mathrm{tr}(\partial_\mu \mathcal{M}^{-1} \, \partial^\mu \mathcal{M}) - \tfrac{1}{24} e \, H^{\mathrm{T}}_{(3)} \mathcal{M} H_{(3)} \ , \tag{6.22}$$

where $\mathcal{M}$ is the $(O(5) \times O(5)) \backslash O(5,5)$ coset matrix defined in section 4.1. Its explicit form is given by (C.4), with G and X as defined above (C.7) in Appendix C. The Bianchi identities and equations of motion that follow from this Lagrangian are

$$dH_{(3)} = 0 \ , \qquad\qquad d * (\mathcal{M} \, H_{(3)}) = 0 \ , \tag{6.23}$$

from which we see that it is consistent to impose the $O(5,5)$-covariant truncation

$$H_{(3)} = \Omega\,\mathcal{M}\, * H_{(3)}\ . \tag{6.24}$$

Here Ω is an $O(5,5)$-invariant metric that again can be considered to be an invariant of $(O(5)\times O(5))$. One of its indices can be raised with the invariant (identity) metric of the maximal compact subgroup to obtain $\Omega^2 = 1$. The subsector of the bosonic Lagrangian for six-dimensional maximal supergravity that we have obtained here agrees with the results obtained in [27], where the complete theory was obtained by direct construction, rather than by dimensional reduction from $D = 11$.

$D = 4$:

The fully-dualised scalar manifold in $D = 4$ has an E_7 global symmetry. The only additional fields of higher degree are the 28 vectors, comprising 21 $A_{(1)ij}$ and 7 $\mathcal{A}^i_{(1)}$. Their associated field strengths, together with their duals, form the 56-dimensional irreducible representation of E_7 [10]. The associated dilaton vectors, $(\vec{a}_{ij}, \vec{b}_i, -\vec{a}_{ij}, -\vec{b}_i)$ are the weight vectors of the 56, with $-\vec{b}_7$ as the highest weight.

The $D = 4$, $N = 8$ supergravity with manifest E_7 global symmetry was obtained by first dimensionally reducing eleven-dimensional supergravity, and then dualising the seven 2-form potentials $A_{(2)i}$ to give rise to additional scalars χ^i [10]. It is also necessary to dualise the twenty-one pseudo-vectors $A_{(1)ij}$ to give twenty-one vectors. Together with the seven Kaluza-Klein vectors, they form a 28-dimensional representation of $SL(8,\mathbb{R})$. The bosonic Lagrangian can then be written as [10]

$$\mathcal{L}_1 = e\,R + \tfrac{1}{4} e\,\mathrm{tr}(\partial_\mu \mathcal{M} \partial^\mu \mathcal{M}) + \tfrac{1}{8} e\, F^{ab}_{\mu\nu}\, *G^{\mu\nu}_{ab}\ , \tag{6.25}$$

where $\mathcal{M}$ parameterises the coset $SU(8)\backslash E_7$, constructed in section 4, and $F^{ab}_{\mu\nu}$, with indices $a,b = (i,8)$, are the field strengths of the twenty-eight vectors. $G^{ab}_{\mu\nu}$ is given by

$$*G^{ab}_{\mu\nu} = -4 \frac{\delta \mathcal{L}}{\delta F^{ab}_{\mu\nu}}\ , \tag{6.26}$$

and is therefore a linear combination of $F^{ab}_{\mu\nu}$ and $*F^{ab}_{\mu\nu}$. At the level of the Lagrangian, the global symmetry is $SL(8,\mathbb{R})$, which extends to E_7 at the level of the equations of motion, where the twenty-eight $F^{ab}_{\mu\nu}$ and twenty-eight $G_{ab\mu\nu}$ form a 56-dimensional representation of E_7. Writing

$$H_{(2)} = \begin{pmatrix} F \\ G \end{pmatrix}\ , \tag{6.27}$$

it was shown that they in fact satisfy the duality relation $H_{(2)} = \Omega\mathcal{M}{*}H_{(2)}$, with [10]

$$\Omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} . \tag{6.28}$$

This allows us to write down an E_7-invariant Lagrangian $\mathcal{L}_2$, where the G fields are regarded as independent of F. The Lagrangian is given by

$$\mathcal{L}_2 = e\,R + \tfrac{1}{4}e\,\mathrm{tr}(\partial_\mu\mathcal{M}\partial^\mu\mathcal{M}^{-1}) - \tfrac{1}{8}H^{\mathrm{T}}_{(2)}\mathcal{M}H_{(2)} \ . \tag{6.29}$$

6.3 Doubled Lagrangians

We now present a general proof that all the equations of motion following from the even-dimensional doubled Lagrangians that we have been discussing here, with a doubled set of potentials for the field strengths of degree $n = \frac{1}{2}D$, are indeed the same as the equations of motion from the original Lagrangians after we impose a universal twisted self-duality constraint. We have already seen that this is true for the equations of motion for the fields of degree n themselves; it remains to be established that the equations of motion for the other fields are the same, we shall now consider them. The structure of the "doubled" Lagrangians is

$$\mathcal{L}_2 = -\tfrac{1}{4\,n!}\,H^{\mathrm{T}}\mathcal{M}H + L(\phi) \ , \tag{6.30}$$

where ϕ denotes all the remaining fields, including the complete set of scalar fields, the metric $g_{\mu\nu}$ *etc.* with Lagrangian $L(\phi)$, and

$$H = \begin{pmatrix} F \\ G \end{pmatrix} . \tag{6.31}$$

Here $F = dA$ is written in terms of the original potentials A, while $G = dB$ is written in terms of the "doubled" potentials B. The fields H satisfy the Bianchi identity $dH = 0$ and equations of motion $d(\mathcal{M}{*}H) = 0$. Here the fields are real and the matrix $\mathcal{M} = \mathcal{V}^{\mathrm{T}}\eta\mathcal{V}$ is symmetrical.

We then impose the twisted self-duality constraint (coined some time ago a silver rule of supergravity)

$$H = \Omega\mathcal{M}{*}H \ . \tag{6.32}$$

Acting with another $*$, and using the fact that $**\,H = (-1)^{n-1}\,H$, this implies that it squares to a multiple of the unit matrix, $(\Omega\mathcal{M})^2 = (-1)^{n-1}\,I$. We also have $(\Omega\eta)^2 = (-1)^{n-1}$. We may use the constraint (6.32) to solve for the field strengths G in terms of F

and $*F$, giving $G = f(\phi)\, F + g(\phi) {*F}$. We may then write a Lagrangian purely in terms of the original fields in the form

$$\mathcal{L}_1 = \tfrac{1}{2\, n!}\, F \cdot {}^*G + L(\phi)\ , \tag{6.33}$$

where G is expressed in terms of F as above. It is obvious that the equations of motion for the $(D/2)$-form field strengths are the same for the Lagrangians (6.33) and (6.30). Note that if the solution for G is substituted into H given in (6.31), it then has the property that in any even dimension:

$$H^{\rm T} \mathcal{M} H = 0\ . \tag{6.34}$$

This can be seen by using (6.32) to re-express $H^{\rm T}\mathcal{M}H$ as the manifestly-vanishing expression $H^{\rm T} \wedge \Omega H$.

With these preliminaries, we are in a position to show that the equations of motion for the other fields ϕ that follow from (6.30), after imposing the constraint (6.32), are the same as the equations of motion that follow from (6.33). To do this we vary (6.34) with respect to ϕ, to get

$$\tfrac{1}{2} H^{\rm T} \frac{\delta \mathcal{M}}{\delta\phi}\, H + \frac{\delta H^{\rm T}}{\delta \phi}\, \mathcal{M} H = 0\ . \tag{6.35}$$

Now, we can use (6.32) in the second term, to give

$$\tfrac{1}{2} H^{\rm T} \frac{\delta \mathcal{M}}{\delta\phi}\, H = -\frac{\delta H^{\rm T}}{\delta \phi}\, \Omega {*H} = -\frac{\delta}{\delta\phi} (F\cdot {*G})\ . \tag{6.36}$$

It is now evident that $\delta\mathcal{L}_1/\delta\phi = \delta\mathcal{L}_2/\delta\phi$, and hence the scalar equations of motion from the two Lagrangians agree.[9]

To conclude this subsection, we present some results on duality in even dimensions. The maximal possible duality symmetry in an even dimension for a given set of N (prior to doubling) field strengths with degree $D/2$ depends on whether $D = 4k$ or $D = 4k + 2$; they are $Sp(2N)$ and $O(N, N)$ respectively [27]. (This generalises the $D = 4$ result in [29].) The maximal coset space for the scalar fields also depends on the spacetime signature. The cosets for the maximal duality symmetries are summarised below in Table 1.

	Lorentzian	Euclidean
$D = 4k$	$U(N)\backslash Sp(2N)$	$GL(N)\backslash Sp(2N)$
$D = 4k + 2$	$O(N) \times O(N)\backslash O(N, N)$	$O(N, C)\backslash O(N, N)$

[9] It is possible to add a linear term $H^{\rm T}X(\phi)$ to the Lagrangian (6.30) provided that $X(\phi)$ satisfies $*X\mathcal{M}\Omega X = 0$ and we modify the self-duality constraint (6.32). The proof is analogous, with appropriate minor modifications.

Table 1: Maximal global symmetry and scalar coset for N $D/2$-form field strengths

In Lorentzian spacetime, the stability group for the scalars is the maximal compact subgroup of the global symmetry group, whilst in Euclidean space, the denominator group is non-compact [30]. Also, in Euclidean space the kinetic terms for the axionic scalars, which couple to the tensor fields, have the opposite sign.

The introduction of the doubled formalism, where the Lagrangian $\mathcal{L}_2$ is invariant under the full duality group, allows us to define conserved Noether currents in the usual way. If we impose the self-duality constraint (6.32) on the current, it defines conserved currents for the original Lagrangian $\mathcal{L}_1$. In the case of $D = 4$, this procedure gives rise to the same currents as defined in [29]. These currents are non-local whenever the dual potentials appear explicitly in the expression. It is worth noting that even for the subgroup which leaves the original Lagrangian $\mathcal{L}_1$ invariant, this procedure does not reproduce the current defined directly from $\mathcal{L}_1$. They differ by the topological current

$$J^{\mu_1} \sim \epsilon^{\mu_1 \cdots \mu_D} \partial_{\mu_2} X_{\mu_3 \cdots \mu_D} \ . \tag{6.37}$$

6.4 Dualisations of higher-degree fields, and global symmetries

We have seen earlier in the paper that a convenient way to discuss the global symmetries of the dimensionally-reduced supergravities is to consider first the symmetries of the scalar manifold. The global symmetries of the scalar sector are themselves dependent on the choice of dualisations, in the sense that one obtains inequivalent symmetry groups if $(D-2)$-form potentials in the direct reduction are not dualised to scalars, or if existing scalars are "undualised" to give $(D-2)$-form potentials. These differences in the global symmetry groups persist even at the level of the equations of motion.

One might be tempted to think that the global symmetries of the scalar manifold would automatically extend to the entire theory, since the higher-degree field strengths transform linearly. However, this is not in general true. What is true is that in the toroidally dimensionally reduced theories coming from eleven-dimensional supergravity, there always exists a choice of dualisations for the higher-degree fields such that the global symmetries of the scalar sector do indeed extend to the full theory. In some cases, this will be true only at the level of the equations of motion, whilst in other cases, the symmetry is realised also at the level of the Lagrangian. What is perhaps more surprising is that there are examples where, even at the level of the equations of motion, the global symmetry of the scalar sector can be broken as a result of performing, or not performing, certain dualisations of the higher-degree fields. The basic reason for this is that it is not in general the case that the field equations and Bianchi identities involve only field strengths; bare potentials can occur too. In such cases the global symmetries must clearly be realised on the potentials themselves, which is possible in terms of local field transformations only if all the potentials associated with a would-be irreducible multiplet have the same degree.

A simple example arises in six dimensions, where in the direct reduction we have a total of $15 = 10 + 5$ vector potentials $A_{(1)ij}$ and $\mathcal{A}^i_{(1)}$, and one 3-form potential $A_{(3)}$. If the latter is dualised to a 1-form, then the sixteen vector potentials can form an irreducible 16 of $O(5,5)$, and the $O(5,5)$ global symmetry of the scalar manifold can then be realised also in the entire theory. Now let us consider the situation where instead the 3-form potential is not dualised. The 4-form field strength and the rest of the fifteen vector potentials could still form a 16-dimensional multiplet if it were the case that the potentials were all covered by derivatives, so that the symmetry transformations could be implemented on the sixteen field strengths. However, if the 3-form field strength is not dualised, the corresponding 4-form field strength $\hat{F}_{(4)}$ is given by (A.29). In these original variables, the $\mathcal{L}_{FFA}$ terms are trilinear and do not involve the Kaluza-Klein vectors $\mathcal{A}^i_{(1)}$. Thus indeed there are no bare vector potentials in the equation of motion for the field strength $\hat{F}_{(4)}$. However, the Bianchi identity for $\hat{F}_{(4)}$ does contain bare Kaluza-Klein potentials. On the other hand, by changing variables to the hatted potentials, where $F_{(4)}$ is given by (A.20), there will now be no Kaluza-Klein vectors in the Bianchi identity. Instead, the $\mathcal{L}_{FFA}$ terms now become complicated and they will involve bare Kaluza-Klein vector potentials. Either way, the occurrence of bare potentials in the equations of motion or Bianchi identities is unavoidable, and in fact there is no possible redefinition of fields that can circumvent the problem. Since one is therefore forced to realise the $O(5,5)$ on the sixteen potentials, rather than their field strengths, it is necessary to dualise $A_{(3)}$ to a vector in order to be able to give a realisation of the global symmetry in terms of local field transformations. Another example of this kind is discussed in section 9, where we observe that the $SL(2,\mathbb{R})$ symmetry of the type IIB theory is lost if one of its 3-form field strengths is dualised to a 7-form.

A contrasting example arises in $D = 7$. We saw in section 6.1 that the $SL(5,\mathbb{R})$ symmetry can be realised in the full theory at the level of the Lagrangian, provided that the 3-form potential $A_{(3)}$ arising from the direct reduction from $D = 11$ is dualised to yield a 2-form potential. There are then five 2-form potentials in total, which form an irreducible 5 of $SL(5,\mathbb{R})$. In fact one can make field redefinitions such that these potentials appear in the Lagrangian only via their field strengths. This means that one can perform arbitrary dualisations of these 3-form field strengths and the resulting theories, at the level of the equations of motion, will still have the unbroken global $SL(5,\mathbb{R})$ symmetry. (For the usual reasons, the symmetry can only be realised as local field transformations on the field strengths, in dualisation choices where 3-form and 4-form field strengths are to be assembled into an irreducible multiplet.)

