Calculus for Business & Social Sciences

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Work Cited

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About this Textbook

In the text's electronic format, the reader is able to navigate throughout the book using hyperlinks. Hyperlinks can be found in the table of contents, figure and table references, section exercises, and bold example references. In the text's pdf format, bookmarks allow for navigation.

Each chapter of the text is formatted in the following manner:

- Each section begins with a list of the learning objectives the reader will be able to demonstrate upon completion of the section.
- Definitions, processes, and theorems are highlighted using shaded boxes for easy identification.
- At the end of most subsections, a try-it exists for the reader to check their understanding of the topic discussed.
- The following icons are used:
  - denotes discovery based on work in the previous example
  - N denotes reminders from the authors to the reader
  - ! denotes common misconceptions and mistakes
- Each section ends with a comprehensive set of exercises.

How to Use Each Chapter for Practice and Review

Try-Its

After most subsections a Try-It provides readers with the opportunity to determine whether or not they can apply the covered concepts to a new problem. The answers to all section Try-Its can be found at the end of the section and before the section exercises begin.

Exercises

Section exercises are broken down into four practice categories: Basic Skills, Intermediate Skills, Mastery, and Communication:

- Basic Skills Practice – These exercises focus on applying a concept or learning objective to basic functions and typically involve one step. Basic Skills exercises are subdivided based on the learning objectives for the section.
- Intermediate Skills Practice – These exercises focus on situations that include a larger variety of functions, as well as multi-step problems. Intermediate Skills exercises are subdivided based on the learning objectives for the section.
- Mastery Practice – These exercises require critical thinking and are not subdivided by learning objective. If readers are able to complete these exercises without use of reference materials, it is assumed they have mastered the learning objectives for the section.
- Communication Practice – These exercises focus on reader's ability to explain concepts in their own words.

Readers are encouraged to begin their practice in a section with the Intermediate Skills Practice. If a reader is struggling with a group of exercises in the Intermediate Skills Practice, then they should go to the corresponding group of exercises in the Basic Skills Practice before moving on to the next group of exercises in the Intermediate Skills Practice. While not every problem in an exercise group needs to be completed by the reader, answers to all exercises (both even and odd) are provided in the back of the text as Appendix A.

Once a reader feels as though they have a solid grasp of the Intermediate Skills Practice, they can move to the Mastery Practice to confirm their understanding.

Appendix A

Appendix A contains the answers to all exercises in the text.

Contents

Chapter 1

1. Limits and Continuity

Suppose you invest \$3000 in an account with an annual interest rate of 3% per year compounded monthly. Recalling that the amount of money in an account in which interest is compounded m times per year is given by $A = P\left(1+\frac{r}{m}\right)^{mt}$, the amount in your account after four years would be $A = 3000\left(1+\frac{0.03}{12}\right)^{(12)(4)} = \$3381.98$. Note that $m = 12$ because interest is compounded monthly, or 12 times per year.

If, instead, the interest was compounded weekly, or $m = 52$ times per year, then the amount in your account after four years would be $A = 300\left(1+\frac{0.03}{52}\right)^{(52)(4)} = \$3382.37$. If the interest was compounded daily ($m = 365$), then your account balance would be $A = 300\left(1+\frac{0.03}{365}\right)^{(365)(4)} = \$3382.47$ after four years.

If we continue this process and consider the interest compounding every hour ($m = 8760$), every minute ($m = 525,600$), or even every second ($m = 31,536,000$), then the number of times per year the interest is compounded is approaching, conceptually, infinitely many, or continuously compounded interest. Recalling that the amount of money in an account in which interest is compounded continuously is given by $A = Pe^{rt}$, if \$3000 is invested into such an account with an annual interest rate of 3% per year, then the amount in the account after four years would be $A = 3000e^{(0.03)(4)} = \$3382.49$.

In mathematical terms, we can say that as m approaches infinity, the amount in the account approaches \$3382.49, or the **limit** is approximately \$3382.49. In fact, it can be shown that for *any* account, as m approaches infinity, $P\left(1+\frac{r}{m}\right)^{mt}$ approaches Pe^{rt}. Even further, the number e can be defined as the limit of $\left(1+\frac{1}{n}\right)^{n}$ as n goes to infinity! A deep understanding of limits is necessary to master important topics in calculus such as continuity, derivatives, and integrals.

In this chapter, we will explore limits graphically, numerically, and algebraically to investigate the behavior of functions at finite numbers and at infinity. We will also use limits to define continuity using calculus.

1.1 Limits: Graphically and Numerically

There may be times when we are interested in exploring some phenomena very near a point, but not necessarily at the point. A cell phone company, for example, may be interested in knowing the number of phones sold near six o'clock, but not necessarily the number of phones sold at six o'clock. In sports, an athlete may be interested in their blood-oxygen levels as they approach a certain point in their workout, but not necessarily the blood-oxygen level at a specific point in their workout.

These are just two real-life examples that help demonstrate the concept of a **limit**, a very important concept in the study of calculus. The limit of a function is the value the function approaches when it is very near a point, but not necessarily at the point. To illustrate this, we will explore limits of functions using their graphs.

Learning Objectives:

In this section, you will learn how to find limits graphically and numerically and determine the corresponding behavior of the function. Upon completion you will be able to:

- Explain the meaning of the mathematical notation representing a limit.
- Estimate both one-sided and two-sided limits graphically.
- Estimate both finite and infinite limits graphically.
- Estimate both one-sided and two-sided limits numerically.
- Estimate both finite and infinite limits numerically.
- Describe the ways in which the limit of a function may not exist.
- Deduce and interpret the behavior of functions using limits.
- Describe the infinite behavior near vertical asymptotes using limit notation.

Determining Limits Graphically

The limit of a function is the value the function approaches when the independent variable is near, *but not necessarily equal to*, a specified number. To illustrate this, let's look at the graph of a function f shown in **Figure 1.1.1**.

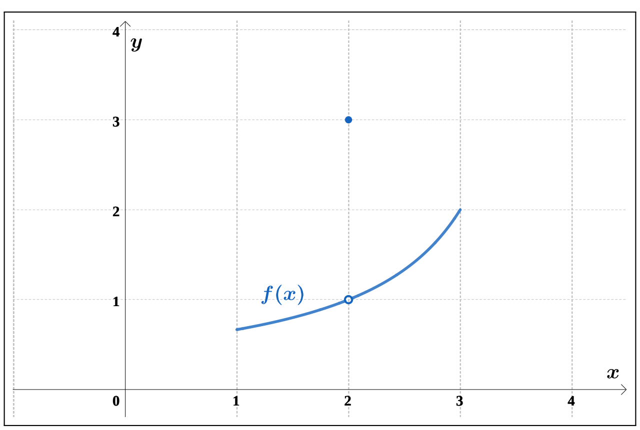


Figure 1.1.1: Graph of f in which $f(2) = 3$, but the values of $f(x)$ near $x = 2$ approach $y = 1$

Suppose we want to determine the value (y-value) of the function *near* $x = 2$. We can do this intuitively by tracing the graph of the function from both the left and right of $x = 2$ and investigating the y-values of the function. When we do this, we see the y-values of the function are heading toward, or approaching, $y = 1$. In this instance, we can predict that the limit of $f(x)$ as x approaches 2 is 1. If we look carefully, however, we see at $x = 2$ there is a filled-in circle graphed at the point $(2,3)$ and an open circle graphed at the point $(2,1)$. The filled-in circle graphed at $(2,3)$ informs us that the value of the function at $x=2$ is 3, or using function notation, $f(2)=3$. The important fact to notice here is that at $x=2$, the limit of $f(x)$ is 1, but the value of the function is 3. This demonstrates the difference between the limit and the value of a function at a specific x-value.

The previous discussion leads us to the general concept of a limit: If the values of $f(x)$ get closer and closer, from both the left and right, to a real number L as we take values of x very close to (but not necessarily equal to) a real number c, then

> we say:
>
> "the limit of $f(x)$, as x approaches c, is L"
>
> and we write:
>
> $$\lim_{x\to c} f(x) = L$$

N *The symbol "→" means "approaches" or "gets very close to."*

For the function whose graph was shown in **Figure 1.1.1**, we predicted that the limit of $f(x)$ as x approached 2 was 1. Thus, using mathematical notation, we write our prediction as $\lim_{x\to 2} f(x) = 1$.

It is very important to remember that

> $f(c)$ is a real number and is the value of the function f **at** $x=c$.

while

> $\lim_{x\to c} f(x)$ is a real number and is the value the function f is approaching when x is **near, but not necessarily at**, $x=c$.

N *Even though the value of a function at a particular x-value may differ from the value of the limit, there are some cases where they will be equal. We will discuss this further when we discuss continuity in* ***Section 1.4.***

Sometimes, using graphs can help us get an intuitive feeling for the limits of functions, but it is important to remember that when we use the graph of a function to find a limit, we are only predicting, or estimating, the limit of the function. Why? Graphs do not provide the precision we need to determine an "exact" limit.

For example, when we predicted the limit of $f(x)$ as x approached 2 in the previous discussion, we could not tell from the graph of f in **Figure 1.1.1** if the limit was 0.98, 0.99, or 1. The limit appeared to be 1, but we cannot say this with 100% certainty. The good news is that we will learn how to use algebra to determine the exact value of a limit, if it exists, in **Section 1.2**. For now, however, we will use the graphs of functions to help us get a good approximation of the limit.

■ **Example 1** Use the graph of f shown in **Figure 1.1.2** to estimate the following limits, if they exist.

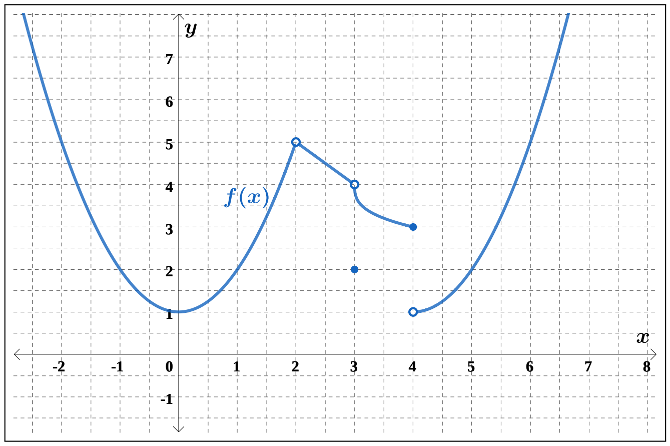


Figure 1.1.2: Graph of a piecewise-defined function f

a. $\lim_{x\to 1} f(x)$

b. $\lim_{x\to 2} f(x)$

c. $\lim_{x\to 3} f(x)$

d. $\lim_{x\to 4} f(x)$

Solution:

a. To determine $\lim_{x\to 1} f(x)$, we need to inspect the values of the function as we get very close to $x = 1$ from both the left and right. When x is very close to 1 on the left, the values of the function are approaching $y = 2$ from below. When x is very close to 1 on the right, the values of the function are approaching $y = 2$ from above. Thus, we predict the limit of $f(x)$ as x approaches 1 is 2, and we write

$$\lim_{x\to 1} f(x) = 2$$

In this example, it happens that the value of the function at $x = 1$ is also 2. In other words, $f(1) = 2$. However, this is not always the case. Remember, when finding limits, we are interested in the values of the function when x is *close* to 1, not when x actually equals 1.

b. To determine $\lim_{x\to 2} f(x)$, we need to inspect the values of $f(x)$ as we get very close to $x = 2$ from both the left and right. When x is very close to 2 on the left, the values of the function appear to be approaching 5 from below. When x is very close to 2 on the right, the values of the function appear to be approaching 5 from below. Thus, we predict the limit of $f(x)$ as x approaches 2 is 5, and we write

$$\lim_{x\to 2} f(x) = 5$$

In this example, it happens that the function is not defined at $x = 2$, and we know this because of the open circle that is graphed at $(2,5)$. Although $f(2)$ is undefined, the limit as x approaches 2 exists because the y-values near $x = 2$ appear to be very close to 5.

c. To find $\lim_{x\to 3} f(x)$, we need to inspect the values of $f(x)$ when the x-values get very close to 3 from both the left and right. When the x-values get very close to 3 from the left, the values of $f(x)$ appear to approach 4 from above. When the x-values get closer to 3 from the right, the values of the function seem to approach 4 from below. Thus, we predict the limit of $f(x)$ as x approaches 3 is 4 and we write

$$\lim_{x\to 3} f(x) = 4$$

In this example, the y-value at $x = 3$ is 2, or $f(3) = 2$, but the limit as x approaches 3 seems to be 4. This is okay because, when finding limits, we are interested in what happens very near $x = 3$, not necessarily what happens at $x = 3$.

d. To find $\lim_{x\to 4} f(x)$, we need to inspect the values of the function when the x-values get very close to 4 from both the left and right. When the x-values get very close to 4 from the left, it appears the values of the function approach 3 from above. When the x-values get very close to 4 from the right, it seems the values of the function approach 1 from above. Because the values of the function approach two different numbers as x approaches 4 from the left and right,

$$\lim_{x\to 4} f(x) \ \textbf{does not exist}$$

■

! *When finding limits of functions, be sure to check the values of the function on* **both the left and right** *of the specified x-value. If the values of f(x) do not approach the same real number L from both the left and right of x = c, where c is a specified real number, then the limit of f(x) as x approaches c* **does not exist**.

Try It # 1:
Use the graph of f shown in **Figure 1.1.3** to estimate the following limits, if they exist.

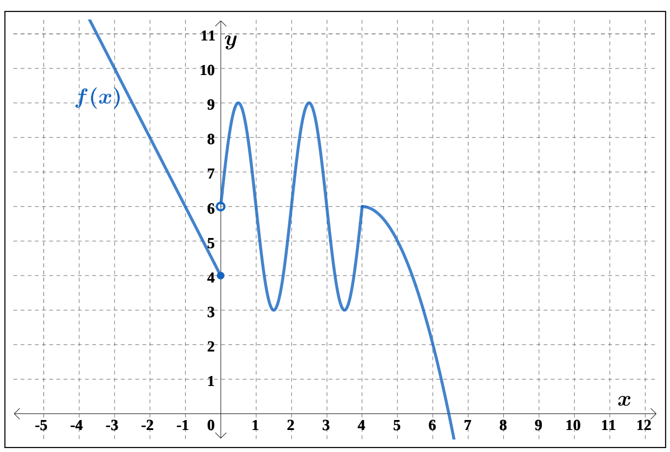


Figure 1.1.3: Graph of a piecewise-defined function f

a. $\lim_{x\to 0} f(x)$

b. $\lim_{x\to -2} f(x)$

c. $\lim_{x\to 4} f(x)$

One-sided Limits

Sometimes, the location, or point, at which we arrive depends on the direction from which we approach that location or point. For example, if we approach Niagara Falls from the upstream side, then we will be 182 feet higher and should be concerned about different dangers than if we approach it from the downstream side. We can transfer this notion to the concept of limits; the values of a function near a particular x-value may depend on the direction from which we approach that x-value. This leads us to the concept of one-sided limits in which we are interested in the values of a function as x approaches a specific value from either the left or the right.

Definition
If L, M, and c are real numbers, then:

The limit from the left, or left-hand limit, of $f(x)$ as x approaches c is equal to L if the values of $f(x)$ get as close to L as we want when x is very close to, but to the left of, c $(x < c)$:

$$\lim_{x\to c^-} f(x) = L.$$

The limit from the right, or right-hand limit, of $f(x)$ as x approaches c is equal to M if the values of $f(x)$ get as close to M as we want when x is very close to, but to the right of, c $(x > c)$:

$$\lim_{x\to c^+} f(x) = M.$$

■

■ **Example 2** Use the graph of f shown in **Figure 1.1.4** to estimate the following one-sided limits.

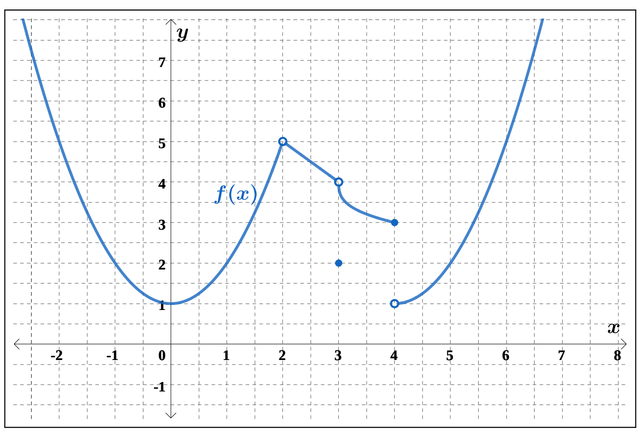


Figure 1.1.4: Graph of a piecewise-defined function f

a. $\lim_{x\to 4^+} f(x)$

b. $\lim_{x\to 4^-} f(x)$

c. $\lim_{x\to 3^+} f(x)$

d. $\lim_{x\to 3^-} f(x)$

Solution:

a. We want to find the limit of $f(x)$ as x approaches 4 from the right. When we look at x-values slightly greater than 4, but very close to 4, it appears the values of $f(x)$ are approaching $y = 1$ from above.

Thus, we predict

$$\lim_{x \to 4^+} f(x) = 1$$

b. Here, we want to find the limit of $f(x)$ as x approaches 4 from the left. When we look at x-values slightly less than 4, but very close to 4, it appears the values of $f(x)$ are approaching $y = 3$ from above. Therefore, we predict

$$\lim_{x \to 4^-} f(x) = 3$$

! *The limit in part **d** of the previous example did not exist because it was a two-sided limit (indicated by $x \to 4$), and the function had to approach the same y-value from both the left and right of $x = 4$. In this example, we are only looking at one-sided limits, and because the function is heading toward finite numbers on both the left and right of $x = 4$, both of the one-sided limits in parts **a** and **b** exist.*

c. We need to find the limit of $f(x)$ as x approaches 3 from the right. When we look at x-values slightly greater than 3, but very close to 3, we see the values of $f(x)$ appear to get close to $y = 4$. Thus, we predict

$$\lim_{x \to 3^+} f(x) = 4$$

d. Here, we need to find the limit of $f(x)$ as x approaches 3 from the left. When we look at x-values slightly less than 3, but very close to 3, we see the values of $f(x)$ appear to get close to $y = 4$. Thus, we predict

$$\lim_{x \to 3^-} f(x) = 4$$

▪

Notice in parts **c** and **d** of the previous example, the left- and right-hand limits both seemed to equal 4. This demonstrates an important relationship between one-sided and two-sided limits. This relationship brings us to an important theorem:

Theorem 1.1 For real numbers c and L, $\lim_{x \to c} f(x) = L$ if and only if $\lim_{x \to c-} f(x) = \lim_{x \to c^+} f(x) = L$.

This theorem has an important consequence that we also saw in the last example:

Theorem 1.2 For some real number c, if $\lim_{x \to c-} f(x) \neq \lim_{x \to c^+} f(x)$, then $\lim_{x \to c} f(x)$ does not exist.

Try It # 2:
Use the graph of f shown in **Figure 1.1.5** to estimate the following one-sided limits.

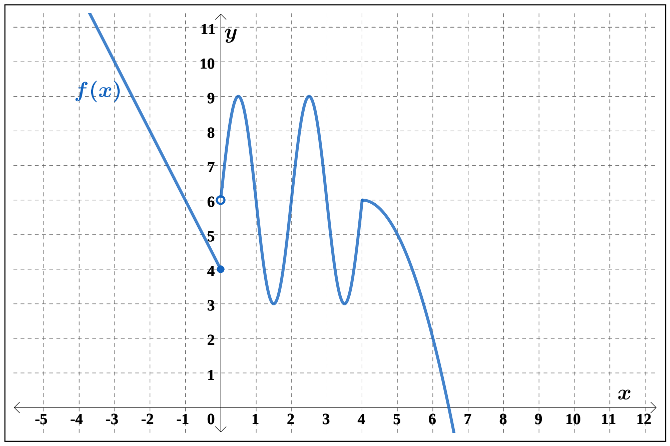


Figure 1.1.5: Graph of a piecewise-defined function f

a. $\lim_{x\to 0^+} f(x)$

b. $\lim_{x\to 0^-} f(x)$

c. $\lim_{x\to 4^+} f(x)$

Determining Limits Numerically

When determining limits graphically, we analyze the y-values of a function as x gets closer to a specified x-value. A graph is very helpful because we can visually inspect, or trace, the graph to help determine the limit, if it exists. But, we can also investigate limits of functions numerically when given their symbolic form, or rule. To estimate a limit numerically, we select x-values very close to the given x-value and then substitute the selected x-values into the rule for the function to analyze its y-values. Let's look at this approach in the form of an example.

▪ **Example 3** Estimate $\lim_{x\to 3} \dfrac{\sqrt{x+1}-2}{x-3}$ numerically, if it exists.

Solution:
First, notice this is a two-sided limit. We know this because in the notation $x \to 3$, there is no super-scripted plus or minus sign. Because it is a two-sided limit, we need to analyze the values of the function as x approaches, or gets very close to, $x = 3$ from both the left and right.

To approximate the limit from the left, we first need to select x-values slightly less than 3, but very close to 3, to substitute into the function. We will choose the four x-values 2.9, 2.99, 2.999, and 2.9999.

To approximate the limit from the right, we need to select x-values slightly greater than 3, but very close to 3, to substitute into the function. We will choose the four x-values 3.1, 3.01, 3.001, and 3.0001.

Next, we will substitute all of these selected x-values into the given function to obtain their corresponding function values (i.e., y-values). The result of each computation, rounded to eight decimal places, is displayed in **Table 1.1**.

$x \to 3^-$	$\dfrac{\sqrt{x+1}-2}{x-3}$	$x \to 3^+$	$\dfrac{\sqrt{x+1}-2}{x-3}$
2.9	0.25158234	3.1	0.24845673
2.99	0.25015645	3.01	0.24984395
2.999	0.25001563	3.001	0.24998438
2.9999	0.25000156	3.0001	0.24999844

Table 1.1: Function values representing $\lim_{x\to 3} \dfrac{\sqrt{x+1}-2}{x-3}$

By examining the y-values of the function as x approaches 3 from the left, we guess the values of the function are approaching 0.25 from above and estimate the limit from the left to be 0.25, or $\lim_{x\to 3^-} \dfrac{\sqrt{x+1}-2}{x-3} = 0.25$.

