

3d $\mathcal{N} = 1$ Chern-Simons-matter theory and localization

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ABSTRACT: We consider the most general, classically-conformal, three-dimensional $\mathcal{N} = 1$ Chern-Simons-matter theory with global symmetry $Sp(2)$ and gauge group $U(N) \times U(N)$. We show that the Lagrangian in the on-shell formulation of the theory admits one more free parameter as compared to the theory formulated in off-shell $\mathcal{N} = 1$ superspace. The theory on T^3 can be formally localized. We partially carry out the localization procedure for the theory on T^3 with periodic boundary conditions. In particular we show that restricting to the saddle points with vanishing gauge connection gives a trivial contribution to the partition function, i.e. the bosonic and fermionic contributions exactly cancel each other.

Contents

1	Introduction	2
2	$\mathcal{N} = 1$ Superconformal Chern-Simons-matter theory	3
2.1	On-Shell	3
2.2	Off-Shell	4
3	Localization	6
3.1	Setup	7
3.2	Deformations	9
3.2.1	Matter Sector	9
3.2.2	Gauge Sector	11
3.3	Gauge Fixing	12
3.4	Saddle Points	13
3.5	One-loop Determinant	15
3.5.1	Determinant from Bosons	16
3.5.2	Determinant from Fermions	20
4	Discussion	25
A	Spinor and gamma-matrix conventions in 3d	26
B	$\mathcal{N} = 1$ Superconformal Symmetry	26
B.1	Poincaré Supersymmetry	26
B.2	Conformal supersymmetry	29
C	Trivial G-structures in 3d	30
D	Lorentz Gauge	32

1 Introduction

Using localization, several exact results have by now been obtained for supersymmetric gauge theories, such as the computation of indices, partition functions and Wilson loops, among others. In many cases these exact computations have provided us with checks of non-trivial dualities, including AdS/CFT.

In the present paper we apply the localization procedure to the most general, classically-conformal, three-dimensional $\mathcal{N} = 1$ Chern-Simons-matter theory with global symmetry $Sp(2)$ and gauge group $U(N) \times U(N)$. Previously, localization had mainly been used to study theories on curved spacetimes with non-trivial R-symmetry [1–11]. We show that the $\mathcal{N} = 1$ theory on a flat three-dimensional torus T^3 can also be formally localized.

The $\mathcal{N} = 1$ theory we consider here is not in general superconformal on the quantum level, except for special points in its moduli space where supersymmetry may be enhanced. In particular the ABJM model [12] is one such special point where supersymmetry is enhanced to $\mathcal{N} = 6$. By considering a classically-conformal $\mathcal{N} = 1$ theory with unequal Chern-Simons (CS) levels which is in a certain sense a small deformation of the ABJM model [12], it was argued in [13] that the theory flows to an RG fixed point in the infrared. These CFT's were then conjectured in [13] to be dual to certain (massive) IIA supergravity solutions [14, 15] which fall within the general class of [16].

The outline of the paper is as follows. In section 2 we give the on-shell formulation of the most general classically-conformal $\mathcal{N} = 1$ $U(N) \times U(N)$ CS-matter theory with $Sp(2)$ global symmetry. We then introduce auxiliary fields and formulate the theory off-shell, as required by the localization procedure. An interesting observation is that the Lagrangian in the on-shell formulation of the theory admits one more free parameter as compared to the theory formulated in off-shell $\mathcal{N} = 1$ superspace.

In section 3 we formulate the theory on a curved manifold. One notable difference from the CS theories with $\mathcal{N} = 2$ supersymmetry studied in [2] is that the requirement of localization excludes positive-curvature manifolds such as S^3 . Formulating the theory on T^3 or the hyperbolic three-dimensional space H_3 preserves superconformal symmetry at the classical level. In this paper we shall focus on the theory on T^3 .¹

We next carry out the localization procedure for the theory on T^3 with periodic boundary conditions. As an illustration of the formalism we compute the contributions to the partition function from the locus of saddle points with vanishing gauge connection. We show that restricting to this locus gives a trivial contribution to the partition function, i.e. the bosonic and fermionic contributions exactly cancel each other. We conclude with a discussion of our results in section 4. Further technical details can be found in the appendices.

¹Demanding that the manifold should be compact, in order to ensure that the partition function is well-defined, leads us to exclude H_3 . Compact quotients thereof may still preserve superconformal symmetry but we shall not examine this possibility here.

2 $\mathcal{N} = 1$ Superconformal Chern-Simons-matter theory

2.1 On-Shell

The general component form of the on-shell $\mathcal{N} = 1$ classically-superconformal CS Lagrangian with $Spin(5) \simeq Sp(2)$ global symmetry and gauge group $U(N) \times U(N)$ is given in [17]:

$$\mathcal{L} = \mathcal{L}_{CS} + \mathcal{L}_{kin} + \mathcal{L}_4 + \mathcal{L}_6, \quad (2.1)$$

where \mathcal{L}_{CS} is the pure CS Lagrangian, \mathcal{L}_{kin} is the matter kinetic term, \mathcal{L}_4 is the quartic interaction and \mathcal{L}_6 is the sextic potential. More specifically,²

$$\mathcal{L}_{CS} = \frac{k_1}{2\pi} \varepsilon^{\mu\nu\rho} \text{tr} \left\{ \frac{1}{2} A_\mu \partial_\nu A_\rho + \frac{i}{3} A_\mu A_\nu A_\rho \right\} - \frac{k_2}{2\pi} \varepsilon^{\mu\nu\rho} \text{tr} \left\{ \frac{1}{2} \hat{A}_\mu \partial_\nu \hat{A}_\rho + \frac{i}{3} \hat{A}_\mu \hat{A}_\nu \hat{A}_\rho \right\}, \quad (2.2)$$

where the normalization above was chosen to facilitate the derivation of the superconformal invariance; A_μ, \hat{A}_μ are gauge fields in the adjoint of $U(N)$. The matter kinetic terms read:

$$\mathcal{L}_{kin} = \frac{1}{2\pi} \text{tr} \left\{ -D^\mu X^A D_\mu X_A + i \tilde{\Psi}_A \gamma^\mu D_\mu \Psi^A \right\}, \quad (2.3)$$

where $A = 1, \dots, 4$ is an $Sp(2)$ index; X_A is in the bifundamental (\bar{N}, N) while X^A is in the (N, \bar{N}) , and similarly for Ψ_A, Ψ^A . The most general quartic interaction terms can be written in the form $\mathcal{L}_4 = \mathcal{L}_{4a} + \mathcal{L}_{4b} + \mathcal{L}_{4c} + \mathcal{L}'$, where:

$$\begin{aligned} \mathcal{L}_{4a} &= \frac{1}{2\pi} i \text{tr} \{ \bar{\alpha}_1 \varepsilon^{ABCD} \tilde{\Psi}_A X_B \Psi_C X_D - \alpha_1 \varepsilon_{ABCD} \tilde{\Psi}^A X^B \Psi^C X^D \} \\ \mathcal{L}_{4b} &= \frac{1}{2\pi} i \text{tr} \{ \alpha_{2,1} \tilde{\Psi}^A \Psi_A X_B X^B - \alpha_{2,2} \tilde{\Psi}_A \Psi^A X^B X_B \} \\ \mathcal{L}_{4c} &= \frac{1}{2\pi} 2i \text{tr} \{ \alpha_{3,1} \tilde{\Psi}_A \Psi^B X^A X_B - \alpha_{3,2} \tilde{\Psi}^B \Psi_A X_B X^A \} \\ \mathcal{L}' &= \frac{1}{2\pi} \text{tr} \{ a_1 \Omega^{AD} \Omega_{BC} \tilde{\Psi}_A \Psi^B X^C X_D + a_2 \Omega_{AD} \Omega^{BC} \tilde{\Psi}^A \Psi_B X_C X^D \\ &\quad + a_3 \Omega^{AC} \Omega^{BD} \tilde{\Psi}_A X_B \Psi_C X_D + \bar{a}_3 \Omega_{AC} \Omega_{BD} \tilde{\Psi}^A X^B \Psi^C X^D \\ &\quad + a_4 \Omega^{AB} \Omega^{CD} \tilde{\Psi}_A X_B \Psi_C X_D + \bar{a}_4 \Omega_{AB} \Omega_{CD} \tilde{\Psi}^A X^B \Psi^C X^D \}. \end{aligned} \quad (2.4)$$

The sextic potential consists of two terms $\mathcal{L}_6 = \mathcal{L}_{pot} + \mathcal{L}''$, where:

$$\begin{aligned} \mathcal{L}_{pot} &= \frac{1}{2\pi} \frac{1}{3} \text{tr} \{ \alpha_{4,1} X^A X_A X^B X_B X^C X_C + \alpha_{4,2} X_A X^A X_B X^B X_C X^C \\ &\quad + 4\alpha_{4,3} X_A X^B X_C X^A X_B X^C - 6\alpha_{4,4} X^A X_B X^B X_A X^C X_C \} \\ \mathcal{L}'' &= \frac{1}{2\pi} \Omega^{BC} \Omega_{DE} \text{tr} \{ n X_B X^A X_C X^D X_A X^E \} \\ &\quad + \frac{1}{2\pi} \Omega^{BC} \Omega_{DE} \text{tr} \{ m X_B X^A X_A X^D X_C X^E \} \\ &\quad + \frac{1}{2\pi} \Omega_{BC} \Omega^{DE} \text{tr} \{ \bar{m} X^B X_A X^A X_D X^C X_E \}. \end{aligned} \quad (2.5)$$

²We follow closely the notation of [17], to which the reader is referred for more details; our spinor notation is explained in appendix A.

Here Ω_{AB} is the $Sp(2)$ -invariant antisymmetric tensor, which satisfies $\Omega^{AB}\Omega_{AC} = \delta_C^B$. As shown in Appendix B, the theory is invariant under the following $\mathcal{N} = 1$ Poincaré supersymmetry:

$$\begin{aligned}
\delta X_A &= i\Omega_{AB}\tilde{\epsilon}\Psi^B \\
\delta X^A &= i\Omega^{AB}\tilde{\epsilon}\Psi_B \\
\delta\Psi_A &= \Omega_{AB}\gamma^\mu\epsilon D_\mu X^B + \{\Omega_{AB}(\alpha_{2,2}X^C X_C X^B \\
&\quad - \alpha_{2,1}X^B X_C X^C) - 2\alpha_3\Omega_{BC}X^B X_A X^C\}\epsilon \\
\delta\Psi^A &= \Omega^{AB}\gamma^\mu\epsilon D_\mu X_B + \{\Omega^{AB}(-\alpha_{2,1}X_C X^C X_B \\
&\quad + \alpha_{2,2}X_B X^C X_C) + 2\alpha_3\Omega^{BC}X_B X^A X_C\}\epsilon \\
\delta A_\mu &= \frac{1}{k_1}[\Omega_{AB}\tilde{\epsilon}\gamma_\mu\Psi^A X^B + \Omega^{AB}X_B\tilde{\Psi}_A\gamma_\mu\epsilon] \\
\delta\hat{A}_\mu &= \frac{1}{k_2}[\Omega_{AB}X^B\tilde{\epsilon}\gamma_\mu\Psi^A + \Omega^{AB}\tilde{\Psi}_A\gamma_\mu\epsilon X_B] ,
\end{aligned} \tag{2.6}$$

provided that the coefficients satisfy the relations:

$$\begin{aligned}
a_1 &= -2i\left(\frac{1}{k_1} + \bar{\alpha}_1\right) , \quad a_2 = 2i\left(\frac{1}{k_2} + \alpha_1\right) , \\
a_3 &= -\bar{a}_3 - i(\alpha_1 - \bar{\alpha}_1) , \quad a_4 = i(\alpha_1 - \bar{\alpha}_1) , \\
\alpha_{2,1} &= -\frac{1}{k_1} - 2\bar{\alpha}_1 , \quad \alpha_{2,2} = -\frac{1}{k_2} - 2\alpha_1 , \quad \alpha_3 = i\bar{a}_3 - \alpha_1 , \\
\alpha_{4,1} &= -3\alpha_{2,2}^2 + 4\alpha_{2,2}\alpha_3 + m , \quad \alpha_{4,2} = -3\alpha_{2,1}^2 + 4\alpha_{2,2}\alpha_3 + m , \\
\alpha_{4,3} &= \alpha_{2,2}\alpha_3 + \frac{m}{4} , \quad \alpha_{4,4} = -\alpha_{2,1}\alpha_{2,2} + 2\alpha_{2,2}\alpha_3 + \frac{m}{2} , \\
\bar{m} &= 4(\alpha_{2,2} - \alpha_{2,1})\alpha_3 + m , \quad n = 4(\alpha_3 - \alpha_{2,2})\alpha_3 - m .
\end{aligned} \tag{2.7}$$

In addition to the CS levels k_1, k_2 , the theory has four independent parameters. One can choose them to be $\alpha_1, \bar{\alpha}_1, \bar{a}_3$ and m .

2.2 Off-Shell

In the previous section we studied the on-shell formulation of the theory. However to carry out the localization procedure one needs off-shell supersymmetry. For that purpose we introduce the auxiliary scalar fields F and the gaugini $\lambda, \hat{\lambda}$ in the scalar and gauge multiplets, respectively. The off-shell action reads:

$$\mathcal{L} = \mathcal{L}_{CS} + \mathcal{L}_{kin} + \mathcal{L}_{potential} , \tag{2.8}$$

where:

$$\begin{aligned}
\mathcal{L}_{CS} &= \frac{k_1}{2\pi}\text{tr}\left\{\varepsilon^{\mu\nu\rho}\left(\frac{1}{2}A_\mu\partial_\nu A_\rho + \frac{i}{3}A_\mu A_\nu A_\rho\right) + \frac{i}{2}\tilde{\lambda}\lambda\right\} \\
&\quad - \frac{k_2}{2\pi}\text{tr}\left\{\varepsilon^{\mu\nu\rho}\left(\frac{1}{2}\hat{A}_\mu\partial_\nu\hat{A}_\rho + \frac{i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\rho\right) + \frac{i}{2}\tilde{\lambda}\hat{\lambda}\right\} ,
\end{aligned} \tag{2.9}$$

$$\mathcal{L}_{kin} = \frac{1}{2\pi} \text{tr} \left\{ -D^\mu X^A D_\mu X_A + i\tilde{\Psi}_A \gamma^\mu D_\mu \Psi^A - F^A F_A \right\} , \quad (2.10)$$

$$\begin{aligned} \mathcal{L}_{\text{potential}} = & \frac{1}{2\pi} \text{tr} \{ i [(-\alpha_{2,1} X_B X^B X_A + \alpha_{2,2} X_A X^B X_B) - 2\alpha_3 \Omega_{AB} \Omega^{CD} X_C X^B X_D] F^A \\ & + i F_A [(-\alpha_{2,1} X^A X_B X^B + \alpha_{2,2} X^B X_B X^A) + 2\alpha_3 \Omega^{AB} \Omega_{CD} X^C X_B X^D] \} \\ & + \frac{1}{2\pi} \text{tr} \{ \Omega_{AB} \tilde{\lambda} \Psi^A X^B - \Omega^{AB} X_B \tilde{\Psi}_A \lambda - \Omega_{AB} X^B \tilde{\Psi}^A \hat{\lambda} + \Omega^{AB} \tilde{\lambda} \Psi_A X_B \} \\ & + \frac{1}{2\pi} \text{tr} \{ i\alpha_{2,1} \Omega^{AD} \Omega_{BC} \tilde{\Psi}_A \Psi^B X^C X_D - i\alpha_{2,2} \Omega_{AD} \Omega^{BC} \tilde{\Psi}^A \Psi_B X_C X^D \\ & - \frac{i}{2} \alpha_{2,2} \Omega^{AB} \Omega^{CD} \tilde{\Psi}_A X_B \Psi_C X_D + \frac{i}{2} \alpha_{2,1} \Omega_{AB} \Omega_{CD} \tilde{\Psi}^A X^B \Psi^C X^D \\ & + i\alpha_3 \Omega^{AC} \Omega^{BD} \tilde{\Psi}_A X_B \Psi_C X_D - i\alpha_3 \Omega_{AC} \Omega_{BD} \tilde{\Psi}^A X^B \Psi^C X^D \\ & - \frac{i}{2} \alpha_{2,1} \Omega^{AD} \Omega^{BC} \tilde{\Psi}_A X_B \Psi_C X_D + \frac{i}{2} \alpha_{2,2} \Omega_{AD} \Omega_{BC} \tilde{\Psi}^A X^B \Psi^C X^D \} \\ & + \frac{1}{2\pi} i \text{tr} \{ \alpha_{2,1} \tilde{\Psi}^A \Psi_A X_B X^B - \alpha_{2,2} \tilde{\Psi}_A \Psi^A X^B X_B \} \\ & + \frac{1}{2\pi} 2i \text{tr} \{ \alpha_3 \tilde{\Psi}_A \Psi^B X^A X_B - \alpha_3 \tilde{\Psi}^B \Psi_A X_B X^A \} . \end{aligned} \quad (2.11)$$