In summary, we note that at the level of the Lagrangian the global symmetries must necessarily be realised on the potentials, since these are the fundamental fields. Consequently, the global symmetry will be broken if some members of an irreducible multiplet are dualised to fields of the dual degree, since then a realisation of the symmetry in terms of local field transformations becomes impossible. At the level of the equations of motion, on the other hand, the symmetries can instead be realised on the field strengths, provided that only the field strengths, and not their bare potentials, appear in the equations of motion and the Bianchi identities. The global symmetry can then be preserved under dualisations, as long as the dualisation of some members of an irreducible multiplet does not result in the appearance of bare potentials for any fields in the rest of the multiplet. Otherwise, the global symmetry will again be broken.

7 R-R dualisation

In the previous sections, we studied the global symmetries of toroidally-compactified eleven-dimensional supergravity. We have seen that the choice of whether or not to dualise certain field strengths can affect the global symmetries of the theories. At the level of the Lagrangian, we saw in section 2 that the undualised theories have a global $GL(11-D,\mathbb{R})\ltimes\mathbb{R}^q$ symmetry, which can be extended to E_{11-D} if all field strengths of ranks $> \frac{1}{2}D$ are dualised. In all cases where the full dualisation is performed, the $GL(11-D,\mathbb{R})$ symmetry is preserved, and now forms a subgroup of the enlarged E_{11-D} symmetry.

Of course, if we do not insist on obtaining a formulation of the theory with the E_{11-D} symmetry, then we can selectively perform dualisations on only a subset of the higher-degree field strengths. In [12] the case was considered where only those fields that correspond to type IIA string Ramond-Ramond fields were dualised. The motivation for this came from perturbative string theory, where the NS-NS gauge potentials, namely the two-form and the metric, couple to the string world-sheet directly, rather than through their field strengths. Thus in terms of a sigma model action we might not wish to dualise these NS-NS fields, in order to have global symmetries that act locally on the potentials that couple directly to the world-sheet. The R-R fields, on the other hand, couple in the world-sheet string action only through their field strengths, and hence these can be freely dualised without compromising the locality of the global symmetry transformations. In fact, the realisation of the perturbative T-duality symmetry $O(10-D, 10-D)$ of the toroidally-compactified type IIA theory requires only that the R-R fields be dualised, while the NS-NS fields can be left undualised [12].

If we insist that the NS-NS gauge potentials are left undualised, then the supergravity theories have the symmetries E_{11-D} for $D = 9$, 8 and 7, but $O(10-D, 10-D)\ltimes\mathbb{R}^q$ for $D \le 6$, where $q = 2^{8-D}$ [12]. The global symmetries in $D \ge 7$ are the same as in the fully dualised theories that we discussed previously, since it is only the R-R potential $A_{(3)}$ that suffers dualisation in these dimensions.

In $D = 6$, the usual $E_5 = O(5,5)$ symmetry of the fully-dualised theory requires the dualisation of all of the five 3-form field strengths, since they and their duals together comprise a 10-dimensional irreducible multiplet under $O(5,5)$. Since one of these 3-forms is an NS-NS field, the requirement that no NS-NS fields be dualised will break the $O(5,5)$ symmetry to $O(4,4)$. In fact the full global symmetry is now $O(4,4)\ltimes\mathbb{R}^8$. (The fact that the NS-NS 2-form potential need no longer be covered by a derivative allows a redefinition of fields in which all 8 R-R axions acquire shift symmetries.) This $O(4,4)\ltimes\mathbb{R}^8$ symmetry is actually a subgroup of E_5. This can be seen from the fact that the scalar sector in $D = 6$ is unaffected by the decision not to dualise the NS-NS fields, and that the global symmetry of the full equations of motion must be equal to or a subgroup of the global symmetry of the scalar sector.

In $D = 5$, the story is similar since the 3-form gauge potential that dualises to a scalar is an R-R field. Thus the scalar Lagrangian with only R-R dualisation is the same as the that for full dualisation, and hence the $O(5,5)\ltimes\mathbb{R}^{16}$ is a subgroup of E_6. In both $D = 5$ and $D = 6$, the E_{11-D} symmetry of the scalar sectors does not extend to the full theories with only R-R dualisations, even at the level of the equations of motion. Only the $O(4,4)\ltimes\mathbb{R}^8$ and $O(5,5)\ltimes\mathbb{R}^{16}$ subgroups remain as global symmetries.

The story changes in $D = 4$ and $D = 3$, in that the $O(6,6)\ltimes\mathbb{R}^{32}$ and $O(7,7)\ltimes\mathbb{R}^{64}$ symmetries for the R-R dualisations are not subgroups of E_7 and E_8. The reason for this can be seen by looking at the scalar sectors of the theories. In these cases, the scalar manifolds with only R-R dualisations are different from the scalar manifolds for the full dualisations, since there are NS-NS $(D-2)$-form potentials that will no longer be dualised to give additional scalars. In fact already in the scalar sectors, one now finds that the global symmetries are instead the $O(6,6)\ltimes\mathbb{R}^{32}$ and $O(7,7)\ltimes\mathbb{R}^{64}$ groups mentioned above. It is easy to see that these cannot be contained in E_7 and E_8, since these allow only the smaller groups $\mathbb{R}^{27}$ and $\mathbb{R}^{36}$ as maximal abelian subgroups.

In the rest of this section, we shall concentrate on studying the scalar sectors of the $D = 4$ and $D = 3$ theories obtained by performing only R-R dualisations. First, we show how to identify the R-R fields and the NS-NS fields. In D dimensions, the field content is given by (2.7). Now in $D = 10$ the metric, the dilaton and the 2-form gauge potential $A_{(2)1}$ are NS-NS fields, while the 3-form gauge potential $A_{(3)}$ and the vector potential $A^1_{(1)}$ are R-R fields. This separation into NS-NS and R-R fields is preserved under the subsequent steps of dimensional reduction. It follows that in the D-dimensional undualised theory the breakdown of the fields into NS-NS and R-R is as follows:

$$\mathrm{NS-NS}: \qquad A_{(2)1} \quad A_{(1)1\alpha} \quad A_{(0)1\alpha\beta} \quad \mathcal{A}^\alpha_{(1)} \quad \mathcal{A}^\alpha_{(0)\beta} \quad \vec{\phi} \quad g_{\mu\nu}\ , \tag{7.1}$$

$$\mathrm{R-R}: \qquad A_{(3)} \quad A_{(2)\alpha} \quad A_{(1)\alpha\beta} \quad A_{(0)\alpha\beta\gamma} \quad \mathcal{A}^1_{(1)} \quad \mathcal{A}^1_{(0)\alpha}\ , \tag{7.2}$$

where we have decomposed the internal index i as $i = (1, \alpha)$.

In $D = 4$, there are seven 2-form gauge potentials $A_{(2)i}$ which could be dualised to scalars, of which $A_{(2)1}$ is an NS-NS field while the six remaining potentials $A_{(2)\alpha}$ are R-R fields. Instead of dualising all seven, as we did in section 3.2, let us now only dualise the six R-R potentials, and so instead of introducing seven Lagrange multipliers for the Bianchi identities (3.13), we now introduce only six multipliers χ^α, for the Bianchi identities $d(\tilde\gamma^i{}_\alpha F_{(3)i}) = 0$. Thus we add Lagrange multiplier terms

$$\mathcal{L}_{\rm LM} = -d\chi^\alpha \wedge \tilde\gamma^i{}_\alpha F_{(3)i} \tag{7.3}$$

to the Lagrangian. We now repeat an analysis analogous to that in section 3.2, except that now we treat only the six fields $F_{(3)\alpha}$ as auxiliary. Solving algebraically for these, we find that $F_{(3)\alpha} = e^{-\vec{a}_\alpha\cdot\vec{\phi}}\, G^\alpha_{(1)}$, where

$$G^\alpha_{(1)} = \tilde\gamma^\alpha{}_\beta \Big(d\chi^\beta + \tfrac{1}{72} A_{(0)k\ell m}\, dA_{(0)npq}\, \epsilon^{\beta k\ell mnpq}\Big)\ . \tag{7.4}$$

(In deriving this, we made use of the fact that $\tilde\gamma^\alpha{}_1 = 0$.) After substituting back into the Lagrangian, the resulting theory is invariant under the transformations

$$\delta A_{(0)ijk} = c_{ijk}\ , \qquad \delta\chi^\alpha = k^\alpha - \tfrac{1}{72}\epsilon^{\alpha ijk\ell mn}\, c_{ijk}\, A_{(0)\ell mn}\ , \tag{7.5}$$

together with the usual $GL(6,\mathbb{R})$ transformations described by $\Lambda^\alpha{}_\beta$, and also those corresponding to the parameters $\Lambda^1{}_\alpha$. Note that the original $GL(7,\mathbb{R})$ breaks down to $GL(6,\mathbb{R})$, since $\chi^{(\alpha)}$ is invariant under $\delta_{\Lambda^1{}_\alpha}$. This invariance can be seen by considering the variation of $\mathcal{L}_{\rm LM}$ in (7.3) under the $\Lambda^1{}_\alpha$ transformations. Noting that $F_{(3)i}$ is invariant, as it is under all $GL(7,{\rm I\!R})$ transformations (since i here is a tangent-space index), and that from (A.15), $\delta\tilde\gamma^1{}_\alpha = \Lambda^1{}_\alpha$ and $\delta\tilde\gamma^\alpha{}_\beta = 0$, we see that if χ^α is invariant then $\mathcal{L}_{\rm LM}$ transforms by

$$\delta\mathcal{L}_{\rm LM} = -d\chi^\alpha\,\wedge F_{(3)1}\,\Lambda^1{}_\alpha\ . \tag{7.6}$$

But $F_{(3)i} = \gamma^j{}_i\,\hat F_{(3)j}$, implying, since $\gamma^j{}_1 = \delta^j_1$, that $F_{(3)1} = \hat F_{(3)1}$. Hence from (A.29) we have that $F_{(3)1} = dA_{(2)1}$ when the vectors are set to zero (as we are assuming here since we are simplifying the discussion by focussing on the scalar sector), and consequently, we see that (7.6) is a total derivative. This shows that the fields χ^α are indeed inert under $\Lambda^1{}_\alpha$ transformations.

The non-trivial commutation relations in this case are

$$[\delta_c,\delta_{c'}] = \delta_k\ ,\qquad k^\alpha = \tfrac{1}{36}\epsilon^{\alpha ijk\ell mn}\,c_{ijk}\,c'{}_{\ell mn}\ . \tag{7.7}$$

Putting all this together, we find a maximal abelian symmetry of dimension 32, corresponding to the parameters

$$\Lambda^1{}_\alpha\ ,\quad k^\alpha\ ,\quad c_{\alpha\beta\gamma}\ . \tag{7.8}$$

Note that this ${\rm I\!R}^{32}$ symmetry corresponds precisely to the shift symmetry of the full set of R-R scalars, namely six $\mathcal{A}^1_{0\alpha}$, six χ^α and twenty $A_{(0)\alpha\beta\gamma}$. Note also that although we simplified the discussion by setting the vectors to zero, the result remains true in the full theory [12]. This is because all the R-R fields can be covered by derivatives simultaneously in $D=10$ already.

From the point of view of the coset construction described in sections 4 and 5, the generator D_1 in the positive-root algebra of E_7, which was associated with the axion dual to the NS-NS potential $A_{(2)1}$, can be rescaled and the scale factor sent to zero. After this contraction of the algebra, we can consistently truncate the generator, resulting in a theory with an $O(6,6)\ltimes{\rm I\!R}^{32}$ global symmetry. This procedure of dualisation can be also understood from point of view of a double coset. With respect to the simple root $\vec r_2 = \vec b_{12}$, the 63 positive roots of E_7 are graded as 1 level-2 root, associated with D_1, 32 level-1 roots and 30 level-0 roots. Thus the double coset can be denoted by $SU(8)\backslash E_7/N_2^{\vec r_2}$, implying that the generator D_1 of the Borel group is contracted and truncated out.

The analysis in $D=3$ is similar. In the undualised theory we have 28 $A_{(1)ij}$ and 8 $\mathcal{A}^i_{(1)}$ potentials, which could be dualised to axionic scalars. If we instead dualise only the R-R subset, namely the $A_{(1)\alpha\beta}$ and $\mathcal{A}^1_{(1)}$ potentials, then the theory will have an $O(7,7)\ltimes{\rm I\!R}^{64}$ global symmetry, with a maximal abelian subalgebra ${\rm I\!R}^{64}$ corresponding to the shift symmetries of the full set of 64 R-R axionic scalars. In fact the remaining undualised 14 NS-NS vectors correspond to the 14 level-2 generators of the Borel group of E_8, with respect to the simple root $\vec{r}_2$, and hence the dualisation is equivalent to the double coset $SO(16)\backslash E_8/N_2^{\vec{r}_2}$.

Note that in $D = 4$ and $D = 3$ the associated $O(6,6)\ltimes \mathbb{R}^{32}$ and $O(7,7)\ltimes \mathbb{R}^{64}$ symmetries are perturbative in nature, from the point of view of string theory. In fact these theories, where only R-R fields have been dualised to give additional axions, do not have any non-perturbative symmetries. Such symmetries can arise in the scalar Lagrangian only when some of the NS-NS $(D-2)$-form gauge potentials are dualised to scalars, in the process of which the scalar manifold is changed.

So far in this paper, we have considered the global symmetries of the maximal supergravities coming directly from the dimensional reduction of eleven-dimensional supergravity. We have also considered how these symmetries are modified when we dualise either the full set of higher-rank field strengths, or alternatively just the R-R subset. In particular, in $D = 4$ or $D = 3$ dimensions various different choices can be made, depending on which set of $(D-2)$-form gauge potentials are dualised to give additional axionic scalars. Each different choice can give rise to a new version of the supergravity, with a different global symmetry, whose positive-root algebra can be understood from the contraction and truncation of the E_{11-D} symmetry that arises in the case of full dualisation. In fact a useful way to discuss the global symmetries in the different dualisations is to begin by considering the fully-dualised theories with the E_{11-D} symmetries, and then pass to the other cases by "undualising" certain fields, or in some circumstances, deliberately raising the rank of field strengths by further dualisations. There are many more possibilities than the no-dualisation, R-R dualisation and full-dualisation examples that we have considered so far. Another example is the following. In $D = 4$ dimensions all 28 vector potentials in the Lagrangian can be covered simultaneously by derivatives, implying that there can be a commuting $\mathbb{R}^{56}$ symmetry in $D = 3$, realised by the the 56 scalars coming from the dimensional reduction of the 28 vectors in $D = 4$. (28 arise as scalars already, and the remaining 28 come from dualising the 28 vectors in $D = 3$.) In this case, the Kaluza-Klein vector arising from the metric in the reduction from $D = 4$ to $D = 3$ can no longer be dualised. In this version of $D = 3$ supergravity we therefore have an $E_7\ltimes \mathbb{R}^{56}$ global symmetry, since the dimensional reduction and the dualisation preserve the E_7 symmetry that was already present in $D = 4$. In $D = 4$ the 28 vectors and their duals formed a 56 of E_7; in $D = 3$ they have reduced to 56 axions that again form a 56 of E_7.