By examining the y-values of the function as x approaches 3 from the right, we guess the values of the function are approaching 0.25 from below and estimate the limit from the right to be 0.25, or $\lim_{x\to 3^+} \dfrac{\sqrt{x+1}-2}{x-3} = 0.25$.

Because it appears the left- and right-hand limits both approach 0.25, we conclude that the two-sided limit seems to approach 0.25. Using notation, we have $\lim_{x\to 3^-} \dfrac{\sqrt{x+1}-2}{x-3} = \lim_{x\to 3^+} \dfrac{\sqrt{x+1}-2}{x-3} = 0.25$, so we predict

$$\lim_{x\to 3} \frac{\sqrt{x+1}-2}{x-3} = 0.25$$

We can look at the graph of $f(x) = \dfrac{\sqrt{x+1}-2}{x-3}$, shown in **Figure 1.1.6**, to confirm that 0.25 is a reasonable estimate for $\lim_{x\to 3} \dfrac{\sqrt{x+1}-2}{x-3}$.

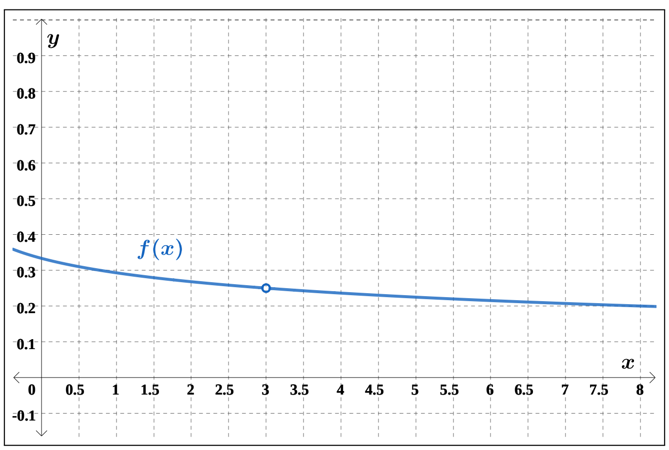


Figure 1.1.6: Graph of the function $f(x) = \dfrac{\sqrt{x+1}-2}{x-3}$

■

Estimating limits numerically is a concrete way to help us understand the behavior of the values of a function when approaching certain x-values. This technique is also helpful when checking the reasonableness of your answer after using other methods for finding limits.

Try It # 3:

Estimate $\lim_{x\to 1} \dfrac{\frac{1}{x}-1}{x-1}$ numerically, if it exists.

Example 4 Given the function f below, estimate the following limits numerically, if they exist.

$$f(x) = \begin{cases} x+1 & x < 2 \\ x^2-4 & 2 \le x < 4 \\ 5x-\pi & x \ge 4 \end{cases}$$

a. $\lim_{x\to 2^-} f(x)$

b. $\lim_{x\to 2^+} f(x)$

Solution:

a. To estimate $\lim_{x\to 2^-} f(x)$ numerically, we need to select x-values that are very close to 2, but slightly less than 2, and substitute them into the function to obtain the corresponding function values. Let's choose the x-values 1.9, 1.99, 1.999, and 1.9999.

Because f is a piecewise-defined function, we need to use the rule within $f(x)$ that is defined for $x < 2$. Thus, we will substitute the selected x-values into the rule $x+1$. The result of each computation is displayed in **Table 1.2**.

$x \to 2^-$	$f(x) = x+1$
1.9	2.9
1.99	2.99
1.999	2.999
1.9999	2.9999

Table 1.2: Values of $f(x) = x+1$ as x approaches 2 from the left

By examining the function values in the table, we predict the limit from the left is 3 and write

$$\lim_{x\to 2^-} f(x) = 3$$

b. To estimate $\lim_{x\to 2^+} f(x)$ numerically, we need to select x-values that are very close to 2, but slightly greater than 2, and substitute them into the function to obtain the corresponding function values. Let's choose the x-values 2.1, 2.01, 2.001, and 2.0001.

Because we are investigating function values that correspond to x-values such that $x > 2$, we need to use the rule within $f(x)$ that is defined for $2 \le x < 4$. Thus, we will substitute the selected x-values into the rule $x^2 - 4$. The result of each computation is displayed in **Table 1.3**.

$x \to 2^+$	$f(x) = x^2 - 4$
2.1	0.41
2.01	0.401
2.001	0.004001
2.0001	0.00040001

Table 1.3: Values of $f(x) = x^2 - 4$ as x approaches 2 from the right

By examining the function values in the table, we predict the limit from the right is 0 and write

$$\lim_{x \to 2^+} f(x) = 0$$

We can look at the graph of f, shown in **Figure 1.1.7**, to confirm the estimates of $\lim_{x \to 2^-} f(x) = 3$ and $\lim_{x \to 2^+} f(x) = 0$ found in parts **a** and **b** are reasonable.

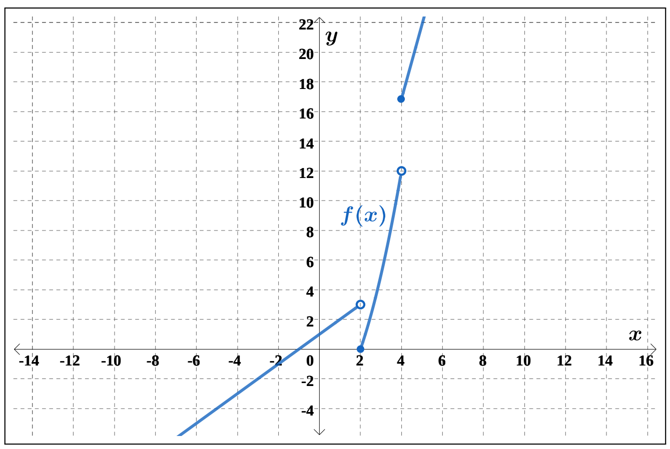


Figure 1.1.7: Graph of the piecewise-defined function f

N *In this example, we estimated one-sided limits of $f(x)$. Because the values of the function approach two different numbers as x approaches 2 from the left and right, we can conclude the two-sided limit, $\lim_{x \to 2} f(x)$, does not exist.*

■

Try It # 4:
Given the function f below, estimate the following limits numerically, if they exist.

$$f(x) = \begin{cases} x+1 & x < 2 \\ x^2 - 4 & 2 \leq x < 4 \\ 5x - \pi & x \geq 4 \end{cases}$$

a. $\lim_{x \to 4^-} f(x)$

b. $\lim_{x \to 4^+} f(x)$

Infinite Limits

So far, our discussion in this section has focused on estimating limits of functions whose values approach a real number L. However, in the field of mathematics, there are many functions whose values do not approach a real number L near certain x-values.

Sometimes, the values of a function may increase without bound and continue to get "more positive" near a specific x-value, while other times, the values of a function may decrease without bound and continue to get "more negative" near a specific x-value. This is the basic concept of **infinite limits**. A classic example of a function with this type of behavior is $f(x) = \frac{1}{x}$ at $x = 0$. Let's take an in-depth look at this classic example.

Suppose we want to find $\lim_{x \to 0} \frac{1}{x}$. Notice this is a two-sided limit, and therefore we need to find the one-sided limits $\lim_{x \to 0^-} \frac{1}{x}$ and $\lim_{x \to 0^+} \frac{1}{x}$. We will start by estimating the one-sided limits graphically and follow-up with estimating them numerically. The graph of f is shown in **Figure 1.1.8**.

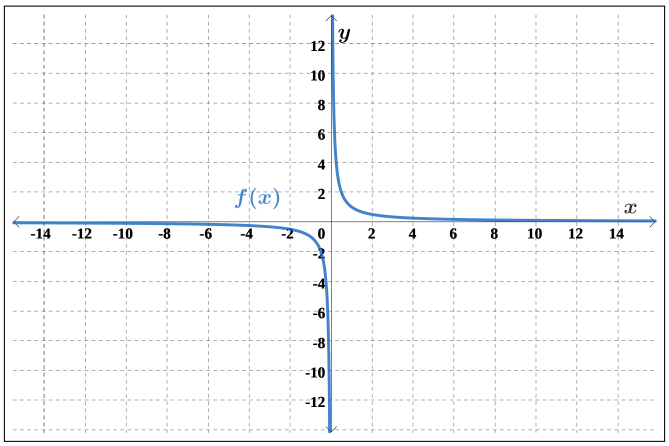


Figure 1.1.8: Graph of $f(x) = \frac{1}{x}$

To find $\lim_{x \to 0^-} f(x)$, we inspect the graph of $f(x) = \frac{1}{x}$ as the x-values approach 0 from the left. As we do this, we see the values of the function decrease without bound, meaning the y-values of the function continue to get "more and more negative" the closer x gets to 0 from the left. Because the x-values can always get closer to 0, without actually equaling 0, it seems the values of the function will become infinitely negative.

What do we mean by this? We will answer this question with another question. If we plot the number -0.001 on a number line, can you find another number between -0.001 and 0? The correct answer is yes, and an example is -0.0001. Now we ask again, can you find a number between -0.0001 and 0? Again, the correct answer is yes, and an example is -0.00001. Hopefully, you are beginning to understand that x can get very close to $x = 0$ from the left, but never actually equal 0. Thus, for the function $f(x) = \frac{1}{x}$, we are dividing 1 by x-values that are getting closer and closer to 0 from the left (i.e., the negative side). We conclude that the values of the function approach negative infinity as x approaches 0 from the left because they are getting "more and more negative".

This means that the limit from the left does not equal a finite number L, so the limit does not exist. Using notation, we write

$$\lim_{x \to 0^-} \frac{1}{x} \text{ does not exist}$$

We can, however, describe the *infinite behavior* of the values of the function near $x = 0$ from the left by using the notation

$$\lim_{x \to 0^-} \frac{1}{x} \to -\infty$$

Now, let's estimate this same left-hand limit numerically. The calculations for six x-values close to 0 on the left are displayed in **Table 1.4**.

$x \to 0^-$	$f(x) = \frac{1}{x}$
-0.1	-10
-0.01	-100
-0.001	-1000
-0.0001	$-10,000$
-0.00001	$-100,000$
-0.000001	$-1,000,000$

Table 1.4: Values of $f(x) = \frac{1}{x}$ as x approaches 0 from the left

When we examine the column for $f(x) = \frac{1}{x}$, we notice the absolute values of the negative numbers are getting larger and larger, and thus, by the nature of negative numbers, the values of the function are getting "more and more negative". In other words, the y-values of the function are decreasing without bound, or becoming infinitely negative.

This confirms our earlier prediction that the limit from the left does not exist. However, we can still use limit notation to describe the infinite behavior of the function as x approaches 0 from the left:

$$\lim_{x \to 0^-} \frac{1}{x} \to -\infty$$

Now, let's estimate $\lim_{x \to 0^+} f(x)$ graphically and then numerically. First, we inspect the graph of $f(x) = \frac{1}{x}$ (see **Figure 1.1.8**) as the x-values approach 0 from the right. As we do this, we see the values of the function increase without bound, meaning the y-values of the function continue to get "more and more positive" the closer x gets to 0 from the right. Because the x-values can always get closer to 0, without actually equaling 0, it seems the values of the function will become infinitely positive.

Thus, we conclude that the values of the function approach positive infinity as x approaches 0 from the right. Also, because the limit from the right does not equal a finite number L, the limit does not exist. Using notation, we write

$$\lim_{x \to 0^+} \frac{1}{x} \text{ does not exist}$$

We can, however, describe the infinite behavior of the values of the function near $x = 0$ from the right by using the notation

$$\lim_{x \to 0^+} \frac{1}{x} \to \infty$$

Now, let's estimate this same right-hand limit numerically. The calculations for six x-values close to 0 on the right are displayed in **Table 1.5**.

$x \to 0^+$	$f(x) = \frac{1}{x}$
0.1	10
0.01	100
0.001	1000
0.0001	10,000
0.00001	100,000
0.000001	1,000,000

Table 1.5: Values of $f(x) = \frac{1}{x}$ as x approaches 0 from the right

When we examine the column for $f(x) = \frac{1}{x}$, we notice the values of the function are increasing without bound because they are getting "more and more positive". This confirms our earlier prediction that the limit from the right does not exist, but the infinite behavior of the function can be denoted

$$\lim_{x\to 0^+} \frac{1}{x} \to \infty$$

In general, if both the left- and right-hand limits of $f(x)$ tend to positive infinity as x approaches a real number c, we can describe the infinite behavior of the two-sided limit by writing $\lim_{x\to c} f(x) \to \infty$. If both the left- and right-hand limits of $f(x)$ tend to negative infinity as x approaches a real number c, we can describe the infinite behavior of the two-sided limit by writing $\lim_{x\to c} f(x) \to -\infty$. If the left- and right-hand limits of $f(x)$ tend to opposite infinities as x approaches a real number c, we cannot describe the infinite behavior using two-sided limit notation. However, we should describe the infinite behavior using one-sided limit notation.

N *It is important to pay careful attention to the notation used for infinite limits. We like to be precise and, at times, overly pedantic; details matter! So, technically speaking, because the values of the function $f(x) = \frac{1}{x}$ do not approach a finite number L as x approaches 0, $\lim_{x\to 0} \frac{1}{x}$ does not exist. And, we use a right arrow (→) in the notation to describe the infinite behavior of the function. We cannot use an equal sign (=) in the notation because the limit is infinite (equal signs are only used when the limit equals a finite number). We write expressions such as $\lim_{x\to c^+} f(x) \to \infty$ or $\lim_{x\to c^+} f(x) \to -\infty$ to describe the infinite behavior of the values of a function when infinite limits occur.*

■ **Example 5** Estimate the following limits of the function $f(x) = \frac{1}{x-1}$ numerically and graphically. If a limit does not exist, state so and use limit notation to describe any infinite behavior.

a. $\lim_{x\to 1^-} \frac{1}{x-1}$

b. $\lim_{x\to 1^+} \frac{1}{x-1}$

c. $\lim_{x\to 1} \frac{1}{x-1}$

■

Solution:

a. We will start by estimating all three limits numerically, and then we will inspect the graph of f to confirm our estimations.

To estimate $\lim_{x\to1^-}\frac{1}{x-1}$ numerically, we must choose x-values very close to 1, but slightly less than 1, to substitute into the function. Let's choose the x-values 0.9, 0.99, 0.999, 0.9999, and 0.99999. The result of substituting these chosen x-values into $f(x)$ is displayed in **Table 1.6**.

$x\to1^-$	$f(x)=\frac{1}{x-1}$
0.9	-10
0.99	-100
0.999	-1000
0.9999	-10,000
0.99999	-100,000

Table 1.6: Values of $f(x)=\frac{1}{x-1}$ as x approaches 1 from the left

Examining the column for $f(x)=\frac{1}{x-1}$, we notice the values of the function are getting "more and more negative" and, thus, are decreasing without bound. So we conclude that the values of the function are approaching negative infinity as x approaches 1 from the left, and we write

$$\lim_{x\to1^-}\frac{1}{x-1} \text{ does not exist}$$

We can describe the infinite behavior by writing

$$\lim_{x\to1^-}\frac{1}{x-1}\to-\infty$$

b. To estimate $\lim_{x\to1^+}\frac{1}{x-1}$ numerically, we must choose x-values very close to 1, but slightly greater than 1, to substitute into the function. Let's choose the x-values 1.1, 1.01, 1.001, 1.0001, and 1.00001. The result of substituting these chosen x-values into $f(x)$ is displayed in **Table 1.7**.

$x\to1^+$	$f(x)=\frac{1}{x-1}$
1.1	10
1.01	100
1.001	1000
1.0001	10,000
1.00001	100,000

Table 1.7: Values of $f(x)=\frac{1}{x-1}$ as x approaches 1 from the right

Examining the column for $f(x)=\frac{1}{x-1}$, we notice the values of the function are getting "more and more positive" and, thus, are increasing without bound. So we conclude that the values of the function are approaching positive infinity as x approaches 1 from the right, and we write

$$\lim_{x\to1^+}\frac{1}{x-1} \text{ does not exist}$$

We can describe the infinite behavior by writing

$$\lim_{x\to 1^+} \frac{1}{x-1} \to \infty$$

c. We can use the results from part **a** or **b** of this problem to conclude

$$\lim_{x\to 1} \frac{1}{x-1} \text{ does not exist}$$

Because the one-sided limits tend to opposite infinities, we must use the following notation to describe the infinite behavior of the function:

$$\lim_{x\to 1^-} \frac{1}{x-1} \to -\infty \text{ and } \lim_{x\to 1^+} \frac{1}{x-1} \to \infty$$

Looking at the graph of $f(x) = \dfrac{1}{x-1}$ shown in **Figure 1.1.9**, we reach similar conclusions when estimating the limits above in parts **a** through **c** graphically.

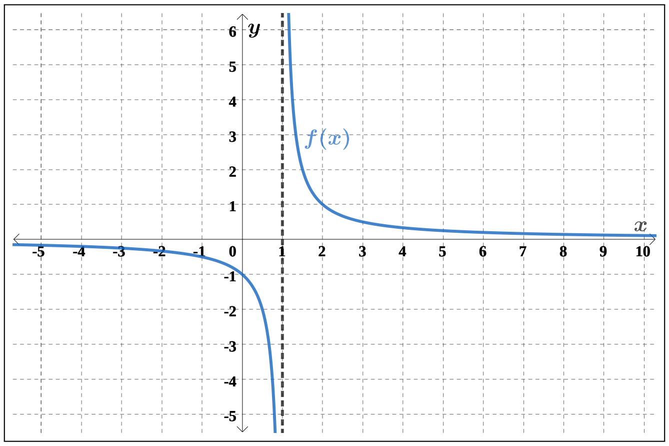


Figure 1.1.9: Graph of $f(x) = \dfrac{1}{x-1}$

Be careful! The reason $\lim_{x\to 1} \frac{1}{x-1}$ *does not exist is because the function values do not approach the same finite number from both the left and right of* $x = 1$. *It is not because the left- and right-hand limits tend to opposite infinities. Even if the left- and right-hand limits tended to the same infinity, the limit would still not exist.*

Try It # 5:

Estimate the following limits of the function $f(x) = \frac{1}{x^2}$ numerically and graphically. If a limit does not exist, state so and use limit notation to describe any infinite behavior.

a. $\lim_{x\to 0^-} \frac{1}{x^2}$

b. $\lim_{x\to 0^+} \frac{1}{x^2}$

c. $\lim_{x\to 0} \frac{1}{x^2}$

If any of the one- or two-sided limits of $f(x)$ as x approaches a real number c are infinite (positive or negative), then the values of $f(x)$ get close to the line $x = c$. We call this line a **vertical asymptote**.

Definition

Let f be a function. If any of the following conditions hold for some real number c, then the line $x = c$ is a **vertical asymptote** of the graph of f.

$$\lim_{x\to c^-} f(x) \to \infty \quad \text{or} \quad \lim_{x\to c^-} f(x) \to -\infty$$

$$\lim_{x\to c^+} f(x) \to \infty \quad \text{or} \quad \lim_{x\to c^+} f(x) \to -\infty$$

$$\lim_{x\to c} f(x) \to \infty \quad \text{or} \quad \lim_{x\to c} f(x) \to -\infty$$

▪

In the previous discussion and example, we estimated $\lim_{x\to 0^+} \frac{1}{x} \to \infty$ and $\lim_{x\to 1^-} \frac{1}{x-1} \to -\infty$, respectively. Thus, we can conclude $x = 0$ is a vertical asymptote of $f(x) = \frac{1}{x}$ and $x = 1$ is a vertical asymptote of $f(x) = \frac{1}{x-1}$. We will discuss how to find vertical asymptotes in **Section 1.3**.

Try It Answers

1. **a.** Does Not Exist
   b. 8
   c. 6

2. **a.** 6
   b. 4
   c. 6

3. -1

4. **a.** 12
   b. $20 - \pi$

5. **a.** Does Not Exist; $\lim_{x\to 0^-} \frac{1}{x^2} \to \infty$

   b. Does Not Exist; $\lim_{x\to 0^+} \frac{1}{x^2} \to \infty$

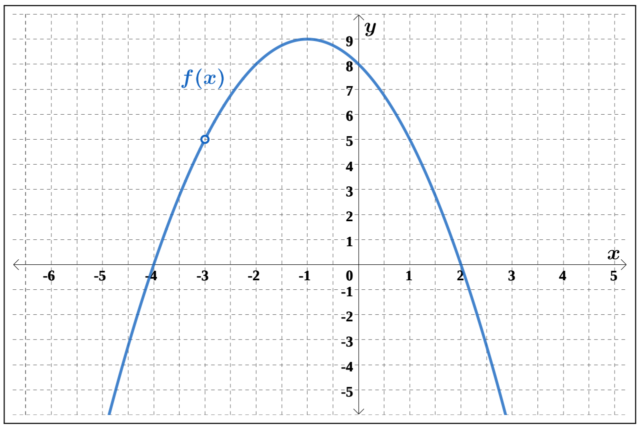
   c. Does Not Exist; $\lim_{x\to 0} \frac{1}{x^2} \to \infty$

EXERCISES

Basic Skills Practice

For Exercises 1 - 4, use the graph of f to estimate each limit. If a limit does not exist, state so and use limit notation to describe any infinite behavior.

1.



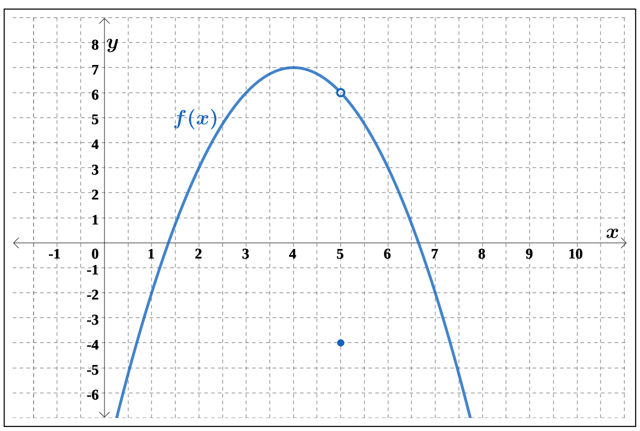
(a) $\lim_{x\to -3^-} f(x)$

(b) $\lim_{x\to -3^+} f(x)$

(c) $\lim_{x\to -3} f(x)$

(d) $f(-3)$

2.