This can be rewritten compactly in superspace formalism, see e.g. (3.8) of [13] which we reproduce here:

$$\begin{aligned} S = & \frac{k_1}{2\pi} S_{CS}(A) - \frac{k_2}{2\pi} S_{CS}(\hat{A}) + \frac{1}{2\pi} \int d^2\theta \text{tr} \{ D_a \Phi_A^\dagger D^a \Phi^A \\ & + (c_1 \Phi_A^\dagger \Phi^A \Phi_B^\dagger \Phi^B + c_2 \Phi_A^\dagger \Phi^B \Phi_B^\dagger \Phi^A + c_3 \Omega^{AB} \Omega_{CD} \Phi_A^\dagger \Phi^C \Phi_B^\dagger \Phi^D) \} , \end{aligned} \quad (2.12)$$

where Φ_A is a superfield, and the connection with the component formulation discussed previously is provided by the relations:

$$c_1 = -i\bar{\alpha}_1 - \frac{i}{2k_1} ; \quad c_2 = i\alpha_1 + \frac{i}{2k_2} ; \quad c_3 = i\alpha_1 + \bar{\alpha}_3 . \quad (2.13)$$

The action is invariant under the off-shell supersymmetry transformations:

$$\begin{aligned} \delta X_A &= i\Omega_{AB} \tilde{\epsilon} \Psi^B \\ \delta X^A &= i\Omega^{AB} \tilde{\epsilon} \Psi_B \\ \delta \Psi_A &= \Omega_{AB} \gamma^\mu \epsilon D_\mu X^B - i\Omega_{AB} F^B \epsilon \\ \delta \Psi^A &= \Omega^{AB} \gamma^\mu \epsilon D_\mu X_B - i\Omega^{AB} F_B \epsilon \\ \delta F_A &= -\Omega_{AB} \tilde{\epsilon} \gamma^\mu D_\mu \Psi^B - iX_A (\tilde{\epsilon} \hat{\lambda}) + i(\tilde{\epsilon} \lambda) X_A \\ \delta F^A &= -\Omega^{AB} \tilde{\epsilon} \gamma^\mu D_\mu \Psi_B - iX^A (\tilde{\epsilon} \lambda) + i(\tilde{\epsilon} \hat{\lambda}) X^A \\ \delta A_\mu &= -i\tilde{\epsilon} \gamma_\mu \lambda \\ \delta \hat{A}_\mu &= -i\tilde{\epsilon} \gamma_\mu \hat{\lambda} \\ \delta \lambda &= -\frac{1}{2} \gamma^{\mu\nu} \epsilon F_{\mu\nu} \\ \delta \hat{\lambda} &= -\frac{1}{2} \gamma^{\mu\nu} \epsilon \hat{F}_{\mu\nu} . \end{aligned} \quad (2.14)$$

We note that besides k_1, k_2 the off-shell theory has only three free parameters, as can be seen from (2.13). This is one fewer parameter than in the on-shell formulation. Specifically, after replacing the auxiliary field F and gaugini $\lambda, \hat{\lambda}$ by the solutions of their respective equations of motion, the Lagrangian (2.8) goes back to (2.1), but with $\alpha_{4,3} = 0$ in \mathcal{L}_{pot} . In other words, for the on-shell theory obtained by starting from (2.8) and then eliminating the auxiliary fields, m is not an independent parameter but is equal to $-4\alpha_{2,2}\alpha_3$, which in its turn can be expressed in terms of $\alpha_1, \bar{\alpha}_1$ and \bar{a}_i . This can be understood from the fact that the sextic potential $X_A X^B X_C X^A X_B X^C$ in \mathcal{L}_{pot} cannot be obtained from the off-shell Lagrangian by replacing F by its solution.

In the following we will put the theory on a curved manifold. More specifically, to go from flat to curved spacetime one needs to:

- covariantize all derivatives,
- introduce additional terms $\frac{1}{3}\Omega_{AB}X^B\gamma^\mu\nabla_\mu\epsilon$ and $\frac{1}{3}\Omega^{AB}X_B\gamma^\mu\nabla_\mu\epsilon$ in the transformations of Ψ_A and Ψ^A , respectively,
- have ϵ satisfy the conformal Killing spinor equation:

$$\nabla_\mu\epsilon = \gamma_\mu\eta, \quad (2.15)$$

where η is some arbitrary spinor,

- add a scalar-curvature coupling term, $-\frac{1}{8}RX^AX_A$, to the Lagrangian.

Explicitly:

$$\begin{aligned} \delta\Psi_A &\rightarrow \delta\Psi_A = \Omega_{AB}\gamma^\mu\epsilon D_\mu X^B + \frac{1}{3}\Omega_{AB}X^B\gamma^\mu\nabla_\mu\epsilon - i\Omega_{AB}F^B\epsilon, \\ \delta\Psi^A &\rightarrow \delta\Psi^A = \Omega^{AB}\gamma^\mu\epsilon D_\mu X_B + \frac{1}{3}\Omega^{AB}X_B\gamma^\mu\nabla_\mu\epsilon - i\Omega^{AB}F_B\epsilon, \end{aligned} \quad (2.16)$$

$$\mathcal{L}_{kin} \rightarrow \mathcal{L}_{kin} = \frac{1}{2\pi}\text{tr} \left\{ -D^\mu X^A D_\mu X_A - \frac{1}{8}RX^AX_A + i\tilde{\Psi}_A\gamma^\mu D_\mu\Psi^A - F^A F_A \right\}. \quad (2.17)$$

The resulting curved-space Lagrangian will be used in the next section.

3 Localization

In order to apply the localization procedure, the theory must be invariant under the action of a fermionic symmetry δ which is nilpotent, $\delta^2 = 0$, or more generally squares to a symmetry of the theory. Deforming the action by a δ -exact term,

$$S \longrightarrow S + t\delta V, \quad (3.1)$$

leaves invariant the expectation values of δ -closed operators. Hence we may take the limit $t \rightarrow \infty$, upon which the theory localizes to the set Σ of critical points of δV [18]. In this

limit the path integral can be performed by restricting S to Σ and computing a one-loop determinant describing the fluctuations normal to Σ . This procedure was first carried out in detail in [1] for the case of SYM on the round S^4 .

In order for the path integral to be well-defined, we will consider the theory in Euclidean signature. All fields are then complexified, while the action becomes a holomorphic functional in the space of complexified fields. This procedure is known under the name of “holomorphic complexification” and ensures that supersymmetry is preserved, see e.g. [19]. Following [1] our strategy will be to choose a path-integration contour in the space of fields, such that when restricted to that contour the deformation δV becomes a sum of positive semi-definite terms. The locus Σ will then be determined by the condition that each term in the sum vanishes.

3.1 Setup

As explained above, in order to apply the localization procedure we need to pass from Lorentzian to Euclidean signature, where all fields become complex. Moreover $\epsilon^{\mu\nu\rho}$ in the CS piece of the Lagrangian becomes $i\epsilon^{\mu\nu\rho}$.

We then deform the action by adding a term $t\delta V$ such that $\delta^2 V = 0$. For theories with $\mathcal{N} \geq 2$ supersymmetry, one can have $\delta^2 = 0$ on all fields of the theory. However, this is not possible for the $\mathcal{N} = 1$ superalgebra. Instead, as we will show later, for $\mathcal{N} = 1$ we can require that δ squares to a transformation in the isometry group of the manifold, which in turn leads to $\delta^2 V = 0$ upon volume integration.

Furthermore we must restrict the supersymmetry parameter ϵ to satisfy the Killing spinor equation:³

$$\nabla_\mu \epsilon = S \gamma_\mu \epsilon , \tag{3.2}$$

where S is in general a complex function. The reason for restricting to this Killing spinor equation instead of the more general one (2.15) is the following. Equation (2.15) would in general imply that δ^2 induces not only a translation, a rotation and a gauge transformation but also a dilatation, which would break the invariance of the deformation δV .

Under the assumption of smoothness, any solution to the Killing spinor equation which is not identically zero is nowhere-vanishing on the manifold. This follows from the fact that (3.2) is a first-order differential equation, hence if the Killing spinor vanishes at any one point it must vanish everywhere.

Given a nowhere-vanishing Killing spinor ϵ , any spinor Ψ can be decomposed as follows:

$$\Psi = \Psi_+ \epsilon + \Psi_- \epsilon^c , \tag{3.3}$$

where Ψ_\pm are anticommuting scalars; our conventions are explained in appendix C. From now on we require the supersymmetry parameters to be commuting. The off-shell Lagrangian given in section 2 remains invariant under supersymmetry with these commuting

³A detailed analysis of this Killing spinor equation in Lorentzian signature is given in section 3 of [20].

parameters. With the above definitions the supersymmetric transformations can be rewritten as:

$$\begin{aligned}
\delta X_A &= ia\Omega_{AB}\Psi_-^B \\
\delta X^A &= ia\Omega^{AB}\Psi_{B-} \\
\delta\Psi_{A-} &= \frac{1}{a}\Omega_{AB}V^\mu D_\mu X^B \\
\delta\Psi_{A+} &= \frac{1}{a}\Omega_{AB}U^\mu D_\mu X^B + S\Omega_{AB}X^B - i\Omega_{AB}F^B \\
\delta\Psi_-^A &= \frac{1}{a}\Omega^{AB}V^\mu D_\mu X_B \\
\delta\Psi_+^A &= \frac{1}{a}\Omega^{AB}U^\mu D_\mu X_B + S\Omega^{AB}X_B - i\Omega^{AB}F_B \\
\delta F_A &= -\Omega_{AB}V^\mu D_\mu \Psi_+^B + \Omega_{AB}U^\mu D_\mu \Psi_-^B \\
&\quad + 3S^*a\Omega_{AB}\Psi_-^B - iaX_A\hat{\lambda}_- + ia\lambda_-X_A \\
\delta F^A &= -\Omega^{AB}V^\mu D_\mu \Psi_{B+} + \Omega^{AB}U^\mu D_\mu \Psi_{B-} \\
&\quad + 3S^*a\Omega^{AB}\Psi_{B-} - iaX^A\lambda_- + ia\hat{\lambda}_-X^A,
\end{aligned} \tag{3.4}$$

and for the gauge multiplets:

$$\begin{aligned}
\delta A_\mu &= -iV_\mu\lambda_+ + iU_\mu\lambda_- \\
\delta \hat{A}_\mu &= -iV_\mu\hat{\lambda}_+ + iU_\mu\hat{\lambda}_- \\
\delta\lambda_+ &= -\frac{1}{2a}i\epsilon^{\mu\nu\rho}U_\rho F_{\mu\nu} \\
\delta\lambda_- &= -\frac{1}{2a}i\epsilon^{\mu\nu\rho}V_\rho F_{\mu\nu} \\
\delta\hat{\lambda}_+ &= -\frac{1}{2a}i\epsilon^{\mu\nu\rho}U_\rho \hat{F}_{\mu\nu} \\
\delta\hat{\lambda}_- &= -\frac{1}{2a}i\epsilon^{\mu\nu\rho}V_\rho \hat{F}_{\mu\nu},
\end{aligned} \tag{3.5}$$

where:

$$\begin{aligned}
a &\equiv \epsilon^\dagger\epsilon = \tilde{\epsilon}\epsilon^c = -\tilde{\epsilon}^c\epsilon, \quad V^\mu \equiv \tilde{\epsilon}\gamma^\mu\epsilon, \\
U^\mu &\equiv \epsilon^\dagger\gamma^\mu\epsilon = -\tilde{\epsilon}\gamma^\mu\epsilon^c, \quad \nabla_\mu\epsilon^c = -S^*\gamma_\mu\epsilon^c.
\end{aligned} \tag{3.6}$$

Note that Ψ_-^A , Ψ_+^A , Ψ_{A-} , Ψ_{A+} , λ_- and λ_+ are anticommuting; so is the supersymmetry transformation δ . With the above setup, we find:

$$\begin{aligned}
\delta^2 X_A &= -iV^\mu D_\mu X_A \\
\delta^2 \Psi_-^A &= -iV^\mu D_\mu \Psi_-^A \\
\delta^2 \Psi_+^A &= -iV^\mu D_\mu \Psi_+^A - 2ia(S - S^*)\Psi_-^A \\
\delta^2 F_A &= -V^\mu D_\mu F_A + V^\mu \partial_\mu S X_A,
\end{aligned} \tag{3.7}$$

and:

$$\begin{aligned}
\delta^2 A_\mu &= -iV^\nu F_{\nu\mu} \\
\delta^2 \hat{A}_\mu &= -iV^\nu \hat{F}_{\nu\mu} \\
\delta^2 \lambda_- &= -iV^\mu D_\mu \lambda_- \\
\delta^2 \lambda_+ &= -iV^\mu D_\mu \lambda_+ - 2ia(S - S^*)\lambda_- \\
\delta^2 \hat{\lambda}_- &= -iV^\mu D_\mu \hat{\lambda}_- \\
\delta^2 \hat{\lambda}_+ &= -iV^\mu D_\mu \hat{\lambda}_+ - 2ia(S - S^*)\hat{\lambda}_- .
\end{aligned} \tag{3.8}$$

Equivalently, written in terms of the original fields, two supersymmetry transformations give:

$$\begin{aligned}
\delta^2 X_A &= -iV^\mu D_\mu X_A \\
\delta^2 \Psi^A &= -iV^\mu D_\mu \Psi^A - iSV^\mu \gamma_\mu \Psi^A \\
\delta^2 F_A &= -iV^\mu D_\mu F_A + V^\mu \partial_\mu S X_A ,
\end{aligned} \tag{3.9}$$

and:

$$\begin{aligned}
\delta^2 A_\mu &= -iV^\nu F_{\nu\mu} \\
\delta^2 \hat{A}_\mu &= -iV^\nu \hat{F}_{\nu\mu} \\
\delta^2 \lambda &= -iV^\mu D_\mu \lambda - iSV^\mu \gamma_\mu \lambda \\
\delta^2 \hat{\lambda} &= -iV^\mu D_\mu \hat{\lambda} - iSV^\mu \gamma_\mu \hat{\lambda} .
\end{aligned} \tag{3.10}$$

As explained in Appendix C, V^μ can be identified as part of the orthonormal frame that trivializes the tangent bundle of the manifold. Therefore, apart from additional terms which can be interpreted as gauge transformations or rotations, δ^2 acting on each field gives a translation along V^μ .

In the next section we will ultimately set $a = 1$ and $S = 0$, upon which the above equations simplify further.

3.2 Deformations

3.2.1 Matter Sector

To localize the matter sector, we first consider the deformation,

$$\delta V = \int \sqrt{g} d^3x \delta[(\delta\Psi_A)^\dagger \Psi_A] , \tag{3.11}$$

where we have *defined*:

$$(\delta\Psi_A)^\dagger \equiv \Omega^{AB} \epsilon^\dagger \gamma^\mu D_\mu X_B + S^* \Omega^{AB} X_B \epsilon^\dagger + i\Omega^{AB} F_B \epsilon^\dagger . \tag{3.12}$$

Note that at generic points in field space $(\delta\Psi_A)^\dagger$ is *not* the adjoint of $\delta\Psi_A$, and δV as defined in (3.11) is a holomorphic functional in the space of complexified fields.