8 Abelian symmetries in maximal supergravities

A global $\mathbb{R}$ symmetry in a toroidally-compactified supergravity corresponds to a constant shift symmetry of an axion. We can choose a basis where this axion is covered by a derivative everywhere in the Lagrangian or the equations of motion. A set of abelian $\mathbb{R}$ symmetries arises when a set of axions can all be covered by derivatives simultaneously. It is of interest to look for the maximal such sets of commuting $\mathbb{R}$ symmetries. One application is for the construction of massive supergravities in lower dimensions, which can be obtained by performing a generalised Scherk-Schwarz reduction in which an axion is allowed an additional linear dependence on the compactification coordinate [31, 32, 33, 34]. Such reductions can be simultaneously performed on each of a set of axions that have commuting $\mathbb{R}$ symmetries, and thus it is useful to identify the maximal such set.

The dimensions of the maximal abelian subgroups for simple Lie groups are well understood by mathematicians. They are given by [17]

$$
\begin{aligned}
&\quad\text{Simply-laced :}\\
A_n:&\quad \Big[\tfrac{1}{4}n(n+2)\Big]\ ,\qquad D_n:\quad \tfrac12 n(n-1)\ ,\\
E_6:&\quad 16\ ,\qquad E_7:\quad 27\ ,\qquad E_8:\quad 36\ .\\
&\\
&\quad\text{Non-simply-laced :}\\
B_n:&\quad 1+\tfrac12 n(n-1)\ ,\quad n\ge 4,\quad \text{with 3 and 5 for } B_2 \text{ and } B_3\ ,\\
C_n:&\quad \tfrac12 n(n+1)\ ,\qquad F_4:\quad 10\ ,\qquad G_2:\quad 3\ .
\end{aligned}
\tag{8.1}
$$

Thus it straightforward to obtain the maximal $\mathbb{R}^q$ symmetries for the fully-dualised maximal supergravities which have E_{11-D} global symmetries, namely $q=\{1,3,6,10,16,27,36\}$ for dimensions $D=\{9,8,7,6,5,4,3\}$. The identification of the sets of axions that realise these maximal abelian $\mathbb{R}$ symmetries was studied in [22, 23], using the method of solvable Lie algebras. The conclusion is that for $D\ge 4$, the maximal $\mathbb{R}$ symmetry can be realised by all the "new" axions in D dimensions that do not exist in $(D+1)$ dimensions. In other words, these are the axions coming from the dimensional reduction of the vectors in $(D+1)$ dimensions. This counting for maximal abelian $\mathbb{R}$ symmetries breaks down in $D=3$, where the 36 axions are given by the dualisations of the 36 vectors, as we saw in section 3.

In this section, we shall investigate maximal abelian $\mathbb{R}$ symmetries (but not necessarily those of maximal dimension) in the toroidal compactifications of eleven-dimensional supergravity. We shall prove that maximal abelian subalgebras of the positive-root algebra correspond precisely to sets of axions that can be simultaneously covered by derivatives everywhere in the Lagrangian. Then the abelian $\mathbb{R}$ symmetries in D-dimensional supergravity can be analysed by studying the abelian subalgebras of the associated positive-root algebra.

8.1 $D \geq 6$

First let us consider $D \geq 6$, where there is no complication from dualisations involving scalars. The axions in these cases are given by $A_{(0)ijk}$ and $\mathcal{A}^i_{0j}$. The scalar Lagrangian is given by (A.4) with all the higher-forms set to zero. The non-linear Kaluza-Klein modifications for $F_{(1)ijk}$ and $\mathcal{F}^i_{(1)j}$ are given in (A.12). To begin with, we work to bilinear order in fields:

$$\begin{aligned} F_{(1)ijk} &= dA_{(0)ijk} - \mathcal{A}^\ell_{(0)i}\, dA_{(0)\ell jk} - \mathcal{A}^\ell_{(0)j}\, dA_{(0)i\ell k} - \mathcal{A}^\ell_{(0)k}\, dA_{(0)ij\ell} + \cdots \ , \\ \mathcal{F}^i_{(1)j} &= d\mathcal{A}^i_{(0)j} - \mathcal{A}^k_{(0)j}\, d\mathcal{A}^i_{0k} + \cdots \ . \end{aligned} \tag{8.2}$$

From these bilinear terms, it is manifest that we cannot cover $A_{(0)\ell jk}$ and $\mathcal{A}^\ell_{(0)i}$ with derivatives simultaneously, implying that their $\mathbb{R}$ symmetries are non-commuting. Similarly, we cannot cover $\mathcal{A}^i_{(0)k}$ and $\mathcal{A}^k_{(0)j}$ with derivatives simultaneously.

The above observation is closely related to the positive-root algebra of the theory. As we saw in section 3, the positive-root algebra can be translated via the Jacobi identities to a set of summation rules for the dilaton vectors of the axions, which are the positive roots of the global symmetry algebra. If the sum of any two dilaton vectors gives rise to a third one, then the associated generators of the positive-root algebra do not commute; otherwise, they do commute. The dilaton vectors for $A_{(0)ijk}$ and $\mathcal{A}^i_{(0)j}$ are $\vec{a}_{ijk}$ and $\vec{b}_{ij}$ respectively. Thus we see that the non-commutativities of the shift symmetries of axions in (8.2) are exactly equivalent to the non-commutativity of the corresponding root vectors, *i.e.*

$$\vec{b}_{ij} + \vec{b}_{jk} = \vec{b}_{ik} \ , \qquad \vec{a}_{ijk} + \vec{b}_{i\ell} = \vec{a}_{\ell jk} \ , \tag{8.3}$$

which is already given in (4.2). Thus in order to have axions with commuting $\mathbb{R}$ symmetries, we must choose a subset whose dilaton vectors correspond to positive roots that commute. In other words, since (8.3) defines the algebra of the root system, we must choose a subset of the axions such that their dilaton vectors cannot, using the summation rules in (8.3), generate any other dilaton vectors for any axions.

The above argument concentrated on the bilinear terms in the non-linear Kaluza-Klein modifications. However, it is clear from the chain structure of the higher-order terms in

$\gamma^i{}_j$ given in (A.13) that any subset of axions that have commuting $\mathbb{R}$ symmetries at the bilinear level will continue to have commuting $\mathbb{R}$ symmetries when the higher-order terms are included as well.

We shall now illustrate this with a few examples. The first non-trivial case occurs in $D=8$, where the global symmetry is given by $E_3 = SL(3,\mathbb{R}) \times SL(2,\mathbb{R})$. The association of the dilaton vectors, positive roots and axions is given in the following table:

Dilaton Vectors	ℓ_1 ℓ_2 ℓ_3	Axions
$\vec{b}_{13}$	0 1 1	$\mathcal{A}^1_{(0)3}$
$\vec{b}_{23}$	0 0 1	$\mathcal{A}^2_{(0)3}$
$\vec{b}_{12}$	0 1 0	$\mathcal{A}^1_{(0)2}$
$\vec{a}_{123}$	1 0 0	$A_{(0)123}$

Table 2: Dilaton vectors, positive roots and axions in $D=8$

The entries in the second column denote the coefficients ℓ_i in the expressions $\sum_i \ell_i\, \vec{r}_i$ for the positive roots. Thus there are two ways to get the maximal abelian commuting $\mathbb{R}$ symmetry, which has three commuting generators:

$$\{\vec{b}_{13}, \vec{b}_{23}, \vec{a}_{123}\} \leftrightarrow \{\mathcal{A}^1_{(0)3}, \mathcal{A}^2_{(0)3}, A_{(0)123}\} \tag{8.4}$$

or

$$\{\vec{b}_{13}, \vec{b}_{12}, \vec{a}_{123}\} \leftrightarrow \{\mathcal{A}^1_{(0)3}, \mathcal{A}^1_{(0)2}, A_{(0)123}\}\ . \tag{8.5}$$

In each case, these sets of three axions can all be covered by derivatives simultaneously in the Lagrangian. In case 1, given by (8.4), all the three axions carry an index 3, implying that they are the new ones arising from the reduction from $D=9$. In this case, there is one R-R axion $\mathcal{A}^1_{(0)3}$, while the other two are NS-NS fields. This set of maximal abelian $\mathbb{R}$ symmetries was also found in [22, 23]. In case 2, given by (8.5), we have only one NS-NS axion, namely $A_{(0)123}$, while the other two are R-R fields. This second $\mathbb{R}^3$ symmetry can be understood more generally from the fact that the theory has a global $\mathbb{R} \times SL(3,\mathbb{R})$ symmetry, which is a subgroup of E_3, where $\mathbb{R}$ is realised on the axion $A_{(0)123}$, and hence commutes with $SL(3,\mathbb{R})$.

Another example that can be presented in detail is in $D=7$. The global symmetry is $E_4 = SL(5,\mathbb{R})$; the dilaton vectors, positive roots and axions are summarised in table 3:

Dilaton Vectors	ℓ_1 ℓ_2 ℓ_3 ℓ_4	Axions
$\vec{a}_{234}$	1 1 1 1	$A_{(0)234}$
$\vec{a}_{134}$	1 0 1 1	$A_{(0)134}$
$\vec{b}_{14}$	0 1 1 1	$\mathcal{A}^1_{(0)4}$
$\vec{a}_{124}$	1 0 0 1	$A_{(0)124}$
$\vec{b}_{24}$	0 0 1 1	$\mathcal{A}^2_{(0)4}$
$\vec{b}_{13}$	0 1 1 0	$\mathcal{A}^1_{(0)3}$
$\vec{b}_{34}$	0 0 0 1	$\mathcal{A}^3_{(0)4}$
$\vec{b}_{23}$	0 0 1 0	$\mathcal{A}^2_{(0)3}$
$\vec{b}_{12}$	0 1 0 0	$\mathcal{A}^1_{(0)2}$
$\vec{a}_{123}$	1 0 0 0	$A_{(0)123}$

Table 3: Dilaton vectors, positive roots and axions in $D = 7$

Sets of axions with abelian $\mathbb{R}$ symmetries can be associated with sets of positive roots that all have a 1 entry in one of the columns of coefficients ℓ_i. This is because there is no coefficient $2 = 1 + 1$ for any of the positive roots. The maximal abelian symmetry is $\mathbb{R}^6$, which can be realised in two different ways, namely by looking at the positive roots with a 1 in column 4 or in column 3. The six axions associated with commuting positive roots determined by column 4 all have an index 4, implying that these are the new axions appearing in the reduction from $D = 8$ to $D = 7$. This set comprises two R-R and four NS-NS axions. This set was also obtained in [22, 23]. The column-3 commuting roots represent a new, alternative, choice for realising the $\mathbb{R}^6$ symmetry. In this case, there are three R-R axions and three NS-NS axions.

Looking instead at column 1, we see that the four axions $A_{(0)ijk}$ have a commuting $\mathbb{R}^4$ symmetry. This is precisely the $\mathbb{R}^4$ in $GL(4,\mathbb{R})\ltimes\mathbb{R}^4$ discussed in section 2. Note that once all the four $A_{(0)ijk}$ axions are covered by derivatives everywhere in the Lagrangian, no further axions can be covered. Although the maximal abelian symmetry is $\mathbb{R}^6$, we cannot extend this $\mathbb{R}^4$ any further. This can be understood from the fact that $\mathbb{R}^4$ is the maximal abelian subalgebra in $GL(4,\mathbb{R})\ltimes\mathbb{R}^4$, which itself is a maximal subalgebra of E_4. Note that as we remarked in section 2, the full $D = 7$ Lagrangian, when the higher-form potentials are not dualised, has a global $GL(4,\mathbb{R})\ltimes\mathbb{R}^4$ symmetry. The extension to an E_4 symmetry can be done at the level of the equations of motion, or at the level of the Lagrangian if the 4-form field strength is dualised to a 3-form field strength.

Finally, looking at column 2, we can see that the four R-R axions, namely $A_{(0)234}$ and $\mathcal{A}^1_{(0)\alpha}$ have commuting $\mathbb{R}^4$ symmetries, which also cannot be extended to $\mathbb{R}^6$. This corresponds to another maximal subalgebra of E_4, namely $O(3,3)\ltimes\mathbb{R}^4$, and $\mathbb{R}^4$ is the maximal abelian subalgebra of $O(3,3)\ltimes\mathbb{R}^4$. This example follows the general pattern where only the R-R fields are dualised [12], which was discussed in section 4.

In $D=6$ there are a total of 20 axions, corresponding to the 20 positive roots of the $E_5=D_5$ group. We shall not present the details, because of the large number of axions that are involved, but the analysis is analogous to the previous two examples. The maximal abelian symmetry for D_5 is $\mathbb{R}^{10}$, and can be realised in two ways. One is the set of axions that all carry an index 5, *i.e.* the new ones in $D=6$. In this case, there are four R-R axions and six NS-NS axions. Another way to realise the $\mathbb{R}^{10}$ is by the set of all the 10 axions $A_{(0)ijk}$, which is also a maximal abelian symmetry of $GL(5,\mathbb{R})\ltimes\mathbb{R}^{10}$. In this case, they also comprise four R-R axions and six NS-NS axions. We can instead easily read off an $\mathbb{R}^8$ symmetry, realised by all the eight R-R axions. This $\mathbb{R}^8$ is the maximal abelian subalgebra of $O(4,4)\ltimes\mathbb{R}^8$.

Another interesting possibility in $D=6$ is to dualise one of the original axions, namely $A_{(0)345}$, to a 4-form potential. By inspecting the list of positive roots and their association with dilatons, analogous to Tables 2 and 3 in $D=8$ and $D=7$, one can see that having removed $A_{(0)345}$ from the set, the remaining positive roots now have an enlarged maximal abelian subalgebra, yielding an $\mathbb{R}^{12}$ symmetry. This is realised as shift symmetries on the 12 axions $A_{(0)1\alpha\beta}$, $A_{(0)2\alpha\beta}$, $\mathcal{A}^1_{(0)\alpha}$ and $\mathcal{A}^2_{(0)\alpha}$, where $\alpha,\beta=3,4,5$. The abelian symmetry in this case is larger than the $\mathbb{R}^8$ of the R-R dualistion or the $\mathbb{R}^{10}$ of the no-dualisation or full-dualisation versions of the theory.