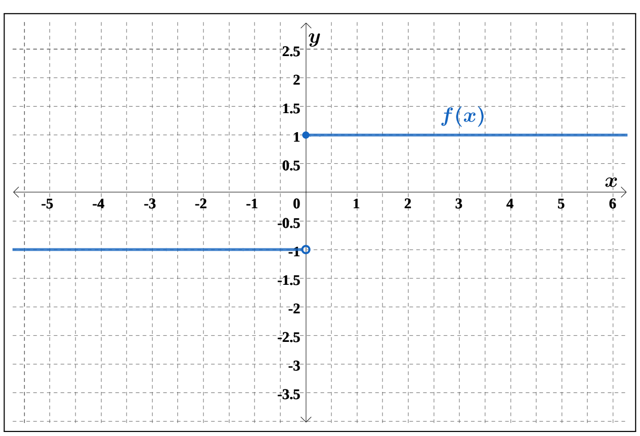
(a) $\lim_{x\to 5^-} f(x)$

(b) $\lim_{x\to 5^+} f(x)$

(c) $\lim_{x\to 5} f(x)$

(d) $f(5)$

3.

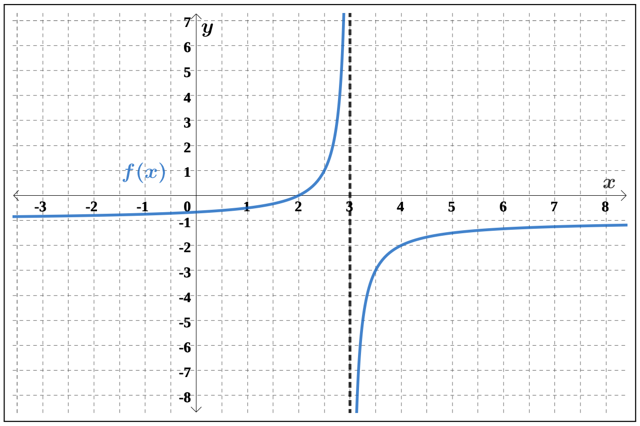


(a) $\lim_{x\to 0^-} f(x)$

(b) $\lim_{x\to 0^+} f(x)$

(c) $\lim_{x\to 0} f(x)$

(d) $f(0)$

4. 

(a) $\lim_{x \to 3^-} f(x)$

(b) $\lim_{x \to 3^+} f(x)$

(c) $\lim_{x \to 3} f(x)$

(d) $f(3)$

For Exercises 5 - 15, complete the table in part (a) and then use it to estimate the limit in part (b) numerically. Round your answers to four decimal places, if necessary.

5. (a)

x	$f(x) = 3x^2$
4.99	
4.999	
4.9999	

(b) $\lim_{x \to 5^-} 3x^2$

6. (a)

x	$f(x) = x^2 + 2x - 7$
-1.99	
-1.999	
-1.9999	

(b) $\lim_{x \to -2^+} \left(x^2 + 2x - 7\right)$

7. (a)

x	$f(x) = \dfrac{x-7}{x+2}$
-3.01	
-3.001	
-3.0001	

(b) $\lim_{x \to -3^-} \dfrac{x-7}{x+2}$

8. (a)

x	$f(x) = \dfrac{2x+3}{3-x}$
6.01	
6.001	
6.0001	

(b) $\lim_{x \to 6^+} \dfrac{2x+3}{3-x}$

9. (a)

x	$f(x) = x^2 + 2x - 4$	x	$f(x) = x^2 + 2x - 4$
0.99		1.01	
0.999		1.001	
0.9999		1.0001	

(b) $\lim_{x \to 1} \left(x^2 + 2x - 4\right)$

10. (a)

x	$f(x) = 2x^2 - 3x + 7$	x	$f(x) = 2x^2 - 3x + 7$
−2.01		−1.99	
−2.001		−1.999	
−2.0001		−1.9999	

(b) $\lim_{x \to -2} \left(2x^2 - 3x + 7\right)$

11. (a)

x	$f(x) = \dfrac{2x+1}{x+3}$	x	$f(x) = \dfrac{2x+1}{x+3}$
2.99		3.01	
2.999		3.001	
2.9999		3.0001	

(b) $\lim_{x \to 3} \dfrac{2x+1}{x+3}$

12. (a)

x	$f(x) = \dfrac{x+2}{x-4}$	x	$f(x) = \dfrac{x+2}{x-4}$
−4.01		−3.99	
−4.001		−3.999	
−4.0001		−3.9999	

(b) $\lim_{x \to -4} \dfrac{x+2}{x-4}$

13. (a)

x	$f(x)=\frac{\sqrt{5-x}}{x}$	x	$f(x)=\frac{\sqrt{5-x}}{x}$
0.99		1.01	
0.999		1.001	
0.9999		1.0001	

(b) $\lim_{x\to 1} \frac{\sqrt{5-x}}{x}$

14. (a)

x	$f(x)=e^{x-3}$	x	$f(x)=e^{x-3}$
2.99		3.01	
2.999		3.001	
2.9999		3.0001	

(b) $\lim_{x\to 3} e^{x-3}$

15. (a)

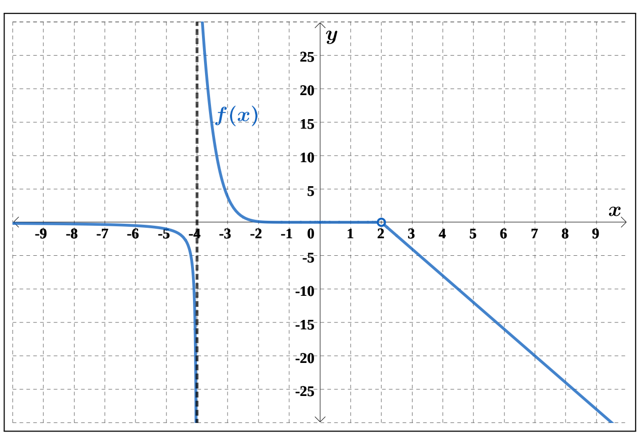
x	$f(x)=\ln(x+6)$	x	$f(x)=\ln(x+6)$
-5.01		-4.99	
-5.001		-4.999	
-5.0001		-4.9999	

(b) $\lim_{x\to -5} \ln(x+6)$

Intermediate Skills Practice

For Exercises 16 - 19, use the graph of f to estimate each limit. If a limit does not exist, state so and use limit notation to describe any infinite behavior.

16.



(a) $\lim_{x\to -4^-} f(x)$

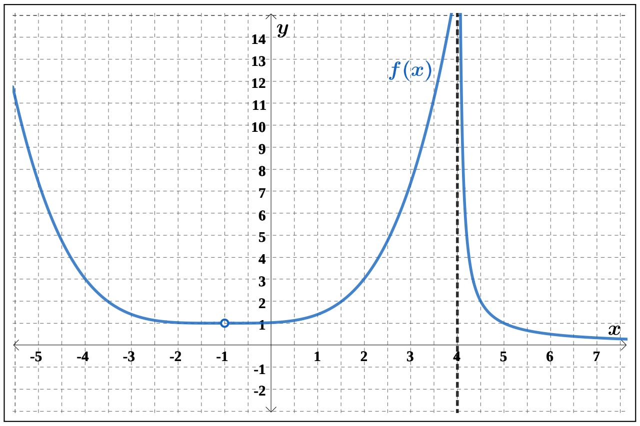
(b) $\lim_{x\to -4^+} f(x)$

(c) $\lim_{x\to -4} f(x)$

(d) $\lim_{x\to 2^-} f(x)$

(e) $\lim_{x\to 2^+} f(x)$

(f) $\lim_{x\to 2} f(x)$

17. 

(a) $\lim_{x\to -1^-} f(x)$

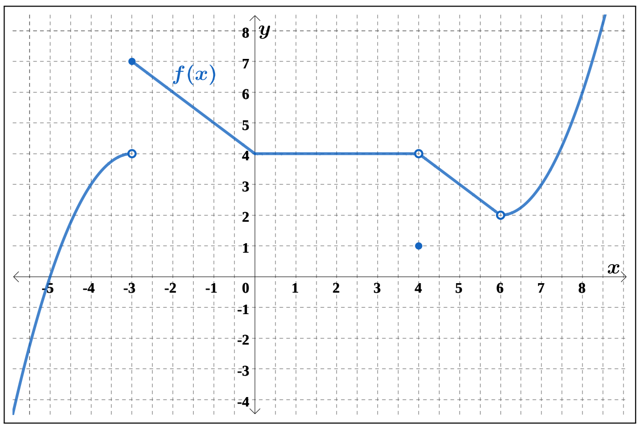
(b) $\lim_{x\to -1^+} f(x)$

(c) $\lim_{x\to -1} f(x)$

(d) $\lim_{x\to 4^-} f(x)$

(e) $\lim_{x\to 4^+} f(x)$

(f) $\lim_{x\to 4} f(x)$

18. 

(a) $\lim_{x\to -3} f(x)$

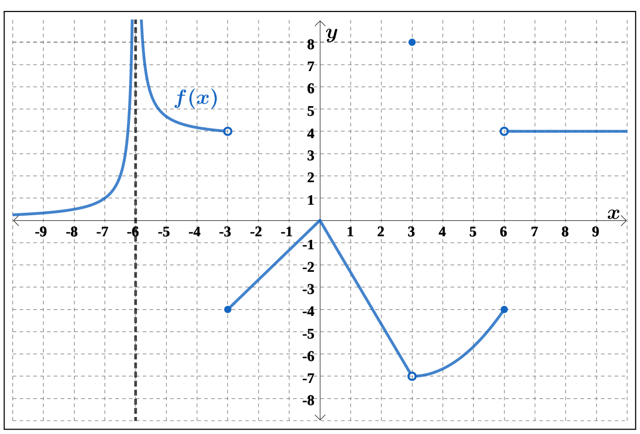
(b) $\lim_{x\to 0^-} f(x)$

(c) $\lim_{x\to 6} f(x)$

(d) $\lim_{x\to 4^+} f(x)$

(e) $\lim_{x\to -3^-} f(x)$

(f) $\lim_{x\to 6^+} f(x)$

19. 

(a) $\lim_{x\to -6} f(x)$

(b) $\lim_{x\to -3^+} f(x)$

(c) $\lim_{x\to 3^-} f(x)$

(d) $\lim_{x\to 6^+} f(x)$

(e) $\lim_{x\to 0} f(x)$

(f) $\lim_{x\to -3^-} f(x)$

For Exercises 20 - 26, complete the table in part (a) and then use it to estimate the limit(s) in part (b) numerically. If the limit does not exist, state so and use limit notation to describe any infinite behavior. Round your answers to four decimal places, if necessary.

20. (a)

x	$f(x)=\dfrac{x^3-x^2-10x-8}{x-4}$	x	$f(x)=\dfrac{x^3-x^2-10x-8}{x-4}$
3.99		4.01	
3.999		4.001	
3.9999		4.0001	

(b) $\displaystyle\lim_{x\to 4}\frac{x^3-x^2-10x-8}{x-4}$

21. (a)

x	$f(x)=\dfrac{x^3-5x^2+2x+8}{x+1}$	x	$f(x)=\dfrac{x^3-5x^2+2x+8}{x+1}$
−1.01		−0.99	
−1.001		−0.999	
−1.0001		−0.9999	

(b) $\displaystyle\lim_{x\to -1}\frac{x^3-5x^2+2x+8}{x+1}$

22. (a)

x	$f(x)=\dfrac{\sqrt{x}-1}{x-1}$	x	$f(x)=\dfrac{\sqrt{x}-1}{x-1}$
0.99		1.01	
0.999		1.001	
0.9999		1.0001	

(b) $\displaystyle\lim_{x\to 1}\frac{\sqrt{x}-1}{x-1}$

23. (a)

t	$f(t)=\dfrac{t+2}{t+7}$	t	$f(t)=\dfrac{t+2}{t+7}$
−7.01		−6.99	
−7.001		−6.999	
−7.0001		−6.9999	

(b) i. $\displaystyle\lim_{t\to -7^-}\frac{t+2}{t+7}$

ii. $\displaystyle\lim_{t\to -7^+}\frac{t+2}{t+7}$

iii. $\displaystyle\lim_{t\to -7}\frac{t+2}{t+7}$

24. (a)

t	$f(t)=\dfrac{2t-9}{(t-3)^2}$	t	$f(t)=\dfrac{2t-9}{(t-3)^2}$
2.99		3.01	
2.999		3.001	
2.9999		3.0001	

(b) i. $\lim\limits_{t\to 3^-} \dfrac{2t-9}{(t-3)^2}$

ii. $\lim\limits_{t\to 3^+} \dfrac{2t-9}{(t-3)^2}$

iii. $\lim\limits_{t\to 3} \dfrac{2t-9}{(t-3)^2}$

25. (a)

x	$f(x)=\dfrac{e^{-x}}{x+2}$	x	$f(x)=\dfrac{e^{-x}}{x+2}$
−2.01		−1.99	
−2.001		−1.999	
−2.0001		−1.9999	

(b) i. $\lim\limits_{x\to -2^-} \dfrac{e^{-x}}{x+2}$

ii. $\lim\limits_{x\to -2^+} \dfrac{e^{-x}}{x+2}$

iii. $\lim\limits_{x\to -2} \dfrac{e^{-x}}{x+2}$

26. (a)

x	$f(x)=\dfrac{\ln(x+4)}{-x}$	x	$f(x)=\dfrac{\ln(x+4)}{-x}$
−0.01		0.01	
−0.001		0.001	
−0.0001		0.0001	

(b) i. $\lim\limits_{x\to 0^-} \dfrac{\ln(x+4)}{-x}$

ii. $\lim\limits_{x\to 0^+} \dfrac{\ln(x+4)}{-x}$

iii. $\lim\limits_{x\to 0} \dfrac{\ln(x+4)}{-x}$

Mastery Practice

For Exercises 27 and 28, sketch the graph of a function f satisfying the given conditions.

27.
- $\lim_{x\to 2} f(x) = 1$
- $\lim_{x\to 4^-} f(x) = 3$
- $\lim_{x\to 4^+} f(x) = 6$
- $f(4)$ is undefined

28.
- $\lim_{x\to 3} f(x) = 5$
- $f(3) = 4$
- $\lim_{x\to -2^+} f(x) \to \infty$
- $\lim_{x\to -2^-} f(x) \to -\infty$

For Exercises 29 - 37, (a) show at least three x-values approaching the given x-value and their corresponding function values in a table. (b) Use the table to estimate the limit numerically. If the limit does not exist, state so and use limit notation to describe any infinite behavior. Round your answers to four decimal places, if necessary.

29. $\lim_{x\to \frac{5}{3}^+} \frac{2x(3x-5)}{x^2-x-1}$

30. $\lim_{t\to 0^+} \frac{|t-5|}{t}$

31. $\lim_{x\to -2} \frac{3x^2-9x+10}{10-7x}$

32. $\lim_{x\to -2^-} \frac{\sqrt{x+8}-2}{x+2}$

33. $\lim_{t\to 0} \frac{t^3-8t^2+16t}{2t^4-8t^3}$

34. $\lim_{x\to 2} \frac{x^2-4}{\ln(3-x)}$

35. $\lim_{x\to -2} \frac{5x^2-9}{x+2}$

36. $\lim_{x\to 3^-} \frac{\sqrt{4-x}}{x^2+1}$

37. $\lim_{x\to 0} \frac{(x+1)^2-e^x}{x}$

Communication Practice

38. Explain the meaning of $\lim_{x\to c} f(x) = L$.

39. Explain the difference between the value of a function and the value of a limit.

40. What is the difference between finding a limit graphically and numerically?

41. What are the two ways in which a limit may fail to exist?

42. If $\lim_{x\to 1^-} f(x) = 5$, does $\lim_{x\to 1} f(x) = 5$? Explain.

43. If $\lim_{x\to 1^-} f(x) = 5$, does $\lim_{x\to 1^+} f(x) = 5$? Explain.

44. If $\lim_{x\to 1} f(x) = 5$, does $\lim_{x\to 1^-} f(x) = 5$? Explain.

45. If $\lim_{x\to c^-} f(x) = L$ and $\lim_{x\to c^+} f(x) = M$, explain the behavior of $f(x)$ at $x = c$ if $L \neq M$.

46. If $\lim_{x\to c^-} f(x) = L$ and $\lim_{x\to c^+} f(x) = M$, explain the behavior of $f(x)$ at $x = c$ if $L = M$.

47. If $\lim_{x\to 2^-} f(x) \to -\infty$ and $\lim_{x\to 2^+} f(x) \to -\infty$, does $\lim_{x\to 2} f(x)$ exist? Explain.

1.2 Limits: Algebraically

In the previous section, we approximated limits of functions graphically and numerically. There are times, however, when limits of functions are difficult to find using these methods. Thus, other techniques to determine limits, and whether they exist, are necessary. More importantly, when we use graphical and numerical approaches, we are only able to estimate the value of a limit, if it exists. The values of the function $f(x) = 0.9x$, for example, may appear to be the same as $g(x) = x$, but in fact they are not. The difference in the values of $f(x) = 0.9x$ and $g(x) = x$ is one-tenth, and this may be very difficult to see from a graphical or numerical standpoint.

In this section, we will use an algebraic approach to find exact limits of functions, if they exist, when we are given the symbolic form, or rule, of a function. We will apply Properties of Limits to a variety of functions to help master the skill of finding limits exactly, but we will not prove any of the Properties because it is beyond the scope of this textbook.

Learning Objectives:
In this section, you will learn how to find limits algebraically and determine the corresponding behavior of the function. Upon completion you will be able to:

- Calculate a limit algebraically using the Properties of Limits.
- Calculate a limit algebraically using the Direct Substitution Property.
- Calculate both one-sided and two-sided limits algebraically.
- Calculate a limit algebraically involving a piecewise-defined function.
- Calculate a limit algebraically that results in infinite behavior.
- Calculate an indeterminate limit by algebraically manipulating the relevant function.
- Deduce and interpret the behavior of functions using limits.
- Describe the infinite behavior near vertical asymptotes using limit notation.

Properties of Limits

At this point, you have had some experience finding limits of functions and probably have a good feel for the concept of a limit. At least we hope you do! Now, we will apply the knowledge you acquired when finding limits graphically and numerically to help you understand some important relationships between limits and the algebra of functions.

To motivate this topic, let's start with the function $f(x) = 3$. The graph of this function is a horizontal line crossing the y-axis at $y = 3$ (see **Figure 1.2.1**).

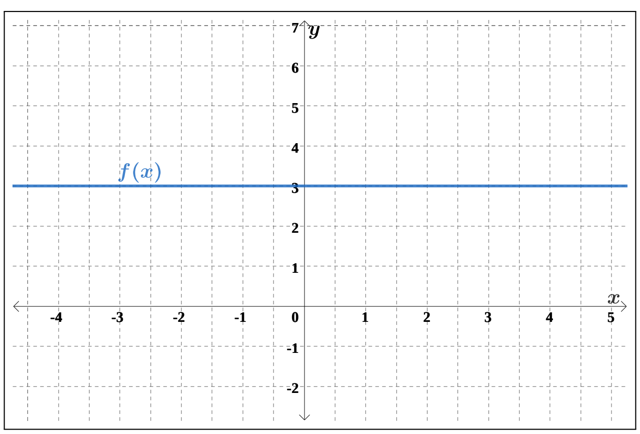


Figure 1.2.1: Graph of the constant function $f(x) = 3$

Suppose we want to find $\lim_{x\to -2} f(x)$. Using the graph, we see the values of the function approach $y = 3$ from both the left and right of $x = -2$ and conclude $\lim_{x\to -2} 3 = 3$. What about $\lim_{x\to 5} f(x)$? Again, using the graph we see the values of the function approach $y = 3$ from both the left and right of $x = 5$ and conclude $\lim_{x\to 5} 3 = 3$.

In fact, because the values of the function are constantly 3, we predict that the limit of $f(x) = 3$ as x approaches any given c, where c is a real number, is 3. Hence, we can write $\lim_{x\to c} 3 = 3$. This idea can be generalized further for any constant function, and the result is shown in the list of Properties of Limits (see Property #1).

There are many other examples that demonstrate the relationship between the algebra of functions and finding limits, but for the sake of time, we will move on and reveal the Properties of Limits. We strongly encourage you to read through the Properties carefully and try to make sense of each one.

Properties of Limits

Let f and g be two functions in which $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = M$, where L, M, and c are real numbers. Then,

1. $\lim_{x\to c} k = k$ for any constant k
2. $\lim_{x\to c} x = c$
3. $\lim_{x\to c} [f(x)+g(x)] = \lim_{x\to c} f(x) + \lim_{x\to c} g(x) = L+M$
4. $\lim_{x\to c} [f(x)-g(x)] = \lim_{x\to c} f(x) - \lim_{x\to c} g(x) = L-M$
5. $\lim_{x\to c} [kf(x)] = k\lim_{x\to c} f(x) = kL$ for any constant k
6. $\lim_{x\to c} [f(x)g(x)] = \left[\lim_{x\to c} f(x)\right]\left[\lim_{x\to c} g(x)\right] = LM$
7. $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)} = \dfrac{L}{M}$ if $M \neq 0$
8. $\lim_{x\to c} [f(x)]^n = \left[\lim_{x\to c} f(x)\right]^n$ where n is a positive integer
9. $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to c} f(x)} = \sqrt[n]{L}$ $L > 0$ when n is even

N *The Properties of Limits also hold for one-sided limits. Thus, for any of the Properties, you can replace $x \to c$ with $x \to c^-$ or $x \to c^+$.*

The Properties of Limits are valuable because they allow us flexibility when finding limits. The Properties allow us to obtain the same result if we perform the algebra first and then find the limit, or if we find the limits first and then perform the algebra.