As explained in section 3, we will choose a path-integration contour \mathcal{C} in the space of fields such that when restricted to \mathcal{C} the deformation δV becomes a sum of positive semi-definite terms. This requirement selects \mathcal{C} as the subspace where the fields satisfy the reality condition:

$$\text{Contour } \mathcal{C} : \quad \begin{aligned} X^{A\dagger} &= X_A, \quad F^{A\dagger} = F_A, \\ A_\mu^\dagger &= A_\mu, \quad \hat{A}_\mu^\dagger = \hat{A}_\mu. \end{aligned} \quad (3.13)$$

Moreover the integrand in (3.11) is given by:

$$\delta[(\delta\Psi_A)^\dagger\Psi_A] = \delta(\delta\Psi_A)^\dagger\Psi_A + (\delta\Psi_A)^\dagger\delta\Psi_A. \quad (3.14)$$

Recall that the supersymmetry transformation δ is anticommuting; the relative sign on the right-hand side is positive since $(\delta\Psi_A)^\dagger$ is bosonic.

Let us now verify that the deformation is δ -closed. From (3.14) we obtain:

$$\begin{aligned} \delta^2[(\delta\Psi_A)^\dagger\Psi_A] &= \delta^2(\delta\Psi_A)^\dagger\Psi_A - \delta(\delta\Psi_A)^\dagger\delta\Psi_A + \delta(\delta\Psi_A)^\dagger\delta\Psi_A + (\delta\Psi_A)^\dagger\delta^2\Psi_A \\ &= \delta^2(\delta\Psi_A)^\dagger\Psi_A + (\delta\Psi_A)^\dagger\delta^2\Psi_A. \end{aligned} \quad (3.15)$$

The second term in the second line can be read off from (3.9). One can obtain the first term in the second line from (3.9) and (3.12):

$$\begin{aligned} \delta^2(\delta\Psi_A)^\dagger\Psi_A &= -iV^\mu D_\mu[(\delta\Psi_A)^\dagger]\Psi_A + iS^*V^\mu(\delta\Psi_A)^\dagger\gamma_\mu\Psi_A \\ &\quad + 2iS^*\Omega^{AB}V_\mu D_\nu X_B \epsilon^\dagger\gamma^{\mu\nu}\Psi_A - 2iS\Omega^{AB}V_\mu D_\nu X_B \epsilon^\dagger\gamma^{\mu\nu}\Psi_A, \end{aligned} \quad (3.16)$$

where we used $\nabla_\mu\epsilon^c = -S^*\gamma_\mu\epsilon^c$ and chose S to be a constant. Finally,

$$\begin{aligned} \delta^2[(\delta\Psi_A)^\dagger\Psi_A] &= -iV^\mu\partial_\mu[(\delta\Psi_A)^\dagger]\Psi_A \\ &\quad + iS^*V^\mu(\delta\Psi_A)^\dagger\gamma_\mu\Psi_A - iSV^\mu(\delta\Psi_A)^\dagger\gamma_\mu\Psi_A \\ &\quad + 2iS^*\Omega^{AB}V_\mu D_\nu X_B \epsilon^\dagger\gamma^{\mu\nu}\Psi_A - 2iS\Omega^{AB}V_\mu D_\nu X_B \epsilon^\dagger\gamma^{\mu\nu}\Psi_A. \end{aligned} \quad (3.17)$$

This vanishes under the volume integration if and only if S is real constant. On the other hand the integrability condition of the Killing spinor (3.2) relates the constant S to the curvature scalar of the manifold:

$$R = -24S^2. \quad (3.18)$$

If S is nonvanishing, this would allow hyperbolic space as a solution. In the following we will discard this possibility and instead demand that the manifold should be compact, in order to ensure that the partition function is well-defined.

On T^3 , the curvature scalar vanishes and so does S . This implies that the Killing spinor is constant and nowhere-vanishing. Moreover, in (3.9) and (3.10), with vanishing S terms, δ^2 gives a translation and a gauge transformation on all fields. δ -exactness and δ -closedness of the deformation are thus guaranteed.

We will henceforth restrict the manifold to be T^3 . We normalize the constant Killing spinor such that $\tilde{\epsilon}\epsilon^c = 1$. The bosonic part of the deformation (3.14) is:

$$\begin{aligned} (\delta\Psi_A)^\dagger\delta\Psi_A &= D_\mu X_A D^\mu X^A + i\epsilon^{\mu\nu\rho}U_\rho D_\mu X_A D_\nu X^A + F_A F^A \\ &\quad + iU^\mu D_\mu X^A F_A - iU^\mu D_\mu X_A F^A, \end{aligned} \quad (3.19)$$

where U_μ is a real unit vector, which we may choose to be along the third direction of T^3 without loss of generality. When restricted to the contour \mathcal{C} , cf. (3.13), the bosonic part of the deformation is positive semi-definite, and the saddle points where it vanishes are given by:

$$D_1 X_A + iD_2 X_A = 0, \quad D_3 X - iF = 0. \quad (3.20)$$

Hence with this deformation alone the theory does not reduce to an ordinary integral with discrete saddle points: one can always choose some nontrivial functions for X_A and F so that (3.20) is satisfied. We therefore add another term $\delta[(\delta\Psi^A)^\dagger\Psi^A]$ to the original deformation:

$$\begin{aligned} (\delta\Psi_A)^\dagger\delta\Psi_A + (\delta\Psi^A)^\dagger\delta\Psi^A &= D_\mu X_A D^\mu X^A + i\epsilon^{\mu\nu\rho} U_\rho D_\mu X_A D_\nu X^A + F_A F^A \\ &\quad + iU^\mu D_\mu X^A F_A - iU^\mu D_\mu X_A F^A \\ &\quad + D_\mu X_A D^\mu X^A - i\epsilon^{\mu\nu\rho} U_\rho D_\mu X_A D_\nu X^A + F_A F^A \\ &\quad - iU^\mu D_\mu X^A F_A + iU^\mu D_\mu X_A F^A \\ &= 2\{D_\mu X_A D^\mu X^A + F_A F^A\}. \end{aligned} \quad (3.21)$$

When restricted to the contour \mathcal{C} , the two terms in the last line are both positive semi-definite, and the critical points are given by:

$$D_\mu X_A = F_A = 0. \quad (3.22)$$

3.2.2 Gauge Sector

A δ -closed deformation for the gauge sector is:

$$\int d^3x \left\{ \delta[(\delta\lambda)^\dagger\lambda] + \delta[(\delta\hat{\lambda})^\dagger\hat{\lambda}] \right\}, \quad (3.23)$$

where we have *defined*:

$$(\delta\lambda)^\dagger \equiv -\epsilon^\dagger \gamma_{\mu\nu} F^{\mu\nu}; \quad (\delta\hat{\lambda})^\dagger \equiv -\epsilon^\dagger \gamma_{\mu\nu} \hat{F}^{\mu\nu}, \quad (3.24)$$

so that the deformation (3.23) is a holomorphic functional of the complexified fields. Note in particular that $(\delta\lambda)^\dagger$ is *not* the adjoint of $\delta\lambda$ at generic points in field space, but only when restricted to the contour \mathcal{C} , cf. (3.13).

The bosonic part of the deformation (3.23) is given by:

$$(\delta\lambda)^\dagger\delta\lambda + (\delta\hat{\lambda})^\dagger\delta\hat{\lambda} = \frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}\hat{F}^{\mu\nu}\hat{F}_{\mu\nu}. \quad (3.25)$$

When restricted to the contour \mathcal{C} this becomes a sum of positive semi-definite terms, with critical points given by:

$$F_{\mu\nu} = \hat{F}_{\mu\nu} = 0. \quad (3.26)$$

3.3 Gauge Fixing

We now introduce the usual ghost and anti-ghost action to fix the infinite degrees of freedom of the gauge fields. The ghost term is not invariant under supersymmetry, so one cannot immediately proceed to do localization. To deal with this, we follow [1, 2], and introduce a new fermionic symmetry Δ :

$$\Delta \equiv \delta_Q + \delta_B , \quad (3.27)$$

where δ_Q stands for supersymmetry and δ_B for BRST transformation.

Under a BRST transformation, we have:

$$\delta_B A_\mu = \partial_\mu C + i[A_\mu, C] , \quad \delta_B \lambda = -i\{\lambda, C\} . \quad (3.28)$$

and similarly for \hat{A} , $\hat{\lambda}$. Here C is the usual anti-commuting ghost field. It transforms under supersymmetry and BRST as:

$$\delta_Q C = 0 , \quad \Delta C = \delta_B C = a_0 - \frac{i}{2}\{C, C\} , \quad \Delta a_0 = 0 , \quad (3.29)$$

where a_0 is a constant ghost-for-ghost field that takes care of the zero mode of C . With this combined transformation, one can verify that:

$$\begin{aligned} \Delta^2 A_\mu &= -iV^\nu F_{\nu\mu} + i[A_\mu, a_0] , \\ \Delta^2 \lambda &= -iV^\mu D_\mu \lambda + i[\lambda, a_0] , \\ \Delta^2 C &= i[C, a_0] . \end{aligned} \quad (3.30)$$

The rest of the ghost complex transforms under Δ as:

$$\begin{aligned} \Delta \bar{C} &= b , \quad \Delta b = -iV \cdot D \bar{C} + i[\bar{C}, a_0] , \\ \Delta \bar{a}_0 &= \bar{C}_0 , \quad \Delta \bar{C}_0 = i[\bar{a}_0, a_0] , \\ \Delta b_0 &= C_0 , \quad \Delta C_0 = [V \cdot A, b_0] + [A_\mu, \partial^\mu (V \cdot A)] - i \square (V \cdot A) + i[b_0, a_0] , \end{aligned} \quad (3.31)$$

where \bar{C} is the anti-ghost, and b is the Lagrangian multiplier; \bar{a}_0, b_0, C_0 and \bar{C}_0 are constant fields needed to fix the zero modes of the ghosts and b .

The gauge-fixing action is:

$$\begin{aligned} & i \int d^3 x \text{tr} \{ \Delta [\bar{C} (\partial^\mu A_\mu + b_0) - C \bar{a}_0] \} \\ &= i \int d^3 x \text{tr} \{ b (\partial^\mu A_\mu + b_0) - \bar{C} (\partial^\mu D_\mu C + \partial^\mu \delta_Q A_\mu + C_0) \\ & \quad - (a_0 - \frac{i}{2}\{C, C\}) \bar{a}_0 + C \bar{C}_0 \} . \end{aligned} \quad (3.32)$$

Note that the ghost, the anti-ghost and the transformation Δ are all anti-commuting. In Appendix D we show that the integration over all fields in the ghost complex gives the

Lorentz gauge. Now this action is invariant under Δ transformation:

$$\begin{aligned}
& \Delta^2[\bar{C}(\partial^\mu A_\mu + b_0) - C\bar{a}_0] \\
&= \Delta^2(\bar{C})(\partial^\mu A_\mu + b_0) + \bar{C}(\partial^\mu \Delta^2(A_\mu) + \Delta^2(b_0)) \\
&\quad - \Delta^2(C)\bar{a}_0 - C\Delta^2(\bar{a}_0) \\
&= \Delta^2(\bar{C})(\partial^\mu A_\mu + b_0) + \bar{C}(\partial^\mu \Delta^2(A_\mu) + \Delta^2(b_0)) \\
&\quad - i[C, a_0]\bar{a}_0 - iC[\bar{a}_0, a_0] .
\end{aligned} \tag{3.33}$$

The last two terms cancel under the trace. The first two can also be shown to cancel:

$$\begin{aligned}
& \int d^3x \text{tr}\{\Delta^2(\bar{C})(\partial^\mu A_\mu + b_0) + \bar{C}(\partial^\mu \Delta^2(A_\mu) + \Delta^2(b_0))\} \\
&= \int d^3x \text{tr}\{(-iV \cdot D\bar{C} + i[\bar{C}, a_0])(\partial^\mu A_\mu + b_0) \\
&\quad + \bar{C}\partial^\mu(-iV^\nu F_{\nu\mu} + i[A_\mu, a_0]) \\
&\quad + \bar{C}([V \cdot A, b_0] + [A_\mu, \partial^\mu(V \cdot A)] - i\Box(V \cdot A) + i[b_0, a_0])\} \\
&= \int d^3x \text{tr}\{i[\bar{C}, a_0](\partial \cdot A + b_0) + i\bar{C}([\partial \cdot A + b_0, a_0]) \\
&\quad - iV \cdot \partial[\bar{C}(\partial \cdot A + b_0)] + [V \cdot A, \bar{C}](\partial \cdot A + b_0) \\
&\quad + \bar{C}[V \cdot A, \partial \cdot A + b_0] + i\bar{C}\Box(V \cdot A) - i\bar{C}\Box(V \cdot A) \\
&\quad \bar{C}[\partial^\mu(V \cdot A), A_\mu] + \bar{C}[A_\mu, \partial^\mu(V \cdot A)]\} \\
&= 0 .
\end{aligned} \tag{3.34}$$

3.4 Saddle Points

For the gauge sector we replace δ by Δ in (3.23) and modify the deformation as follows:

$$\begin{aligned}
\Delta V_{\text{gauge}} &= \int dx^3 \Delta \text{tr}\left\{\frac{1}{2}\epsilon^\dagger \gamma_{\mu\nu} F_{\mu\nu} \lambda\right\} \\
&= \int dx^3 \text{tr}\left\{\frac{1}{2}F_{\mu\nu} F^{\mu\nu} - i\tilde{\lambda}\not{D}\lambda\right\} ,
\end{aligned} \tag{3.35}$$

and similarly for the hatted fields. This deformation is Δ -exact and Δ -closed. For the matter sector, Δ is defined to be the same as δ , and the deformation is:

$$\begin{aligned}
\Delta V_{\text{matter}} &= \int dx^3 \text{tr}\{\Delta[(\Delta\Psi^A)^\dagger\Psi^A + (\Delta\Psi_A)^\dagger\Psi_A]\} \\
&= 2 \int dx^3 \text{tr}\{D_\mu X_A D^\mu X^A + F_A F^A - i\tilde{\Psi}^A \not{D}\Psi_A \\
&\quad + \Omega^{AB}\tilde{\lambda}X_B\Psi_A + \Omega_{AB}\tilde{\lambda}X^B\Psi^A - \Omega^{AB}X_B\tilde{\lambda}\Psi_A - \Omega_{AB}X^B\tilde{\lambda}\Psi^A\} .
\end{aligned} \tag{3.36}$$

The gauge sector localizes to:

$$F_{\mu\nu} = 0 ; \quad \lambda = 0 , \tag{3.37}$$

where we have restricted to the contour \mathcal{C} , cf. (3.13). In particular the saddle points of the gauge field correspond to flat gauge connections over the Euclidean three-torus. For a

simply-connected gauge group $\pi_1(G) = 0$, such as $G = SU(N) \times SU(N)$, this implies that:

$$A_\mu = c_\mu^i H_i , \quad (3.38)$$

where c^i 's are constants and $\{H_i\}$, $i = 1, \dots, \text{rank}(G)$, is the Cartan subalgebra of G . This can be seen as follows (see e.g. [21, 22]): Since A_μ is a flat connection there exists a group element $U \in G$ such that $A_\mu = -i\partial_\mu U U^{-1}$, at least locally. I.e. U need not be globally defined but is allowed to undergo G -valued jumps as we wind around each of the three circles of the torus. More explicitly, suppose we have a square torus of radius L parameterized by $\{x^\mu \in [0, L]\}$. The group element $U(x^1, x^2, x^3)$ obeys nontrivial, in general, boundary conditions which may be parameterized as follows,

$$\begin{aligned} U(x^1 + L, x^2, x^3) &= U(x^1, x^2, x^3)\Omega_1 ; \\ U(x^1, x^2 + L, x^3) &= U(x^1, x^2, x^3)\Omega_2 ; \\ U(x^1, x^2, x^3 + L) &= U(x^1, x^2, x^3)\Omega_3 , \end{aligned} \quad (3.39)$$

for some constant $\Omega_\mu \in G$. In addition, for consistency, Ω_μ must mutually commute. Indeed going once around the circle parameterized by x^μ and then once around the circle parameterized by x^ν must produce the same jump in U as when going first around the x^ν direction and then along x^μ . This implies, taking (3.39) into account,

$$[\Omega_\mu, \Omega_\nu] = 0 . \quad (3.40)$$

For a unitary group G , as is the case in the present paper, this implies that Ω_μ can be put in the form:

$$\Omega_\mu = \exp(iLc_\mu^j H_j) , \quad (3.41)$$

up to similarity transformation. Recalling the relation between A_μ and U we are thus led to the result cited in (3.38), provided we can show that for any set of mutually commuting Ω_μ 's we can always construct a group element $U \sim \exp(ix^\mu c_\mu^j H_j)$ obeying (3.39).