8.2 $3\le D\le 5$

In these lower dimensions, the theories contain $(D-2)$-form gauge potentials that can be dualised to give additional axions. We shall show that the abelian $\mathbb{R}$ symmetries in these cases are also governed by the algebras of the positive-root systems. In $D=5$, let us consider the scalar Lagrangian where the 3-form gauge potential is dualised to a scalar. It follows from (3.6) that the axions $A_{(0)ijk}$ and $A_{(0)\ell mn}$ cannot be covered by derivatives simultaneously if $ijk\ell mn$ are all different, implying that the corresponding shift symmetries are non-commuting. This non-commutativity is precisely equivalent to the non-commutativity implied by the sum rules for their associated dilaton vectors, given by (4.25). In $D=4$, it follows from (3.16) that the non-commutativity of the $\mathbb{R}$ symmetries of the axions $\{A_{(0)ijk}, A_{(0)\ell mn}\}$ is described by the sum rules for their associated positive roots, given by (3.16). The story is similar in $D = 3$, where the non-commutativity for the axions $\{A_{(0)ijk}, A_{(0)\ell mn}\}$ and for $\{A_{(0)ijk}, \lambda_{jk}\}$, which can be seen from (3.31) and (3.32), is also implied by the sum rules for their dilaton vectors, given by (4.33).

Having established the equivalence of the non-commutativity of the axionic shift symmetries and the sum rules for the associated positive roots in $3 \le D \le 5$, the task of finding the maximal sets of axions that can be simultaneously covered by derivatives becomes equivalent to that of finding the maximal numbers of commuting positive roots. It is straightforward to verify that this leads to exactly the same counting of commuting $\mathbb{R}$ symmetries as we obtained in section 2, with an identical identification of the axions on which the symmetries are realised.

The above discussion was focussed on the versions of the supergravities where all the $(D-2)$-form potentials are dualised to scalars. However, the same methods can also be applied to the non-dualised or partially dualised cases. The key point is that if one of the $(D-2)$-forms is not dualised, then its associated dilaton vector should not be included as a positive root. For example in $D = 5$, if the 3-form gauge potential is not dualised, then $-\vec{a}$ is not a positive root in the system, and hence the sum rule $\vec{a}_{ijk} + \vec{a}_{\ell mn} = -\vec{a}$ no longer implies that the sum of $\vec{a}_{ijk}$ and $\vec{a}_{\ell mn}$ gives rise to another positive root. The consequent commutativity implies that there will be a global $\mathbb{R}^{20}$ symmetry, which is the maximal abelian subalgebra of $GL(6,\mathbb{R})\ltimes\mathbb{R}^{20}$, the symmetry group of the undualised theory. A similar analysis applies to $D = 4$ and $D = 3$, where the maximal abelian $\mathbb{R}$ symmetries can be read off from the sum rules for the dilaton vectors for any choice of dualisations, including for example the no-dualisation or R-R dualisation possibilities.

In summary, we have shown that we can read off the abelian symmetries from the E_{11-D} positive-root algebra of the fully-dualised theories. The maximal abelian symmetries (of largest dimension for each particular dualisation choice) of the various versions, including non-dualised, fully-dualised and R-R dualised cases, are given in Table 4 below.

Dim.	10	9	8	7	6	5	4	3
No dual	1	2	3	4	10	20	35	56
R-R dual	1	2	3	6	8	16	32	64
Full dual	1	2	3	6	10	16	27	36
Max.	1	2	3	6	12	21	35	64

Table 4: Maximal $\mathbb{R}$ symmetries

We have also listed in the row denoted "Max" the largest maximal abelian symmetry that can be achieved in any of the various versions of the theories. The $\mathbb{R}^{12}$ and $\mathbb{R}^{21}$ cases in $D=6$ and $D=5$ occur in the direct dimensional reduction of type IIB supergravity, which we shall discuss in the next section.

9 Dimensional reduction of type IIB supergravity

As is well known, the toroidal dimensional reductions of type IIB supergravity are equivalent to those of the type IIA theory, and are simply related by field redefinitions. In fact there are two different routes that one can follow in the descent from the type IIB theory to $D=9$. This is because there is a self-dual 5-form field strength in type IIB, which, under Kaluza-Klein reduction, can give either a 5-form field strength or a 4-form field strength (but not both) in $D=9$, depending upon how the reduction is performed.[10] If the latter choice is made, the fields in $D=9$ are precisely those of the $D=9$ reduction from $D=11$, modulo simple local field redefinitions. If, on the other hand, the reduction to $D=9$ is expressed in terms of the 5-form field strength, the theory is then related by non-local field redefinitions involving the 4-form potential. In fact, it is the theory one would get from $D=11$ by choosing to perform an inverse dualisation of the 3-form potential after reduction to $D=9$. Subsequent direct dimensional reductions without any dualisations will then continue to yield versions of the lower-dimensional supergravities that can be interpreted as specific inverse dualisations of the supergravities that we discussed in the previous sections.

The global symmetries of these direct type IIB reductions will be $(SL(2,\mathbb{R})\times GL(9-D,\mathbb{R}))\ltimes\mathbb{R}^r$, where the $SL(2,\mathbb{R})$ factor is directly inherited from the $SL(2,\mathbb{R})$ symmetry of type IIB in $D=10$, the $GL(9-D,\mathbb{R})$ is the usual global symmetry from toroidal reduction as discussed in section 2. (The symmetry is $GL(9-D,\mathbb{R})$ rather than $GL(10-D,\mathbb{R})$ because of a breaking of the symmetry by the self-duality condition on the 5-form in $D=10$.) The abelian factor $\mathbb{R}^r$ describes the maximal commuting shift symmetries of the axions coming from the antisymmetric tensors. The values of r in each dimension are $r=\{0,1,3,6,12,21,35,57\}$ in $D=\{10,9,8,7,6,5,4,3\}$. We shall discuss the origin of these $\mathbb{R}$ symmetries later in the section. One reason for looking at these reductions of the type IIB theory is that they can sometimes give rise to maximal abelian shift symmetries that are larger than the ones obtained via the type IIA route. For example, as we mentioned

[10]More precisely, the self-duality of the 5-form field strength in $D=10$ implies that its 5-form and 4-form reductions are related by Hodge dualisation in $D=9$. The Lagrangian in $D=9$ can be formulated using the potential for either one or the other of these as the fundamental field.

in section 8 there is a version of six-dimensional supergravity in which there are 12 abelian symmetries, exceeding the 10, 8 and 10 abelian symmetries of the non-dualised, R-R dualised and fully-dualised versions from $D = 11$. This new version is in fact nothing but the direct reduction of the type IIB theory, although it can of course instead be understood as a specific inverse dualisation of the type IIA reduction.

The bosonic sector of type IIB supergravity comprises the metric, a dilaton ϕ, an axion χ, two 2-form potentials $A^{(i)}_{(2)}$, and a 4-form potential whose associated field strength is self dual. The 4-form potential, χ and $A^{(2)}_{(2)}$ are R-R fields, and the remainder are NS-NS. Owing to the self-duality of the 5-form field strength, there is no simple way to write a covariant Lagrangian for these fields alone. However by adding extra degrees of freedom, namely by removing the self-duality condition, one can write a Lagrangian whose equation of motion yield the type IIB equations after imposing the self-duality constraint as a consistent truncation [28]. (A similar technique was used in section 6 when we constructed Lagrangians for the higher-degree fields in even dimensions.) Thus our starting point is the Lagrangian

$$\begin{aligned}
\mathcal{L} &= eR + \tfrac14 e\,\mathrm{tr}(\partial_\mu \mathcal{M}^{-1}\,\partial^\mu \mathcal{M}) - \tfrac{1}{12} e\, H^T_{(3)}\,\mathcal{M}\,H_{(3)} - \tfrac{1}{240} e\, H^2_{(5)} \\
&\quad - \tfrac{1}{2\sqrt2}\epsilon_{ij}\, B_{(4)}\wedge dA^{(i)}_{(2)}\wedge dA^{(j)}_{(2)}\ , \\
&= eR - \tfrac12 e\,(\partial\phi)^2 - \tfrac12 e\, e^{2\phi}\,(\partial\chi)^2 - \tfrac{1}{12} e\, e^{-\phi}\,(F^{(1)}_{(3)})^2 - \tfrac{1}{12} e\, e^{\phi}\,(F^{(2)}_{(3)})^2 \\
&\quad - \tfrac{1}{240} e\, H^2_{(5)} - \tfrac{1}{2\sqrt2}\epsilon_{ij}\, B_{(4)}\wedge dA^{(i)}_{(2)}\wedge dA^{(j)}_{(2)}\ , \qquad (9.1)
\end{aligned}$$

where

$$\mathcal{M} = \begin{pmatrix} e^{-\phi} + \chi^2\, e^{\phi} & -\chi\, e^{\phi} \\ -\chi\, e^{\phi} & e^{\phi} \end{pmatrix}\ , \qquad H_{(3)} = \begin{pmatrix} dA^{(1)}_{(2)} \\ dA^{(2)}_{(2)} \end{pmatrix}\ . \qquad (9.2)$$

The field strengths appearing in (9.1) are defined as follows:

$$F^{(1)}_{(3)} = dA^{(1)}_{(2)}\ , \qquad F^{(2)}_{(3)} = dA^{(2)}_{(2)} - \chi\, dA^{(1)}_{(2)}\ , \qquad H_{(5)} = dB_{(4)} + \tfrac{1}{2\sqrt2}\epsilon_{ij} A^{(i)}_{(2)}\wedge dA^{(j)}_{(2)}\ . \quad (9.3)$$

The Lagrangian is manifestly $SL(2,\mathbb{R})$-invariant.

The equations of motion following from (9.1) are

$$\begin{aligned}
& R_{\mu\nu} = \tfrac12 \partial_\mu\phi\,\partial_\nu\phi + \tfrac12 e^{2\phi}\,\partial_\mu\chi\,\partial_\nu\chi + \tfrac{1}{48}(H^2_{\mu\nu} - \tfrac{1}{10} H^2_{(5)}\, g_{\mu\nu}) \qquad (9.4)\\
& + \tfrac14 e^{-\phi}((F^{(1)}_{(3)})^2_{\mu\nu} - \tfrac{1}{12}(F^{(1)}_{(3)})^2 g_{\mu\nu}) + \tfrac14 e^{\phi}((F^{(2)}_{(3)})^2_{\mu\nu} - \tfrac{1}{12}(F^{(2)}_{(3)})^2 g_{\mu\nu})\ , \\
& d * H_{(5)} = \tfrac{1}{2\sqrt2}\epsilon_{ij}\, F^{(i)}_{(3)}\wedge F^{(j)}_{(3)}\ , \qquad (9.5)\\
& d * (e^{-\phi}\, F^{(1)}_{(3)} - e^{\phi}\,\chi\, F^{(2)}_{(3)}) = -\tfrac{1}{\sqrt2} H_{(5)}\wedge (F^{(2)}_{(3)} + \chi\, F^{(1)}_{(3)})\ , \qquad (9.6)\\
& d * (e^{\phi}\, F^{(2)}_{(3)}) = \tfrac{1}{\sqrt2} H_5 \wedge F^{(1)}_{(3)}\ , \qquad (9.7)\\
& \nabla^\mu (e^{2\phi}\,\partial_\mu\chi) = -\tfrac16\, e^{\phi}\, F^{(1)}_{\mu\nu\rho}\, F^{(2)\mu\nu\rho}\ , \qquad (9.8)\\
& \Box\phi = e^{2\phi}\,(\partial\chi)^2 + \tfrac{1}{12}\, e^{-\phi}\,(F^{(1)}_{(3)})^2 - \tfrac{1}{12}\, e^{\phi}\,(F^{(2)}_{(3)})^2\ , \qquad (9.9)
\end{aligned}$$

where $\Box\phi \equiv \nabla^\mu \partial_\mu \phi$.

Note that the equation (9.5) for $H_{(5)}$ implies, since we also have the Bianchi identity $dH_{(5)} = \frac{1}{2\sqrt{2}}\epsilon_{ij}\, F_{(3)}^{(i)} \wedge F_{(3)}^{(j)}$, that we can consistently impose the self-duality condition $H_{(5)} = {*H_{(5)}}$. After doing this, the equations (9.4)-(9.9) become precisely the field equations of type IIB supergravity [35]. Note also that the self-duality constraint preserves the $SL(2,\mathbb{R})$ symmetry, since H_5 is a singlet under $SL(2,\mathbb{R})$. The equations of motion for the 3-forms $F_3^{(i)}$ can be written more elegantly as

$$d*(\mathcal{M}\, H_{(3)}) = -\frac{1}{\sqrt{2}} H_{(5)} \wedge \Omega H_{(3)}\ , \qquad\qquad \Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\ . \tag{9.10}$$

Let us now consider some possible dual formulations of the type IIB theory, and their global symmetries. One possibility involves replacing the axion χ by an 8-form potential. Such a dualisation is discussed in the context of a pure (ϕ, χ) matter system in Appendix B, where it is shown that the original global $SL(2,\mathbb{R})$ symmetry of the scalar manifold is broken to a global $\mathbb{R}$ symmetry. Another possibility is to consider dualising one or both of the 2-form potentials $A_{(2)}^{(i)}$. These both enter the field equations and Bianchi identities only via their field strengths $F_{(3)}^{(i)}$. However, as we discussed earlier in the paper, a set of fields cannot necessarily be simultaneously dualised merely because they appear in the equations of motion and Bianchi identities only through their field strengths. Rather, we use the (sufficient) property that their bare potentials be absent also in the Lagrangian. This criterion is not satisfied simultaneously by the potentials $A_{(2)}^{(i)}$, on account of the "Chern-Simons" modification to the field strength $H_{(5)}$, given in (9.3). Although the simultaneous dualisation of the two potentials $A_{(2)}^{(i)}$ is therefore not possible in this way, we can nevertheless dualise either one or the other. Having done so, the bare potential of the undualised field will appear in the new equations of motion and Bianchi identities. Consequently, the original $SL(2,\mathbb{R})$ global symmetry will be broken, since it can no longer be realised in terms of any local transformations on the new set of fields. In fact the unbroken global symmetry will be $\mathbb{R}$ if the NS-NS 2-form $A_{(2)}^{(1)}$ is dualised, or $\mathbb{R}\ltimes\mathbb{R}$ if instead the R-R 2-form $A_{(2)}^{(2)}$ is dualised.