For example, Property #3 states that the limit of a sum of two functions is equal to the sum of the limits of the two functions. Property #6 states that the limit of a product of two functions is the same as the product of the limits of the two functions. To start exploring the Properties of Limits, we will rely on our knowledge of finding limits graphically.

■ **Example 1** Given the graphs of f and g shown in **Figure 1.2.2**, find the following limits.

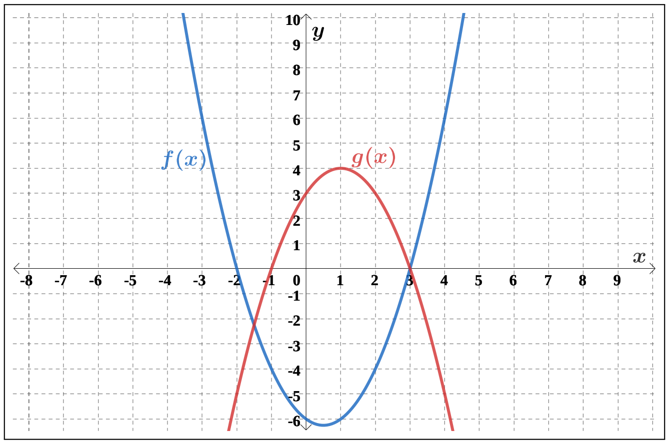


Figure 1.2.2: Graph of two functions f and g

a. $\lim_{x\to 1}[f(x)+g(x)]$

b. $\lim_{x\to -1}[f(x)g(x)]$

c. $\lim_{x\to 0}\frac{2f(x)}{g(x)}$

Solution:

a. Notice the graphs of f and g do not have any "jumps", so the limits of both functions exist everywhere. Hence, we can use the Properties of Limits to find the indicated limit. Applying Property #3, we get

$$\lim_{x\to 1}[f(x)+g(x)] = \lim_{x\to 1} f(x) + \lim_{x\to 1} g(x)$$

Looking at the graphs of f and g to identify the y-values of each function as x approaches 1 from both the left and right, we see

$$\lim_{x\to 1} f(x) = -6 \text{ and } \lim_{x\to 1} g(x) = 4$$

Thus,

$$\begin{aligned}\lim_{x\to 1}[f(x)+g(x)] &= \lim_{x\to 1} f(x) + \lim_{x\to 1} g(x)\\ &= -6+4\\ &= -2\end{aligned}$$

b. Again, the graphs of f and g do not have any "jumps", so the limits of both functions exist everywhere. Hence, we can use the Properties of Limits to find the indicated limit. Applying Property #6, we get

$$\lim_{x\to -1}[f(x)g(x)] = \left[\lim_{x\to -1} f(x)\right]\left[\lim_{x\to -1} g(x)\right]$$

Looking at the graphs of f and g to identify the y-values of each function as x approaches -1 from both the left and right, we see

$$\lim_{x\to -1} f(x) = -4 \text{ and } \lim_{x\to -1} g(x) = 0$$

Thus,

$$\begin{aligned}\lim_{x\to -1}\left[f(x)g(x)\right] &= \left[\lim_{x\to -1} f(x)\right]\left[\lim_{x\to -1} g(x)\right]\\ &= (-4)(0)\\ &= 0\end{aligned}$$

c. Recall that we must find $\lim_{x\to 0}\frac{2f(x)}{g(x)}$. Notice this limit involves a function that is a quotient, or ratio, of two functions. So we need to be careful! To use Property #7 of the Properties of Limits, the limit of the function in the numerator must exist and the limit of the function in the denominator must exist and be nonzero.

Let's start by checking the limit of the numerator. Because we know the limit of $f(x)$ exists everywhere, we can apply Property #5 to find the limit of the function in the numerator:

$$\lim_{x\to 0}\left[2f(x)\right] = 2\lim_{x\to 0} f(x)$$

Looking at the graph of f to identify the y-value the function is heading toward as x approaches 0 from both the left and right, we see

$$\lim_{x\to 0} f(x) = -6$$

Thus,

$$\begin{aligned}\lim_{x\to 0}\left[2f(x)\right] &= 2\lim_{x\to 0} f(x)\\ &= 2(-6)\\ &= -12\end{aligned}$$

Because $\lim_{x\to 0}\left[2f(x)\right] = -12$, the limit of the numerator exists.

Now, let's find the limit of the denominator, $\lim_{x\to 0} g(x)$, which we know exists. Looking at the graph of g to identify the y-value the function is heading toward as x approaches 0 from both the left and right, we see $\lim_{x\to 0} g(x) = 3$. Thus, the limit of the denominator exists and is nonzero. Now, we know that we can find the indicated limit using the Properties of Limits. Applying Property #7 gives

$$\begin{aligned}\lim_{x\to 0}\frac{2f(x)}{g(x)} &= \frac{\lim_{x\to 0}\left[2f(x)\right]}{\lim_{x\to 0} g(x)}\\ &= \frac{2\lim_{x\to 0} f(x)}{\lim_{x\to 0} g(x)}\\ &= \frac{-12}{3}\\ &= -4\end{aligned}$$

■

Try It # 1:
Given the graphs of f and g in shown **Figure 1.2.3**, find the following limits.

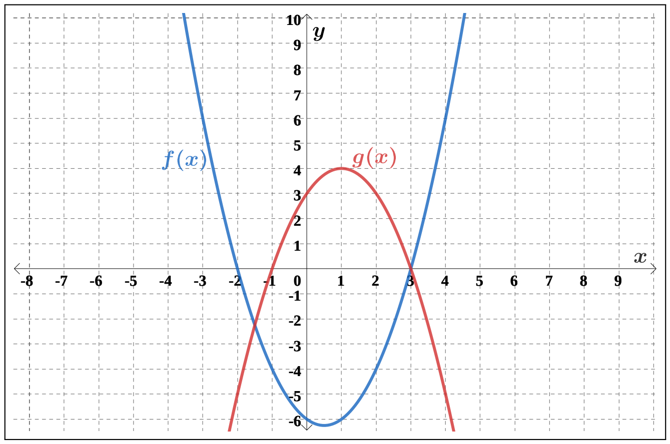


Figure 1.2.3: Graph of two functions f and g

a. $\lim_{x\to 4}[f(x)g(x)]$

b. $\lim_{x\to 4}\dfrac{f(x)}{g(x)}$

c. $\lim_{x\to -2}[g(x)]^3$

d. $\lim_{x\to -3}\sqrt{f(x)-2}$

Direct Substitution

In the previous example, we used graphs of functions, along with the Properties of Limits, to find limits of functions. However, sometimes we may need to find limits of functions given the symbolic form, or rule, of the function. The good news is that finding limits of functions given their rule is fairly straightforward, particularly when the functions are polynomials.

Suppose we want to find $\lim_{x\to 2}\left(2x^2+3\right)$. The function we are finding the limit of is a polynomial, so its graph will not have any "jumps". Therefore, the limit of the function exists. By the same reasoning, the limit of each term in the function exists, so we can use the Properties of Limits to find the indicated limit.

Applying the Properties of Limits at each step we get

$$\begin{aligned}
\lim_{x\to 2}\left(2x^2+3\right) &= \lim_{x\to 2}\left(2x^2\right)+\lim_{x\to 2}3 && \text{Property \#3}\\
&= 2\lim_{x\to 2}x^2+\lim_{x\to 2}3 && \text{Property \#5}\\
&= 2\left(\lim_{x\to 2}x\right)^2+\lim_{x\to 2}3 && \text{Property \#8}\\
&= 2(2)^2+3 && \text{Properties \#2 and \#1}\\
&= 2(4)+3\\
&= 11
\end{aligned}$$

Notice that if $f(x) = 2x^2 + 3$ and we substitute $x = 2$ into $f(x)$, we get $2(2)^2 + 3 = 11$. This is the same value we obtained when calculating $\lim_{x\to 2}\left(2x^2+3\right)$ using the Properties of Limits! In other words,

$$\lim_{x\to 2}\left(2x^2+3\right) = 11 = f(2)$$

This is not a coincidence, and we will soon learn why.

Now, let's see what happens when we attempt to apply the Properties of Limits to find the limit of a ratio of two polynomials at a specific x-value. Suppose we want to find $\lim_{x\to -3}\frac{x^2}{x+6}$. Because we are finding the limit of a ratio of two functions, we need to be careful. We have to make sure the limits of the functions in the numerator and denominator exist and that the limit of the function in the denominator is nonzero before we can apply Property #7.

As stated previously, we know the limits of the functions in the numerator and denominator exist because they are polynomials and their graphs do not have any "jumps". Likewise, the limit of each term in both functions exists. Therefore, we can apply the Properties of Limits to find the limits of the functions in the numerator and denominator.

Let's start with the numerator:

$$\begin{aligned}\lim_{x\to -3} x^2 &= \left(\lim_{x\to -3} x\right)^2 && \text{Property \#8}\\ &= (-3)^2 && \text{Property \#2}\\ &= 9\end{aligned}$$

Now, we will find the limit of the function in the denominator:

$$\begin{aligned}\lim_{x\to -3}(x+6) &= \lim_{x\to -3} x + \lim_{x\to -3} 6 && \text{Property \#3}\\ &= -3+6 && \text{Properties \#2 and \#1}\\ &= 3\end{aligned}$$

Because $\lim_{x\to -3} x^2$ and $\lim_{x\to -3}(x+6)$ exist and $\lim_{x\to -3}(x+6)$ is nonzero, we can use the Properties of Limits to conclude

$$\begin{aligned}\lim_{x\to -3}\frac{x^2}{x+6} &= \frac{\lim_{x\to -3} x^2}{\lim_{x\to -3}(x+6)} && \text{Property \#7}\\ &= \frac{9}{3}\\ &= 3\end{aligned}$$

Notice that if $h(x) = \frac{x^2}{x+6}$, then $h(-3) = 3$. As in the previous example, it is no coincidence that $h(-3) = 3$ and $\lim_{x\to -3}\frac{x^2}{x+6} = 3$. This leads us to the next very useful property:

Direct Substitution Property

If P and Q are polynomials and c is any real number, then

$$\lim_{x\to c} P(x) = P(c) \qquad \text{and} \qquad \lim_{x\to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}$$

as long as $Q(c)$ is nonzero.

The underlying principles for the Direct Substitution Property are the Properties of Limits. Thus, when we find limits using the Direct Substitution Property, we are finding the *exact* value of a limit, not an estimate of the limit like we did when executing graphical and numerical methods. This is exciting, isn't it?

You have now been exposed to the tools required to find the exact value of a limit of a function, if it exists. Now, let's use these tools to master the skill of finding limits exactly!

Example 2 Use the Properties of Limits to find the following limits algebraically, and verify your answer using the Direct Substitution Property.

a. $\lim_{x\to 2}\left(5x^3 - x^2 + 3\right)$

b. $\lim_{x\to -1} \frac{x^4 - 7x}{x^2 + 3x}$

Solution:

a. The function we are finding the limit of is a polynomial, so its graph will not have any "jumps". Therefore, the limit of the function exists. Likewise, the limit of each term in the function exists, so we can use the Properties of Limits to find the indicated limit.

Applying the Properties of Limits we get

$$\begin{aligned}
\lim_{x\to 2}\left(5x^3 - x^2 + 3\right) &= \lim_{x\to 2}\left(5x^3\right) - \lim_{x\to 2} x^2 + \lim_{x\to 2} 3 && \text{Properties \#4 and \#3}\\
&= 5\lim_{x\to 2} x^3 - \lim_{x\to 2} x^2 + \lim_{x\to 2} 3 && \text{Property \#5}\\
&= 5\left(\lim_{x\to 2} x\right)^3 - \left(\lim_{x\to 2} x\right)^2 + \lim_{x\to 2} 3 && \text{Property \#8}\\
&= 5(2)^3 - 2^2 + 3 && \text{Properties \#2 and \#1}\\
&= 39
\end{aligned}$$

We can verify this answer using the Direct Substitution Property because the function we are taking the limit of is a polynomial. Substituting $x = 2$ into the rule $5x^3 - x^2 + 3$ gives

$$\begin{aligned}
\lim_{x\to 2}\left(5x^3 - x^2 + 3\right) &= 5(2)^3 - (2)^2 + 3\\
&= 39
\end{aligned}$$

b. Because we are finding the limit of a ratio of two function, we first need to determine if $\lim_{x\to -1}\left(x^4 - 7x\right)$ and $\lim_{x\to -1}\left(x^2 + 3x\right)$ exist and then make sure $\lim_{x\to -1}\left(x^2 + 3x\right) \neq 0$.

As stated previously, we know both of these limits exist because the functions are polynomials and their graphs do not have any "jumps". Likewise, the limit of each term in the function exists. Therefore, we can apply the Properties of Limits to find both limits.

Let's start with the numerator:

$$\begin{aligned}\lim_{x\to-1}\left(x^4-7x\right) &= \lim_{x\to-1} x^4 - \lim_{x\to-1}(7x) && \text{Property \#4}\\ &= \left(\lim_{x\to-1} x\right)^4 - 7\lim_{x\to-1} x && \text{Property \#8 and \#5}\\ &= (-1)^4-7(-1) && \text{Properties \#2}\\ &= 1+7\\ &= 8\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to-1}\left(x^2+3x\right) &= \lim_{x\to-1} x^2 + \lim_{x\to-1}(3x) && \text{Property \#3}\\ &= \left(\lim_{x\to-1} x\right)^2 + 3\lim_{x\to-1} x && \text{Properties \#8 and \#5}\\ &= (-1)^2+3(-1) && \text{Property \#2}\\ &= 1-3\\ &= -2\end{aligned}$$

Because $\lim_{x\to-1}\left(x^4-7x\right)=8$ and $\lim_{x\to-1}\left(x^2+3x\right)=-2$, both limits exist and the limit of the function in the denominator is nonzero. Thus, we can use Properties of Limits to conclude

$$\begin{aligned}\lim_{x\to-1}\frac{x^4-7x}{x^2+3x} &= \frac{\lim_{x\to-1}\left(x^4-7x\right)}{\lim_{x\to-1}\left(x^2+3x\right)} && \text{Property \#7}\\ &= \frac{8}{-2}\\ &= -4\end{aligned}$$

We will verify this answer by attempting to apply the Direct Substitution Property. We say "attempting" because we can only use the Direct Substitution Property to find the limit of a rational function if the value of the function in the denominator is nonzero.

To find the value of the function in the denominator of this rational function at $x=-1$, we substitute $x=-1$ into the rule x^2+3x. Doing so gives $(-1)^2+3(-1)=-2$.

Thus, the value of the function in the denominator when $x=-1$ is nonzero, so we can apply the Direct Substitution Property:

$$\lim_{x\to-1}\frac{x^4-7x}{x^2+3x}=\frac{(-1)^4-7(-1)}{(-1)^2+3(-1)}=\frac{1+7}{1-3}=\frac{8}{-2}=-4$$

Hence, we have verified $\lim_{x\to-1}\frac{x^4-7x}{x^2+3x}=-4$.

▪

Try It # 2:
Use the Properties of Limits to find the following limits algebraically, and verify your answer using the Direct Substitution Property.

a. $\lim_{x \to 7} \left(5x - x^2\right)$

b. $\lim_{x \to -4} \dfrac{x^3 - 2}{2x^2}$

After all the work we have done so far, it should not be hard to convince you that the Direct Substitution Property saves us a lot of time and effort. All the details and algebraic steps necessary to find limits using the Properties of Limits can be time consuming, so moving forward, we will use the Direct Substitution Property when possible. Remember, the underlying principles of the Direct Substitution Property are the Properties of Limits, so be sure you keep these properties in the back of your mind.

■ **Example 3** Given the function f below, find the following limits algebraically, if they exist.

$$f(x) = \begin{cases} 4x+7 & x < 2 \\ (x-3)^2 & x \geq 2 \end{cases}$$

a. $\lim_{x \to 2^-} f(x)$

b. $\lim_{x \to 2^+} f(x)$

c. $\lim_{x \to 2} f(x)$

d. $\lim_{x \to 5} f(x)$

e. $\lim_{x \to -4} f(x)$

Solution:

a. Because this is a one-sided limit, $x \to 2^-$, we only need to consider x-values on the left of $x = 2$. Thus, we only need to consider the x-values less than 2. This means that we need to use the first rule of the piecewise-defined function: $4x+7$. Because this rule is a polynomial, we can apply the Direct Substitution Property and substitute $x = 2$ into the rule $4x+7$. This gives

$$\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} (4x+7) \\ &= 4(2)+7 \\ &= 15 \end{aligned}$$

b. Because this is a one-sided limit, $x \to 2^+$, we only need to consider $x > 2$. This means we need to use the second rule of the piecewise-defined function: $(x-3)^2$. Because this rule is a polynomial, we can apply the Direct Substitution Property and substitute $x = 2$ into $(x-3)^2$. When doing so, we get

$$\begin{aligned} \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} (x-3)^2 \\ &= (2-3)^2 \\ &= (-1)^2 \\ &= 1 \end{aligned}$$

c. From parts **a** and **b**, we know $\lim_{x\to 2^-} f(x) = 15$ and $\lim_{x\to 2^+} f(x) = 1$. Thus, the left- and right-hand limits are not equal, so $\lim_{x\to 2} f(x)$ does not exist.

d. To find $\lim_{x\to 5} f(x)$, we need to investigate the values of the function *near* $x = 5$ on both the left and right. Because 5 is greater than 2, we use the second rule of the piecewise-defined function, $(x-3)^2$, which will give us the values of the function near $x = 5$ on both the left and right. Because $(x-3)^2$ is a polynomial, we can apply the Direct Substitution Property:

$$\begin{aligned}\lim_{x\to 5} f(x) &= \lim_{x\to 5}(x-3)^2 \\ &= (5-3)^2 \\ &= (2)^2 \\ &= 4\end{aligned}$$

e. To find $\lim_{x\to -4} f(x)$, we need to investigate the values of the function when x is *near* -4 on both the left and right. Because -4 is less than 2, we use the first rule of the piecewise-defined function: $4x+7$. We can apply the Direct Substitution Property because $4x+7$ is a polynomial:

$$\begin{aligned}\lim_{x\to -4} f(x) &= \lim_{x\to -4}(4x+7) \\ &= 4(-4)+7 \\ &= -9\end{aligned}$$

■

Try It # 3:
Given the function g below, find the following limits algebraically, if they exist.

$$g(x) = \begin{cases} x^2-4 & x \le 0 \\ -4 & x > 0 \end{cases}$$

a. $\lim_{x\to 0^-} g(x)$

b. $\lim_{x\to 0^+} g(x)$

c. $\lim_{x\to 0} g(x)$

d. $\lim_{x\to 3} g(x)$

e. $\lim_{x\to -3} g(x)$

■ **Example 4** Find $\lim_{x\to 1} \sqrt{3x^2-1}$ algebraically, if it exists.

Solution:
In this example, we are not finding the limit of a polynomial or a rational function, so we cannot use the Direct Substitution Property. Thus, we will have to use the Properties of Limits to find this limit. Starting with Property #8, we have

$$\lim_{x\to 1} \sqrt{3x^2-1} = \sqrt{\lim_{x\to 1}\left(3x^2-1\right)}$$

We could continue with the Properties of Limits and calculate $\lim_{x\to 1}(3x^2-1)$, but because $3x^2-1$ is a polynomial, we can apply the Direct Substitution Property:

$$\begin{aligned}\lim_{x\to 1}\left(3x^2-1\right) &= 3(1)^2-1\\ &= 2\end{aligned}$$

Thus,

$$\begin{aligned}\lim_{x\to 1}\sqrt{3x^2-1} &= \sqrt{\lim_{x\to 1}\left(3x^2-1\right)}\\ &= \sqrt{2}\end{aligned}$$

■

Try It # 4:

Find $\lim_{x\to 4}\sqrt[3]{\frac{1}{2}x^3+x}$ algebraically, if it exists.

To find $\lim_{x\to 1}\sqrt{3x^2-1}$ in the previous example, notice we could have substituted $x=1$ into the function $\sqrt{3x^2-1}$ and obtained the same answer of $\sqrt{2}$. So it seems we did not necessarily need to apply the Properties of Limits. This is not just a coincidence, which is good news! Substituting a specified x-value into any type of function will give the value of the corresponding limit, as long as there are no domain issues.

Thus, when finding a limit algebraically, your first attempt should be to substitute the given x-value into the function. If you obtain a real number, then you have found the limit. In other words, we are recommending that when you are trying to find $\lim_{x\to c}f(x)$ algebraically, first try to find $f(c)$. If $f(c)$ is a real number, then you can conclude $\lim_{x\to c}f(x)=f(c)$.

When using this approach for piecewise-defined functions, be very careful! If the x-value at which you are evaluating a limit is the "cutoff" number of a piecewise-defined function, you must be sure to check both one-sided limits (i.e., the limit from the left and the limit from the right).

Restrictions on Limit Properties and Direct Substitution

Recall that Property #7 of the Properties of Limits states the following:

If L, M, and c are any real numbers and $\lim_{x\to c}f(x)=L$ and $\lim_{x\to c}g(x)=M$, then

$$\lim_{x\to c}\frac{f(x)}{g(x)}=\frac{\lim_{x\to c}f(x)}{\lim_{x\to c}g(x)}=\frac{L}{M}\quad\text{if } M\neq 0.$$

There are three cases that may arise when evaluating limits of quotients, which are directly related to Property #7. Let's take an in-depth look at each case.

Case 1 (L is any real number, $M\neq 0$):

If $\lim_{x\to c}f(x)=L$ and $\lim_{x\to c}g(x)\neq 0$, then $\lim_{x\to c}\frac{f(x)}{g(x)}=\frac{L}{M}$.

In Case 1, if $L=0$, then $\lim_{x\to c}\frac{f(x)}{g(x)}=0$ because zero divided by a nonzero number is always zero.

Case 2 $(L \neq 0, M = 0)$:

If $\lim_{x\to c} f(x) \neq 0$ and $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c} \frac{f(x)}{g(x)}$ **does not exist**.