The proof of the last step proceeds by showing that there is no obstruction in constructing an element $U(x^1, x^2, x^3)$ on the edges of a cube of side L such that (3.39) is satisfied. Then U can be continued on the faces of the cube provided $\pi_1(G) = 0$, and finally in the interior provided $\pi_2(G) = 0$, which holds true for $G = SU(N) \times SU(N)$.

An important observation is that the constants c_μ^i should be understood as periodic variables with periodic identification,

$$c_\mu^i \sim c_\mu^i + \frac{2\pi}{L} . \quad (3.42)$$

This can be seen by performing a gauge transformation generated by $U = \exp(\frac{2\pi i}{L} x^\mu H_i)$, which shifts A_μ in accordance with (3.42). On the other hand the element U thus defined is periodic⁴, i.e. as we wind around the x^μ direction of the torus it forms a closed loop in group space. But since the group is simply connected U may be continuously deformed to

⁴We are adopting the normalization $\exp(2\pi i H_i) = 1$.

the identity, and the gauge transformation generated by U should act trivially on all fields of the theory. We thus arrive at the identification (3.42).

It follows from the above that the c_μ^i 's can be constrained to take values in $[0, \frac{2\pi}{L}]$. In particular taking the infinite-volume limit of the torus, $L \rightarrow \infty$, we conclude that the only solution to (3.37) is the trivial flat connection $A_\mu = 0$. Of course on \mathbb{R}^3 there is no obstruction to gauging away any flat connection of the form (3.38). The point is that we can formally reproduce this result by considering \mathbb{R}^3 as the infinite-volume limit of T^3 .

The case of $G = U(N) \times U(N)$ presents one crucial difference: $\pi_1(U(N)) \cong \mathbb{Z}$ and thus G is not simply connected. By considering the decomposition of the algebra-valued connection along the G -generators it is not very difficult to see that we may still put the most general flat connection in the form (3.38),

$$A_\mu = c_\mu^i H_i + d_\mu J + e_\mu K , \quad (3.43)$$

where the first term on the right-hand side is as in the case of $SU(N) \times SU(N)$; d_μ , e_μ are constants; J , K are the two additional $u(1)$ Cartan generators coming from the decomposition:

$$u(N) \oplus u(N) \cong su(N) \oplus su(N) \oplus u(1) \oplus u(1) . \quad (3.44)$$

Now the previous argument which allowed us to conclude that c_μ^i are periodic does not go through for the variables d_μ , e_μ . The reason is that the gauge transformations generated by $U = \exp(\frac{2\pi i}{L} x^\mu J)$ and $U = \exp(\frac{2\pi i}{L} x^\mu K)$ form closed loops in the group space which are not contractible to the identity. Hence the gauge transformations generated by U need not act trivially on all fields of the theory.

In particular our argument that in the infinite-volume limit the only flat connection is the trivial one, does not go through in this case without additional assumptions. If we wish to recover $A = 0$ as the unique (up to gauge transformations) solution to (3.43) in the infinite-volume limit, we must impose by hand that $U = \exp(\frac{2\pi i}{L} x^\mu J)$ and $U = \exp(\frac{2\pi i}{L} x^\mu K)$ act trivially on all fields of the theory.

Finally, the matter sector localizes to the following field configurations:

$$F_A = 0 ; \quad \Psi_A = \Psi^A = 0 ; \quad X_A = \text{const} , \quad (3.45)$$

where we have restricted to the contour \mathcal{C} , cf. (3.13).

3.5 One-loop Determinant

We will now compute the one-loop determinant from the quadratic fluctuations around the following saddle points,

$$\begin{aligned} A_\mu = 0 ; \quad \lambda = 0 ; \\ F_A = 0 ; \quad \Psi_A = \Psi^A = 0 ; \quad X_A = \text{const} , \end{aligned} \quad (3.46)$$

and similarly for \hat{A} , $\hat{\lambda}$. I.e. we will ignore the contributions from non-vanishing flat gauge connections, as discussed in the previous section.

The full path integral is of the form:

$$\int d\varphi \exp\{iS + iS_{\text{g.f.}} - t(\Delta V_{\text{gauge}} + \frac{1}{2}\Delta V_{\text{matter}})\}, \quad (3.47)$$

where $iS_{\text{g.f.}}$ is the gauge-fixing action (3.32), and $\int d\varphi$ stands for integrations over all fields and ghosts; ΔV_{gauge} contains deformations for both hatted and unhatted gauge multiplets.

Next we expand the fields around the saddle points:

$$X_A \rightarrow X_A^0 + \frac{1}{\sqrt{t}}X'_A, \quad \phi \rightarrow 0 + \frac{1}{\sqrt{t}}\phi. \quad (3.48)$$

Here X_A^0 is a constant field and X'_A represents the nonzero mode of X_A ; ϕ stands for all fields other than X_A . The path integral (3.47) is t -independent thanks to localization. On the other hand, taking $t \rightarrow \infty$ allows us to keep only the quadratic terms in the deformation:

$$\begin{aligned} & t(\Delta V_{\text{gauge}} + \frac{1}{2}\Delta V_{\text{matter}}) \\ &= \int dx^3 \text{tr}\{\frac{1}{2}F_{\mu\nu}^A F^{A\mu\nu} - i\tilde{\lambda}\not{\partial}\lambda\} + \int dx^3 \text{tr}\{\frac{1}{2}\hat{F}_{\mu\nu}^A \hat{F}^{A\mu\nu} - i\tilde{\lambda}\not{\partial}\hat{\lambda}\} \\ &+ \int dx^3 \text{tr}\{\partial_\mu X'_A \partial^\mu X'^A + X^{0A} A_\mu A^\mu X_A^0 + X_A^0 \hat{A}_\mu \hat{A}^\mu X^{0A} - 2X_A^0 \hat{A}_\mu X^{0A} A^\mu \\ &+ F_A F^A - i\tilde{\Psi}^A \not{\partial}\Psi_A + \Omega^{AB} \tilde{\lambda} X_B^0 \Psi_A + \Omega_{AB} \tilde{\lambda} X^{0B} \Psi^A - \Omega^{AB} X_B^0 \tilde{\lambda} \Psi_A - \Omega_{AB} X^{0B} \tilde{\lambda} \Psi^A\}, \end{aligned} \quad (3.49)$$

where $F_{\mu\nu}^A \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the linearized field strength; some terms have been eliminated using Lorentz gauge.

3.5.1 Determinant from Bosons

We start with the calculation of the one-loop determinant of the bosonic part. Under Lorentz gauge, we have:

$$\begin{aligned} & \int d^3x \text{tr}\{\frac{1}{2}F_{\mu\nu}^A F^{A\mu\nu}\} + \int d^3x \text{tr}\{\frac{1}{2}\hat{F}_{\mu\nu}^A \hat{F}^{A\mu\nu}\} + \int d^3x \text{tr}\{\partial_\mu X'_A \partial^\mu X'^A \\ &+ X^{A0} A_\mu A^\mu X_A^0 + X_A^0 \hat{A}_\mu \hat{A}^\mu X^{A0} - 2X_A^0 \hat{A}_\mu X^{A0} A^\mu + F_A F^A\} \\ &= \int d^3x \text{tr}\{-A_\mu \square A^\mu\} + \int d^3x \text{tr}\{-\hat{A}_\mu \square \hat{A}^\mu\} + \int d^3x \text{tr}\{-X'_A \square X'^A \\ &+ X^{A0} A_\mu A^\mu X_A^0 + X_A^0 \hat{A}_\mu \hat{A}^\mu X^{A0} - 2X_A^0 \hat{A}_\mu X^{A0} A^\mu + F_A F^A\}. \end{aligned} \quad (3.50)$$

On T^3 with periodic conditions, any field φ can be expanded in terms of Fourier modes:

$$\varphi = \sum_{\vec{n}} \varphi_{\vec{n}} \exp\{i2\pi\vec{n} \cdot \vec{x}\}, \quad (3.51)$$

where $\vec{n} = (n_x, n_y, n_z)$ and each n_μ runs over all integers. In addition, for the gauge field the Lorentz gauge implies that for each \vec{n} ,

$$n_x A_{x,\vec{n}} + n_y A_{y,\vec{n}} + n_z A_{z,\vec{n}} = 0. \quad (3.52)$$

Let us first assume $n_z \neq 0$. (We will come back to the case $n_z = 0$ in the following). Then the previous equation can be used to eliminate $A_{z,\vec{n}}$ via:

$$A_{z,\vec{n}} = -\frac{n_x}{n_z}A_{x,\vec{n}} - \frac{n_y}{n_z}A_{y,\vec{n}}. \quad (3.53)$$

The gauge fields are in the adjoint representation, $A_\mu = A_\mu^a t_a$, where the generators t_a are normalized so that $\text{tr}\{t_a t_b\} = \delta_{ab}$. The gauge kinetic action becomes:

$$\begin{aligned} & \int d^3x \text{tr}\{-A_\mu \square A^\mu\} \\ &= \int d^3x \sum_a \sum_{\vec{n}, n_z \neq 0} 4\pi^2 \vec{n}^2 \left\{ \left(\frac{n_x^2 + n_z^2}{n_z^2} \right) A_{x,-\vec{n}}^a A_{x,\vec{n}}^a + \left(\frac{n_y^2 + n_z^2}{n_z^2} \right) A_{y,-\vec{n}}^a A_{y,\vec{n}}^a \right. \\ & \quad \left. + \frac{n_x n_y}{n_z^2} A_{x,-\vec{n}}^a A_{y,\vec{n}}^a + \frac{n_x n_y}{n_z^2} A_{y,-\vec{n}}^a A_{x,\vec{n}}^a \right\}. \end{aligned} \quad (3.54)$$

By symmetrizing \vec{n} and $-\vec{n}$, for each pair of $(\vec{n}, -\vec{n})$ and each a , this can be written in matrix notation as follows:

$$\begin{matrix} A_{x,\vec{n}}^a \\ A_{x,-\vec{n}}^a \\ A_{y,\vec{n}}^a \\ A_{y,-\vec{n}}^a \end{matrix} \begin{pmatrix} A_{x,\vec{n}}^a & A_{x,-\vec{n}}^a & A_{y,\vec{n}}^a & A_{y,-\vec{n}}^a \\ 0 & \frac{n_x^2 + n_z^2}{n_z^2} & 0 & \frac{n_x n_y}{n_z^2} \\ \frac{n_x^2 + n_z^2}{n_z^2} & 0 & \frac{n_x n_y}{n_z^2} & 0 \\ 0 & \frac{n_x n_y}{n_z^2} & 0 & \frac{n_y^2 + n_z^2}{n_z^2} \\ \frac{n_x n_y}{n_z^2} & 0 & \frac{n_y^2 + n_z^2}{n_z^2} & 0 \end{pmatrix} \times 4\pi^2 (\vec{n} \cdot \vec{n}). \quad (3.55)$$

Similarly, for each $(\vec{n}, -\vec{n})$ and a, b , the potentials involving the gauge fields are:

$$\begin{aligned} X^{A0} A^a \cdot A^b t_a t_b X_A^0 &: \Gamma \times X^{A0} t_a t_b X_A^0, \\ X_A^0 \hat{A}^a \cdot \hat{A}^b \hat{t}_a \hat{t}_b X^{A0} &: \Gamma \times X_A^0 \hat{t}_a \hat{t}_b X^{A0}, \\ -2X_A^0 \hat{A}_\mu^a \hat{t}_a X^{A0} A^{b\mu} t_b &: \Gamma \times -2X_A^0 \hat{t}_a X^{A0} t_b, \end{aligned} \quad (3.56)$$

where:

$$\Gamma \equiv \begin{pmatrix} 0 & \frac{n_x^2 + n_z^2}{n_z^2} & 0 & \frac{n_x n_y}{n_z^2} \\ \frac{n_x^2 + n_z^2}{n_z^2} & 0 & \frac{n_x n_y}{n_z^2} & 0 \\ 0 & \frac{n_x n_y}{n_z^2} & 0 & \frac{n_y^2 + n_z^2}{n_z^2} \\ \frac{n_x n_y}{n_z^2} & 0 & \frac{n_y^2 + n_z^2}{n_z^2} & 0 \end{pmatrix}. \quad (3.57)$$

The matter fields are in the bifundamental representation of the gauge group $U(N) \times U(N)$. Moreover $X = \sum_{(\rho, \hat{\rho})} X^{(\rho, \hat{\rho})} |\rho\rangle \otimes |\hat{\rho}\rangle$, where $|\rho\rangle$, $|\hat{\rho}\rangle$ are representatives of the weights in each weight space; we choose the normalization so that $\langle \rho | \rho' \rangle = \delta_{\rho, \rho'}$ and $\langle \hat{\rho} | \hat{\rho}' \rangle = \delta_{\hat{\rho}, \hat{\rho}'}$, in

some gauge-invariant contraction of the relevant color indices. We then have:

$$\begin{aligned}
X^{A0} t_a t_b X_A^0 &= \sum_{(\rho, \hat{\rho})} \sum_{(\rho', \hat{\rho}')} X^{A0(\rho, \hat{\rho})} \langle \hat{\rho} | \otimes \langle \rho | t_a t_b | \rho' \rangle | \hat{\rho}' \rangle X_A^{0(\rho', \hat{\rho}')} \\
&= \sum_{\rho, \rho', \hat{\rho}, \hat{\rho}'} X^{A0(\rho, \hat{\rho})} \langle \rho | t_a | \rho'' \rangle \langle \rho'' | t_b | \rho' \rangle X_A^{0(\rho', \hat{\rho}')} \\
&= \sum_{\rho, \rho', \hat{\rho}, \hat{\rho}'} X^{A0(\rho, \hat{\rho})} \sigma_a^{(\rho, \rho'')} \sigma_b^{(\rho'', \rho')} X_A^{0(\rho', \hat{\rho}')} , \\
X_A^0 \hat{t}_a \hat{t}_b X^{A0} &= \sum_{\hat{\rho}, \hat{\rho}', \rho, \rho''} X_A^{0(\rho, \hat{\rho})} \hat{\sigma}_a^{(\hat{\rho}, \hat{\rho}'')} \hat{\sigma}_b^{(\hat{\rho}'', \hat{\rho}')} X^{A0(\rho, \hat{\rho}')} , \\
X_A^0 \hat{t}_a X^{A0} t_b &= \sum_{\rho, \rho', \hat{\rho}, \hat{\rho}'} X_A^{0(\rho, \hat{\rho})} \hat{\sigma}_a^{(\hat{\rho}, \hat{\rho}')} X^{A0(\rho', \hat{\rho}')} \sigma_b^{(\rho', \rho)} ,
\end{aligned} \tag{3.58}$$

where $\sigma_a^{(\rho, \rho')} \equiv \langle \rho | t_a | \rho' \rangle$ and we used the fact that $\sum_{\rho} |\rho\rangle \langle \rho| = \mathbb{1}$. We then define the following matrices:

$$\begin{aligned}
\mathbb{B}_{ab} &= X^{A0} t_a t_b X_A^0 , \\
\mathbb{C}_{ab} &= X_A^0 \hat{t}_a \hat{t}_b X^{A0} , \\
\mathbb{D}_{ab} &= -X_A^0 \hat{t}_a X^{A0} t_b ,
\end{aligned} \tag{3.59}$$

and the deformations that are quadratic in gauge fields can be represented as:

$$\begin{matrix} A & \hat{A} \\ A \begin{pmatrix} \mathbb{B} + 4\pi^2(\vec{n} \cdot \vec{n}) \times \mathbb{1} & \mathbb{D}^{Tr} \\ \hat{A} \begin{pmatrix} \mathbb{D} & \mathbb{C} + 4\pi^2(\vec{n} \cdot \vec{n}) \times \mathbb{1} \end{pmatrix} \end{pmatrix} \otimes \Gamma . \end{matrix} \tag{3.60}$$