We now turn to the toroidal dimensional reduction of the type IIB theory. Normally, when a field strength of degree n is dimensionally reduced on a circle, it yields two lower-dimensional field strengths, of degrees n and $n-1$. However, in a case such as that of the 5-form $H_{(5)}$ in the type IIB theory, its reduction on a circle gives just one or the other of the lower-dimensional fields, on account of the self-duality constraint. (In other words, the 5-form and 4-form field strengths that we would obtain in $D = 9$ are not independent, once the self-duality in $D = 10$ is imposed, but are related by nine-dimensional Hodge duality.

Either one or the other can be chosen as the independent reduced field in $D = 9$.) We shall choose the reduction scheme in which the 4-form potential for the 5-form field strength is taken as the fundamental field in $D = 9$. The resulting axionic field strength content in each lower dimension is given in Table 5 below, where $F_{(n)}$ denotes an n-form field strength coming from the field strengths already present in $D = 10$, while $\mathcal{F}_{(n)}$ denotes one coming from the dimensional reduction of the metric.

D	NS-NS					R-R				
	$F_{(3)}$	$F_{(2)}$	$F_{(1)}$	$\mathcal{F}_{(2)}$	$\mathcal{F}_{(1)}$	$F_{(5)}$	$F_{(4)}$	$F_{(3)}$	$F_{(2)}$	$F_{(1)}$
10	1	0	0	0	0	$\frac{1}{2}$	0	1	0	0
9	1	1	0	1	0	1	0	1	1	1
8	1	2	1	2	1	1	1	1	2	2
7	1	3	3	3	3	1	2	2	3	4
6	1	4	6	4	6	1	3	4	5	7
5	1	5	10	5	10	–	4	7	9	12
4	1	6	15	6	15	–	–	11	16	21
3	–	7	21	7	21	–	–	–	27	37

Table 5. Field strengths in the type IIB reductions

First of all, we note that if we fully dualise all higher-degree field strengths to lower-degree ones, then in $D \le 9$ the counting of fields indeed reduces to that of the fully-dualised reductions of $D = 11$ supergravity. In this section, we shall instead study the global symmetries of the theories whose field contents are given in the table.

Following this reduction scheme, the scalar sectors in $D \ge 7$ are identical to those in the direct type IIA reductions, and thus the scalar manifolds again have global E_{11-D} symmetries in these dimensions. In $D = 6$, the total number of scalars resulting from the direct reduction of the type IIB theory is 24, comprising 5 dilatons and 19 axions. The Lagrangian has the global symmetry $(SL(2,\mathbb{R}) \times GL(3,\mathbb{R})) \ltimes \mathbb{R}^{12}$, implying in particular that there are 12 commuting axionic shift symmetries. If we were to dualise the 5-form field strength in $D = 6$, we would get one further axion, and the symmetry of the new scalar manifold would become $E_5 = O(5,5)$, which has an $\mathbb{R}^{10}$ maximal abelian symmetry. Note that the axion dual to the 4-form potential is $A_{(0)345}$ in the type IIA notation. It is easy to verify in the type IIA reductions of the previous sections that if $A_{(0)345}$ is dualised in $D = 6$, the axions $A_{(0)1\alpha\beta}$, $A_{(0)2\alpha\beta}$, $\mathcal{A}^1_{(0)\alpha}$ and $\mathcal{A}^2_{(0)\alpha}$ can indeed all acquire commuting shift symmetries. (In fact in $D = 6$ the scalar Lagrangian has an $(SL(2,\mathbb{R}) \times GL(4,\mathbb{R})) \ltimes \mathbb{R}^{12}$

symmetry, since $H_{(5)}$ is a singlet under $GL(4,\mathbb{R})$. However, this $GL(4,\mathbb{R})$ symmetry is broken to $GL(3,\mathbb{R})$ in the full theory, since the three 4-form field strengths can form an irreducible representation under $GL(3,\mathbb{R})$ only. If we were to dualise these three 4-forms, while keeping the 5-form undualised, then we would instead have a global $(SL(2,\mathbb{R}) \times GL(4,\mathbb{R}))\ltimes\mathbb{R}^{12}$ symmetry in the full theory.)

In $D=5$, we see from Table 5 that there are four 4-form field strengths that can be dualised to give additional scalars. In the notation of the $D=11$ reduction, these scalars would be $A_{(0)\alpha\beta\gamma}$, where the indices range over $3,4,5,6$. The global symmetry of the type IIB version can therefore be understood by taking the fully-dualised $D=11$ version with its E_6 global symmetry, and then inversely dualising the scalars $A_{(0)\alpha\beta\gamma}$. The maximal abelian symmetry is then $\mathbb{R}^{21}$, realised by the axions $\mathcal{A}^1_{(0)\alpha}$, $\mathcal{A}^2_{(0)\alpha}$, $A_{(0)1\alpha\beta}$, $A_{(0)2\alpha\beta}$ and χ, where the last axion is the dual of $A_{(3)}$ in the $D=11$ reduction.

A similar analysis can be given in $D=4$ and $D=3$. In four dimensions there are twelve 3-form field strengths, which can be viewed as coming from the inverse dualisation of the axions $A_{(0)\alpha\beta\gamma}$, χ^1 and χ^2 of the fully-dualised $D=11$ reduction. (The axions χ^i were themselves the dualisations of the 2-forms $A_{(2)i}$ in the $D=11$ reduction; see section 3.) In the $D=11$ notation, the type IIB version then has a maximal abelian $\mathbb{R}^{35}$ algebra that corresponds to simultaneous shift symmetries for the axions $\mathcal{A}^1_{(0)\alpha}$, $\mathcal{A}^2_{(0)\alpha}$, $A_{(0)1\alpha\beta}$, $A_{(0)2\alpha\beta}$ and χ^α, where now the indices range over $3,4,5,6,7$. Finally, in $D=3$ we have 41 2-form field strengths, which can be viewed in the notation of the $D=11$ fully-dualised reduction as the inverse dualisation of $A_{(0)\alpha\beta\gamma}$, $\chi^{1\alpha}$, $\chi^{2\alpha}$, χ^{12}, χ_1, χ_2 and χ_α (see section 3). In the $D=11$ notation, the 57 maximal abelian $\mathbb{R}$ symmetries are associated with simultaneous shifts for $\mathcal{A}^1_{(0)\alpha}$, $\mathcal{A}^2_{(0)\alpha}$, $A_{(0)1\alpha\beta}$, $A_{(0)2\alpha\beta}$, $\chi^{\alpha\beta}$.

Note that in all dimensions, in the direct reductions of the type IIB theory, the original $SL(2,\mathbb{R})$ symmetry acts on the 1 and 2 indices of the $D=11$ notation, and leaves invariant the α indices. The α indices rotate under $GL(9-D,\mathbb{R})$. In all cases, the symmetries of the type IIB reductions are described by deleting the simple roots $\vec{r}_1=\vec{a}_{123}$ and $\vec{r}_3=\vec{b}_{23}$ in the E_{11-D} Dynkin diagram given in Table 1.

10 Conclusions

In this paper, we have studied the global symmetries of the maximal D-dimensional supergravities, obtained by toroidal dimensional reduction from $D=11$. If one simply performs a direct dimensional reduction, without dualising any of the gauge fields, the resulting theory in D dimensions has a global $GL(11-D,\mathbb{R})\ltimes\mathbb{R}^q$ symmetry, where the first factor has its origin in the general coordinate invariance in $D=11$, restricted to the internal compactified dimensions. The abelian factor $\mathbb{R}^q$ comes from the original local gauge symmetry of the 4-form field strength in $D=11$, and is realised on the $q=\frac{1}{6}(11-D)(10-D)(9-D)$ axions that come from the reduction of its 3-form potential. In all dimensions, $GL(11-D,\mathbb{R})\ltimes\mathbb{R}^q$ acts on the various potentials in the theory, and it is a global symmetry of the action. In $D=8$ the equations of motion actually have a larger global symmetry, namely $SL(3,\mathbb{R})\times SL(2,\mathbb{R})$, where in the case of the 3-form potential, the symmetry is now taken to act instead on its field strength F_4. In fact the $SL(2,\mathbb{R})$ factor in the enlarged symmetry group acts as a kind of electric/magnetic duality symmetry, under which F_4 and its dual form a doublet. In dimensions $D\le 7$, it is natural to consider alternative formulations of the theories, in which all potentials whose field strengths have degrees greater than $\frac{1}{2}D$ are dualised to give fields of lower degrees. If this is done, one obtains theories that can also have enlarged global symmetries, namely E_{11-D} in its maximally non-compact version. In odd dimensions these are symmetries of the action, while in even dimensions they are symmetries only of the equations of motion, owing to the need to put the field strengths of degree $\frac{1}{2}D$ and their duals into a single irreducible multiplet under E_{11-D}.

It is important to appreciate that the theories obtained by performing dualisations of the kind described above are non-locally related to their original undualised versions. This is because the potentials for the dualised fields are related to the potentials for the undualised fields by non-local field transformations. Thus at least in cases where the global symmetry is realised at the level of the action, where the potentials are the fundamental fields, it should come as no surprise that the global symmetries can be affected by the process of dualisation. More precisely it is the choice of the realisation of the symmetry that dictates its form; roughly speaking the E_{11-D} group can act more or less faithfully. We discussed a simple example of this in Appendix B.1, where the global $SL(2,\mathbb{R})$ symmetry of a dilaton/axion system was shown to be broken if the axion was dualised to a $(D-2)$-form potential; the $SL(2,\mathbb{R})$ symmetry acts via non-linear transformations on the dilaton and axion themselves, and it cannot be realised in terms of local transformations on the dilaton and $(D-2)$-form potential. Similarly even for higher-degree fields, where the global symmetries act linearly, at the level of the action they must act on the potentials themselves, and so they cannot be implemented in terms of local field transformations of the fundamental variables if some members of a would-be irreducible multiplet under the global symmetry are dualised to have the conjugate degree.

What is perhaps more surprising is that even at the level of the equations of motion, the global symmetries can also be affected by the choice of dualisation for the higher-degree field strengths. In simple situations one can become accustomed to the idea that only field strengths, and not bare potentials, appear in equations of motion and Bianchi identities. In theories where such is the case then indeed, provided one views the field strengths rather than their potentials as the fundamental quantities on which the symmetries act, the symmetry need not be affected provided that bare potentials are still absent after the dualisation. In more complicated situations, however, including many of the supergravity theories that we have been considering, the appearance of bare potentials in the equations of motion or Bianchi identities can be unavoidable, for certain choices of dualisation. In cases where this happens, one would then again be forced to realise the symmetry on the potentials themselves, and this would not be possible in terms of local field transformations if some of the potentials in a would-be irreducible multiplet had degrees dual to the rest. Thus in particular it is not always true that the sectors of the theories involving the higher-degree field strengths, on which the symmetries act linearly, will necessarily respect the global symmetry of the scalar manifold. We discussed an example in $D = 6$, where the failure to dualise the 3-form potential to give another vector leads to a breaking of the $O(5,5)$ global symmetry of the scalar sector, even at the level of the equations of motion.

The upshot of the above considerations is that in general there is no unique answer for the global symmetry group of D-dimensional maximal supergravity; it depends upon what choices of dualisations are made. Put another way, in such cases there is not a unique maximal supergravity; rather, there exist inequivalent theories, related to one another by non-local field transformations, which have different global symmetries. In this paper, we considered some specific choices of dualisations that could, in some sense, be considered "natural." As well as the non-dualised versions obtained by direct dimensional reduction, and the fully-dualised versions with the E_{11-D} symmetries, another natural choice is when only those fields which from the ten-dimensional type II string point of view are Ramond-Ramond fields are allowed to be dualised. The motivation for considering this is that in perturbative string theory the NS-NS fields couple to the worldsheet via their potentials, whilst the R-R fields have only field-strength couplings. Thus if the global symmetries are to act locally on the fundamental fields of perturbative string theory, then no dualisations of NS-NS fields should be permitted. Under these circumstances, we saw that the global symmetries can be different from either of the previous two possibilities. A final natural class of theories that we considered were the ones obtained by the direct dimensional reduction of type IIB supergravity. One has a choice, in the reduction to nine dimensions, as to whether the self-dual 5-form in $D = 10$ will be described in terms of a 3-form or a 4-form potential in $D = 9$. If the former is chosen, the theory immediately coincides, after simple local field transformations, with the reduction of eleven-dimensional supergravity to $D = 9$. If instead the latter choice is made, the theory in $D = 9$ is non-locally related to the direct reduction of eleven-dimensional supergravity, and it provides a natural starting point for further dimensional reduction. Of course it, and its lower-dimensional descendants, can all be viewed as certain specific "inverse dualisations" of the usual $D = 11$ reductions, but their natural type IIB origin endows these versions of the supergravities with a preferred status.

It turns out that the fully-dualised versions of the supergravities, which have E_{11-D} global symmetries, enjoy a preferred rôle and provide a convenient way of discussing the global symmetries for all other possible versions, which can then all be viewed as specific "inverse dualisations." The reason for this stems from the fact that it is these fully-dualised versions that maximise the numbers of scalar fields, and while the higher-degree fields in the theory need not necessarily respect the global symmetry of the scalar manifold, the latter certainly sets an "upper bound" on the global symmetry of the entire theory. We showed that in the fully-dualised versions, the axionic scalars are in one-to-one correspondence with the positive-root generators of E_{11-D}. The symmetries of the scalar manifolds in the various other dualisations can then be understood in terms of the Coxeter-Dynkin diagrams for the E_{11-D} algebras. Inverse dualisations of certain axions implies the removal of the associated positive roots from the algebra. In particular, we showed that the non-dualised, R-R dualised and type IIB dualised versions of the supergravities correspond respectively to the deletion of the simple roots $\vec{r}_1$, $\vec{r}_2$ and $(\vec{r}_1, \vec{r}_3)$ respectively, as described below Diagram 1. As well as determining the global symmetries of the scalar manifolds, these considerations also allowed us to find the dimensions of the maximal abelian subalgebras in the various versions of the supergravities. These determine the number of simultaneous commuting shift symmetries of the axions in the theories.

In the sense described above, one might think of the fully-dualised versions of the supergravities as having the "largest" symmetries of all the possible versions. However, this viewpoint should be treated with some caution because versions involving fewer dualisations can have global symmetry groups that are neither contained in nor contain E_{11-D}, and so there is no one version of the theory in a given dimension that could be said to encompass the others. Perhaps the lesson that should be drawn from these considerations is that there is a richer variety of supergravities than has sometimes been appreciated, and that sometimes the apparently simple sounding question of “what is the symmetry” can have a rather involved answer.