Let's think about Case 2 a bit more. If the values of the function in the denominator, g, approach zero as x approaches c, and the values of the function in the numerator, f, approach a nonzero constant L as x approaches c, then we know L is being divided by very small numbers on the interval $(-1,0)$ or $(0,1)$. This means the ratio $\frac{L}{M}$ will become either infinitely positive or infinitely negative. Think of the ratio $\frac{4}{0.001} = 4000$, for example. If we choose the denominator so it is closer to zero, say 0.0001, then the ratio $\frac{4}{0.0001} = 40,000$ becomes even larger. To convince yourself even more, find the ratios $\frac{4}{0.00001}$ and $\frac{4}{0.000001}$. Do you see the quotient becomes even larger as the denominator gets smaller? We hope you do!

This concrete example should help convince you that when we find ourselves in Case 2 while evaluating limits, the limit will not exist and the values of the function of at least one of the one-sided limits will approach positive or negative infinity. This means that, in this case, there will be a vertical asymptote at $x = c$. In summary:

Case 2 with more detail:

If $\lim_{x\to c} f(x) \neq 0$ and $\lim_{x\to c} g(x) = 0$, then $\lim_{x\to c} \frac{f(x)}{g(x)}$ does not exist and at least one of the following is true:

$$\lim_{x\to c^+} \frac{f(x)}{g(x)} \to \infty \quad \text{or} \quad \lim_{x\to c^+} \frac{f(x)}{g(x)} \to -\infty$$

$$\lim_{x\to c^-} \frac{f(x)}{g(x)} \to \infty \quad \text{or} \quad \lim_{x\to c^-} \frac{f(x)}{g(x)} \to -\infty$$

and the line $x = c$ is a vertical asymptote.

Before we move on to the last case, Case 3, let's get more familiar with Case 2 by looking at an example.

■ **Example 5** Find $\lim_{x\to -6} \frac{3x^4 - 2x + 7}{x+6}$ algebraically. If the limit does not exist, state so and use limit notation to describe any infinite behavior.

Solution:

Recall that to find a limit algebraically, we should first try to substitute the given x-value into the function (note that because this is a rational function, we would technically be attempting to use the Direct Substitution Property). However, before we can attempt direct substitution, we must first check that the value of the function in the denominator is nonzero.

Because the value of the function in the denominator when $x = -6$ is $-6 + 6 = 0$, we cannot use the Direct Substitution Property. Therefore, we need to find the limits of the functions in the numerator and denominator separately to try and determine the overall behavior of the function.

Let's start by finding the limit of the function in the numerator. Because the function in the numerator is a polynomial and does not have any domain issues, we can use direct substitution (or, in this case, the Direct Substitution Property) to find the limit:

$$\begin{aligned}\lim_{x\to -6}(3x^4 - 2x + 7) &= 3(-6)^4 - 2(-6) + 7 \\ &= 3907\end{aligned}$$

Now, we will find the limit of the function in the denominator. Note that because the function in the denominator is a polynomial, we know the limit will equal the function value, which we already calculated (remember, this is the Direct Substitution Property):

$$\begin{aligned}\lim_{x\to-6}(x+6) &= -6+6\\ &= 0\end{aligned}$$

We see the limits of both the numerator and denominator exist, but the limit of the denominator is zero. Because of this, we find ourselves in Case 2. Thus, we know

$$\lim_{x\to-6}\frac{3x^4-2x+7}{x+6} \quad \text{does not exist}$$

and there is a vertical asymptote at $x=-6$.

Because there is a vertical asymptote at $x=-6$, we must continue and use limit notation to describe the infinite behavior near the vertical asymptote. Let's look at the graph of the function (see **Figure 1.2.4**) to help us understand the behavior of the values of the function near $x=-6$:

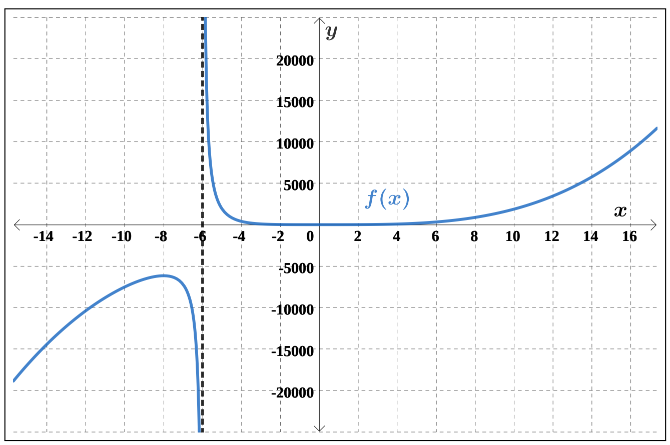


Figure 1.2.4: Graph of $f(x)=\dfrac{3x^4-2x+7}{x+6}$

It appears the values of the function on the right of $x=-6$ increase without bound to positive infinity and the values of the function on the left of $x=-6$ decrease without bound to negative infinity. Thus, we can describe the infinite behavior of the values of the function near $x=-6$ as

$$\lim_{x\to-6^-}\frac{3x^4-2x+7}{x+6}\to-\infty \quad\text{and}\quad \lim_{x\to-6^+}\frac{3x^4-2x+7}{x+6}\to\infty$$

■

N *When finding the limit of a ratio of two functions as in the previous example, rather than checking to see if the value of the function in the denominator is nonzero to use direct substitution, from this point forward we will simply start by finding the limits of the functions in the numerator and denominator separately, if they exist. This careful approach will lead us to the correct answer regardless of the value of the function in the denominator.*

Try It # 5:
Find $\lim\limits_{x \to 5} \dfrac{x^2 - x}{x - 5}$ algebraically. If the limit does not exist, state so and use limit notation to describe any infinite behavior.

Finally, we will investigate the last case, Case 3, that may arise when finding limits of rational functions:

Case 3 ($L = 0, M = 0$):
If $\lim\limits_{x \to c} f(x) = 0$ and $\lim\limits_{x \to c} g(x) = 0$, then $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)}$ cannot be determined and further algebraic manipulation is necessary to convert the limit to an expression where Case 1 or Case 2 applies.

For a quick moment, let's go back to the days in grade school when we learned about dividing numbers. We learned that 12 divided by 4 is 3 because 3 times 4 is 12. Thus, we can *determine* the quotient, $\frac{12}{4}$, by using the inverse operation of multiplication and asking ourselves, what number times 4 equals 12?

The instance of $\frac{0}{0}$ is a bit more problematic because we *cannot determine* the quotient. Let's think about this more in-depth. When trying to find the quotient q for $\frac{0}{0}$, we can use the inverse operation of multiplication and ask ourselves, what number times zero equals zero? Thus, we are trying to find the value of q such that $q \cdot 0 = 0$. Well, this is easy, right? Oh wait, it may not be as easy as we think because we can choose any real number for q.

To further convince you, let's choose $q = 7$ because $7 \cdot 0 = 0$. Is 7 the only solution for q? No, because $q = 8$, $q = 9$, and $q = 10$ also work. There are infinitely many solutions for q. Thus, it seems we cannot determine a single real number for q, and so we cannot determine the quotient $\frac{0}{0}$.

The previous discussion reveals a very important definition:

Definition
If $\lim\limits_{x \to c} f(x) = 0$ and $\lim\limits_{x \to c} g(x) = 0$, then $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)}$ is **indeterminate** (or of **indeterminate form**). ▪

Be careful! If a limit is of indeterminate form, it does not mean that the limit does not exist. Rather, it means the limit *cannot be determined* from the limits of the individual functions (hence the reason it is said to be indeterminate). To determine the limit, we generally need to use algebraic techniques to manipulate the function, which will also help us identify any holes or vertical asymptotes of the function as well as the values of the function near these special locations.

There are a number of algebraic techniques we may use, and the technique we execute depends on the given function. If the function is a ratio of two polynomials (i.e., a rational function), then we will generally try factoring the numerator and denominator and determine if there are common factors to divide. Other techniques that are often used include simplifying complex fractions, multiplying by the conjugate, and rewriting an absolute value function as a piecewise-defined function. After using one of these techniques to manipulate the function, we will try to find the limit again (of the resulting function). You will see these techniques in action in the next few examples.

▪ **Example 6** Find $\lim\limits_{x \to 1} \dfrac{x^2 - 1}{x - 1}$ algebraically. If the limit does not exist, state so and use limit notation to describe any infinite behavior.

Solution:
Rather than checking the value of the function in the denominator to see if we can apply the Direct Substitution Property to this rational function, we will go straight to finding the limits of the numerator and denominator separately to see if both limits exist and if the limit of the function in the denominator is nonzero (recall this is the approach we will take from this point forward when dealing with a ratio of any types of functions).

Let's start by checking the limit of the numerator. Because the function in the numerator is a polynomial, we will use direct substitution to find the limit:

$$\begin{aligned}\lim_{x\to 1}\left(x^2-1\right) &= 1^2-1\\ &= 0\end{aligned}$$

Finding the limit of the denominator gives

$$\begin{aligned}\lim_{x\to 1}(x-1) &= 1-1\\ &= 0\end{aligned}$$

Both of these limits exist, but because the limit of each function is zero, the original limit is of the indeterminate form $\frac{0}{0}$. Thus, we are in Case 3 and must use one of the previously listed algebraic techniques to manipulate the function. The functions in the numerator and denominator are polynomials, so we will factor each polynomial and then divide the common factor, $x-1$:

$$\begin{aligned}\lim_{x\to 1}\frac{x^2-1}{x-1} &= \lim_{x\to 1}\frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}}\\ &= \lim_{x\to 1}(x+1)\end{aligned}$$

After algebraically manipulating the function, notice we have a linear function, $x+1$. Thus, we can use direct substitution to find this limit:

$$\begin{aligned}\lim_{x\to 1}(x+1) &= 1+1\\ &= 2\end{aligned}$$

Hence,

$$\lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1}(x+1) = 2$$

Let's verify this result by looking at the graph of $f(x)=\dfrac{x^2-1}{x-1}$ shown in **Figure1.2.5**:

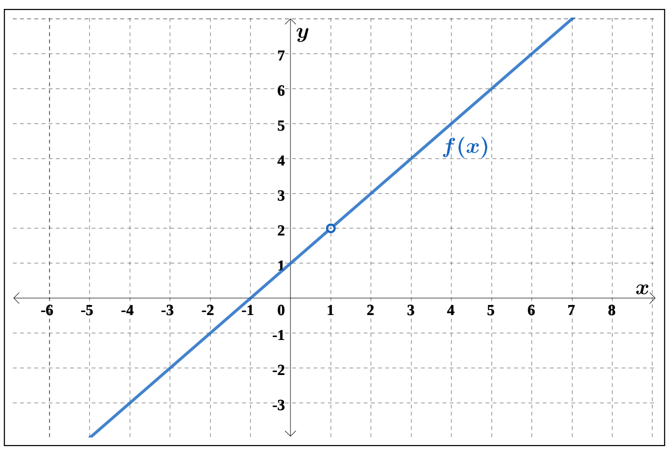


Figure 1.2.5: Graph of $f(x)=\dfrac{x^2-1}{x-1}$

We see the original function, $f(x)=\dfrac{x^2-1}{x-1}$, has a hole at $x=1$. We know this because we divided the expression $x-1$ from the denominator when we simplified the original function. The y-value of the hole is equal to the value of $\lim_{x\to 1}(x+1)=2$, and thus, the function has a hole at the point $(1,2)$.

■

The result in the previous example demonstrates one of the function behaviors that can result when working with an indeterminate limit. If, after using algebraic techniques to manipulate the function, a common factor can be divided and a finite number results when finding the limit again, then the value of the limit is the y-value of the hole at the relevant x-value.

Let's investigate this further to help us understand why using algebraic techniques to manipulate functions enables us to determine indeterminate limits. To do this, you'll need to recall the meaning of a limit. These techniques work because when finding limits, we are interested in the y-value the function is *heading toward* as x approaches a specific x-value, not the y-value of the function at the specific x-value. Oftentimes, in these indeterminate cases, the x-value of the limit is not in the domain of the given function. We use algebraic techniques to manipulate the function to help us determine what is happening *near* the x-value. Remember, with limits, we are interested in values of the function *near* $x = c$, *not necessarily at* $x = c$, which works well in indeterminate cases because $x = c$ is often not in the domain of the given function.

In the previous example, for instance, when calculating $\lim_{x\to 1} f(x)$ where $f(x) = \dfrac{x^2-1}{x-1}$, we used the simplified function, $g(x) = x+1$, to determine the y-value $f(x)$ was *heading toward* as x approached 1. Note that the functions $f(x) = \dfrac{x^2-1}{x-1}$ and $g(x) = x+1$ are *different* functions; $x = 1$ is not in the domain of f, but it is in the domain of g. So we know f and g behave the same *near* $x = 1$, but differently at $x = 1$. More specifically, the graph of f has a hole at $x = 1$ (see **Figure 1.2.6**), but the graph of g does not (see **Figure 1.2.7**).

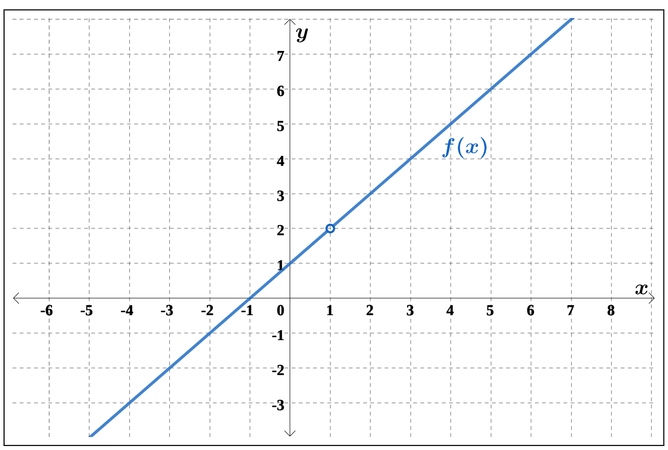


Figure 1.2.6: Graph of $f(x) = \dfrac{x^2-1}{x-1}$

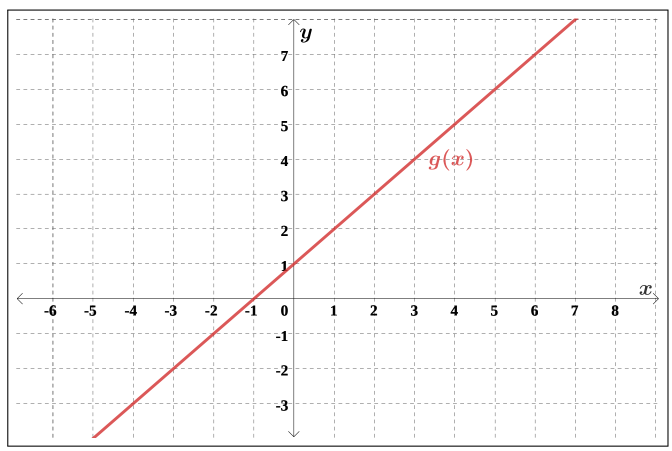


Figure 1.2.7: Graph of $g(x) = x+1$

This is so interesting, isn't it? Think about what we just discovered; even though the original function f and the simplified function g are different functions, they both have the same limit values everywhere. Why? A limit gives us the y-value the function is heading toward, not the actual value of the function. Both $f(x)$ and $g(x)$ approach the y-value 2 when x approaches 1. They have the same limit values everywhere even though f is undefined at $x = 1$ and g is defined at $x = 1$, that is $g(1) = 2$.

Thus, manipulating the function when we have an indeterminate limit allows us to find the value of the limit, if it exists, because the simplified function will have the same limit as the original function.

When manipulating a function with an indeterminate limit, we always write the mathematical notation for the limit ("lim") at each step of our work. This is because the limits of each simplified version of the function are equal, but the functions themselves may not be. For instance, in the previous example,

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1}(x+1)$$

However,

$$\frac{x^2-1}{x-1} \neq x-1 \text{ when } x=1$$

Example 7 Find $\lim_{x\to 4} \frac{x-4}{x^2-8x+16}$ algebraically. If the limit does not exist, state so and use limit notation to describe any infinite behavior.

Solution:
Because this function consists of a ratio of two functions, we need make sure the limits of the functions in the numerator and denominator exist and that the limit of the function in the denominator is nonzero. Then, we can determine whether the limit results in Case 1, 2, or 3. Let's start by determining the limit of function in the numerator, if it exists:

$$\begin{aligned}\lim_{x\to 4}(x-4) &= 4-4 \\ &= 0\end{aligned}$$

Now, let's investigate the limit of the function in the denominator:

$$\begin{aligned}\lim_{x\to 4}\left(x^2-8x+16\right) &= 4^2-8(4)+16 \\ &= 16-32+16 \\ &= 0\end{aligned}$$

Because the limits of the functions in the numerator and denominator equal zero, the limit is of the indeterminate form $\frac{0}{0}$. Thus, we are in Case 3 and must use one of the previously listed algebraic techniques to manipulate the function. Notice the functions in the numerator and denominator are both polynomials, so we will try factoring each polynomial and divide any common factors:

$$\lim_{x\to 4} \frac{x-4}{x^2-8x+16} = \lim_{x\to 4} \frac{x-4}{(x-4)(x-4)}$$

Dividing the common factor of $x-4$ from the numerator and denominator gives

$$\begin{aligned}\lim_{x\to 4} \frac{x-4}{(x-4)(x-4)} &= \lim_{x\to 4} \frac{\cancel{x-4}}{\cancel{(x-4)}(x-4)} \\ &= \lim_{x\to 4} \frac{1}{x-4}\end{aligned}$$

The resulting function, $g(x) = \frac{1}{x-4}$ is, again, a ratio of two functions. Thus, we should find the limits of the numerator and denominator separately.

Let's find the limit of the function in the numerator:

$$\lim_{x\to 4} 1 = 1$$

Finding the limit of the function in the denominator using direct substitution gives

$$\begin{aligned}\lim_{x\to 4}(x-4) &= 4-4\\ &= 0\end{aligned}$$

Both limits exist, but the limit of the function in the numerator is 1 and the limit of the function in the denominator is 0. Therefore, we are in Case 2. This means

$$\lim_{x\to 4}\frac{x-4}{x^2-8x+16} = \lim_{x\to 4}\frac{1}{x-4} \quad \text{does not exist}$$

Furthermore, this also tells us both the original and simplified functions have a vertical asymptote at $x=4$ (notice $x=4$ is not in the domain of the original function or the simplified function).

Because the function has a vertical asymptote at $x=4$, we should continue and describe the infinite behavior near $x=4$ using limit notation. We can examine the graph of the original function, f, to determine the infinite behavior near $x=4$ (see **Figure1.2.8**). Note that we could have graphed the simplified function, g, instead to determine the infinite behavior near $x=4$ because it will behave the same as f.

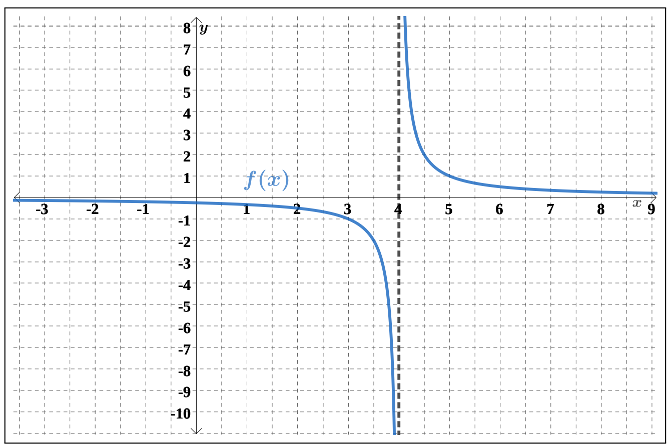


Figure 1.2.8: Graph of $f(x) = \dfrac{x-4}{x^2-8x+16}$

As x approaches 4 from the left, $x\to 4^-$, the function is heading toward negative infinity. As x approaches 4 from the right, $x\to 4^+$, the function is heading toward positive infinity. We can describe the infinite behavior of the function near $x=4$ using the following limits:

$$\lim_{x\to 4^-}\frac{x-4}{x^2-8x+16}\to -\infty \quad\text{and}\quad \lim_{x\to 4^+}\frac{x-4}{x^2-8x+16}\to \infty$$

N *Because the function is heading toward opposite infinities from the left and right of $x=4$, we have to write two separate limits to describe the infinite behavior. However, if the function headed toward the same infinity (both positive or both negative), we could have simply written one limit as $x\to 4$ to describe the infinite behavior.*

■

The last two examples demonstrate that, ultimately, the limit of a function that results from the algebraic manipulation of a ratio of two functions will either be a finite number, in which case the limit exists (see Case 1), or it will not exist (see Case 2).

Try It # 6:

Find $\lim\limits_{x\to 3} \dfrac{x^2-3x}{2x^2-5x-3}$ algebraically. If the limit does not exist, state so and use limit notation to describe any infinite behavior.

■ **Example 8** Find $\lim\limits_{x\to 1} \dfrac{\dfrac{1}{x+1}-\dfrac{1}{2}}{x-1}$ algebraically. If the limit does not exist, state so and use limit notation to describe any infinite behavior.

Solution:

Because this function consists of a ratio of two functions, we need to make sure the limits of the functions in the numerator and denominator exist and that the limit of the function in the denominator is nonzero. Then, we can determine whether the limit results in Case 1, 2, or 3.

Let's start by investigating the limit of the function in the numerator. We can use direct substitution for this limit because there will not be any domain issues:

$$\begin{aligned}\lim_{x\to 1}\left(\frac{1}{x+1}-\frac{1}{2}\right) &= \frac{1}{1+1}-\frac{1}{2}\\ &= \frac{1}{2}-\frac{1}{2}\\ &= 0\end{aligned}$$

Now, let's investigate the limit of the function in the denominator:

$$\begin{aligned}\lim_{x\to 1}(x-1) &= 1-1\\ &= 0\end{aligned}$$

Because the limits of the two functions equal zero, the limit is of the indeterminate form $\frac{0}{0}$. Thus, we are in Case 3 and must use one of the previously listed algebraic techniques to manipulate the function.

Notice the given function is a complex fraction (i.e., a fraction within a fraction). To simplify this complex fraction, we will use algebra to subtract the two expressions in the numerator and then simplify the complex fraction by dividing the expression in the numerator by the expression in the denominator.