The determinant of the tensor product of two matrices A and B is given by:

$$\det(A \otimes B) = (\det A)^{\dim B} (\det B)^{\dim A} . \tag{3.61}$$

Therefore, when $n_z \neq 0$, we have:

$$\begin{aligned}
\det(A, \hat{A})|_{n_z \neq 0} &= \prod_{(\vec{n}, -\vec{n}), n_z \neq 0} \{(\det \mathbb{A})^4 \times (\prod_a \det \Gamma)^2\} \\
&= \prod_{(\vec{n}, -\vec{n}), n_z \neq 0} \{(\det \mathbb{A})^4 \times (\prod_a \frac{(\vec{n} \cdot \vec{n})^2}{n_z^4})^2\} \\
&= \prod_{(\vec{n}, -\vec{n}), n_z \neq 0} \{(\det[\frac{\mathbb{A}}{4\pi^2(\vec{n} \cdot \vec{n})}]^4 \times (\prod_a \frac{(4\pi^2)^4 (\vec{n} \cdot \vec{n})^6}{n_z^4})^2\} \\
&= \prod_{\vec{n}, n_z \neq 0} \{(\det[\frac{\mathbb{A}}{4\pi^2(\vec{n} \cdot \vec{n})}]^2 \times (\prod_a \frac{16\pi^4 (\vec{n} \cdot \vec{n})^3}{n_z^2})^2\} ,
\end{aligned} \tag{3.62}$$

where:

$$\mathbb{A} \equiv \begin{pmatrix} \mathbb{B} + 4\pi^2(\vec{n} \cdot \vec{n}) \times \mathbb{1} & \mathbb{D}^{Tr} \\ \mathbb{D} & \mathbb{C} + 4\pi^2(\vec{n} \cdot \vec{n}) \times \mathbb{1} \end{pmatrix} . \tag{3.63}$$

For the case where $n_z = 0$, but n_x or n_y are not equal to zero, the procedure is similar. The determinant coming from integrating over A_μ reads:

$$\begin{aligned} \det(A, \hat{A}) &= \prod_{\vec{n}} \left(\det \left(\frac{\mathbb{A}}{4\pi^2(\vec{n} \cdot \vec{n})} \right) \right)^2 \prod_a \left\{ \prod_{\vec{n}, n_z \neq 0} \left[16\pi^4 \frac{(\vec{n} \cdot \vec{n})^3}{n_z^2} \right]^2 \right. \\ &\quad \times \left. \prod_{\vec{n}, n_z=0, n_x \neq 0} \left[16\pi^4 \frac{(\vec{n} \cdot \vec{n})^3}{n_x^2} \right]^2 \prod_{\vec{n}, n_z=n_x=0, n_y \neq 0} \left[16\pi^4 \frac{(\vec{n} \cdot \vec{n})^3}{n_y^2} \right]^2 \right\}. \end{aligned} \quad (3.64)$$

The contribution to the one-loop determinant coming from the terms involving gauge fields is thus:

$$\begin{aligned} Z_{1-loop}(A, \hat{A}) &= \frac{\prod_a \left\{ \prod_{\vec{n}, n_z \neq 0} n_z^2 \prod_{\vec{n}, n_z=0, n_x \neq 0} n_x^2 \prod_{\vec{n}, n_z=n_x=0, n_y \neq 0} n_y^2 \right\}}{\prod_a \prod_{\vec{n}} 16\pi^4 (\vec{n} \cdot \vec{n})^3} \\ &\quad \times \prod_{\vec{n}} \left(\det \left[\frac{\mathbb{A}}{4\pi^2(\vec{n} \cdot \vec{n})} \right] \right)^{-1}. \end{aligned} \quad (3.65)$$

One may worry about regularizing the numerator. However, we note that the gauge-fixing delta function also gives a Jacobian factor to the one-loop determinant. Indeed in the ghost action we have:

$$\begin{aligned} &\exp \left\{ i \int d^3x \operatorname{tr} (b \partial^\mu A_\mu) \right\} \\ &= \exp \left\{ i 2\pi \sum_{\vec{n}} \sum_a b_{-\vec{n}}^a (\vec{n} \cdot \vec{A}_{\vec{n}}^a) \right\}. \end{aligned} \quad (3.66)$$

After integrating out $b_{\vec{n}}^a$ we obtain:

$$\prod_{\vec{n}} \prod_a \delta(\vec{n} \cdot \vec{A}_{\vec{n}}^a). \quad (3.67)$$

This product of delta functions imposes the gauge-fixing Lorentz condition and, upon integrating out A_μ , \hat{A}_μ , gives a Jacobian factor which cancels the numerator of (3.65).

The integral over F_A simply contributes an overall constant factor. Finally we are left with the integration over X'_A :

$$\begin{aligned} &\int d^3x \operatorname{tr} \{ -X'_A \square X'^A \} \\ &= \sum_{\vec{n}} 4\pi^2 \vec{n}^2 \operatorname{tr} \{ X'_{A, -\vec{n}} X'^A_{\vec{n}} \} \\ &= \sum_{\vec{n}} 2\pi^2 \vec{n}^2 \operatorname{tr} \{ X'_{A, -\vec{n}} X'^A_{\vec{n}} + X'^A_{\vec{n}} X'_{A, -\vec{n}} \} \\ &= \sum_{(\vec{n}, -\vec{n})} 2\pi^2 \vec{n}^2 \operatorname{tr} \{ X'_{A, -\vec{n}} X'^A_{\vec{n}} + X'_{A, \vec{n}} X'^A_{-\vec{n}} \\ &\quad + X'^A_{\vec{n}} X'_{A, -\vec{n}} + X'^A_{-\vec{n}} X'_{A, \vec{n}} \}. \end{aligned} \quad (3.68)$$

This integration is Gaussian, and the corresponding determinant is:

$$\det X'_A = \prod_A \prod_{(\rho, \hat{\rho})} \prod_{\vec{n}} (2\pi^2 \vec{n}^2)^2, \quad (3.69)$$

where $(\rho, \hat{\rho})$ runs over the weights of the bifundamental representation. Therefore, the total contribution of the bosonic part to the one-loop determinant reads:

$$\begin{aligned}
Z_{1-loop}(Boson) &= \frac{1}{\{\prod_a \prod_{\vec{n}} 16\pi^4 (\vec{n}^2)^3\} |_{A, \hat{A}} \{\prod_A \prod_{(\rho, \hat{\rho})} \prod_{\vec{n}} 2\pi^2 \vec{n}^2\} |_{X'}} \\
&\quad \times \prod_{\vec{n}} (\det[\frac{\mathbb{A}}{4\pi^2 (\vec{n} \cdot \vec{n})}])^{-1} \\
&= \frac{1}{\{\prod_{\vec{n}} [16\pi^4 (\vec{n}^2)^3]^d\} \{\prod_A \prod_{\vec{n}} (2\pi^2 \vec{n}^2)^{w^2}\}} \\
&\quad \times \prod_{\vec{n}} (\det[\frac{\mathbb{A}}{4\pi^2 (\vec{n} \cdot \vec{n})}])^{-1}.
\end{aligned} \tag{3.70}$$

Here d is the dimension of the gauge group and w is the dimension of its fundamental representation. For $U(N)$ in particular we have $d = N^2$, $w = N$.

3.5.2 Determinant from Fermions

The fermionic part of the deformation is:

$$\begin{aligned}
&\int dx^3 \text{tr}\{-i\tilde{\lambda}\not{\partial}\lambda\} + \int dx^3 \text{tr}\{-i\tilde{\lambda}\not{\partial}\hat{\lambda}\} + \int dx^3 \text{tr}\{-i\tilde{\Psi}^A \not{\partial}\Psi_A \\
&\quad + \Omega^{AB} \tilde{\lambda} X_B^0 \Psi_A + \Omega_{AB} \tilde{\lambda} X^{0B} \Psi^A - \Omega^{AB} X_B^0 \tilde{\lambda} \Psi_A - \Omega_{AB} X^{0B} \tilde{\lambda} \Psi^A\}.
\end{aligned} \tag{3.71}$$

Using the expansion $\lambda = \lambda_+ \epsilon + \lambda_- \epsilon^c$ for the gaugino kinetic term, cf. appendix C, we have:

$$\begin{aligned}
&\int dx^3 \text{tr}\{-i\tilde{\lambda}\not{\partial}\lambda\} \\
&= \int dx^3 \text{tr}\{-i(\lambda_+ V \cdot \partial\lambda_+ - \lambda_- \bar{V} \cdot \partial\lambda_- - \lambda_- U \cdot \partial\lambda_+ - \lambda_+ U \cdot \partial\lambda_-)\} \\
&= 2\pi \sum_a \sum_{\vec{n}} \{V \cdot \vec{n} \lambda_{+, -\vec{n}}^a - \bar{V} \cdot \vec{n} \lambda_{-, -\vec{n}}^a - U \cdot \vec{n} \lambda_{-, -\vec{n}}^a \lambda_{+, \vec{n}}^a \\
&\quad - U \cdot \vec{n} \lambda_{+, -\vec{n}}^a \lambda_{-, \vec{n}}^a\} \\
&= 2\pi \sum_a \sum_{(\vec{n}, -\vec{n})} \{(V \cdot \vec{n} \lambda_{+, -\vec{n}}^a - \bar{V} \cdot \vec{n} \lambda_{-, -\vec{n}}^a - U \cdot \vec{n} \lambda_{-, -\vec{n}}^a \lambda_{+, \vec{n}}^a \\
&\quad - U \cdot \vec{n} \lambda_{+, -\vec{n}}^a \lambda_{-, \vec{n}}^a) + (-V \cdot \vec{n} \lambda_{+, \vec{n}}^a - \bar{V} \cdot \vec{n} \lambda_{-, \vec{n}}^a + \bar{V} \cdot \vec{n} \lambda_{-, \vec{n}}^a \lambda_{+, -\vec{n}}^a \\
&\quad + U \cdot \vec{n} \lambda_{-, \vec{n}}^a \lambda_{+, -\vec{n}}^a + U \cdot \vec{n} \lambda_{+, \vec{n}}^a \lambda_{-, -\vec{n}}^a)\},
\end{aligned} \tag{3.72}$$

where we symmetrized the indices $+$, $-$ and \vec{n} , $-\vec{n}$ of the gaugini in the last equation. For each pair of $(\vec{n}, -\vec{n})$ and each a , this can be written in matrix notation as:

$$\begin{aligned}
&\lambda_{+, \vec{n}}^a \begin{pmatrix} \lambda_{+, \vec{n}}^a & \lambda_{-, \vec{n}}^a & \lambda_{+, -\vec{n}}^a & \lambda_{-, -\vec{n}}^a \\ 0 & 0 & -2\pi V \cdot \vec{n} & 2\pi U \cdot \vec{n} \\ 0 & 0 & 2\pi U \cdot \vec{n} & 2\pi \bar{V} \cdot \vec{n} \\ 2\pi V \cdot \vec{n} & -2\pi U \cdot \vec{n} & 0 & 0 \\ -2\pi U \cdot \vec{n} & -2\pi \bar{V} \cdot \vec{n} & 0 & 0 \end{pmatrix} \\
&\lambda_{-, \vec{n}}^a \begin{pmatrix} \lambda_{+, \vec{n}}^a & \lambda_{-, \vec{n}}^a & \lambda_{+, -\vec{n}}^a & \lambda_{-, -\vec{n}}^a \\ 0 & 0 & -2\pi V \cdot \vec{n} & 2\pi U \cdot \vec{n} \\ 0 & 0 & 2\pi U \cdot \vec{n} & 2\pi \bar{V} \cdot \vec{n} \\ 2\pi V \cdot \vec{n} & -2\pi U \cdot \vec{n} & 0 & 0 \\ -2\pi U \cdot \vec{n} & -2\pi \bar{V} \cdot \vec{n} & 0 & 0 \end{pmatrix}.
\end{aligned} \tag{3.73}$$

Similarly for the matter fermion kinetic term:

$$\begin{aligned}
& 2\pi \sum_{(\vec{n}, -\vec{n})} \text{tr}\{(V \cdot \vec{n} \Psi_{+,-\vec{n}}^A \Psi_{A+,\vec{n}} - \bar{V} \cdot \vec{n} \Psi_{-,-\vec{n}}^A \Psi_{A-,\vec{n}} - U \cdot \vec{n} \Psi_{-,-\vec{n}}^A \Psi_{A+,\vec{n}} \\
& - U \cdot \vec{n} \Psi_{+,-\vec{n}}^A \Psi_{A-,\vec{n}}) + (-V \cdot \vec{n} \Psi_{+,\vec{n}}^A \Psi_{A+,-\vec{n}} + \bar{V} \cdot \vec{n} \Psi_{-,\vec{n}}^A \Psi_{A-,-\vec{n}} \\
& + U \cdot \vec{n} \Psi_{-,\vec{n}}^A \Psi_{A+,-\vec{n}} + U \cdot \vec{n} \Psi_{+,\vec{n}}^A \Psi_{A-,-\vec{n}})\} \\
& = \pi \sum_{(\vec{n}, -\vec{n})} \text{tr}\{(V \cdot \vec{n} \Psi_{+,-\vec{n}}^A \Psi_{A+,\vec{n}} - \bar{V} \cdot \vec{n} \Psi_{-,-\vec{n}}^A \Psi_{A-,\vec{n}} - U \cdot \vec{n} \Psi_{-,-\vec{n}}^A \Psi_{A+,\vec{n}} \\
& - U \cdot \vec{n} \Psi_{+,-\vec{n}}^A \Psi_{A-,\vec{n}}) + (-V \cdot \vec{n} \Psi_{+,\vec{n}}^A \Psi_{A+,-\vec{n}} + \bar{V} \cdot \vec{n} \Psi_{-,\vec{n}}^A \Psi_{A-,-\vec{n}} \\
& + U \cdot \vec{n} \Psi_{-,\vec{n}}^A \Psi_{A+,-\vec{n}} + U \cdot \vec{n} \Psi_{+,\vec{n}}^A \Psi_{A-,-\vec{n}})\} + (-1) \Psi^A \leftrightarrow \Psi_A .
\end{aligned} \tag{3.74}$$

The last term arises due to the symmetrization of Ψ^A and Ψ_A . When decomposed into the weight spaces, this becomes:

$$\begin{aligned}
& \pi \sum_{(\rho, \hat{\rho})} \sum_{(\vec{n}, -\vec{n})} \{(V \cdot \vec{n} \Psi_{+,-\vec{n}}^{A(\rho, \hat{\rho})} \Psi_{A+,\vec{n}}^{(\rho, \hat{\rho})} - \bar{V} \cdot \vec{n} \Psi_{-,-\vec{n}}^{A(\rho, \hat{\rho})} \Psi_{A-,\vec{n}}^{(\rho, \hat{\rho})} - U \cdot \vec{n} \Psi_{-,-\vec{n}}^{A(\rho, \hat{\rho})} \Psi_{A+,\vec{n}}^{(\rho, \hat{\rho})} \\
& - U \cdot \vec{n} \Psi_{+,-\vec{n}}^{A(\rho, \hat{\rho})} \Psi_{A-,\vec{n}}^{(\rho, \hat{\rho})}) + (-V \cdot \vec{n} \Psi_{+,\vec{n}}^{A(\rho, \hat{\rho})} \Psi_{A+,-\vec{n}}^{(\rho, \hat{\rho})} + \bar{V} \cdot \vec{n} \Psi_{-,\vec{n}}^{A(\rho, \hat{\rho})} \Psi_{A-,-\vec{n}}^{(\rho, \hat{\rho})} \\
& + U \cdot \vec{n} \Psi_{-,\vec{n}}^{A(\rho, \hat{\rho})} \Psi_{A+,-\vec{n}}^{(\rho, \hat{\rho})} + U \cdot \vec{n} \Psi_{+,\vec{n}}^{A(\rho, \hat{\rho})} \Psi_{A-,-\vec{n}}^{(\rho, \hat{\rho})})\} + (-1) \Psi^A \leftrightarrow \Psi_A .
\end{aligned} \tag{3.75}$$