Acknowledgement

We are grateful to E. Bergshoeff, M.J. Duff, M. Duflo, S. Ferrara, V. Kac, B. Kostant and K.S. Stelle for discussions, and were supported in part by EC contract ERBCHH-BGT9220176.

A Lagrangian of D-dimensional maximal supergravity

We begin by establishing some notation and conventions. The components of an n'th-rank antisymmetric tensor $F_{(n)}$ are related to its expression as an n-form according to

$$F_{(n)} = \frac{1}{n!}\, F_{\mu_1\cdots\mu_n}\, dx^{\mu_1}\wedge\cdots dx^{\mu_n}\ . \tag{A.1}$$

The Hodge dual operation $*$ in D dimensions is defined by

$$*\,(dx^{\mu_1}\wedge\cdots dx^{\mu_n}) = \frac{1}{m!}\,\varepsilon_{\nu_1\cdots\nu_m}{}^{\mu_1\cdots\mu_n}\, dx^{\nu_1}\wedge\cdots dx^{\nu_m}\ , \tag{A.2}$$

where $m = D - n$ and $\varepsilon_{\mu_1\cdots\mu_D}$ is the Levi-Civita tensor. From these definitions, it follows that the components of the m'th rank antisymmetric tensor $G_{(m)}$ dual to $F_{(n)}$ are given by

$$G_{\mu_1\cdots\mu_m} = \frac{1}{n!}\,\varepsilon_{\mu_1\cdots\mu_m}{}^{\nu_1\cdots\nu_n}\, F_{\nu_1\cdots\nu_n}\ . \tag{A.3}$$

Following the discussion and notations of section 2.1, the Lagrangian for the bosonic D-dimensional toroidal compactification of eleven-dimensional supergravity (with any dualisation) then takes the form [16]

$$\begin{aligned}\mathcal{L} \;=\;& eR - \tfrac{1}{2}e\,(\partial\vec{\phi})^2 - \tfrac{1}{48}e\,e^{\vec{a}\cdot\vec{\phi}}\,F_{(4)}^2 - \tfrac{1}{12}e\sum_i e^{\vec{a}_i\cdot\vec{\phi}}\,(F_{(3)i})^2 - \tfrac{1}{4}e\sum_{i<j} e^{\vec{a}_{ij}\cdot\vec{\phi}}\,(F_{(2)ij})^2 \\ & -\tfrac{1}{4}e\sum_i e^{\vec{b}_i\cdot\vec{\phi}}\,(\mathcal{F}^i_{(2)})^2 - \tfrac{1}{2}e\sum_{i<j<k} e^{\vec{a}_{ijk}\cdot\vec{\phi}}\,(F_{(1)ijk})^2 - \tfrac{1}{2}e\sum_{i<j} e^{\vec{b}_{ij}\cdot\vec{\phi}}\,(\mathcal{F}^i_{(1)j})^2 + \mathcal{L}_{FFA}\ ,\end{aligned} \tag{A.4}$$

where the "dilaton vectors" $\vec{a}$, $\vec{a}_i$, $\vec{a}_{ij}$, $\vec{a}_{ijk}$, $\vec{b}_i$, $\vec{b}_{ij}$ are constants that characterise the couplings of the dilatonic scalars $\vec{\phi}$ to the various gauge fields. They are given by [16]

$$\begin{array}{lll} & F_{MNPQ} & \text{vielbein} \\ 4-\text{form}: & \vec{a} = -\vec{g}\ , & \\ 3-\text{forms}: & \vec{a}_i = \vec{f}_i - \vec{g}\ , & \\ 2-\text{forms}: & \vec{a}_{ij} = \vec{f}_i + \vec{f}_j - \vec{g}\ , & \vec{b}_i = -\vec{f}_i\ , \\ 1-\text{forms}: & \vec{a}_{ijk} = \vec{f}_i + \vec{f}_j + \vec{f}_k - \vec{g}\ , & \vec{b}_{ij} = -\vec{f}_i + \vec{f}_j\ , \end{array} \tag{A.5}$$

where the vectors $\vec{g}$ and $\vec{f}_i$ have $(11-D)$ components in D dimensions, and are given by

$$\begin{aligned}
\vec{g} &= 3(s_1, s_2, \ldots, s_{11-D}) \ , \\
\vec{f}_i &= \Big(\underbrace{0, 0, \ldots, 0}_{i-1}, (10-i)s_i, s_{i+1}, s_{i+2}, \ldots, s_{11-D}\Big) \ ,
\end{aligned} \tag{A.6}$$

where $s_i = \sqrt{2/((10-i)(9-i))}$. It is easy to see that they satisfy

$$\vec{g}\cdot\vec{g} = \tfrac{2(11-D)}{D-2}, \qquad \vec{g}\cdot\vec{f}_i = \tfrac{6}{D-2} \ , \qquad \vec{f}_i\cdot\vec{f}_j = 2\delta_{ij} + \tfrac{2}{D-2} \ . \tag{A.7}$$

Note also that

$$\sum_i \vec{f}_i = 3\,\vec{g} \ . \tag{A.8}$$

In fact the vectors $\vec{f}_i$ can be written as $\vec{f}_i = \sqrt{2}\vec{e}_i + \alpha\vec{g}$, where $\alpha = (3-\sqrt{D-2})/(11-D)$ and $\vec{e}_i$ are orthonormal vectors, *i.e.* $\vec{e}_i\cdot\vec{e}_j = \delta_{ij}$. It follows from (A.8) that we have $\vec{g} = \sqrt{2/(D-2)}\sum_i \vec{e}_i$. A useful identity that follows from (A.7) is that

$$\sum_i (\vec{f}_i\cdot\vec{x})^2 = 2\vec{x}^2 + (\vec{g}\cdot\vec{x})^2 \ , \tag{A.9}$$

where $\vec{x}$ is an arbitrary vector. Note that the D-dimensional metric is related to the eleven-dimensional one by

$$ds_{11}^2 = e^{\frac13\vec{g}\cdot\vec{\phi}}\, ds_D^2 + \sum_i e^{2\vec{\gamma}_i\cdot\vec{\phi}}\,(h^i)^2 \ , \tag{A.10}$$

where $\vec{\gamma}_i = \frac16\vec{g} - \frac12\vec{f}_i$, and

$$h^i = dz^i + \mathcal{A}_1^i + \mathcal{A}_{0j}^i\, dz^j \ . \tag{A.11}$$

The naive field strengths are associated with the gauge potentials in the obvious way; for example $F_{(4)}$ is the field strength for $A_{(3)}$, $F_{(3)i}$ is the field strength for $A_{(2)i}$, *etc.* In general, the field strengths appearing in the kinetic terms are not simply the exterior derivatives of their associated potentials, but have non-linear Kaluza-Klein modifications as well. On the other hand the terms included in $\mathcal{L}_{FFA}$, which denotes the dimensional reduction of the $F_{(4)}\wedge F_{(4)}\wedge A_{(3)}$ term in $D=11$, are best expressed purely in terms of the potentials and their exterior derivatives. The complete details of all the field strengths appearing in this Appendix, in the notation we are using here, were obtained in [16]. (We correct some sign errors here.) The field strengths are given by

$$\begin{aligned}
F_{(4)} &= \tilde{F}_{(4)} - \gamma^i{}_j\, \tilde{F}_{(3)i}\wedge\mathcal{A}_{(1)}^j + \tfrac12\gamma^i{}_k\,\gamma^j{}_\ell\,\tilde{F}_{(2)ij}\wedge\mathcal{A}_{(1)}^k\wedge\mathcal{A}_{(1)}^\ell \\
&\quad -\tfrac16\gamma^i{}_\ell\,\gamma^j{}_m\,\gamma^k{}_n\,\tilde{F}_{(1)ijk}\wedge\mathcal{A}_{(1)}^\ell\wedge\mathcal{A}_{(1)}^m\wedge\mathcal{A}_{(1)}^n \ , \\
F_{(3)i} &= \gamma^j{}_i\,\tilde{F}_{(3)j} + \gamma^j{}_i\,\gamma^k{}_\ell\,\tilde{F}_{(2)jk}\wedge\mathcal{A}_{(1)}^\ell + \tfrac12\gamma^j{}_i\,\gamma^k{}_m\,\gamma^\ell{}_n\,\tilde{F}_{(1)jk\ell}\wedge\mathcal{A}_{(1)}^m\wedge\mathcal{A}_{(1)}^n \ ,
\end{aligned}$$

$$
\begin{aligned}
F_{(2)ij} &= \gamma^k{}_i\,\gamma^\ell{}_j\,\tilde F_{(2)k\ell} - \gamma^k{}_i\,\gamma^\ell{}_j\,\gamma^m{}_n\,\tilde F_{(1)k\ell m}\wedge\mathcal{A}^n_{(1)}\ , \\
F_{(1)ijk} &= \gamma^\ell{}_i\,\gamma^m{}_j\,\gamma^n{}_k\,\tilde F_{(1)\ell mn}\ , \\
\mathcal{F}^i_{(2)} &= \tilde{\mathcal{F}}^i_{(2)} - \gamma^j{}_k\,\tilde{\mathcal{F}}^i_{(1)j}\wedge\mathcal{A}^k_{(1)}\ , \\
\mathcal{F}^i_{(1)j} &= \gamma^k{}_j\,\tilde{\mathcal{F}}^i_{(1)k}\ ,
\end{aligned} \tag{A.12}
$$

where the tilded quantities represent the unmodified pure exterior derivatives of the corresponding potentials, and $\gamma^i{}_j$ is defined by

$$
\gamma^i{}_j = [(1+\mathcal{A}_0)^{-1}]^i{}_j = \delta^i_j - \mathcal{A}^i_{(0)j} + \mathcal{A}^i_{(0)k}\,\mathcal{A}^k_{(0)j} + \cdots\ . \tag{A.13}
$$

Recalling that $\mathcal{A}^i_{(0)j}$ is defined only for $j>i$ (and vanishes if $j\le i$), we see that the series terminates after a finite number of terms. We also define here the inverse of $\gamma^i{}_j$, namely $\tilde\gamma^i{}_j$ given by

$$
\tilde\gamma^i{}_j = \delta^i_j + \mathcal{A}^i_{(0)j}\ . \tag{A.14}
$$

Note that the upper index on $\tilde\gamma^i{}_j$ is a tangent-space index, while the lower is a world index. Conversely, the upper index on $\gamma^i{}_j$ is a world index, while the lower is a tangent-space index. These characteristics reflect themselves in their $GL(11-D,\mathbb{R})$ transformations, since these act only on world indices. Thus from (2.17) we have that

$$
\delta\gamma^i{}_j = -\Lambda^i{}_k\,\gamma^k{}_j\ ,\qquad \delta\tilde\gamma^i{}_j = \Lambda^k{}_j\,\tilde\gamma^i{}_k\ . \tag{A.15}
$$

The term $\mathcal{L}_{FFA}$ in (A.4) is the dimensional reduction of the $\tilde F_{(4)}\wedge\tilde F_{(4)}\wedge A_{(3)}$ term in $D=11$, and is given in lower dimensions by [16]

$$
\begin{aligned}
D=10:&\quad \tfrac12\tilde F_{(4)}\wedge\tilde F_{(4)}\wedge A_{(2)}\ , \\
D=9:&\quad \Big(\tfrac14\tilde F_{(4)}\wedge\tilde F_{(4)}\wedge A_{(1)ij} - \tfrac12\tilde F_{(3)i}\wedge\tilde F_{(3)j}\wedge A_{(3)}\Big)\epsilon^{ij}\ , \\
D=8:&\quad \Big(\tfrac1{12}\tilde F_{(4)}\wedge\tilde F_{(4)}A_{(0)ijk} - \tfrac16\tilde F_{(3)i}\wedge\tilde F_{(3)j}\wedge A_{(2)k} - \tfrac12\tilde F_{(4)}\wedge\tilde F_{(3)i}\wedge A_{(1)jk}\Big)\epsilon^{ijk}\ , \\
D=7:&\quad \Big(\tfrac16\tilde F_{(4)}\wedge\tilde F_{(3)i}A_{(0)jkl} - \tfrac14\tilde F_{(3)i}\wedge\tilde F_{(3)j}\wedge A_{(1)kl} + \tfrac18\tilde F_{(2)ij}\wedge\tilde F_{(2)kl}\wedge A_{(3)}\Big)\epsilon^{ijkl}\ , \\
D=6:&\quad \Big(\tfrac1{12}\tilde F_{(4)}\wedge\tilde F_{(2)ij}A_{(0)klm} - \tfrac1{12}\tilde F_{(3)i}\wedge\tilde F_{(3)j}A_{(0)klm} + \tfrac18\tilde F_{(2)ij}\wedge\tilde F_{(2)kl}\wedge A_{(2)m}\Big)\epsilon^{ijklm}\ , \\
D=5:&\quad \Big(\tfrac1{12}\tilde F_{(3)i}\wedge\tilde F_{(2)jk}A_{(0)lmn} + \tfrac1{48}\tilde F_{(2)ij}\wedge\tilde F_{(2)kl}\wedge A_{(1)mn} \\
&\qquad -\tfrac1{72}\tilde F_{(1)ijk}\wedge\tilde F_{(1)lmn}\wedge A_{(3)}\Big)\epsilon^{ijklmn}\ , \\
D=4:&\quad \Big(\tfrac1{48}\tilde F_{(2)ij}\wedge\tilde F_{(2)kl}A_{(0)mnp} - \tfrac1{72}\tilde F_{(1)ijk}\wedge\tilde F_{(1)lmn}\wedge A_{(2)p}\Big)\epsilon^{ijklmnp}\ , \\
D=3:&\quad -\tfrac1{144}\,\tilde F_{(1)ijk}\wedge\tilde F_{(1)lmn}\wedge A_{(1)pq}\,\epsilon^{ijklmnpq}\ , \\
D=2:&\quad -\tfrac1{1296}\,\tilde F_{(1)ijk}\wedge\tilde F_{(1)lmn}A_{(0)pqr}\,\epsilon^{ijklmnpqr}\ .
\end{aligned} \tag{A.16}
$$

Here, and elsewhere in the paper, we commonly omit the Hodge $*$ symbol when writing the Wess-Zumino terms in the Lagrangian. It is then understood that $dA_{(m)} \wedge dB_{(n)} \wedge C_{(p)}$ represents a contribution

$$\mathcal{L}_{\rm WZ} = \frac{1}{m!\, n!\, p!} \epsilon^{\mu_1 \cdots \mu_{m+1}\, \nu_1 \cdots \nu_{n+1}\, \rho_1 \cdots \rho_p}\, \partial_{\mu_1} A_{\mu_2 \cdots \mu_{m+1}}\, \partial_{\nu_1} B_{\nu_2 \cdots \nu_{n+1}}\, C_{\rho_1 \cdots \rho_p}\ . \tag{A.17}$$