Let's start by finding a common denominator for the fractions in the numerator and then subtract the fractions. For now, we will hold off from doing anything with the denominator:

$$\lim_{x\to 1}\frac{\frac{1}{x+1}-\frac{1}{2}}{x-1}=\lim_{x\to 1}\frac{\left(\frac{2}{2}\right)\left(\frac{1}{x+1}\right)-\left(\frac{1}{2}\right)\left(\frac{x+1}{x+1}\right)}{x-1}$$

$$=\lim_{x\to 1}\frac{\frac{2}{2(x+1)}-\frac{x+1}{2(x+1)}}{x-1}$$

$$=\lim_{x\to 1}\frac{\frac{2-(x+1)}{2(x+1)}}{x-1}$$

$$=\lim_{x\to 1}\frac{\frac{2-x-1}{2x+2}}{x-1}$$

$$=\lim_{x\to 1}\frac{\frac{-x+1}{2x+2}}{x-1}$$

At this point, we need to simplify the complex fraction. To do this, we will multiply the numerator by the reciprocal of the denominator. That is, we will will multiply $\frac{-x+1}{2x+2}$ by $\frac{1}{x-1}$:

$$\lim_{x\to 1}\frac{\frac{-x+1}{2x+2}}{x-1}=\lim_{x\to 1}\left[\left(\frac{-x+1}{2x+2}\right)\left(\frac{1}{x-1}\right)\right]$$

To create common factors, we need to factor -1 from $x-1$ to get $-(x+1)$. Thus, we have

$$\lim_{x\to 1}\left[\left(\frac{-x+1}{2x+2}\right)\left(\frac{1}{x-1}\right)\right]=\lim_{x\to 1}\left[\left(\frac{-(x-1)}{2x+2}\right)\left(\frac{1}{x-1}\right)\right]$$

We can now divide the common factor $x-1$:

$$\lim_{x\to 1}\left[\left(\frac{-(x-1)}{2x+2}\right)\left(\frac{1}{x-1}\right)\right]=\lim_{x\to 1}\left[\left(\frac{-\cancel{(x-1)}}{2x+2}\right)\left(\frac{1}{\cancel{x-1}}\right)\right]$$

$$=\lim_{x\to 1}\frac{-1}{2x+2}$$

! *When dividing the factor $x-1$ in the numerator by the factor $x-1$ in the denominator, be careful not to lose the -1 that remains in the numerator.*

The resulting function, $g(x)=\frac{-1}{2x+2}$ is, again, a ratio of two functions. Thus, we should find the limits of the numerator and denominator separately.

Let's find the limit of the function in the numerator:

$$\lim_{x\to 1}(-1) = -1$$

Finding the limit of the function in the denominator using direct substitution gives

$$\begin{aligned}\lim_{x\to 1}(2x+2) &= 2(1)+2\\ &= 4\end{aligned}$$

Both limits exist and are nonzero. Thus, we can apply the Properties of Limits and divide the limits (specifically, Property #7):

$$\begin{aligned}\lim_{x\to 1}\frac{-1}{2x+2} &= \frac{\lim\limits_{x\to 1}(-1)}{\lim\limits_{x\to 1}(2x+2)}\\ &= \frac{-1}{2(1)+2}\\ &= -\frac{1}{4}\end{aligned}$$

Hence,

$$\lim_{x\to 1}\frac{\frac{1}{x+1}-\frac{1}{2}}{x-1} = \lim_{x\to 1}\frac{-1}{2x+2} = -\frac{1}{4}$$

Because the limit exists as x approaches 1, but $x = 1$ is not in the domain of the given function, $f(x) = \dfrac{\frac{1}{x+1}-\frac{1}{2}}{x-1}$, we know $f(x)$ has a hole at the point $\left(1, -\frac{1}{4}\right)$. ■

Try It # 7:

Find $\lim\limits_{x\to -3}\dfrac{\frac{1}{x+2}+1}{x+3}$ algebraically. If the limit does not exist, state so and use limit notation to describe any infinite behavior.

N *Even though we have been using direct substitution and subsequent ideas (i.e., Cases 1, 2, and 3) for two-sided limits, they hold just as well for one-sided limits. We will see this in the next example.*

■ **Example 9** Find $\lim\limits_{x\to 41^+}\dfrac{\sqrt{x-5}-6}{x-41}$ algebraically. If the limit does not exist, state so and use limit notation to describe any infinite behavior.

Solution:

First, notice the given function is a ratio of two functions. So we need to make sure the limits of the two functions exist and that the limit of the function in the denominator is nonzero. We will first attempt to find the limits of the functions in the numerator and denominator using direct substitution:

$$\begin{aligned}\lim_{x\to 41^+}\left(\sqrt{x-5}-6\right) &= \sqrt{41-5}-6\\ &= \sqrt{36}-6\\ &= 6-6\\ &= 0\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to 41^+}(x-41) &= 41-41\\ &= 0\end{aligned}$$

Because the limits of the two functions equal zero, the limit is of the indeterminate form $\frac{0}{0}$. Thus, we are in Case 3 and we must use one of the previously listed algebraic techniques to manipulate the function.

Notice the function in the numerator contains a square root. When this happens, we usually try to manipulate the function using a technique called "multiplying by the conjugate". To use this technique, we will multiply the numerator and denominator of the given function by the conjugate of the numerator. The numerator is $\sqrt{x-5}-6$, and its conjugate is $\sqrt{x-5}+6$. Notice that to create the conjugate, we change the operation between the terms in the numerator. Thus, instead of subtracting $\sqrt{x-5}$ and 6, we add $\sqrt{x-5}$ and 6. Multiplying both the numerator and denominator by the conjugate gives

$$\lim_{x\to 41^+}\frac{\sqrt{x-5}-6}{x-41} = \lim_{x\to 41^+}\left[\frac{\left(\sqrt{x-5}-6\right)}{(x-41)}\frac{\left(\sqrt{x-5}+6\right)}{\left(\sqrt{x-5}+6\right)}\right]$$

Multiplying the numerators (using FOIL) and multiplying the denominators (writing them as a product) gives

$$\lim_{x\to 41^+}\left[\frac{\left(\sqrt{x-5}-6\right)}{(x-41)}\frac{\left(\sqrt{x-5}+6\right)}{\left(\sqrt{x-5}+6\right)}\right] = \lim_{x\to 41^+}\frac{\left(\sqrt{x-5}\right)\left(\sqrt{x-5}\right)+(6)\left(\sqrt{x-5}\right)-(6)\left(\sqrt{x-5}\right)-(6)(6)}{(x-41)\left(\sqrt{x-5}+6\right)}$$

The reason we multiplied by the conjugate is so we can eliminate the square root from the numerator. Notice when we add the square root terms in the middle of the numerator, we get zero. Continuing to simplify the expression gives

$$= \lim_{x\to 41^+}\frac{\left(\sqrt{x-5}\right)\left(\sqrt{x-5}\right)+\cancel{(6)\left(\sqrt{x-5}\right)}-\cancel{(6)\left(\sqrt{x-5}\right)}-(6)(6)}{(x-41)\left(\sqrt{x-5}+6\right)}$$

$$= \lim_{x\to 41^+}\frac{\left(\sqrt{x-5}\right)^2-36}{(x-41)\left(\sqrt{x-5}+6\right)}$$

$$= \lim_{x\to 41^+}\frac{(x-5)-36}{(x-41)\left(\sqrt{x-5}+6\right)}$$

$$= \lim_{x\to 41^+}\frac{x-41}{(x-41)\left(\sqrt{x-5}+6\right)}$$

Now, we see a common factor of $x-41$ in both the numerator and denominator. Dividing the common factor gives

$$\lim_{x\to 41^+} \frac{x-41}{(x-41)\left(\sqrt{x-5}+6\right)} = \lim_{x\to 41^+} \frac{\cancel{x-41}}{\cancel{(x-41)}\left(\sqrt{x-5}+6\right)}$$

$$= \lim_{x\to 41^+} \frac{1}{\sqrt{x-5}+6}$$

Notice we still need to find the limit of a ratio of two functions. We know the limit of the function in this numerator exists because $\lim_{x\to 41^+} 1 = 1$, but we need to investigate the limit of the function in this denominator to make sure it exists and is nonzero. Attempting direct substitution gives

$$\begin{aligned}\lim_{x\to 41^+}\left(\sqrt{x-5}+6\right) &= \sqrt{41-5}+6\\ &= \sqrt{36}+6\\ &= 6+6\\ &= 12\end{aligned}$$

Now, we know the limits of the functions in the numerator and denominator exist and that the limit of the function in the denominator is nonzero. Thus, we can use the Properties of Limits and divide the limits:

$$\lim_{x\to 41^+} \frac{1}{\sqrt{x-5}+6} = \frac{\lim_{x\to 41^+} 1}{\lim_{x\to 41^+}\left(\sqrt{x-5}+6\right)}$$

$$= \frac{1}{12}$$

Hence,

$$\lim_{x\to 41^+} \frac{\sqrt{x-5}-6}{x-41} = \lim_{x\to 41^+} \frac{1}{\sqrt{x-5}+6} = \frac{1}{12}$$

Because the limit from the right exists at $x = 41$, but $x = 41$ is not in the domain of the given function, we know the function has a hole at the point $\left(41, \frac{1}{12}\right)$.

■

Try It # 8:

Find $\lim_{x\to -1} \frac{\sqrt{x+2}-1}{x+1}$ algebraically. If the limit does not exist, state so and use limit notation to describe any infinite behavior.

■ **Example 10** Find $\lim_{x\to 5} \frac{|5-x|}{25-x^2}$ algebraically. If the limit does not exist, state so and use limit notation to describe any infinite behavior.

Solution:

Because this function consists of a ratio of two functions, we first need to determine if the limits of the functions in the numerator and denominator exist and make sure the limit of the function in the denominator is nonzero. Using direct substitution to find the limits of each gives

$$\begin{aligned}\lim_{x\to 5}|5-x| &= |5-5| \\ &= |0| \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to 5}\left(25-x^2\right) &= 25-5^2 \\ &= 25-25 \\ &= 0\end{aligned}$$

Notice both limits equal zero, so the limit is of the indeterminate form $\frac{0}{0}$. We must use an algebraic technique to manipulate the function. In this case, the numerator has an absolute value function. We will manipulate the function by first rewriting the absolute value function as a piecewise-defined function.

To write $f(x) = \dfrac{|5-x|}{25-x^2}$ as a piecewise-defined function, we first find where the argument of the absolute value function is nonnegative:

$$\begin{aligned}5-x &\geq 0 \\ 5 \geq x, &\quad \text{or} \quad x \leq 5\end{aligned}$$

This gives us the first cutoff for the piecewise-defined function. Because the first rule corresponds to where the argument of the absolute value function is nonnegative (i.e., zero or positive), we simply remove the absolute value signs because we know that for any x-value less than or equal to 5, the function is guaranteed to be nonnegative:

$$f(x) = \begin{cases} \dfrac{5-x}{25-x^2} & x \leq 5 \\ & \end{cases}$$

The next rule in our piecewise-defined function corresponds to where the argument of the absolute value function is negative, which happens when $x > 5$ (note that we do not include $x = 5$ because we already have equality with the first cutoff). For this rule, to create an equivalent expression for $|5-x|$, we must multiply $5-x$ by -1 to ensure the output values will be positive. Think about this a bit more. Because $5-x$ is negative for all $x > 5$, we would have to multiply $5-x$ by -1 to get a positive result, which is how the original absolute value function would behave. Thus, we know $|5-x| = -(5-x)$ when $x > 5$. Therefore, the final piecewise-defined function is

$$f(x) = \begin{cases} \dfrac{5-x}{25-x^2} & x \leq 5 \\[2ex] \dfrac{-(5-x)}{25-x^2} & x > 5 \end{cases}$$

Now, the given function is in a format in which we can find the limit.

Because we are looking for the two-sided limit as $x \to 5$ and $x = 5$ is the cutoff number of the piecewise-defined function, we must find both the left- and right-hand limits. If both of the one-sided limits approach the same finite number, we know the limit exists.

We will start by looking at the limit from the left. Note that the left-hand limit corresponds to x-values less than, but very close to, $x = 5$. So we will use the first rule of the piecewise-defined function, $\dfrac{5-x}{25-x^2}$, to find this limit.

Notice as $x \to 5^-$, the limits of the functions in the numerator and denominator both exist and equal zero, so the limit is of the indeterminate form $\frac{0}{0}$. We must use our algebraic skills to manipulate the function. Because both the numerator and denominator are polynomials, we will factor the polynomials and divide the common factor:

$$\begin{aligned}\lim_{x\to 5^-}\frac{5-x}{25-x^2} &= \lim_{x\to 5^-}\frac{5-x}{(5-x)(5+x)}\\ &= \lim_{x\to 5^-}\frac{\cancel{5-x}}{\cancel{(5-x)}(5+x)}\\ &= \lim_{x\to 5^-}\frac{1}{5+x}\end{aligned}$$

Notice $\lim\limits_{x\to 5^-} 1 = 1$ and $\lim\limits_{x\to 5^-}(5+x) = 5+5 = 10$. Thus, we can apply the Properties of Limits and divide the limits of the numerator and denominator:

$$\begin{aligned}\lim_{x\to 5^-}\frac{1}{5+x} &= \frac{\lim\limits_{x\to 5^-} 1}{\lim\limits_{x\to 5^-}(5+x)}\\ &= \frac{1}{10}\end{aligned}$$

Hence,

$$\lim_{x\to 5^-}\frac{|5-x|}{25-x^2} = \lim_{x\to 5^-}\frac{1}{5+x} = \frac{1}{10}$$

Now, we must calculate the limit from the right of $x = 5$, and x-values to right of (or greater than) $x = 5$ will correspond to the second rule of the piecewise-defined function, $\dfrac{-(5-x)}{25-x^2}$.

Notice as $x \to 5^+$, the limits of the functions in the numerator and denominator both exist and equal zero, so the limit is of the indeterminate form $\frac{0}{0}$. We must use our algebraic skills to manipulate the function. Because both the numerator and denominator are polynomials, we will factor the polynomials and divide the common factor:

$$\begin{aligned}\lim_{x\to 5^+}\frac{-(5-x)}{25-x^2} &= \lim_{x\to 5^+}\frac{-(5-x)}{(5-x)(5+x)}\\ &= \lim_{x\to 5^+}\frac{-\cancel{(5-x)}}{\cancel{(5-x)}(5+x)}\\ &= \lim_{x\to 5^+}\frac{-1}{5+x}\end{aligned}$$

Notice $\lim\limits_{x\to 5^+}(-1) = -1$ and $\lim\limits_{x\to 5^+}(5+x) = 5+5 = 10$. Thus, we can apply the Properties of Limits and divide the limits of the numerator and denominator:

$$\lim_{x\to 5^+} \frac{-1}{5+x} = \frac{\lim_{x\to 5^+}(-1)}{\lim_{x\to 5^+}(5+x)}$$

$$= \frac{-1}{10}$$

Therefore,

$$\lim_{x\to 5^+} \frac{|5-x|}{25-x^2} = \lim_{x\to 5^+} \frac{-1}{5+x} = -\frac{1}{10}$$

Because $\frac{1}{10} \neq \frac{-1}{10}$, we know $\lim_{x\to 5^-} \frac{|5-x|}{25-x^2} \neq \lim_{x\to 5^+} \frac{|5-x|}{25-x^2}$. Thus,

$$\lim_{x\to 5} \frac{|5-x|}{25-x^2} \text{ does not exist}$$

Note that this function does not have infinite behavior at $x = 5$. This limit does not exist because the left- and right-hand limits are not equal.

If the limit of a function does not exist, that does not necessarily mean the function has a vertical asymptote at the relevant x-value. In the previous example, the limit does not exist, but it is because the left- and right-hand limits are not equal (not because it has a vertical asymptote). In order to have a vertical asymptote, the limit must result in a nonzero number divided by zero.

Try It # 9:

Find $\lim_{x\to -7} \frac{x+7}{|x+7|}$ algebraically. If the limit does not exist, state so and use limit notation to describe any infinite behavior.

Try It Answers

1. **a.** -30
   b. $-6/5$
   c. -125
   d. 2
2. **a.** -14
   b. $-33/16$
3. **a.** -4
   b. -4
   c. -4
   d. -4
   e. 5

4. $\sqrt[3]{36}$

5. Does Not Exist; $\lim_{x\to 5^-} \frac{x^2-x}{x-5} \to -\infty$ and $\lim_{x\to 5^+} \frac{x^2-x}{x-5} \to \infty$

6. $3/7$

7. -1

8. $1/2$

9. Does Not Exist

EXERCISES

Basic Skills Practice

For Exercises 1 - 6, find the limit algebraically.

1. $\lim_{x\to 2} 5$

2. $\lim_{x\to -5} x$

3. $\lim_{x\to 3} \left(3x^2 - 2x + 1\right)$

4. $\lim_{x\to -2} \dfrac{x+2}{7x-3}$

5. $\lim_{x\to 2} \dfrac{3x+2}{(x-4)^3}$

6. $\lim_{x\to 3} \dfrac{2x^2-5}{e^{x-3}}$

7. Given the information below, find the following limits.

$$\lim_{x\to 6} f(x) = 4 \qquad \lim_{x\to 6} g(x) = 9 \qquad \lim_{x\to 6} h(x) = 6$$

   (a) $\lim_{x\to 6} [f(x) - g(x)]$

   (b) $\lim_{x\to 6} [f(x)h(x)]$

   (c) $\lim_{x\to 6} \left[g(x) + \frac{1}{3}h(x)\right]$

   (d) $\lim_{x\to 6} \dfrac{[h(x)]^3}{2}$

8. Given the information below, find the following limits.

$$\lim_{x\to 2} f(x) = -3 \qquad \lim_{x\to 2} g(x) = 5 \qquad \lim_{x\to 2} h(x) = 4$$

(a) $\lim_{x\to 2} [2f(x) - 3g(x)]$

(b) $\lim_{x\to 2} \frac{g(x)-1}{f(x)}$

(c) $\lim_{x\to 2} \sqrt{6h(x)+1}$

(d) $\lim_{x\to 2} [3f(x)+g(x)]^4$

For Exercises 9 - 12, find the limit algebraically, if it exists.

9. $\lim_{x\to 4} \frac{2x}{x-4}$

10. $\lim_{x\to -2} \frac{x^2-5}{3x+6}$

11. $\lim_{x\to -5} \frac{x^2-x-6}{2x^2+3x-35}$

12. $\lim_{x\to 1} \frac{x^3-4x+9}{x^4-2x^2+1}$

For Exercises 13 - 16, find the limit algebraically.

13. $\lim_{x\to 6} \frac{x(x-6)^2}{x-6}$

14. $\lim_{x\to 0} \frac{2x(x+3)^3}{x}$

15. $\lim_{x\to 1} \frac{2x(x-1)^3}{(x+2)(x-1)}$

16. $\lim_{x\to -4} \frac{(x-2)(x+4)}{(x+4)(x-5)}$

For Exercises 17 - 20, find the limit algebraically, if it exists.

17. $\lim_{x\to 1} \frac{2x(x-1)}{(x-1)(x-1)}$

18. $\lim_{x\to -6} \frac{(x+4)(x+6)}{x(x+6)^2}$

19. $\lim_{x\to 2} \frac{(x-2)(3x+7)}{(x+5)(x-2)^3}$

20. $\lim_{x\to -4} \frac{(10-5x)(x+4)^2}{(x+4)^3(x-8)}$

For Exercises 21 - 26, find the limit algebraically, if it exists.

21. $\lim_{x\to 4} f(x)$, where $f(x) = \begin{cases} x-5 & x \leq 6 \\ 2x & x > 6 \end{cases}$

22. $\lim_{x\to 0} f(x)$, where $f(x) = \begin{cases} 8 & x \leq -1 \\ 4x-3 & x > -1 \end{cases}$

23. $\lim_{x\to -2^-} f(x)$, where $f(x) = \begin{cases} 10 & x \leq -2 \\ -9x^2 & x > -2 \end{cases}$

24. $\lim_{x\to 7^+} f(x)$, where $f(x) = \begin{cases} -x & x \leq 7 \\ 5x+6 & x > 7 \end{cases}$

25. $\lim_{x\to 6} f(x)$, where $f(x) = \begin{cases} 4x & x \leq 6 \\ 9-x & x > 6 \end{cases}$

26. $\lim_{x\to -3} f(x)$, where $f(x) = \begin{cases} x^2 & x \leq -3 \\ x+12 & x > -3 \end{cases}$

Intermediate Skills Practice

For Exercises 27 - 31, find the limit algebraically.