For each pair of weights $(\rho, \hat{\rho})$ and each pair of $(\vec{n}, -\vec{n})$, these terms can be written with the help of two matrices:

$$\begin{aligned}
& \begin{matrix} \Psi_{+,\vec{n}}^{A(\rho, \hat{\rho})} \\ \Psi_{-,\vec{n}}^{A(\rho, \hat{\rho})} \\ \Psi_{+,-\vec{n}}^{A(\rho, \hat{\rho})} \\ \Psi_{-,-\vec{n}}^{A(\rho, \hat{\rho})} \end{matrix} \begin{pmatrix} \Psi_{A+,\vec{n}}^{(\rho, \hat{\rho})} & \Psi_{A-,\vec{n}}^{(\rho, \hat{\rho})} & \Psi_{A+,-\vec{n}}^{(\rho, \hat{\rho})} & \Psi_{A-,-\vec{n}}^{(\rho, \hat{\rho})} \\ 0 & 0 & -\pi V \cdot \vec{n} & \pi U \cdot \vec{n} \\ 0 & 0 & \pi U \cdot \vec{n} & \pi \bar{V} \cdot \vec{n} \\ \pi V \cdot \vec{n} & -\pi U \cdot \vec{n} & 0 & 0 \\ -\pi U \cdot \vec{n} & -\pi \bar{V} \cdot \vec{n} & 0 & 0 \end{pmatrix} ,
\end{aligned} \tag{3.76}$$

and:

$$\begin{aligned}
& \begin{matrix} \Psi_{A+,\vec{n}}^{(\rho, \hat{\rho})} \\ \Psi_{A-,\vec{n}}^{(\rho, \hat{\rho})} \\ \Psi_{A+,-\vec{n}}^{(\rho, \hat{\rho})} \\ \Psi_{A-,-\vec{n}}^{(\rho, \hat{\rho})} \end{matrix} \begin{pmatrix} \Psi_{+,\vec{n}}^{A(\rho, \hat{\rho})} & \Psi_{-,\vec{n}}^{A(\rho, \hat{\rho})} & \Psi_{+,-\vec{n}}^{A(\rho, \hat{\rho})} & \Psi_{-,-\vec{n}}^{A(\rho, \hat{\rho})} \\ 0 & 0 & -\pi V \cdot \vec{n} & \pi U \cdot \vec{n} \\ 0 & 0 & -\pi V \cdot \vec{n} & \pi U \cdot \vec{n} \\ \pi V \cdot \vec{n} & -\pi U \cdot \vec{n} & 0 & 0 \\ -\pi U \cdot \vec{n} & -\pi \bar{V} \cdot \vec{n} & 0 & 0 \end{pmatrix} .
\end{aligned} \tag{3.77}$$

Similarly, the Yukawa interactions can be written as:

$$\begin{aligned}
& \int dx^3 \text{tr} \{ \Omega^{AB} \tilde{\lambda} X_B^0 \Psi_A + \Omega_{AB} \tilde{\lambda} X^{0B} \Psi^A - \Omega^{AB} X_B^0 \tilde{\lambda} \Psi_A - \Omega_{AB} X^{0B} \tilde{\lambda} \Psi^A \} \\
&= a \sum_{\vec{n}} \text{tr} \{ (\Psi_{A+, -\vec{n}} \lambda_{-, \vec{n}} - \Psi_{A-, -\vec{n}} \lambda_{+, \vec{n}}) \Omega^{AB} X_B^0 + (\Psi_{+, -\vec{n}}^A \hat{\lambda}_{-, \vec{n}} - \Psi_{-, -\vec{n}}^A \hat{\lambda}_{+, \vec{n}}) \Omega_{AB} X^{0B} \\
&\quad - \Omega^{AB} X_B^0 (\hat{\lambda}_{+, -\vec{n}} \Psi_{A-, \vec{n}} - \hat{\lambda}_{-, -\vec{n}} \Psi_{A+, \vec{n}}) - \Omega_{AB} X^{0B} (\lambda_{+, -\vec{n}} \Psi_{-, \vec{n}}^A - \lambda_{-, -\vec{n}} \Psi_{+, \vec{n}}^A) \} \\
&= a \sum_{(\vec{n}, -\vec{n})} \text{tr} \{ (\Psi_{A+, -\vec{n}} \lambda_{-, \vec{n}} - \Psi_{A-, -\vec{n}} \lambda_{+, \vec{n}}) \Omega^{AB} X_B^0 + (\Psi_{+, -\vec{n}}^A \hat{\lambda}_{-, \vec{n}} - \Psi_{-, -\vec{n}}^A \hat{\lambda}_{+, \vec{n}}) \Omega_{AB} X^{0B} \\
&\quad - \Omega^{AB} X_B^0 (\hat{\lambda}_{+, -\vec{n}} \Psi_{A-, \vec{n}} - \hat{\lambda}_{-, -\vec{n}} \Psi_{A+, \vec{n}}) - \Omega_{AB} X^{0B} (\lambda_{+, -\vec{n}} \Psi_{-, \vec{n}}^A - \lambda_{-, -\vec{n}} \Psi_{+, \vec{n}}^A) \\
&\quad + (\Psi_{A+, \vec{n}} \lambda_{-, -\vec{n}} - \Psi_{A-, \vec{n}} \lambda_{+, -\vec{n}}) \Omega^{AB} X_B^0 + (\Psi_{+, \vec{n}}^A \hat{\lambda}_{-, -\vec{n}} - \Psi_{-, \vec{n}}^A \hat{\lambda}_{+, -\vec{n}}) \Omega_{AB} X^{0B} \\
&\quad - \Omega^{AB} X_B^0 (\hat{\lambda}_{+, \vec{n}} \Psi_{A-, -\vec{n}} - \hat{\lambda}_{-, \vec{n}} \Psi_{A+, -\vec{n}}) - \Omega_{AB} X^{0B} (\lambda_{+, \vec{n}} \Psi_{-, -\vec{n}}^A - \lambda_{-, \vec{n}} \Psi_{+, -\vec{n}}^A) \} .
\end{aligned} \tag{3.78}$$

Each term, such as $\text{tr} \{ \Psi_{A+, -\vec{n}} \lambda_{-, \vec{n}} \Omega^{AB} X_B^0 \}$ for example, can be written in terms of the algebra representations as follows:

$$\begin{aligned}
& \sum_{(\rho, \hat{\rho})} \sum_{(\rho', \hat{\rho}')} \sum_a \Psi_{A+, -\vec{n}}^{(\rho, \hat{\rho})} \langle \hat{\rho} | \langle \rho | \lambda_{-, \vec{n}}^a t_a | \rho' \rangle | \hat{\rho}' \rangle \Omega^{AB} X_B^{0(\rho', \hat{\rho}')} \\
&= \sum_{\rho, \rho', \hat{\rho}} \sum_a \Psi_{A+, -\vec{n}}^{(\rho, \hat{\rho})} \lambda_{-, \vec{n}}^a \sigma_a^{(\rho, \rho')} \Omega^{AB} X_B^{0(\rho', \hat{\rho}')} ,
\end{aligned} \tag{3.79}$$

where $\sigma_a^{(\rho, \rho')} \equiv \langle \rho | t_a | \rho' \rangle$ ($\hat{\sigma}_a^{(\hat{\rho}, \hat{\rho}')} \equiv \langle \hat{\rho} | \hat{t}_a | \hat{\rho}' \rangle$). Therefore the matrix elements for each $\Psi_A^{(\rho, \hat{\rho})}$ and each λ^a are:

$$\begin{aligned}
& \Psi_{A+, \vec{n}}^{(\rho, \hat{\rho})} \begin{pmatrix} \lambda_{+, \vec{n}}^a & \lambda_{-, \vec{n}}^a & \lambda_{+, -\vec{n}}^a & \lambda_{-, -\vec{n}}^a \\ 0 & 0 & 0 & [\sigma X] \\ 0 & 0 & -[\sigma X] & 0 \\ 0 & [\sigma X] & 0 & 0 \\ -[\sigma X] & 0 & 0 & 0 \end{pmatrix}, \quad \lambda_{+, \vec{n}}^a \begin{pmatrix} \Psi_{A+, \vec{n}}^{(\rho, \hat{\rho})} & \Psi_{A-, \vec{n}}^{(\rho, \hat{\rho})} & \Psi_{A-, \vec{n}}^{(\rho, \hat{\rho})} & \Psi_{A-, -\vec{n}}^{(\rho, \hat{\rho})} \\ 0 & 0 & 0 & [\sigma X] \\ 0 & 0 & -[\sigma X] & 0 \\ 0 & [\sigma X] & 0 & 0 \\ -[\sigma X] & 0 & 0 & 0 \end{pmatrix},
\end{aligned} \tag{3.80}$$

where $[\sigma X] \equiv \frac{1}{2} \sigma_a^{(\rho, \rho')} \Omega^{AB} X_B^{0(\rho', \hat{\rho})}$ and λ, Ψ are symmetrized. This explains the factor $\frac{1}{2}$ in each entry. A summation over ρ' is understood in $\sigma_a^{(\rho, \rho')} \Omega^{AB} X_B^{0(\rho', \hat{\rho})}$.

The fermionic part of the deformation for each pair of $(\vec{n}, -\vec{n})$ can be written in matrix notation as:

$$\begin{aligned}
& \begin{matrix} \lambda^a & \hat{\lambda}^{a'} & \Psi^{A(\rho, \hat{\rho})} & \Psi_A^{(\rho, \hat{\rho})} \\ \lambda^a & \hat{\lambda}^{a'} & \Psi^{A(\rho, \hat{\rho})} & \Psi_A^{(\rho, \hat{\rho})} \\ \Psi^{A(\rho, \hat{\rho})} & \Psi_A^{(\rho, \hat{\rho})} & \Psi^{A(\rho, \hat{\rho})} & \Psi_A^{(\rho, \hat{\rho})} \\ \Psi_A^{(\rho, \hat{\rho})} & \Psi^{A(\rho, \hat{\rho})} & \Psi^{A(\rho, \hat{\rho})} & \Psi_A^{(\rho, \hat{\rho})} \end{matrix} \begin{pmatrix} M & 0 & (X\sigma)_A & -(\sigma X)^A \\ 0 & M & -(\widehat{\sigma X})_A & (\widehat{X\sigma})^A \\ (X\sigma)_A & -(\widehat{\sigma X})_A & 0 & N \\ -(\sigma X)^A & (\widehat{X\sigma})^A & N & 0 \end{pmatrix},
\end{aligned} \tag{3.81}$$

where:

$$\begin{aligned}
M = 2N &\equiv \begin{pmatrix} 0 & 0 & -2\pi V \cdot \vec{n} & 2\pi U \cdot \vec{n} \\ 0 & 0 & 2\pi U \cdot \vec{n} & 2\pi \bar{V} \cdot \vec{n} \\ 2\pi V \cdot \vec{n} & -2\pi U \cdot \vec{n} & 0 & 0 \\ -2\pi U \cdot \vec{n} & -2\pi \bar{V} \cdot \vec{n} & 0 & 0 \end{pmatrix}, \\
(\sigma X)^A &\equiv \frac{1}{2} \sigma_a^{(\rho, \rho')} \Omega^{AB} X_B^{0(\rho', \hat{\rho})} \times \mathbb{S}, \\
(X\sigma)_A &\equiv \frac{1}{2} \Omega_{AB} X^{B0(\rho', \hat{\rho})} \sigma_a^{(\rho', \rho)} \times \mathbb{S}, \\
\widehat{(\sigma X)}_A &\equiv \frac{1}{2} \hat{\sigma}_a^{(\hat{\rho}, \hat{\rho}')} \Omega_{AB} X^{B0(\rho, \hat{\rho}')} \times \mathbb{S}, \\
\widehat{(X\sigma)}^A &\equiv \frac{1}{2} \Omega^{AB} X_B^{0(\rho, \hat{\rho}')} \hat{\sigma}_a^{(\hat{\rho}', \hat{\rho})} \times \mathbb{S},
\end{aligned} \tag{3.82}$$

and:

$$\mathbb{S} \equiv \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{3.83}$$

As before we have $a, a' = 1, \dots, d$; $\rho, \hat{\rho} = 1, \dots, w$; $A = 1, \dots, 4$, where d is the dimension of the gauge group and w is the dimension of its fundamental representation. Therefore (3.81) is a $2d + 8w^2$ by $2d + 8w^2$ block matrix: each entry is given by one of the above four by four matrices.

The matrix (3.81) can be partitioned into four blocks:

$$\begin{pmatrix} A_{8d \times 8d} & B_{8d \times 32w^2} \\ C_{32w^2 \times 8d} & D_{32w^2 \times 32w^2} \end{pmatrix} := \begin{pmatrix} M & 0 & (X\sigma)_A & -(\sigma X)^A \\ 0 & M & -\widehat{(\sigma X)}_A & \widehat{(X\sigma)}^A \\ (X\sigma)_A & -\widehat{(\sigma X)}_A & 0 & N \\ -(\sigma X)^A & \widehat{(X\sigma)}^A & N & 0 \end{pmatrix}, \tag{3.84}$$

so that the determinant reads:⁵

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \det D \det[\mathbb{1} - D^{-1}CA^{-1}B]. \tag{3.85}$$

The determinants $\det A$ and $\det D$ are straightforward to compute:

$$\det A = (\det M)^{2d} = [16\pi^4 (\vec{n}^2)^2]^{2d}, \tag{3.86}$$

$$\det D = \prod_A (\det N)^{2w^2} = \prod_A [\pi^4 (\vec{n}^2)^2]^{2w^2}. \tag{3.87}$$

⁵We use the notation $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}$ for the matrices in the bosonic sector, while the matrices A, B, C, D are used for the fermion fields. We hope this does not cause any confusion with the $Sp(2)$ indices.

Their combined contribution to the one-loop determinant is:

$$\prod_{\vec{n}} \{(4\pi^2 \vec{n}^2)^d \prod_A (\pi^2 \vec{n}^2)^{w^2}\} . \quad (3.88)$$

Furthermore the integrations over the ghosts and anti-ghosts for the two gauge groups contribute $(\det \square)^2 = \{\prod_{\vec{n}} (4\pi^2 \vec{n}^2)^d\}^2$. When combined with (3.88) this gives:

$$\prod_{\vec{n}} \{(4\pi^2 \vec{n}^2)^{3d} \prod_A (\pi^2 \vec{n}^2)^{w^2}\} . \quad (3.89)$$

Up to a constant factor, this partially cancels the one-loop determinant from the boson sector, (3.70). We are thus left with only X^0 -dependent contributions from both boson and fermion sectors.