The expressions for the non-linear Kaluza-Klein modified field strengths can be simplified considerably by introducing redefined potentials $\hat{\mathcal{A}}^i_{(1)}$, $\hat{A}_{(1)ij}$, $\hat{A}_{(2)i}$ and $\hat{A}_{(3)}$, given by solving:

$$\begin{aligned}
&\hat{\mathcal{A}}^i_{(1)} = \gamma^i{}_j\, \mathcal{A}^j_{(1)}\ , \qquad A_{(1)ij} = \hat{A}_{(1)ij} + A_{(0)ijk}\, \hat{\mathcal{A}}^k_{(1)}\ ,\\
&A_{(2)i} = \hat{A}_{(2)i} - \hat{A}_{(1)ij} \wedge \hat{\mathcal{A}}^j_{(1)} + \tfrac12 A_{(0)ijk}\, \hat{\mathcal{A}}^j_{(1)} \wedge \hat{\mathcal{A}}^k_{(1)}\ ,\\
&A_{(3)} = \hat{A}_{(3)} + \hat{A}_{(2)i} \wedge \hat{\mathcal{A}}^i_{(1)} + \tfrac12 \hat{A}_{(1)ij} \wedge \hat{\mathcal{A}}^i_{(1)} \wedge \hat{\mathcal{A}}^j_{(1)} + \tfrac16 A_{(0)ijk}\, \hat{\mathcal{A}}^i_{(1)} \wedge \hat{\mathcal{A}}^j_{(1)} \wedge \hat{\mathcal{A}}^k_{(1)}\ .
\end{aligned} \tag{A.18}$$

The various Kaluza-Klein modified field strengths are given by

$$\begin{aligned}
\mathcal{F}^i_{(2)} &= \tilde\gamma^i{}_j\, \hat{\mathcal{F}}^j_{(2)}\ , \qquad F_{(2)ij} = \gamma^k{}_i\, \gamma^\ell{}_j\, \hat{F}_{(2)k\ell}\ ,\\
F_{(3)i} &= \gamma^j{}_i\, \hat{F}_{(3)j}\ , \qquad F_{(4)} = \hat{F}_{(4)}\ ,
\end{aligned} \tag{A.19}$$

where $\tilde\gamma^i{}_j$ is the inverse of $\gamma^i{}_j$, given by (A.14), and

$$\begin{aligned}
\hat{\mathcal{F}}^i_{(2)} &= d\hat{\mathcal{A}}^i_{(1)}\ , \qquad \hat{F}_{(2)ij} = d\hat{A}_{(1)ij} + A_{(0)ijk}\, \hat{\mathcal{F}}^k_{(2)}\ ,\\
\hat{F}_{(3)i} &= d\hat{A}_{(2)i} + \hat{A}_{(1)ij} \wedge \hat{\mathcal{F}}^j_{(2)}\ , \qquad \hat{F}_{(4)} = d\hat{A}_{(3)} + \hat{A}_{(2)i} \wedge \hat{\mathcal{F}}^i_{(2)}\ .
\end{aligned} \tag{A.20}$$

The non-dualised D-dimensional Lagrangian obtained by the direct reduction of eleven-dimensional supergravity is then given by

$$\mathcal{L} = e\, R + \mathcal{L}_{\rm scalar} + \mathcal{L}_{(2)} + \mathcal{L}_{(3)} + \mathcal{L}_{(4)} + \mathcal{L}_{FFA}\ , \tag{A.21}$$

where $\mathcal{L}_{\rm scalar}$ is the kinetic Lagrangian for the scalar sector, $\mathcal{L}_{(2)}$, $\mathcal{L}_{(3)}$ and $\mathcal{L}_{(4)}$ are the kinetic Lagrangians for the 2-form, 3-form and 4-form field strengths, and $\mathcal{L}_{FFA}$ represents the remaining Wess-Zumino terms. The kinetic terms are given by [10]

$$\begin{aligned}
\mathcal{L}_{\rm scalar} &= -\tfrac14 e\, g^{ik}\, g^{j\ell}\, \partial_\mu g_{ij}\, \partial^\mu g_{k\ell} + \tfrac14 e \Big(\frac{\partial\omega}{\omega}\Big)^2\\
&\quad -\tfrac{1}{12} e\, \omega g^{i\ell}\, g^{jm}\, g^{kn}\, \partial_\mu A_{(0)ijk}\, \partial^\mu A_{(0)\ell mn}\ ,
\end{aligned} \tag{A.22}$$

$$\mathcal{L}_{(2)} = -\tfrac18 e\, \omega\, g^{ik}\, g^{j\ell}\, \hat{F}_{\mu\nu ij}\, \hat{F}^{\mu\nu}{}_{k\ell} - \tfrac14 e\, g_{ij}\, \hat{\mathcal{F}}^i_{\mu\nu}\, \hat{\mathcal{F}}^{\mu\nu j}\ , \tag{A.23}$$

$$\mathcal{L}_{(3)} = -\tfrac{1}{12} e\, \omega\, g^{ij}\, \hat{F}_{\mu\nu\rho i}\, \hat{F}^{\mu\nu\rho}_j\ , \tag{A.24}$$

$$\mathcal{L}_{(4)} = -\tfrac{1}{48} e\, \omega\, \hat{F}_{\mu\nu\rho\sigma}\, \hat{F}^{\mu\nu\rho\sigma}\ . \tag{A.25}$$

The metrics g^{ij} and g_{ij} are defined by

$$g^{ij} \equiv \sum_k \gamma^i{}_k\, \gamma^j{}_k\, e^{\vec f_k\cdot\vec\phi}\ , \qquad g_{ij} \equiv \sum_k \tilde\gamma^k{}_i\, \tilde\gamma^k{}_j\, e^{-\vec f_k\cdot\vec\phi}\ , \tag{A.26}$$

and

$$\omega \equiv (\det (g_{ij}))^{\frac{1}{3}} = e^{-\vec{g}\cdot\vec{\phi}} \ . \tag{A.27}$$

Note that the summation index k in both of the equations in (A.26) is a tangent-space index. The first two terms in $\mathcal{L}_{\rm scalar}$ come from the dimensional reduction of the eleven-dimensional metric, and are equal to $-\frac{1}{2}(\partial\vec{\phi})^2 - \frac{1}{2}\sum_{i<j} e^{\vec{b}_{ij}\cdot\vec{\phi}}\,(\mathcal{F}^i_{(1)j})^2$. The derivation of $\mathcal{L}_{\rm scalar}$ involves the use of the identity (A.9), and the fact that $\gamma^i{}_j$ and $\tilde{\gamma}^i{}_j$ vanish when $i > j$. As we shall show in Appendix C, the first two terms in $\mathcal{L}_{\rm scalar}$ describe an $O(11-D,\mathbb{R})\backslash GL(11-D,\mathbb{R})$ coset, which is the scalar manifold of the dimensional reduction of the pure gravity sector of the eleven-dimensional theory.

A general remark is in order here about the distinction between internal world indices and tangent-space indices. We have used the same kind of index, $i, j, k\ldots$, for both of these, because in many of the discussions it would have been inconvenient to have to make the distinction. However, it is useful to take note of which indices are of which kind. Let us therefore, just within the confines of this paragraph, introduce the notation that $i, j, k\ldots$ represent world indices and $a, b, c\ldots$ represent tangent-space indices. Then the indices on the various fields we have been using in this paper are as follows:

$$\begin{array}{lllll} A_{(2)i}\ , & A_{(1)ij}\ , & A_{(0)ijk}\ , & \mathcal{A}^a_{(1)}\ , & \mathcal{A}^a_{(0)i}\ ,\\ \tilde{F}_{(3)i}\ , & \tilde{F}_{(2)ij}\ , & \tilde{F}_{(1)ijk}\ , & \tilde{\mathcal{F}}^a_{(2)}\ , & \tilde{\mathcal{F}}^a_{(1)i}\ ,\\ F_{(3)a}\ , & F_{(2)ab}\ , & F_{(1)abc}\ , & \mathcal{F}^a_{(2)}\ , & \mathcal{F}^a_{(1)b}\ . \end{array} \tag{A.28}$$

In addition, we have $\gamma^i{}_a$ and $\tilde{\gamma}^a{}_i$.

In terms of the hatted potentials, we saw that the non-linear Kaluza-Klein modifications (A.20) for the higher-degree field strengths became very simple. However, this is achieved at the price of making the $\mathcal{F}_{FFA}$ terms in (A.16) very much more complicated, with up to seventh powers of fields arising. In low dimensions, this is not a problem in the fully-dualised formulations, since in fact these terms conspire to cancel against others arising from the dualisations. In higher dimensions an intermediate redefinition of fields seems to be more useful, which does not change the structure of the $\mathcal{L}_{FFA}$ terms, but which nonetheless considerably simplifies the non-linear Kaluza-Klein modifications to the various field strengths. This can be done by again introducing $\hat{\mathcal{A}}^i_{(1)}$ as in (A.18), but now writing the hatted field strengths defined in (A.19) in terms of the original gauge potentials $A_{(3)}$, $A_{(2)i}$, $A_{(1)ij}$ and $A_{(0)ijk}$, rather than the hatted potentials defined in (A.18). In terms of these potentials we have

$$\hat{\mathcal{F}}^i_{(2)} \;=\; d\hat{\mathcal{A}}^j_{(1)}\ , \qquad \hat{F}_{(2)ij} = dA_{(1)ij} - dA_{(0)ijk}\wedge\hat{\mathcal{A}}^k_{(1)}\ ,$$

$$\hat{F}_{(3)i} = dA_{(2)i} + dA_{(1)ij} \wedge \hat{\mathcal{A}}^j_{(1)} + \tfrac{1}{2} dA_{(0)ijk} \wedge \hat{\mathcal{A}}^j_{(1)} \wedge \hat{\mathcal{A}}^k_{(1)} \ , \tag{A.29}$$

$$\hat{F}_{(4)} = dA_{(3)} - dA_{(2)i} \wedge \hat{\mathcal{A}}^i_{(1)} + \tfrac{1}{2} dA_{(1)ij} \wedge \hat{\mathcal{A}}^i_{(1)} \wedge \hat{\mathcal{A}}^j_{(1)} - \tfrac{1}{6} dA_{(0)ijk} \wedge \hat{\mathcal{A}}^i_{(1)} \wedge \hat{\mathcal{A}}^j_{(1)} \wedge \hat{\mathcal{A}}^k_{(1)} \ .$$

The kinetic terms in the Lagrangian are still given by (A.21), while the $\mathcal{L}_{FFA}$ terms are given by (A.16).

B $O(2)\backslash SL(2,\mathbb{R})$ coset manifold

In this appendix, we illustrate the details of the coset construction for the scalar manifolds in section 4 by considering the example of the two-dimensional $SL(2,\mathbb{R})/O(2)$ manifold. We beging by introducing the Cartan generator H, and the raising and lowering operators $E_\pm$, which may be taken to be

$$H = \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \qquad E_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \qquad E_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ . \tag{B.1}$$

Then we can parameterise the coset as

$$\mathcal{V} = e^{\frac{1}{2}\phi\, H}\, e^{\chi\, E_+} = \begin{pmatrix} e^{\frac{1}{2}\phi} & \chi\, e^{\frac{1}{2}\phi} \\ 0 & e^{-\frac{1}{2}\phi} \end{pmatrix} \ . \tag{B.2}$$

Then we see that

$$d\mathcal{V}\,\mathcal{V}^{-1} = \begin{pmatrix} \frac{1}{2}d\phi & e^{\phi}\, d\chi \\ 0 & -\frac{1}{2}d\phi \end{pmatrix} = \tfrac{1}{2}d\phi\, H + e^{\phi}\, d\chi\, E_+ \tag{B.3}$$

Clearly we can now write the Lagrangian as

$$\mathcal{L} = \tfrac{1}{4}\mathrm{tr}\Big(\partial\mathcal{M}^{-1}\,\partial\mathcal{M})\Big) = -\tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}e^{2\phi}\,(\partial\chi)^2 \ , \tag{B.4}$$

where $\mathcal{M} = \mathcal{V}^{\mathrm{T}}\,\mathcal{V}$. This Lagrangian is obviously invariant under $SL(2,\mathbb{R})$ transformations $\mathcal{V} \to \mathcal{V}'' = \mathcal{V}\,U$, where U is a constant $SL(2,\mathbb{R})$ matrix of the form

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ , \qquad ad - bc = 1 \ . \tag{B.5}$$

However, it is clear that $\mathcal{V}''$ is no longer in the upper triangular gauge (B.2). We can perform a *local* compensating transformation, however, to get a $\mathcal{V}'$ which *is* back in upper-triangular gauge:

$$\mathcal{V}' = \mathcal{O}\,\mathcal{V}'' = \mathcal{O}\,\mathcal{V}\,U \ , \tag{B.6}$$

where $\mathcal{O}$ is a field-dependent $SL(2,\mathbb{R})$ matrix in the $O(2)$ subgroup, satisfying $\mathcal{O}^{\mathrm{T}}\,\mathcal{O} = 1$. After a little algebra, we find that $\mathcal{O}$ is given by

$$\mathcal{O} = (c^2 + e^{2\phi}\,(c\,\chi + a)^2)^{-1/2} \begin{pmatrix} (c\,\chi + a)\, e^{\phi} & c \\ c & (c\,\chi + a)\, e^{\phi} \end{pmatrix} \ . \tag{B.7}$$

Thus for a given global U, with constant components a, b, c, d, there is a local $\mathcal{O}$, with field-dependent and U-dependent components, which restores the upper-triangular gauge. This means that $\mathcal{V}$ defined in (B.2) parameterises elements in the coset $O(2)\backslash SL(2, \mathbb{R})$.

B.1 Dualisation of scalars: an $SL(2, \mathbb{R})$ example

In section 3, we show the dualisation of the $(D-2)$-form gauge potentials to scalars in $D = 5$, 4, and 3 has the effect of changing the global symmetry $GL(11-D, \mathbb{R}) \ltimes \mathbb{R}^q$ of the undualised theory to E_6, E_7 and E_8 respectively. These examples, however, are rather complicated, and it is difficult to study exactly how the symmetry alters under the dualisation. We focussed principally on how the maximal abelian $\mathbb{R}$ symmetries are altered under the dualisations. Here, we shall present a simpler example, namely a D-dimensional scalar Lagrangian with a global $SL(2, \mathbb{R})$ symmetry. It contains the metric, a dilaton ϕ and an axion χ. The Lagrangian is given by

$$e^{-1}\mathcal{L} = R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}e^{2\phi}\,(\partial\chi)^2\ . \tag{B.8}$$

This is precisely of the form of the scalar Lagrangian for type IIB supergravity when $D = 10$. The theory has the global $SL(2, \mathbb{R})$ symmetry

$$\tau \longrightarrow \tau' = \frac{a\tau + b}{c\tau + d}\ , \tag{B.9}$$

where $\tau = \chi + \mathrm{i}e^{-\phi}$, and $ad - bc = 1$.