27. $\lim_{x\to 0} \left(0.8x^2 + \sqrt[3]{x} + \pi\right)$

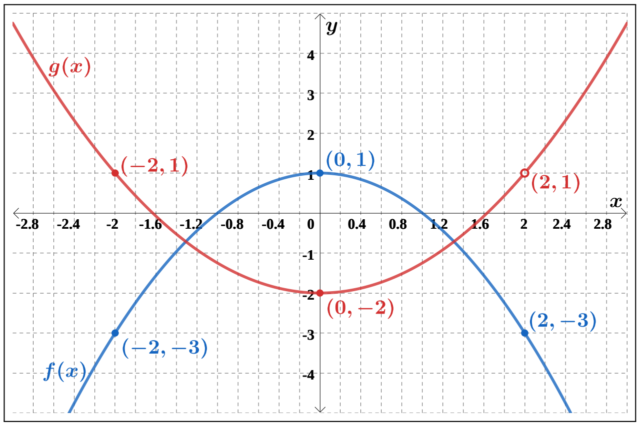
28. $\lim_{x\to -7^+} \frac{x^2-2x+7}{-4x-1}$

29. $\lim_{x\to -4} \left|-5x^2-3x+9\right|$

30. $\lim_{x\to 2^-} \frac{4x^2-6x}{e^{x-2}}$

31. $\lim_{x\to e} \left(x^3 + \ln(2x-e)\right)$

32. Given the graphs of f and g shown below, find the following limits.



(a) $\lim_{x\to-2}[f(x)+g(x)]$

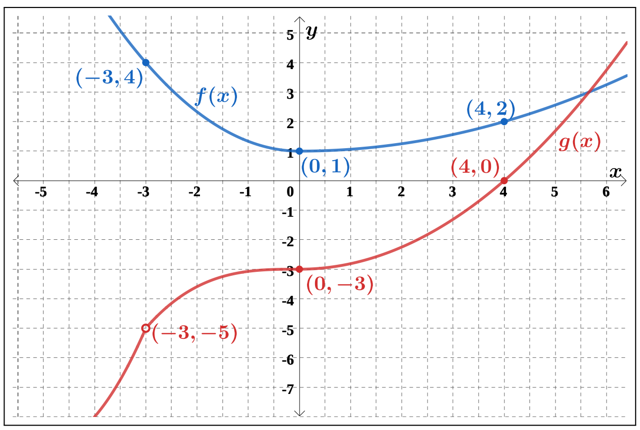
(b) $\lim_{x\to0}\frac{f(x)}{g(x)}$

(c) $\lim_{x\to2}[f(x)]^3$

(d) $\lim_{x\to0^+}\frac{\sqrt{2-f(x)}}{5g(x)}$

(e) $\lim_{x\to2^-}\frac{2-f(x)}{g(x)}$

33. Given the graphs of f and g shown below, find the following limits.



(a) $\lim_{x\to-3}[g(x)-f(x)]$

(b) $\lim_{x\to0}\sqrt[9]{f(x)}$

(c) $\lim_{x\to4}\left[[f(x)]^3+\sqrt{g(x)}\right]$

(d) $\lim_{x\to-3^-}[4f(x)-3g(x)]+5$

(e) $\lim_{x\to0^+}\frac{[9-f(x)]^2}{g(x)-7}$

For Exercises 34 - 41, find the limit algebraically. If the limit does not exist, state so and use limit notation to describe any infinite behavior.

34. $\lim_{x\to-1}\frac{x^2-x-2}{x+1}$

35. $\lim_{x\to1}\frac{9x+2}{x-1}$

36. $\lim_{x\to7}\frac{x^2-6x-7}{x-7}$

37. $\lim_{x\to-3}\frac{x^2+3x}{x^2+6x+9}$

38. $\lim_{x\to4}\frac{x^2-x-12}{x^2-9x+20}$

39. $\lim_{x\to 2} \dfrac{x^2-4x+7}{(x-2)^2}$

40. $\lim_{x\to 2} \dfrac{x^2-x-2}{x^2-4x+4}$

41. $\lim_{x\to -2} \dfrac{x^2+3x+2}{x^2-2x-8}$

For Exercises 42 - 53, find the limit algebraically. If the limit does not exist, state so and use limit notation to describe any infinite behavior.

42. $\lim_{x\to 6} f(x)$, where $f(x) = \begin{cases} x^2-8x+1 & x<5 \\ \sqrt{x-5} & x\geq 5 \end{cases}$

43. $\lim_{x\to -5} f(x)$, where $f(x) = \begin{cases} 4e^{x+5} & x<-3 \\ \frac{3}{5}x^2-7x & x\geq -3 \end{cases}$

44. $\lim_{x\to -2^+} f(x)$, where $f(x) = \begin{cases} 3(x-1)^2+1 & x\leq -2 \\ 5x^2-x+3 & x>-2 \end{cases}$

45. $\lim_{x\to 6^-} f(x)$, where $f(x) = \begin{cases} \ln(7-x)+5x & x<6 \\ 2^x-6 & x\geq 6 \end{cases}$

46. $\lim_{x\to 3} f(x)$, where $f(x) = \begin{cases} x^2-x+4 & x\leq 3 \\ 2x+1 & x>3 \end{cases}$

47. $\lim_{x\to 2} f(x)$, where $f(x) = \begin{cases} e^{4x-5} & x\leq 2 \\ e^{27-12x} & x>2 \end{cases}$

48. $\lim_{x\to -1} f(x)$, where $f(x) = \begin{cases} \dfrac{x^2-8x}{4-x^2} & x\leq -1 \\ -2x^3+6x+7 & x>-1 \end{cases}$

49. $\lim_{x\to -7} f(x)$, where $f(x) = \begin{cases} \sqrt[3]{15+x} & x\leq -7 \\ \ln(8+x)-2x & x>-7 \end{cases}$

50. $\lim_{x\to 0} f(x)$, where $f(x) = \begin{cases} 5^{4x-3} & x \leq -2 \\ \dfrac{x^2-x}{x^2+2x} & x > -2 \end{cases}$

51. $\lim_{x\to 4} f(x)$, where $f(x) = \begin{cases} \dfrac{3x^3-4x^2+2x}{x^2-3x-4} & x < 7 \\ \dfrac{e^{-2x}}{20-3x^2} & x > 7 \end{cases}$

52. $\lim_{x\to -6^-} f(x)$, where $f(x) = \begin{cases} \dfrac{x^2+3x-18}{x^2+12x+36} & x \leq -6 \\ \ln(x+7) & x > -6 \end{cases}$

53. $\lim_{x\to 2^+} f(x)$, where $f(x) = \begin{cases} x^2+5 & x \leq 2 \\ \dfrac{x^2+x-6}{x-2} & x > 2 \end{cases}$

For Exercises 54 - 61, find the limit algebraically.

54. $\lim_{x\to -2} \dfrac{\frac{2}{x}+1}{x+2}$

55. $\lim_{x\to -4} \dfrac{\frac{8}{x}+2}{x+4}$

56. $\lim_{x\to 1} \dfrac{\frac{1}{x}-1}{x-1}$

57. $\lim_{x\to 5} \dfrac{\frac{10}{x}-2}{x-5}$

58. $\lim_{x\to 16} \dfrac{\sqrt{x}-4}{x-16}$

59. $\lim_{x\to 1} \dfrac{1-\sqrt{x}}{x-1}$

60. $\lim_{x\to 25} \dfrac{x-25}{\sqrt{x}-5}$

61. $\lim_{x \to 4} \frac{x-4}{2-\sqrt{x}}$

Mastery Practice

62. Given the information below, find the following limits, if they exist.

$$\lim_{x \to 3^-} f(x) = 1 \qquad \lim_{x \to 3^+} f(x) = -6 \qquad f(3) = 6 \qquad \lim_{x \to 3^-} g(x) = 7 \qquad \lim_{x \to 3^+} g(x) = -4 \qquad g(3) = 5$$

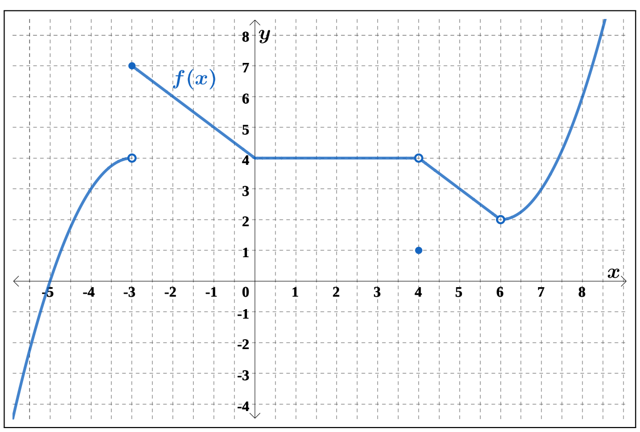
(a) $\lim_{x \to 3^-} \left[3[f(x)]^2 - g(x)\right]$

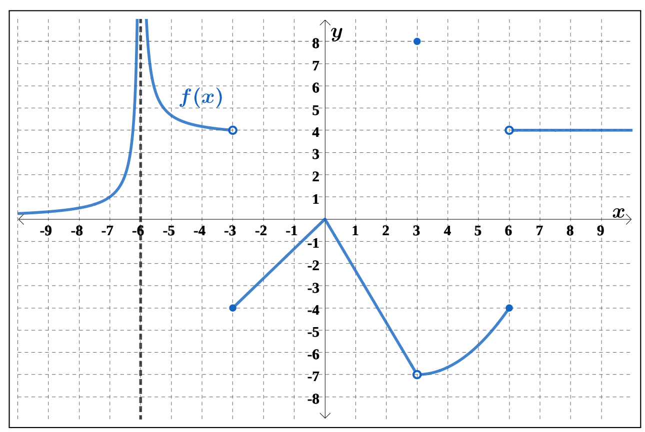
(b) $\lim_{x \to 3^+} [17 - g(x)]$

(c) $\lim_{x \to 3^+} \frac{g(x)}{f(x)}$

(d) $\lim_{x \to 3^-} \frac{g(x)+2}{f(x)-1}$

63. Given the graphs of f and g shown below, find the following limits, if they exist.





(a) $\lim_{x \to -3^-} [4f(x) - g(x)]$

(b) $\lim_{x \to 0} \frac{f(x)-10}{[g(x)]^2}$

(c) $\lim_{x \to 3} \left[[f(x)]^2 - 2g(x)\right]$

(d) $\lim_{x \to 6^-} \frac{g(x)}{f(x)-1}$

For Exercises 64 - 67, find the limit algebraically, if it exists.

64. $\lim_{x\to 7^-} \dfrac{|x-7|}{x-7}$

65. $\lim_{x\to -2} \dfrac{x+2}{|x+2|}$

66. $\lim_{x\to 3} \dfrac{9-x^2}{|3-x|}$

67. $\lim_{x\to 6^+} \dfrac{|6-x|}{36-x^2}$

For Exercises 68 - 91, find the limit algebraically. If the limit does not exist, state so and use limit notation to describe any infinite behavior.

68. $\lim_{x\to \frac{1}{2}} \dfrac{6x^2+11x-7}{8x^2-10x+3}$

69. $\lim_{x\to 4^+} \left(2x^3-\sqrt{x}+\dfrac{5}{8}\right)$

70. $\lim_{x\to 1^-} \dfrac{\dfrac{8}{x+5}-\dfrac{4}{x+2}}{x-1}$

71. $\lim_{t\to 5} \dfrac{25-t^2}{|5-t|}$

72. $\lim_{x\to -3^-} \dfrac{4e^{x-9}}{x^2-6x-27}$

73. $\lim_{x\to 18^+} \dfrac{\sqrt{x+7}-5}{x-18}$

74. $\lim_{t\to 1} \dfrac{5t^4-3t^3-2t^2}{4t^3-7t^2+3t}$

75. $\lim_{x\to 2} \dfrac{3x^2-11x+10}{x^3-4x^2+4x}$

76. $\lim_{x\to 8} \dfrac{\dfrac{2}{x}-\dfrac{1}{x-4}}{x-8}$

77. $\lim_{x\to 9} \dfrac{\sqrt{x}-3}{x-9}$

78. $\lim_{t\to -4} \dfrac{\ln(t+6)}{t^2+8t+16}$

79. $\lim_{x\to 6^-} \dfrac{|x-6|}{x-6}$

80. $\lim_{x\to 7} \dfrac{\dfrac{2}{x+1}-\dfrac{x-6}{x-3}}{x-7}$

81. $\lim_{x\to 3^+} \dfrac{2x^3-5x^2-3x}{3x^4-10x^3+3x^2}$

82. $\lim_{x\to 0} \dfrac{x\cdot 2^x}{14x^3-35x^2}$

83. $\lim_{t\to -3} \dfrac{t+3}{\sqrt{t+4}-1}$

84. $\lim_{x\to -9} \dfrac{x+9}{|x+9|}$

85. $\lim_{x\to -\frac{2}{3}} \dfrac{12x^2+17x+6}{6x^2-11x-10}$

86. $\lim_{x\to 1} \dfrac{\ln(2x+5)}{\sqrt[3]{22x^2-2x+7}}$

87. $\lim_{t\to -3} \dfrac{\dfrac{7}{t}+\dfrac{2t-8}{t-3}}{t+3}$

88. $\lim_{x\to 10^+} \dfrac{|10-x|}{100-x^2}$

89. $\lim_{x\to 2^-} \dfrac{-4x^2}{\ln(3-x)}$

90. $\lim_{t\to 9} \dfrac{t-9}{\sqrt{t}-3}$

91. $\lim_{x\to -4} \dfrac{7x-6}{(e^{2x+8}-1)^2}$

For Exercises 92 - 98, find each limit or function value algebraically. If a limit does not exist, state so and use limit notation to describe any infinite behavior.

92. $f(x) = \begin{cases} \dfrac{2x^2-3x-2}{2x+1} & x < -\frac{1}{2} \\ 2x+7 & x > -\frac{1}{2} \end{cases}$

(a) $\lim\limits_{x\to 4} f(x)$

(b) $f\left(-\frac{1}{2}\right)$

(c) $\lim\limits_{x\to -\frac{1}{2}^-} f(x)$

93. $f(x) = \begin{cases} \sqrt{13-x} & x \le -3 \\ \dfrac{x^2+5x+6}{2x^2+x-15} & x > -3 \end{cases}$

(a) $\lim\limits_{x\to -3^+} f(x)$

(b) $\lim\limits_{x\to -12} f(x)$

(c) $\lim\limits_{x\to \frac{5}{2}} f(x)$

94. $f(x) = \begin{cases} \dfrac{2x^2+9x-5}{x^2+7x+10} & x < -5 \\ 10 & x = -5 \\ \dfrac{2x-3}{-4x+1} & x > -5 \end{cases}$

(a) $\lim\limits_{x\to \frac{1}{4}} f(x)$

(b) $f(-5)$

(c) $\lim\limits_{x\to -5} f(x)$

95. $f(x) = \begin{cases} 2x-1 & x \le -7 \\ \dfrac{3x-8}{4-x} & -7 < x < 1 \\ \dfrac{2x^2+x-3}{x^2-5x+4} & x \ge 1 \end{cases}$

(a) $\lim\limits_{x\to 1} f(x)$

(b) $\lim\limits_{x\to -7^+} f(x)$

(c) $\lim\limits_{x\to 4} f(x)$

96. $f(x) = \begin{cases} 8^{x+5}+6x & x < -5 \\ \dfrac{x^2-4x-12}{x^2+4x+4} & -5 < x < 6 \\ \dfrac{\ln(7-x)}{-4x^3+9x} & 6 \le x < 7 \end{cases}$

(a) $\lim\limits_{x\to -2} f(x)$

(b) $\lim\limits_{x\to -5^-} f(x)$

(c) $\lim\limits_{x\to 6} f(x)$

97. $f(x) = \begin{cases} \dfrac{x^2+4x-32}{x^2-8x+16} & x < 4 \\ e^{x-2} & 4 \le x < 12 \\ \dfrac{\sqrt[3]{x+15}}{6x-70} & x \ge 12 \end{cases}$

(a) $f(4)$

(b) $\lim\limits_{x\to 12} f(x)$

(c) $\lim\limits_{x\to 4^-} f(x)$

98. $f(x) = \begin{cases} \dfrac{x^2+x-6}{x^2+3x} & x < 0 \\ \dfrac{1}{\log_2(8-x)} & 0 < x < 6 \\ \dfrac{x^2-32}{4e^{6-x}} & x \geq 6 \end{cases}$

(a) $\lim_{x \to 6} f(x)$

(b) $\lim_{x \to 0} f(x)$

(c) $\lim_{x \to -3} f(x)$

99. Find the following limit, if it exists: $\lim_{x \to a} \dfrac{2x(x-a)(3x+7)}{(x-a)(x+2)}$, where a is any nonzero real number.

100. Find the following limit, if it exists: $\lim_{x \to -a} \dfrac{(2x+7)(x+a)}{(x-a)(x+a)}$, where a is any nonzero real number.

101. Find the following limit, if it exists: $\lim_{h \to 0} \dfrac{\frac{1}{a+h} - \frac{1}{a}}{h}$, where a is any nonzero real number.

102. If $f(x) = \begin{cases} a(x-b)^2 + c & x < b \\ a(x-b) + c & x > b \end{cases}$, where a, b, and c are real numbers, find each of the following, if they exist:

(a) $\lim_{x \to b^-} f(x)$

(b) $\lim_{x \to b^+} f(x)$

(c) $\lim_{x \to b} f(x)$

(d) $f(b)$

103. If $f(x) = \begin{cases} \dfrac{|x|}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$, find each of the following, if they exist:

(a) $\lim_{x \to 0^-} f(x)$

(b) $\lim_{x \to 0^+} f(x)$

(c) $\lim_{x \to 0} f(x)$

(d) $f(0)$

Communication Practice

104. What is the difference between finding a limit graphically, numerically, and algebraically?

105. When calculating a limit algebraically, what technique should you try first?

106. If after substituting x to calculate a limit algebraically you get an indeterminate form, what should you try to do next to calculate the limit?

107. You use direct substitution to find $\lim_{x\to a} f(x)$, where f is a rational function. If the result is of the indeterminate form $\frac{0}{0}$, what are the possibilities for the behavior of $f(x)$ at $x = a$?

108. You use direct substitution to find $\lim_{x\to 7} f(x)$, where f is a rational function. If the limit of the function in the numerator is nonzero and the limit of the function in the denominator equals zero, can it be concluded that $x = 7$ is a vertical asymptote? Why or why not?

109. You use direct substitution to find $\lim_{x\to a} f(x)$, where f is a rational function. If the result is of the indeterminate form $\frac{0}{0}$, can it be concluded that there is a hole in the graph of $f(x)$ at $x = a$? Why or why not?

110. If $f(x) = \dfrac{(x+1)(x-2)}{x-2}$ and $g(x) = x+1$, does $f(x) = g(x)$? Explain.

1.3 Limits: At Infinity and Infinite

In the previous sections, we investigated the values of functions near specific x-values such as $x = c$, where c is a real number. In other words, we found limits such as $\lim_{x\to 2} x^2 = 4$ and $\lim_{x\to 2} \frac{x}{x-1} = 2$. In this section, we will find limits like $\lim_{x\to\infty} x^2$ and $\lim_{x\to\infty} \frac{x}{x-1}$. Compare $\lim_{x\to 2} x^2$ to $\lim_{x\to\infty} x^2$ and $\lim_{x\to 2} \frac{x}{x-1}$ to $\lim_{x\to\infty} \frac{x}{x-1}$. What difference do you notice?

Hopefully, you noticed that in one limit x is approaching 2 and in the other limit x is approaching positive infinity. That is, instead of $x \to 2$, we have $x \to \infty$. So instead of investigating the values of a function near a specific x-value, we will now investigate the values of a function as the x-values get very large in the positive direction (i.e., as $x \to \infty$) and very small in the negative direction (i.e., as $x \to -\infty$).

When looking at the graph of a function, this corresponds to observing the function values on the far right of the graph to find the limit as $x \to \infty$ and observing the function values on the far left of the graph to find the limit as $x \to -\infty$.

This concept, known as **limits at infinity**, is frequently used in finance and economics to determine the long term behavior of various mathematical models. Sometimes investment companies want to predict the value of an investment after a long period of time, and to do this, they have to determine the value of the investment as time, t, gets large (i.e., as $t \to \infty$).

Learning Objectives:
In this section, you will learn how to find limits at infinity and determine if functions have any corresponding horizontal asymptotes. You will also learn how to find holes and vertical asymptotes in the graphs of rational functions and distinguish between the two. Upon completion you will be able to:

- Identify limits at infinity graphically.
- Calculate limits at infinity algebraically of polynomials.
- Calculate limits at infinity algebraically of rational functions.
- Calculate limits at infinity algebraically of functions involving exponential functions.
- Calculate limits at infinity to determine the end behavior of polynomials.
- Calculate limits at infinity to find horizontal asymptotes of rational functions.
- Calculate limits at infinity to find horizontal asymptotes of functions involving exponential functions.
- Locate holes of rational functions algebraically.
- Locate vertical asymptotes of rational functions algebraically.
- Describe the infinite behavior near vertical asymptotes using limit notation.

Limits at Infinity

Recall that $\lim_{x\to c} f(x) = L$ means that the y-values of $f(x)$ get close to the finite number L when x gets close to c, a real number, from both the left and right. We will now consider what happens to the values of a function as x gets "close to" infinity (or negative infinity). In other words, we will find limits of the form $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. These limits are referred to as limits at infinity.

We can estimate limits at infinity graphically and numerically, and we can also calculate limits at infinity exactly using algebraic techniques. We will explore all three methods in this section.

Let's consider the function $f(x) = 2 + \frac{1}{x}$. To find $\lim_{x\to\infty} f(x)$, we need to investigate the y-values as x gets "close to" infinity. This means we need to find values of the function as x increases without bound. Thus, to estimate $\lim_{x\to\infty} f(x)$ numerically, we will substitute increasingly large x-values into the function and observe the corresponding y-values of the function.

For instance, we may choose to evaluate $f(x)$ at $x = 100$, $x = 1000$, and $x = 10,000$. Evaluating $f(x)$ at these x-values gives $f(100) = 2.01$, $f(1000) = 2.001$, and $f(10,000) = 2.0001$. It appears that as $x \to \infty$, the values of the function approach $y = 2$. So we estimate that $\lim_{x\to\infty} f(x) = 2$.

Similarly, we can find $\lim_{x\to-\infty} f(x)$ numerically by substituting x-values that are getting "close to" negative infinity. We might choose $x = -100$, -1000, and $-10,000$. Evaluating f at these x-values gives $f(-100) = 1.99$, $f(-1000) = 1.999$, and $f(-10,000) = 1.9999$. Again, we can make a reasonable guess that $\lim_{x\to-\infty} f(x) = 2$.

We can verify the numeric estimates found above by investigating the limits graphically as well (see **Figure 1.3.1**). To do this, we look at the values of the function on the far right of the graph to find $\lim_{x\to\infty} f(x)$ and on the far left of the graph to find $\lim_{x\to-\infty} f(x)$:

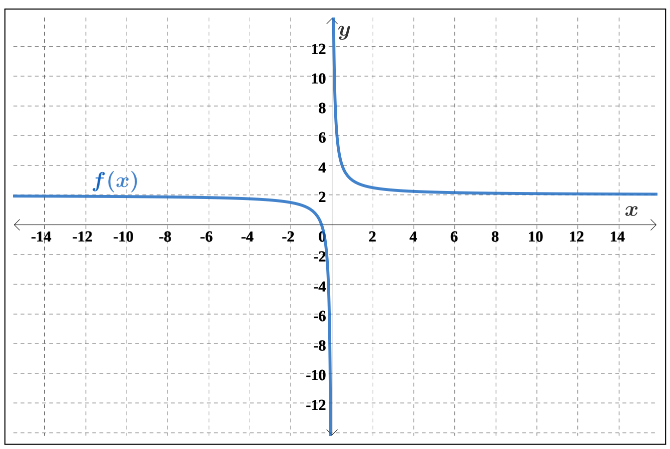


Figure 1.3.1: Graph of $f(x) = 2 + \frac{1}{x}$

By inspection, it seems that the values of the function approach $y = 2$ as $x \to -\infty$ and $x \to \infty$. So we predict $\lim_{x\to\infty} f(x) = 2$ and $\lim_{x\to-\infty} f(x) = 2$.