Inserting the localization conditions (3.46) into the off-shell Lagrangian (2.8) gives a vanishing classical contribution. Therefore the partition function is given purely by the one-loop determinant:

$$Z = \int \prod_A \prod_{(\rho, \hat{\rho})} dX_A^{0(\rho, \hat{\rho})} \prod_B \prod_{(\rho', \hat{\rho}')} dX^{B0(\rho', \hat{\rho}')} \frac{\prod_{(\vec{n}, -\vec{n})} \{\det[\mathbb{1} - D^{-1}CA^{-1}B]\}^{\frac{1}{2}}}{\prod_{\vec{n}} \det[\frac{\mathbb{A}}{4\pi^2(\vec{n} \cdot \vec{n})}]} . \quad (3.90)$$

We now make use of the Sylvester identity:

$$\det[\mathbb{1} - D^{-1}CA^{-1}B] = \det[\mathbb{1} - BD^{-1}CA^{-1}] , \quad (3.91)$$

where the matrix on the left-hand side above is $32w^2 \times 32w^2$, while the matrix on the right-hand side is $8d \times 8d$. Using the definitions in (3.82) and (3.83), one can show that:

$$\begin{aligned} & \det[\mathbb{1} - BD^{-1}CA^{-1}] \\ &= \det[\mathbb{1} + C^{Tr} D^{-1}CA^{-1}] \\ &= \det[\mathbb{1} + \begin{pmatrix} \mathbb{B} & \mathbb{D}^{Tr} \\ \mathbb{D} & \mathbb{C} \end{pmatrix} \otimes \frac{\mathbb{S}N^{-1}\mathbb{S}M^{-1}}{2}] \\ &= \det[\mathbb{1} + \begin{pmatrix} \mathbb{B} & \mathbb{D}^{Tr} \\ \mathbb{D} & \mathbb{C} \end{pmatrix} \otimes \frac{\mathbb{1}_{4 \times 4}}{4\pi^2(\vec{n} \cdot \vec{n})}] \\ &= \{\det[\frac{\mathbb{A}}{4\pi^2(\vec{n} \cdot \vec{n})}]\}^4 . \end{aligned} \quad (3.92)$$

Putting this back into the one-loop determinant, we see that the fermion and boson determinants cancel exactly against each other.

4 Discussion

We have partially carried out the localization procedure for the $\mathcal{N} = 1$ Chern-Simons-matter theory on T^3 with periodic boundary conditions. In particular we computed the contributions to the partition function from the locus of saddle points with vanishing gauge connection. As expected, restricting to this locus gives a trivial contribution to the partition function, i.e. the bosonic and fermionic contributions exactly cancel each other. Indeed evaluating the partition function on the flat torus at the trivial vacuum (vanishing gauge connection) simply counts the degrees of freedom of the theory, and for a supersymmetric theory one expects a complete cancellation. Of course the full partition function should receive contributions also from saddle points with nonvanishing flat gauge connections, which we have not computed here. We hope to return to this in the future.

Another potentially interesting direction in which this paper may be generalized is by allowing for a more general Killing spinor equation than the eq. (2.15) which was used for the present analysis. This may be achieved by coupling to a supergravity background and could provide additional possibilities for spaces on which the theory localizes.

The authors of [23] considered Euclidean 4d $\mathcal{N} = 1$ theories without R-symmetry, and concluded that no localization is possible in this case. Our results are not in contradiction with their conclusions. Indeed it is possible to construct 3d theories without R-symmetry by dimensional reduction and further truncation of 4d theories *with* R-symmetry.

Our results have the following implication for the partition function of the ABJM model on T^3 .⁶ Our analysis of the saddle points shows that the classical CS action vanishes on the locus of flat gauge connections on T^3 , cf. (3.43). Since the one-loop determinant around the saddle points does not introduce any dependence on the two CS levels, it follows by the localization argument that the partition function is independent of the level $k \equiv k_1 = -k_2$. Hence we may compute the partition function in the limit $k \rightarrow \infty$ with N fixed, which corresponds to vanishing 't Hooft coupling. In this limit the matter sector becomes free and decouples from the CS action. Therefore the resulting partition function factorizes into a pure supersymmetric CS partition function and a free matter piece. The latter is trivial, i.e. the bosonic and fermionic contributions exactly cancel each other. Moreover our localization results can be applied to the pure CS partition function to show that the contribution from the saddle points with vanishing gauge connection is also trivial. As mentioned above, this is consistent with what one expects for a supersymmetric theory.

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⁶Note that placing the ABJM model on T^3 breaks conformal invariance: this can be seen directly from the fact that the superconformal transformation parameter η of section B.2 is not well-defined on the torus, being linear in the coordinates.

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A Spinor and gamma-matrix conventions in 3d

The charge conjugation matrix in three dimensions satisfies:

$$C^{Tr} = -C; \quad (C\gamma^\mu)^{Tr} = C\gamma^\mu; \quad C^* = -C^{-1} . \quad (\text{A.1})$$

For any spinor ψ and in any spacetime signature we define:

$$\tilde{\psi} \equiv \psi^{Tr} C^{-1} . \quad (\text{A.2})$$

Moreover in Euclidean signature we define:

$$\psi^c \equiv C\psi^* . \quad (\text{A.3})$$

It follows that,

$$\psi^\dagger = -\tilde{\psi}^c ; \quad (\psi^c)^c = -\psi . \quad (\text{A.4})$$

The irreducible spinor representation in three Euclidean dimensions is two-dimensional complex (pseudoreal).

The Gamma matrices in Euclidean signature are taken to obey:

$$(\gamma_\mu)^\dagger = \gamma_\mu . \quad (\text{A.5})$$

Antisymmetric products of Gamma matrices are defined by:

$$\gamma_{\mu_1 \dots \mu_n}^{(n)} \equiv \gamma_{[\mu_1 \dots \mu_n]} . \quad (\text{A.6})$$

In Euclidean signature the Hodge-dual of an antisymmetric product of gamma matrices is given by:

$$\star \gamma_{(n)} = (-1)^{\frac{1}{2}n(n-1)} \gamma_{(3-n)} . \quad (\text{A.7})$$

B $\mathcal{N} = 1$ Superconformal Symmetry

B.1 Poincaré Supersymmetry

In this subsection we show the invariance of the on-shell Lagrangian (2.1) under the Poincaré supersymmetry.

The most general Poincaré supersymmetry transformations read:

$$\begin{aligned}
\delta X_A &= i\Omega_{AB}\tilde{\epsilon}\Psi^B \\
\delta X^A &= i\Omega^{AB}\tilde{\epsilon}\Psi_B \\
\delta\Psi_A &= \Omega_{AB}\gamma^\mu\epsilon D_\mu X^B + \delta_3\Psi_A \\
\delta\Psi^A &= \Omega^{AB}\gamma^\mu\epsilon D_\mu X_B + \delta_3\Psi^A \\
\delta A_\mu &= \frac{1}{k_1}[\Omega_{AB}\tilde{\epsilon}\gamma_\mu\Psi^A X^B + \Omega^{AB}X_B\tilde{\Psi}_A\gamma_\mu\epsilon] \\
\delta\hat{A}_\mu &= \frac{1}{k_2}[\Omega_{AB}X^B\tilde{\epsilon}\gamma_\mu\Psi^A + \Omega^{AB}\tilde{\Psi}_A\gamma_\mu\epsilon X_B] ,
\end{aligned} \tag{B.1}$$

where the variation δ_3 will be determined in the following.

- The variation of \mathcal{L}_{CS} cancels against the variation of the matter fields in \mathcal{L}_{kin} .
- The variation of the gauge fields in the spinor kinetic term in \mathcal{L}_{kin} cancels against the variation of the bosonic fields in \mathcal{L}_4 , iff:

$$\begin{aligned}
\frac{2}{k_1} + 2\alpha_1 - ia_1 + 2i\bar{a}_4 &= 0 , & -\frac{2}{k_2} - 2\bar{\alpha}_1 - ia_2 + 2ia_4 &= 0 , \\
\frac{1}{k_1} + 2\alpha_1 + 2i\bar{a}_4 + \alpha_{2,1} &= 0 , & -\frac{1}{k_2} - 2\bar{\alpha}_1 + 2ia_4 - \alpha_{2,2} &= 0 , \\
\frac{1}{k_2} + 2\alpha_1 + \alpha_{2,2} &= 0 , & \frac{1}{k_1} + 2\bar{\alpha}_1 + \alpha_{2,1} &= 0 , \\
2\alpha_1 - 2i\bar{a}_3 + 2\alpha_{3,2} &= 0 , & 2\bar{\alpha}_1 + 2ia_3 + 2\alpha_{3,1} &= 0 , \\
2\alpha_1 - 2i\bar{a}_3 + 2\alpha_{3,1} &= 0 , & 2\bar{\alpha}_1 + 2ia_3 + 2\alpha_{3,2} &= 0 ,
\end{aligned} \tag{B.2}$$

or:

$$\begin{aligned}
a_1 &= -2i\left(\frac{1}{k_1} + \bar{\alpha}_1\right) , & a_2 &= 2i\left(\frac{1}{k_2} + \alpha_1\right) , \\
a_3 &= -\bar{a}_3 - i(\alpha_1 - \bar{\alpha}_1) , & a_4 &= \bar{a}_4 = i(\alpha_1 - \bar{\alpha}_1) , \\
\alpha_{2,1} &= -\frac{1}{k_1} - 2\bar{\alpha}_1 , & \alpha_{2,2} &= -\frac{1}{k_2} - 2\alpha_1 , \\
\alpha_{3,1} &= \alpha_{3,2} = i\bar{a}_3 - \alpha_1 = -ia_3 - \bar{\alpha}_1 ,
\end{aligned} \tag{B.3}$$

where all parameters are expressed in terms of $k_1, k_2, \alpha_1, \bar{\alpha}_1$ and \bar{a}_3 . In the following we will set $\alpha_3 \equiv \alpha_{3,1} = \alpha_{3,2}$, and use a_4 instead of \bar{a}_4 .

- The variation of the gauge fields in the boson kinetic terms in \mathcal{L}_{kin} , together with the variation of the fermion fields in \mathcal{L}_4 without $\delta_3\Psi$, cancel against the $\delta_3\Psi$ variation of the fermion kinetic terms in \mathcal{L}_{kin} , iff:

$$\begin{aligned}
\delta_3\Psi_A &= \{\Omega_{AB}(\alpha_{2,2}X^C X_C X^B - \alpha_{2,1}X^B X_C X^C) - 2\alpha_3\Omega_{BC}X^B X_A X^C\}\epsilon , \\
\delta_3\Psi^A &= \{\Omega^{AB}(-\alpha_{2,1}X_C X^C X_B + \alpha_{2,2}X_B X^C X_C) + 2\alpha_3\Omega^{BC}X_B X^A X_C\}\epsilon .
\end{aligned} \tag{B.4}$$

- The $\delta_3\Psi$ variation of \mathcal{L}_4 cancels against the variation of \mathcal{L}_6 , iff:

$$\begin{aligned}
& 2i\alpha_1\alpha_{2,1} - i\alpha_{2,1}^2 - a_1\alpha_{2,1} - 2a_4\alpha_{2,1} - i\alpha_{4,2} + P = 0 , \\
& 2i\bar{\alpha}_1\alpha_{2,2} - i\alpha_{2,2}^2 + a_2\alpha_{2,2} + 2a_4\alpha_{2,2} - i\alpha_{4,1} + \bar{P} = 0 , \\
& - 2i\bar{\alpha}_1\alpha_{2,1} - a_2\alpha_{2,1} - 2a_4\alpha_{2,1} + 2i\alpha_{4,4} - \bar{P} = 0 , \\
& - 2i\alpha_1\alpha_{2,2} + a_1\alpha_{2,2} + 2a_4\alpha_{2,2} + 2i\alpha_{4,4} - P = 0 , \\
& 2i\alpha_1\alpha_{2,1} - 2i\alpha_3\alpha_{2,1} + 2\bar{a}_3\alpha_{2,1} - i\bar{m} + P = 0 , \\
& 2i\bar{\alpha}_1\alpha_{2,2} - 2i\alpha_3\alpha_{2,2} - 2a_3\alpha_{2,2} - im + \bar{P} = 0 , \\
& - 2i\alpha_1\alpha_{2,2} + 2i\alpha_3\alpha_{2,2} - 2\bar{a}_3\alpha_{2,2} + im - P = 0 , \\
& - 2i\bar{\alpha}_1\alpha_{2,1} + 2i\alpha_3\alpha_{2,1} + 2a_3\alpha_{2,1} + i\bar{m} - \bar{P} = 0 , \\
& - 2i\alpha_1\alpha_{2,1} - a_2\alpha_{2,1} + 2i\alpha_{4,4} - P = 0 , \\
& - 2i\bar{\alpha}_1\alpha_{2,2} + a_1\alpha_{2,2} + 2i\alpha_{4,4} - \bar{P} = 0 , \\
& 2i\alpha_1\alpha_{2,2} - i\alpha_{2,2}^2 + a_2\alpha_{2,2} - i\alpha_{4,1} + P = 0 , \\
& 2i\bar{\alpha}_1\alpha_{2,1} - i\alpha_{2,1}^2 - a_1\alpha_{2,1} - i\alpha_{4,2} + \bar{P} = 0 , \\
& 4i\alpha_{2,2}\alpha_3 + im - P = 0 , \quad 4i\alpha_{2,1}\alpha_3 + i\bar{m} - \bar{P} = 0 , \\
& 4i\alpha_{2,1}\alpha_3 + i\bar{m} - P = 0 , \quad 4i\alpha_{2,2}\alpha_3 + im - \bar{P} = 0 , \\
& 2i\alpha_{2,1}\alpha_{2,2} + 2i\alpha_{4,4} - P = 0 , \quad 2i\alpha_{2,1}\alpha_{2,2} + 2i\alpha_{4,4} - \bar{P} = 0 , \\
& 4i\alpha_3^2 - in - P = 0 , \quad 4i\alpha_3^2 - in - \bar{P} = 0 , \\
& 4i\alpha_{4,3} - P = 0 , \quad 4i\alpha_{4,3} - \bar{P} = 0 ,
\end{aligned} \tag{B.5}$$

and:

$$\begin{aligned}
& 2a_1\alpha_3 + 4a_4\alpha_3 + 4\bar{a}_3\alpha_3 + i\bar{m} + in = 0 , \\
& 2a_2\alpha_3 - im + 2a_1\alpha_3 + 4a_4\alpha_3 + i\bar{m} = 0 , \\
& - 2a_2\alpha_3 - 4a_4\alpha_3 - 4a_3\alpha_3 + im + in = 0 , \\
& 2a_1\alpha_3 + i\bar{m} + 2a_2\alpha_3 + 4a_4\alpha_3 - im = 0 ,
\end{aligned} \tag{B.6}$$

where:

$$\begin{aligned}
P &= -4i\alpha_1\alpha_3 + 2a_1\alpha_3 + 4a_4\alpha_3 + i\bar{m} , \\
\bar{P} &= -4i\bar{\alpha}_1\alpha_3 - 2a_2\alpha_3 - 4a_4\alpha_3 + im ,
\end{aligned} \tag{B.7}$$

and we made use of the identities:

$$\varepsilon_{ABCD} = \Omega_{AB}\Omega_{CD} - \Omega_{AC}\Omega_{BD} + \Omega_{AD}\Omega_{BC} ; \quad \varepsilon_{ABCD}\Omega^{EF} = 24\Omega_{[AB}\delta_C^E\delta_D^F] . \tag{B.8}$$

After some further manipulation of these equations, taking (B.3) into account, we find:

$$\begin{aligned}
a_1 &= -2i\left(\frac{1}{k_1} + \bar{\alpha}_1\right), & a_2 &= 2i\left(\frac{1}{k_2} + \alpha_1\right), \\
a_3 &= -\bar{a}_3 - i(\alpha_1 - \bar{\alpha}_1), & a_4 &= i(\alpha_1 - \bar{\alpha}_1), \\
\alpha_{2,1} &= -\frac{1}{k_1} - 2\bar{\alpha}_1, & \alpha_{2,2} &= -\frac{1}{k_2} - 2\alpha_1, \\
\alpha_3 &= i\bar{a}_3 - \alpha_1 = -ia_3 - \bar{\alpha}_1, \\
\alpha_{4,1} &= -3\alpha_{2,2}^2 + 4\alpha_{2,2}\alpha_3 + m, \\
\alpha_{4,2} &= -3\alpha_{2,1}^2 + 4\alpha_{2,1}\alpha_3 + m, \\
\alpha_{4,3} &= \alpha_{2,2}\alpha_3 + \frac{m}{4}, \\
\alpha_{4,4} &= -\alpha_{2,1}\alpha_{2,2} + 2\alpha_{2,2}\alpha_3 + \frac{m}{2}, \\
\bar{m} &= 4(\alpha_{2,2} - \alpha_{2,1})\alpha_3 + m \\
n &= 4(\alpha_3 - \alpha_{2,2})\alpha_3 - m.
\end{aligned} \tag{B.9}$$

• Let us also mention that the requirement that the total Lagrangian should be real (which we do not need to impose in the present paper) would imply the following additional conditions on the parameters:

$$\begin{aligned}
&\alpha_{2,1}, \alpha_{2,2}, \alpha_3, \alpha_{4,1}, \alpha_{4,2}, \alpha_{4,3}, \alpha_{4,4}, m, \bar{m}, n \text{ are real,} \\
&a_1, a_2 \text{ are imaginary,} \\
&(a_3 - i\bar{\alpha}_1)^* = \bar{a}_3 + i\alpha_1, \quad (a_4 + i\bar{\alpha}_1)^* = -i\alpha_1, \quad (i\bar{\alpha}_1)^* = a_4 - i\alpha_1.
\end{aligned} \tag{B.10}$$

When we combine the above reality conditions with (B.9), we see that α_1 and $\bar{\alpha}_1$ are real and a_4, a_3, \bar{a}_3 are imaginary.