Since the axion χ appears in the Lagrangian only through its derivative, it follows that it can be obtained by dualising a $(D-2)$-form gauge potential A_{D-2}. Consider a theory with the metric, a dilaton and such a potential A_{D-2}, with Lagrangian given by

$$e^{-1}\mathcal{L} = R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2(D-1)!}F_{(D-1)}^2 e^{-2\phi}\ , \tag{B.10}$$

where $F_{(D-1)} = dA_{(D-2)}$. If we dualise the gauge potential $A_{(D-2)}$, we recover the scalar Lagrangian (B.8). However this original undualised Lagrangian (B.10) has just one scalar, which has an $\mathbb{R}$ global symmetry, namely

$$\phi \longrightarrow \phi' = \phi + c\ , \qquad A_{(D-2)} \longrightarrow A'_{(D-2)} = e^{2c}A_{(D-2)}\ . \tag{B.11}$$

There is also a local gauge symmetry for the gauge field $A_{(D-2)}$.

The $SL(2, \mathbb{R})$ symmetry of the two-scalar system (B.8) involves three nontrivial transformations, namely the shift symmetry $\chi' = \chi + \text{const.}$ of the axion; the inversion $\tau' = -1/\tau$; and a shift symmetry $\phi' = \phi + \text{const.}$ of the dilaton, together with the necessary rescaling of the axion. In the original undualised theory (B.10), only the last of these symmetries is preserved: The constant shift symmetry of χ simply becomes obsolete, since there is no field on which to realise it. In fact, the global shift symmetry is replaced by the local gauge symmetry of $F_{(D-1)}$ in (B.10). The $\tau \to -1/\tau$ symmetry in $SL(2,\mathbb{R})$ also breaks down, owing to its non-linearity; it can, however, be viewed as an on-shell symmetry in the original theory (B.10). It should be emphasised here that the $SL(2,\mathbb{R})$ symmetry is absent not only in the Lagrangian (B.10), but also in its equations of motion.

It is of interest to follow the various symmetries in detail in the dualisation procedure, in order to see when and how they are altered. To dualise the potential in the Lagrangian (B.10), we introduce the Lagrange multiplier χ for the $(D-1)$-form field strength $F_{D-1} = dA_{D-2}$, to enforce the Bianchi identity $dF_{D-1} = 0$. This leads to the first-order Lagrangian

$$e^{-1}\mathcal{L} = R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2(D-1)!}F_{(D-1)}^2 e^{-2\phi} + \chi * dF_{(D-1)} \ . \tag{B.12}$$

The relevant equations of motion that follow from this are given by

$$\Box\phi + \frac{1}{(D-1)!}e^{-2\phi}F_{D-1}^2 = 0\ , \quad dF_{(D-1)} = 0\ , \quad *F_{(D-1)} = e^{2\phi}d\chi\ . \tag{B.13}$$

Note that in this first-order formalism, the $SL(2,\mathbb{R})$ symmetry is still present, with the three transformation rules given by

$$\chi \to \chi + c\ , \quad e^{\phi} \to e^{\phi}\ , \quad F_{(D-1)} \to F_{(D-1)}\ , \tag{B.14}$$

$$e^{\phi} \to \lambda^2 e^{\phi}\ , \quad F_{(D-1)} \to \lambda^2 F_{(D-1)}\ , \quad \chi \to \lambda^{-2}\chi\ , \tag{B.15}$$

$$\tau \to -\frac{\alpha^2}{\tau}\ , \quad *F_{(D-1)} \to \frac{1}{\alpha^2}\Big((\chi^2 + e^{-2\phi})*F_{(D-1)} - 2\chi d\phi - 2d\chi\Big)\ . \tag{B.16}$$

It is a matter of straightforward computation to verify that the Lagrangian (B.12) is invariant under these. At the level of the equations of motion, the invariance under (B.14) and (B.15) is manifest. The transformation (B.16) is more complicated, sending the third first-order equation in (B.13) into itself, while transforming the other two equations into two independent combinations of the original equations of motion.

If we integrate out the auxiliary field F_{D-1}, it gives rise to the two-scalar system (ϕ,χ) with Lagrangian (B.8), and the $SL(2,\mathbb{R})$ symmetry is preserved. However, if we instead integrate χ, the resulting theory, whose Lagrangian is given by (B.10), no longer has an $SL(2,\mathbb{R})$ global symmetry; but instead only a global $\mathbb{R}$ symmetry. It does, however, have an additional local gauge symmetry for $F_{(D-1)}$. The reason for the loss of the global $SL(2,\mathbb{R})$ symmetry can be seen from (B.16). This transformation involves the undifferentiated χ field, and thus cannot be implemented in terms of local transformations either for the field strength $F_{(D-1)}$ or for its gauge potential. Such a reduction of the global symmetry is not surprising, since in the first-order formalism there is an auxiliary field, which can allow the symmetry to be "artificially" enlarged. Integrating out the auxiliary fields then leads to a reduction of the symmetry. The Lagrangian (B.8) maintains the full global symmetry, while losing the local gauge invariance. The Lagrangian (B.10), on the other hand, maintains the local gauge symmetry but loses part of the global symmetry.

It should be noted that there is more than one way to write down a first-order Lagrangian that can lead to (B.8) and (B.10). In the first-order formulation (B.12) we chose χ and $F_{(D-1)}$ as the fundamental fields. We could instead choose $B_{(D-2)}$ and $F_{(1)}$ as the fundamental fields, with the Lagrangian now talking the form

$$e^{-1}\mathcal{L} = R - \tfrac{1}{2}(\partial\phi)^2 - \tfrac{1}{2}F_{(1)}^2 e^{2\phi} + B_{(D-2)} \wedge dF_{(1)} \ . \tag{B.17}$$

In this case, even the first-order Lagrangian has only an $\mathbb{R}$ global symmetry, corresponding to a shift of the dilaton ϕ.

Note that the global $\mathbb{R}$ symmetry of the original theory (B.10) is a subgroup of the $SL(2,\mathbb{R})$ symmetry of the two-scalar system. This feature will in general be the case if the Lagrangian has no $\mathcal{L}_{FFA}$ type topological terms. In a supergravity theory, however, the presence of such an $\mathcal{L}_{FFA}$ term can have the effect that the global symmetry group for the theory where an axion is dualised may no longer be contained in the global symmetry group of the theory where it is left undualised.

C $(O(n)\times O(n))\backslash O(n,n)$ coset manifolds

Consider the $2n \times 2n$ matrix

$$\Omega = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & \delta_B^A \\ \delta_A^B & 0 \end{pmatrix} \ , \tag{C.1}$$

which is left invariant by the $O(n,n)$ infinitesimal transformations generated by[11]

$$L = \begin{pmatrix} U & V \\ \widetilde{V} & -U^T \end{pmatrix} \longrightarrow \begin{pmatrix} U_A{}^B & V_{AB} \\ \widetilde{V}^{AB} & -U_B{}^A \end{pmatrix} \ , \tag{C.2}$$

where U is an arbitrary real matrix, and $V = -V^T$, $\widetilde{V} = -\widetilde{V}^T$. There is a manifest $GL(n,\mathbb{R})$ subgroup of $O(n,n)$ generated by matrices (C.2) with $V = \widetilde{V} = 0$. Also, the

[11]This formulation is related by a simple change of basis to the more familiar one where the metric is $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ and the $O(N)\times O(N)$ subgroup is manifest.

maximal compact subgroup $O(n)\times O(n)$ is generated by matrices (C.2) with $U=-U^T$ and $\widetilde{V}=V$. We can parameterise the coset $(O(n)\times O(n))\backslash O(n,n)$ in terms of upper triangular matrices of the form

$$\mathcal{V}=\begin{pmatrix} S & R \\ 0 & (S^{-1})^T \end{pmatrix}\,, \tag{C.3}$$

where, in order to satisfy the condition $\mathcal{V}\,\Omega\,\mathcal{V}^T=\Omega$, we must have $R\,S^T+S\,R^T=0$.

From $\mathcal{V}$, we may construct the matrix $\mathcal{M}=\mathcal{V}^T\,\mathcal{V}$, giving

$$\mathcal{M}=\begin{pmatrix} S^T\,S & S^T\,R \\ R^T\,S & (S^T\,S)^{-1}+R^T\,R \end{pmatrix} \longrightarrow \begin{pmatrix} (S^T\,S)_{AB} & (S^T\,R)_A{}^B \\ (R^T\,S)^A{}_B & ((S^T\,S)^{-1})^{AB}+(R^T\,R)^{AB} \end{pmatrix}\,. \tag{C.4}$$

Defining $(S^T\,S)_{AB}=G_{AB}$ and $(S^{-1}\,R)^{AB}=-X^{AB}$, and noting that $G_{AB}=G_{BA}$ and $X^{AB}=-X^{BA}$, we can write $\mathcal{M}$ as

$$\mathcal{M}=\begin{pmatrix} G & -G\,X \\ X\,G & G^{-1}-X\,G\,X \end{pmatrix} \longrightarrow \begin{pmatrix} G_{AB} & -G_{AC}\,X^{CB} \\ X^{AC}\,G_{CB} & G^{AB}-X^{AC}\,G_{CD}\,X^{DB} \end{pmatrix}\,. \tag{C.5}$$

The $(O(n)\times O(n))\backslash O(n,n)$ coset Lagrangian can then be written as

$$\begin{aligned} \mathcal{L} &= \tfrac18 e\,\mathrm{tr}\,(\partial_\mu\mathcal{M}^{-1}\,\partial^\mu\mathcal{M})=\tfrac18 e\,\mathrm{tr}\,(\Omega\,\partial_\mu\mathcal{M}\,\Omega\,\partial^\mu\mathcal{M})\,,\\ &= -\tfrac14 e\,G_{AC}\,G_{BD}\Big(\partial_\mu G^{AB}\,\partial^\mu G^{CD}+\partial_\mu X^{AB}\,\partial^\mu X^{CD}\Big)\,. \end{aligned} \tag{C.6}$$

Note that if we set the fields X^{AB} to zero, we get the coset Lagrangian for $O(n)\backslash GL(n,\mathbb{R})$.

The above formalism can be used to describe the scalar Lagrangians of various supergravity theories. For example, the global symmetry of the supergravity describing the low-energy limit of the heterotic string in $D>4$ dimensions is $O(10-D,10-D)$, which is the perturbative T-duality group. (This supergravity theory is obtained from the type IIA supergravity by setting all the R-R fields to zero.) Another example, which we shall study in detail, is the maximal supergravity in $D=6$, which has $E_5=O(5,5)$ global symmetry. The complete scalar Lagrangian coming from the dimensional reduction of eleven-dimensional supergravity is given in (A.22). Defining G_{ij} and X^{ij} by $g_{ij}=\omega\,G_{ij}$ and $A_{(0)ijk}=\frac12\epsilon_{ijk\ell m}\,X^{\ell m}$, we find that (A.22) in $D=6$ reduces to

$$\mathcal{L}_{\rm scalar}=-\tfrac14 e\,G_{ik}\,G_{j\ell}\Big(\partial_\mu G^{ij}\,\partial^\mu G^{k\ell}+\partial_\mu X^{ij}\,\partial^\mu X^{k\ell}\Big)\,, \tag{C.7}$$

which is precisely of the form (C.6) with $n=5$.

The first two terms in (A.22) correspond to a scalar coset manifold with a $GL(11-D,\mathbb{R})$ symmetry, and could be cast into the form of the $GL(n,\mathbb{R})$-invariant Lagrangian (C.6) with X^{AB} set to zero, by making an appropriate Weyl rescaling of g_{ij}. The $GL(11-D,\mathbb{R})$

symmetry is augmented by the inclusion of the third term in (A.22), describing the scalar Lagrangian for the axions coming from the dimensional reduction of the antisymmetric tensor in $D = 11$. We have already seen that in $D = 6$, the inclusion of the third term enlarges $GL(5,\mathbb{R})$ to $O(5,5)$. In $D = 7$, the enlarged symmetry group is $SL(5,\mathbb{R})$. We can see this from (A.22) by making the redefinition $A_{(0)ijk} = \epsilon_{ijk\ell}\, X^\ell$ and defining a 5×5 metric G_{AB} by

$$
\begin{aligned}
G_{AB} &= \begin{pmatrix} \frac{1}{\omega}\, g_{ij} & \frac{1}{\omega}\, g_{ik}\, X^k \\ \frac{1}{\omega}\, g_{jk}\, X^k & \omega + \frac{1}{\omega}\, g_{k\ell}\, X^k\, X^\ell \end{pmatrix} , \\
G^{AB} &= \begin{pmatrix} \omega\, g^{ij} + \frac{1}{\omega}\, X^i\, X^j & -\frac{1}{\omega}\, X^i \\ -\frac{1}{\omega}\, X^j & \frac{1}{\omega} \end{pmatrix} .
\end{aligned}
\tag{C.8}
$$

Note that G_{AB} has unit determinant. Substituting this into the general $GL(n,\mathbb{R})$-invariant Lagrangian (C.6) where X^{AB} is set to zero, we find that it precisely gives the scalar Lagrangian (A.22) for $D = 7$. Thus the $SL(5,\mathbb{R})$ symmetry is made manifest.

In $D = 8$, we define $A_{(0)ijk} = \epsilon_{ijk}\,\chi$ and $g_{ij} = \omega\, G_{ij}$, and substitute these into (A.22). Noting that G_{ij} has unit determinant here, and making the further redefinition $\omega = e^{-\phi}$, we find that the complete scalar Lagrangian becomes

$$
\mathcal{L}_{\rm scalar} = -\tfrac{1}{4} e\, G_{ik}\, G_{j\ell}\, \partial_\mu G^{ij}\, \partial^\mu G^{k\ell} - \tfrac{1}{2} e\, (\partial\phi)^2 - \tfrac{1}{2} e\, e^{2\phi}\, (\partial\chi)^2 \ . \tag{C.9}
$$

Owing to the fact that G_{ij} has unit determinant we see that the first term describes the coset $O(3)\backslash SL(3,\mathbb{R})$, and the remaining two terms describe the coset $O(2)\backslash SL(2,\mathbb{R})$ as discussed in Appendix B.

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