From the above it follows that the on-shell theory has four independent parameters besides the CS levels k_1, k_2 . They can be chosen to be $\alpha_1, \bar{\alpha}_1, \bar{a}_3$ and m .

B.2 Conformal supersymmetry

Provided (B.9) holds, the action possesses an additional conformal supersymmetry. To show this, we follow [17] and replace the parameter ϵ of the Poincaré supersymmetry by $x_\mu \gamma^\mu \eta$, while adding to the spinor variations the terms:

$$\begin{aligned}
\delta' \Psi_A &= \Omega_{AB} X^B \eta, \\
\delta' \Psi^A &= \Omega^{AB} X_B \eta.
\end{aligned} \tag{B.11}$$

Most terms in the Lagrangian are then invariant by virtue of the Poincaré supersymmetry. The term coming from the derivative acting on x of $x_\mu \gamma^\mu \eta$ in $\delta_3 \Psi$ of the fermion kinetic Lagrangian cancels with $\delta' \Psi$ of \mathcal{L}_4 , if (B.9) holds. Finally terms generated by the remaining variations of the fermions in the fermionic kinetic terms cancel against the boson transformations in the bosonic kinetic Lagrangian and the variations of the CS terms.

C Trivial G -structures in 3d

The existence of a nowhere-vanishing (commuting) spinor ϵ on a Riemannian three-manifold implies the existence of a trivial G -structure, i.e. the trivialization of the tangent bundle. In this section we will explore in detail the implications of this trivialization.

Since ϵ is assumed nowhere-vanishing we can take it to be normalized:

$$\epsilon^\dagger \epsilon = \tilde{\epsilon} \epsilon^c = -\tilde{\epsilon}^c \epsilon = 1 , \quad (\text{C.1})$$

where we used the formulas in Appendix A. On the other hand,

$$\tilde{\epsilon} \epsilon = 0 , \quad (\text{C.2})$$

due to the antisymmetry of the charge conjugation matrix, cf. (A.1). furthermore we can define the following ϵ -bilinear one-forms:⁷

$$U_\mu \equiv \epsilon^\dagger \gamma_\mu \epsilon = -\tilde{\epsilon}^c \gamma_\mu \epsilon = -\tilde{\epsilon} \gamma_\mu \epsilon^c , \quad (\text{C.3})$$

where we took (A.4) into account, and:

$$V_\mu \equiv \tilde{\epsilon} \gamma_\mu \epsilon . \quad (\text{C.4})$$

It can be seen that U is real whereas V is complex:

$$\bar{V}_\mu = -\tilde{\epsilon}^c \gamma_\mu \epsilon^c = \epsilon^\dagger \gamma_\mu \epsilon^c . \quad (\text{C.5})$$

The Fierz identities can be conveniently written in terms of the bilinears above:

$$\begin{aligned} \epsilon \tilde{\epsilon}^c &= -\frac{1}{2}(\mathbb{1} + U^\mu \gamma_\mu) ; & \epsilon^c \tilde{\epsilon} &= \frac{1}{2}(\mathbb{1} - U^\mu \gamma_\mu) \\ \epsilon \tilde{\epsilon} &= \frac{1}{2}V^\mu \gamma_\mu ; & \epsilon^c \tilde{\epsilon}^c &= -\frac{1}{2}\bar{V}^\mu \gamma_\mu . \end{aligned} \quad (\text{C.6})$$

Using the above, the following relations can be shown:

$$U^2 = \Re V^2 = \Im V^2 = 1 ; \quad U \cdot \Re V = U \cdot \Im V = \Re V \cdot \Im V = 0 , \quad (\text{C.7})$$

where we have defined $A^2 \equiv A^\mu A_\mu$, $A \cdot B \equiv A^\mu B_\mu$ and $V = \Re V + i\Im V$. In other words the triplet $(U, \Re V, \Im V)$ is a globally-defined orthonormal frame thus trivializing the (co)tangent bundle of the manifold.

Let us also mention the following useful identities which can similarly be shown by fierzing:

$$\begin{aligned} \gamma_\mu \epsilon &= U_\mu \epsilon + V_\mu \epsilon^c \\ \gamma_\mu \epsilon^c &= \bar{V}_\mu \epsilon - U_\mu \epsilon^c . \end{aligned} \quad (\text{C.8})$$

From these we also obtain:

$$\begin{aligned} U^\mu \gamma_\mu \epsilon &= \epsilon ; & U^\mu \gamma_\mu \epsilon^c &= -\epsilon^c \\ \frac{1}{2}V^\mu \gamma_\mu \epsilon^c &= \epsilon ; & \frac{1}{2}\bar{V}^\mu \gamma_\mu \epsilon &= \epsilon^c \\ V^\mu \gamma_\mu \epsilon &= \bar{V}^\mu \gamma_\mu \epsilon^c = 0 . \end{aligned} \quad (\text{C.9})$$

⁷Since we are assuming the existence of a Riemannian metric on our manifold, we can convert vectors to one-forms and vice-versa.

Spinor and tensor decomposition

Spinors on the manifold can be expanded on the basis of ϵ, ϵ^c . Explicitly, for any spinor λ we have:

$$\lambda = \lambda_+ \epsilon + \lambda_- \epsilon^c, \quad (\text{C.10})$$

where λ_{\pm} are scalar coefficients given by:

$$\lambda_+ = \tilde{\lambda} \epsilon^c; \quad \lambda_- = \tilde{\epsilon} \lambda. \quad (\text{C.11})$$

The notation is motivated by the fact that we may define a chirality operator:

$$\gamma \equiv U^\mu \gamma_\mu, \quad (\text{C.12})$$

which indeed squares to one as follows from (C.7). Moreover ϵ, ϵ^c are chiral, antichiral respectively with respect to γ , as can be seen from (C.9).

Forms and tensors can be decomposed using the orthonormal frame provided by (U, V) . For example any one-form A can be decomposed as follows:

$$A = A_\perp U + A_+ V + A_- \bar{V}, \quad (\text{C.13})$$

where A_\perp, A_\pm are scalar coefficients given by:

$$A_\perp = U \cdot A; \quad A_+ = \frac{1}{2} \bar{V} \cdot A; \quad A_- = \frac{1}{2} V \cdot A. \quad (\text{C.14})$$

The notation is motivated by the fact that one-forms can be decomposed into the subspaces parallel and orthogonal to U , which we may call the vertical and horizontal subspaces respectively. The horizontal subspace can then be further decomposed into directions parallel and orthogonal to V (equivalently: orthogonal and parallel to \bar{V}), which we may consider as the holomorphic and antiholomorphic directions respectively.

D Lorentz Gauge

In this section we give the details of the integration over the ghost complex. First note that in (3.32) the integration over b and b_0 can be performed independently:

$$\begin{aligned}
& \int \prod_x db(x)db_0 \exp^{i \int dx^3 \text{tr}\{b(\partial^\mu A_\mu + b_0)\}} \\
&= \int \prod_x db'(x)db_0 \exp^{i \int dx^3 \text{tr}\{b'(x)(\partial^\mu A_\mu + b_0)\}} \int db' \exp^{i \int dx^3 \text{tr}\{b'(\partial^\mu A_\mu + b_0)\}} \\
&= \int \prod_x db'(x)db_0 \exp^{i \int dx^3 \text{tr}\{b'(x)(\partial^\mu A_\mu + b_0)\}} \int db' \exp^{i \int dx^3 \text{tr}\{b'b_0\}} \\
&= \frac{1}{\text{Vol}} \int \prod_x db'(x)db_0 \exp^{i \int dx^3 \text{tr}\{b'(x)(\partial^\mu A_\mu + b_0)\}} \delta(b_0) \\
&= \int \prod_x db'(x)db_0 \exp^{i \int dx^3 \text{tr}\{b'(x)(\partial^\mu A_\mu + b_0)\}} \int db' \exp^{i \int dx^3 \text{tr}\{b'b_0\}} \\
&= \frac{1}{\text{Vol}} \int \prod_x db'(x) \exp^{i \int dx^3 \text{tr}\{b'(x)\partial^\mu A_\mu\}} ,
\end{aligned} \tag{D.1}$$

where Vol denotes the volume of T^3 , and we decompose $b(x) = b'(x) + b'$; b' is a constant field: it is the zero mode of $b(x)$. The remaining integration over $b'(x)$ imposes the Lorentz gauge condition.

Next we integrate over a_0 , then \bar{a}_0 :

$$\begin{aligned}
& \int d\bar{a}_0 da_0 \exp^{i \int dx^3 \text{tr}\{-(a_0 - \frac{i}{2}\{C,C\})\bar{a}_0\}} \\
&= \frac{1}{\text{Vol}} \int d\bar{a}_0 \exp^{i \int dx^3 \text{tr}\{\frac{i}{2}\{C,C\}\bar{a}_0\}} \delta(\bar{a}_0) \\
&= \frac{1}{\text{Vol}} .
\end{aligned} \tag{D.2}$$

The remaining integrations read:

$$\begin{aligned}
& \int \prod_x d\bar{C}(x) dC(x) dC_0 d\bar{C}_0 \exp^{i \int dx^3 \text{tr} \{ -\bar{C}(\partial^\mu D_\mu C + \partial^\mu \delta_Q A_\mu + C_0) + C\bar{C}_0 \}} \\
&= \int \prod_x dC(x) dC_0 d\bar{C}_0 d\bar{C}'(x) \exp^{i \int dx^3 \text{tr} \{ -\bar{C}'(x)(\partial^\mu D_\mu C + \partial^\mu \delta_Q A_\mu + C_0) + C\bar{C}_0 \}} \\
&\quad \times \int d\bar{C}' \exp^{i \int dx^3 \text{tr} \{ -\bar{C}'(\partial^\mu D_\mu C + \partial^\mu \delta_Q A_\mu + C_0) \}} \\
&= \int \prod_x dC(x) dC_0 d\bar{C}_0 \int d\bar{C}'(x) \exp^{i \int dx^3 \text{tr} \{ -\bar{C}'(x)(\partial^\mu D_\mu C + \partial^\mu \delta_Q A_\mu + C_0) + C\bar{C}_0 \}} \\
&\quad \times \int d\bar{C}' \exp^{i \int dx^3 \text{tr} \{ -\bar{C}' C_0 \}} \tag{D.3} \\
&= \int \prod_x dC(x) dC_0 d\bar{C}_0 \int \bar{C}'(x) \exp^{i \int dx^3 \text{tr} \{ -\bar{C}'(x)(\partial^\mu D_\mu C + \partial^\mu \delta_Q A_\mu + C_0) + C\bar{C}_0 \}} \\
&\quad \times \text{Vol} \delta(C_0) \\
&= \text{Vol} \int \prod_x dC(x) d\bar{C}_0 \int d\bar{C}'(x) \exp^{i \int dx^3 \text{tr} \{ -\bar{C}'(x)(\partial^\mu D_\mu C + \partial^\mu \delta_Q A_\mu) + C\bar{C}_0 \}} \\
&= \text{Vol} \int \prod_x dC'(x) d\bar{C}_0 \int d\bar{C}'(x) \exp^{i \int dx^3 \text{tr} \{ -\bar{C}'(x)(\partial^\mu D_\mu C'(x) + \partial^\mu \delta_Q A_\mu) + C'(x)\bar{C}_0 \}} \\
&\quad \times \int dC' \exp^{i \int dx^3 \text{tr} \{ -i\bar{C}'(x)[\partial^\mu A_\mu, C'] + C'\bar{C}_0 \}} .
\end{aligned}$$

Note that the expression above is multiplied by an overall factor $\delta(\partial \cdot A)$, therefore we can set $\partial \cdot A$ to zero and integrate over C' :

$$\begin{aligned}
& \text{Vol}^2 \int \prod_x dC'(x) d\bar{C}'(x) d\bar{C}_0 \exp^{i \int dx^3 \text{tr} \{ -\bar{C}'(x)(\partial^\mu D_\mu C'(x) + \partial^\mu \delta_Q A_\mu) + C'(x)\bar{C}_0 \}} \\
&\quad \times \delta(\bar{C}_0) \tag{D.4} \\
&= \text{Vol}^2 \int \prod_x dC'(x) d\bar{C}'(x) \exp^{i \int dx^3 \text{tr} \{ -\bar{C}'(x)(\partial^\mu D_\mu C'(x) + \partial^\mu \delta_Q A_\mu) \}} .
\end{aligned}$$

Absorbing $\delta_Q A_\mu$ into $C'(x)$, restricting to the saddle point $A = 0$ and integrating over C' and \bar{C}' , the last line gives $\det \square$.

To see that $\delta_Q A_\mu$ can indeed be absorbed into $C'(x)$ it suffices to show that there is a C'' such that:

$$\partial^\mu D_\mu C'(x) + \partial^\mu \delta_Q A_\mu = \partial^\mu D_\mu C'' . \tag{D.5}$$

Equivalently in form notation:

$$d^\dagger(\delta_Q A + d_A \delta C) = 0 , \tag{D.6}$$

where we have set $\delta C := C' - C''$, $d_A := d + i[A, \cdot]$. The Hodge decompositions of δC , $\delta_Q A$ are as follows:

$$\delta C = \delta C_{(h)} + d^\dagger \delta C_{(1)} ; \quad \delta_Q A = \delta_Q A_{(h)} + d \delta_Q A_{(0)} + d^\dagger \delta_Q A_{(2)} , \tag{D.7}$$

where the numerical subscripts indicate the rank of the corresponding form and $\delta C_{(h)}$, $\delta_Q A_{(h)}$ are harmonic zero-, one-forms respectively; in particular $\delta C_{(h)}$ is constant. Similarly for the gauge field we expand:

$$A = d^\dagger A_{(2)} + A_{(h)} . \quad (\text{D.8})$$

The fact that there is no exact piece in the decomposition above is due to the Lorentz gauge, $d^\dagger A = 0$. Furthermore equation (D.6) is equivalent to the statement that there exist a two-form u and a harmonic one-form w_h such that:

$$d_A \delta C + \delta_Q A = d^\dagger u + w_{(h)} . \quad (\text{D.9})$$

On the other hand, taking the expansions (D.7),(D.8) into account, the left-hand side of (D.6) reads:

$$d(\delta C + \delta_Q A_{(0)}) + i d^\dagger ([A_{(2)}, \delta C] + [A_{(h)}, \delta C_{(1)}]) + i [A_{(h)}, \delta C_{(h)}] . \quad (\text{D.10})$$

It follows that (D.9), is solved for:

$$\delta C = -\delta_Q A_{(0)} ; \quad u = i([A_{(2)}, \delta C] + [A_{(h)}, \delta C_{(1)}]) ; \quad w_{(h)} = i[A_{(h)}, \delta C_{(h)}] . \quad (\text{D.11})$$

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