Before we learn how to calculate limits at infinity (i.e., $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$) algebraically, let's practice estimating these limits graphically. Note that we may also refer to finding limits at infinity as finding the **end behavior** of a function because we are observing the function values, or behavior, at each "end" of the function.

▪ **Example 1** Given the graph of f shown in **Figure 1.3.2**, estimate $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$. If a limit does not exist because the function has infinite behavior, use limit notation to describe the infinite behavior.

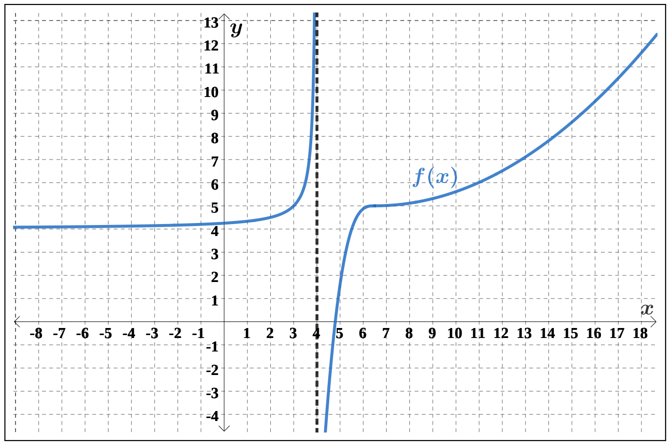


Figure 1.3.2: Graph of a function f

Solution:
Looking at the graph, we see that as x gets very small in the negative direction (i.e., as $x \to -\infty$), the y-values of the values of the function seem to approach $y = 4$. Thus, we predict

$$\lim_{x\to-\infty} f(x) = 4$$

Investigating the values of the function as $x \to \infty$, it seems the values of the function do not approach a finite value. Instead, the values of the function seem to grow increasingly large as x gets large. Thus, we predict $\lim_{x\to\infty} f(x)$ does not exist. However, when finding limits at infinity (i.e., end behavior) of a function, we are interested in the specific behavior of the y-values of the function. Therefore, we describe the infinite behavior as

$$\lim_{x\to\infty} f(x) \to \infty$$

▪

As seen in the previous example, $\lim_{x\to-\infty} f(x)$ *does not necessarily equal* $\lim_{x\to\infty} f(x)$*. In general, a function does not have to have the same end behavior on both ends.*

Now that you have a feel for the concept of limits at infinity, we will discuss how to find $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$ algebraically for different classes of functions. We will start with the nicest class of functions we have: polynomials. Then, we will learn how to find limits at infinity of rational functions and functions involving exponential functions.

Polynomials

Suppose $f(x) = x^3$, $g(x) = x^5$, and $h(x) = x^8$. If we evaluate these functions at increasingly large values of x, say at $x = 1000$ and then $x = 10,000$, we see the values of all three functions approach positive infinity. In other words, $f(x) \to \infty$, $g(x) \to \infty$, and $h(x) \to \infty$. However, if we evaluate these functions at values of x approaching negative infinity, say at $x = -1000$ and then $x = -10,000$, we see that $f(x) \to -\infty$ and $g(x) \to -\infty$, but $h(x) \to \infty$. Why the difference?

It is because f and g are odd degree polynomials and h is an even degree polynomial. It turns out that when determining the end behavior of polynomial functions, we only need to consider two aspects of the polynomial: its degree and leading coefficient.

When analyzing the degree of a polynomial, we only need to determine whether the degree is odd or even. If the degree of the polynomial is even, the limits as $x \to \infty$ and $x \to -\infty$ will behave the same way; they will either both tend to positive infinity or both tend to negative infinity. For an odd degree polynomial, the end behaviors are opposite of each other; the function will tend toward positive infinity on one end and negative infinity on the other.

Whether a polynomial function tends to positive or negative infinity also depends on the sign of its leading coefficient. For instance, as $x \to \infty$, the values of a polynomial with a positive leading coefficient tend toward positive infinity (whether the polynomial is of even or odd degree). However, if the polynomial has a negative leading coefficient, the function will tend toward negative infinity as $x \to \infty$ (whether the polynomial is of even or odd degree).

For example, the leading coefficient of $p(x) = 3x^2$ is positive. Thus, as $x \to \infty$, $p(x) \to \infty$. Because p is an even degree polynomial, the end behavior of the function on each end is the same. Therefore, as $x \longrightarrow -\infty$, $p(x) \to \infty$. On the other hand, if we change the leading coefficient so it is negative, say $q(x) = -3x^2$, then the end behavior is still the same on both ends because it is an even degree polynomial, but because the leading coefficient is negative, the function approaches negative infinity on both ends: $\lim\limits_{x\to-\infty} q(x) \to -\infty$ and $\lim\limits_{x\to\infty} q(x) \to -\infty$.

We summarize the previous information in **Table 1.8**:

Degree	**Leading Coefficient**	$x \to \infty$	$x \to -\infty$
even	positive	$\lim\limits_{x\to\infty} p(x) \to \infty$	$\lim\limits_{x\to-\infty} p(x) \to \infty$
even	negative	$\lim\limits_{x\to\infty} p(x) \to -\infty$	$\lim\limits_{x\to-\infty} p(x) \to -\infty$
odd	positive	$\lim\limits_{x\to\infty} p(x) \to \infty$	$\lim\limits_{x\to-\infty} p(x) \to -\infty$
odd	negative	$\lim\limits_{x\to\infty} p(x) \to -\infty$	$\lim\limits_{x\to-\infty} p(x) \to \infty$

Table 1.8: End behaviors of polynomial functions given the degree and leading coefficient

The end behavior of a polynomial function is always positive or negative infinity. This means these limits ***do not exist****, but remember our goal in this section is to use limit notation to describe the infinite behavior.*

Example 2 Find the following limits. If a limit does not exist because the function has infinite behavior, use limit notation to describe the infinite behavior.

a. $\lim\limits_{x\to\infty} \left(4x^5 - 3x^3 + 2\right)$

b. $\lim\limits_{x\to-\infty} \left(\left(x^5 - 3x^2 + 12\right)\left(x^3\right)\right)$

c. $\lim\limits_{x\to-\infty} \left(x + x^2 - x^{12}\right)$

Solution:

a. The function we are finding the limit of is a polynomial, so $\lim\limits_{x\to\infty} \left(4x^5 - 3x^3 + 2\right)$ does not exist because the function has infinite behavior as $x \to \infty$. Thus, we need to describe the infinite behavior, and to do so we only need to find the sign of the leading coefficient (we need not be concerned with the degree of the polynomial because $x \to \infty$). The leading coefficient is 4, which is positive, so we know as $x \to \infty$ the values of the function tend to positive infinity. Therefore,

$$\lim_{x\to\infty} (4x^5 - 3x^3 + 2) \to \infty$$

b. The function we are finding the limit of is a polynomial, so $\lim_{x\to-\infty}\left(\left(x^5-3x^2+12\right)\left(x^3\right)\right)$ does not exist because the function has infinite behavior as $x\to-\infty$. We can, however, describe the infinite behavior. To do so, we need to find both the degree and the leading coefficient of the polynomial. We may be tempted to say the degree of the polynomial is 5, but we need to be careful! We have to multiply the factors x^5-3x^2+12 and x^3 before we can determine the degree:

$$\left(x^5-3x^2+12\right)\left(x^3\right)=x^8-3x^5+12x^3$$

Now, we see the degree is 8, an even number, and the leading coefficient is 1, a positive number. Using **Table 1.8**, we see

$$\lim_{x\to-\infty}\left(\left(x^5-3x^2+12\right)\left(x^3\right)\right)\to\infty$$

c. The function we are finding the limit of is a polynomial, so $\lim_{x\to-\infty}\left(x+x^2-x^{12}\right)$ does not exist. We have to be careful when determining the infinite behavior. The first term of a polynomial may not always be the degree (or highest power) of the polynomial. The degree of this polynomial is 12, which is even. The leading coefficient is -1, which is negative. Using **Table 1.8**, we see

$$\lim_{x\to-\infty}\left(x+x^2-x^{12}\right)\to-\infty$$

■

Try It # 1:
Find the following limits. If a limit does not exist because the function has infinite behavior, use limit notation to describe the infinite behavior.

a. $\lim_{x\to\infty}\left(x^5-3x^2+12\right)$

b. $\lim_{x\to-\infty}\left(\left(-2x^4\right)\left(x^2-x^{12}\right)\right)$

Rational Functions

Our next class of functions to learn how to find limits at infinity of, or end behavior of, is rational functions. Recall that a rational function is the ratio of two polynomial functions, as long as the polynomial in the denominator is not the zero polynomial. To explore the end behavior of rational functions, we will start with an example and then analyze the results to discuss a general method for finding limits at infinity of rational functions.

■ **Example 3** Estimate $\lim_{x\to\infty}\frac{1}{x^2}$. If the limit does not exist because the function has infinite behavior, use limit notation to describe the infinite behavior.

Solution:
Let's start by estimating this limit numerically in **Table 1.9**:

$x\to\infty$	$f(x)=\frac{1}{x^2}$
10	0.01
100	0.0001
1000	0.000001
10,000	0.00000001

Table 1.9: Values of $f(x)=\frac{1}{x^2}$ for increasingly large x-values

Looking at the function values in the table, we predict

$$\lim_{x\to\infty} \frac{1}{x^2} = 0$$

Let's examine more closely why this is. If we have one pizza and divide it evenly among 100 (or 10^2) people, that's not a very big slice of pizza for each person. If we increase the number of people at our already sad pizza party to 10,000 (which would give us 100^2 people), each person gets an even smaller slice! If billions of people join in, each person only gets a few atoms of the pizza, which is basically none at all! We say that this limit is zero pizza per person, or (less relevant to life and more relevant to this problem) $\lim\limits_{x\to\infty} \frac{1}{x^2} = 0$.

■

The previous example is a particular case of an important theorem we will use throughout this section:

Theorem 1.3 If n is a positive integer, then $\lim\limits_{x\to\infty} \frac{1}{x^n} = 0$ and $\lim\limits_{x\to-\infty} \frac{1}{x^n} = 0$.

This theorem also helps us develop our approach: To find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$ when f is a rational function, we will divide every term in the numerator and the denominator by x^p, where p is the highest power of x that appears *in the denominator*. Any terms of the form $\frac{1}{x^n}$, where n is a positive integer, will tend to zero, and then we can consider the behavior of the remaining terms.

■ **Example 4** Find $\lim\limits_{x\to\infty} \frac{6x^5+2x}{3x^5+4x^2+18}$. If the limit does not exist because the function has infinite behavior, use limit notation to describe the infinite behavior.

Solution:
Recall from **Section 1.2** that we must be careful when finding the limit of a quotient, or ratio, of two functions. This includes finding limits at infinity of a ratio of two functions. The Properties of Limits apply to limits at infinity as well, and to use Property #7 and divide the limits of the numerator and denominator, we must check that both limits exist and that the limit of the function in the denominator is nonzero.

If we observe the behavior of the numerator and denominator separately as x gets large, we see $\lim\limits_{x\to\infty}\left(6x^5+2x\right) \to \infty$ and $\lim\limits_{x\to\infty}\left(3x^5+4x^2+1\right) \to \infty$ based on our previous discussion about limits at infinity of polynomials.

Unfortunately, this means we cannot determine the overall behavior of the rational function as it is currently because the limit is of the indeterminate form $\frac{\infty}{\infty}$. We will have to algebraically manipulate the function to find the limit, if it exists.

Our strategy will be to divide both the numerator and denominator by x^5. We specifically choose x^5 because it is part of the term in the denominator that is "getting the largest" when $x \to \infty$. In other words, it is the highest power of x in the denominator. Dividing both the numerator and the denominator by x^5 will not change the value of the function because we will be dividing by 1:

$$\lim_{x\to\infty} \frac{6x^5+2x}{3x^5+4x^2+18} = \lim_{x\to\infty} \frac{\dfrac{6x^5+2x}{x^5}}{\dfrac{3x^5+4x^2+18}{x^5}}$$

$$= \lim_{x\to\infty} \frac{\dfrac{6x^5}{x^5}+\dfrac{2x}{x^5}}{\dfrac{3x^5}{x^5}+\dfrac{4x^2}{x^5}+\dfrac{18}{x^5}}$$

$$= \lim_{x\to\infty} \frac{6+\dfrac{2}{x^4}}{3+\dfrac{4}{x^3}+\dfrac{18}{x^5}}$$

Notice the resulting function is also a quotient, or ratio, of two functions. Hence, to use the Properties of Limits and divide the limits of the numerator and denominator, we must again check that both limits exist and that the limit of the function in the denominator is nonzero.

Luckily, although the limit of the numerator of the resulting quotient may or may not exist, our strategy of dividing by the highest power of x in the denominator ensures the limit of the denominator of the resulting quotient will always exist, and furthermore, be nonzero.

Thus, to check for existence of a limit at infinity of a rational function after we have performed the relevant division, we only need to check for existence of the limit of the resulting numerator. In any case, though, to determine the value of a limit at infinity of a rational function (if the limit exists) or any infinite behavior of the function (if the limit does not exist), we still need to check the limits of both the numerator and denominator of the quotient that results after performing division.

Let's start by finding the behavior of the numerator. We will apply **Theorem 1.3** where appropriate:

$$\lim_{x\to\infty}\left(6+\frac{2}{x^4}\right) = \lim_{x\to\infty}\left(\overset{6}{\cancel{6}}+\overset{0}{\cancel{\frac{2}{x^4}}}\right)$$

$$= 6$$

Next, we will determine the behavior of the denominator:

$$\lim_{x\to\infty}\left(3+\frac{4}{x^3}+\frac{18}{x^5}\right) = \lim_{x\to\infty}\left(\overset{3}{\cancel{3}}+\overset{0}{\cancel{\frac{4}{x^3}}}+\overset{0}{\cancel{\frac{18}{x^5}}}\right)$$

$$= 3$$

We see both limits exist and the limit of the function in the denominator is nonzero. Now, we can finish finding the limit of the resulting quotient by using the Properties of Limits and dividing the limits of the numerator and denominator (specifically, by using Property #7):

$$\lim_{x\to\infty} \frac{6+\dfrac{2}{x^4}}{3+\dfrac{4}{x^3}+\dfrac{18}{x^5}} = \frac{\lim\limits_{x\to\infty}\left(6+\dfrac{2}{x^4}\right)}{\lim\limits_{x\to\infty}\left(3+\dfrac{4}{x^3}+\dfrac{18}{x^5}\right)}$$

$$= \frac{6}{3}$$

$$= 2$$

Hence,

$$\lim_{x\to\infty} \frac{6x^5+2x}{3x^5+4x^2+18} = \lim_{x\to\infty} \frac{6+\dfrac{2}{x^4}}{3+\dfrac{4}{x^3}+\dfrac{18}{x^5}} = 2$$

We can check our answer by looking at the graph of the function $f(x) = \dfrac{6x^5+2x}{3x^5+4x^2+18}$ shown in **Figure 1.3.3**.

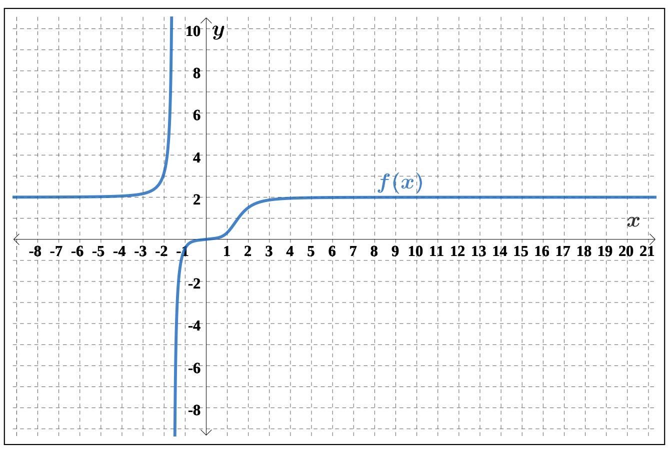


Figure 1.3.3: Graph of $f(x) = \dfrac{6x^5+2x}{3x^5+4x^2+18}$

Indeed, as $x \to \infty$, the limit equals 2.

■

Try It # 2:

Find $\lim\limits_{x\to-\infty} \dfrac{6x^5+2x}{3x^5+4x^2+18}$. If the limit does not exist because the function has infinite behavior, use limit notation to describe the infinite behavior.

Looking at the graph of the function in the previous example (**Figure 1.3.3**), we see the function approaches $y = 2$ as $x \to \infty$ and $x \to -\infty$. Graphically speaking, the function approaches the *line* $y = 2$. This behavior of approaching a line produces an asymptote, and in this case, it is a **horizontal asymptote**.

Definition
If $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, where L is a finite number, we say the line $y = L$ is a **horizontal asymptote** of the graph of f. ▪

N *We will discuss horizontal asymptotes and how to find them later in this section.*

▪ **Example 5** Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for each of the following functions. If a limit does not exist because the function has infinite behavior, use limit notation to describe the infinite behavior.

a. $f(x) = \dfrac{x^4 - 2x^3 + 13x}{7x^2 + x^6 - 4x^8}$

b. $f(x) = \dfrac{-2x^7 + 3x^6 - 1}{x^6 - 3x^4 + 122}$

Solution:

a. Regardless of whether we are looking for $\lim_{x\to\infty} f(x)$ or $\lim_{x\to-\infty} f(x)$, the strategy is the same: divide both the numerator and denominator by the highest power of x in the denominator.

Let's start by finding $\lim_{x\to\infty} f(x)$. The highest power of x in the denominator is x^8, so we start by dividing both the numerator and denominator by x^8:

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{x^4 - 2x^3 + 13x}{7x^2 + x^6 - 4x^8} \\
&= \lim_{x\to\infty} \frac{\dfrac{x^4 - 2x^3 + 13x}{x^8}}{\dfrac{7x^2 + x^6 - 4x^8}{x^8}} \\
&= \lim_{x\to\infty} \frac{\dfrac{x^4}{x^8} - \dfrac{2x^3}{x^8} + \dfrac{13x}{x^8}}{\dfrac{7x^2}{x^8} + \dfrac{x^6}{x^8} - \dfrac{4x^8}{x^8}} \\
&= \lim_{x\to\infty} \frac{\dfrac{1}{x^4} - \dfrac{2}{x^5} + \dfrac{13}{x^7}}{\dfrac{7}{x^6} + \dfrac{1}{x^2} - 4}
\end{aligned}$$

Again, we have to be careful when finding the limit of this resulting quotient. We must investigate the behavior of the numerator and denominator separately.

Let's start by finding the behavior of the numerator. We will apply **Theorem 1.3** where appropriate:

$$\begin{aligned}
\lim_{x\to\infty} \left(\frac{1}{x^4} - \frac{2}{x^5} + \frac{13}{x^7} \right) &= \lim_{x\to\infty} \left(\overset{0}{\cancel{\frac{1}{x^4}}} - \overset{0}{\cancel{\frac{2}{x^5}}} + \overset{0}{\cancel{\frac{13}{x^7}}} \right) \\
&= 0
\end{aligned}$$

Next, we will determine the behavior of the denominator:

$$\lim_{x\to\infty}\left(\frac{7}{x^6}-\frac{1}{x^2}-4\right)=\lim_{x\to\infty}\left(\cancelto{0}{\frac{7}{x^6}}-\cancelto{0}{\frac{1}{x^2}}-\cancelto{4}{4}\right)$$

$$=-4$$

We see both limits exist and the limit of the function in the denominator is nonzero. Now, we can finish finding the limit using the Properties of Limits and dividing the limits of the numerator and denominator:

$$\lim_{x\to\infty}\frac{\frac{1}{x^4}-\frac{2}{x^5}+\frac{13}{x^7}}{\frac{7}{x^6}+\frac{1}{x^2}-4}=\frac{\lim_{x\to\infty}\left(\frac{1}{x^4}-\frac{2}{x^5}+\frac{13}{x^7}\right)}{\lim_{x\to\infty}\left(\frac{7}{x^6}+\frac{1}{x^2}-4\right)}$$

$$=\frac{0}{-4}$$

$$=0$$

Hence,

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{-2x^7+3x^6-1}{x^6-3x^4+122}=\lim_{x\to\infty}\frac{\frac{1}{x^4}-\frac{2}{x^5}+\frac{13}{x^7}}{\frac{7}{x^6}+\frac{1}{x^2}-4}=0$$

Similarly, to find $\lim\limits_{x\to-\infty}f(x)$, we divide both the numerator and denominator by the highest power of x in the denominator, x^8:

$$\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\frac{x^4-2x^3+13x}{7x^2+x^6-4x^8}$$

$$=\lim_{x\to-\infty}\frac{\frac{x^4-2x^3+13x}{x^8}}{\frac{7x^2+x^6-4x^8}{x^8}}$$

$$=\lim_{x\to-\infty}\frac{\frac{x^4}{x^8}-\frac{2x^3}{x^8}+\frac{13x}{x^8}}{\frac{7x^2}{x^8}+\frac{x^6}{x^8}-\frac{4x^8}{x^8}}$$

$$=\lim_{x\to-\infty}\frac{\frac{1}{x^4}-\frac{2}{x^5}+\frac{13}{x^7}}{\frac{7}{x^6}+\frac{1}{x^2}-4}$$

Again, we have to be careful when finding the limit of this resulting quotient. We must investigate the behavior of the numerator and denominator separately.

Let's start by finding the behavior of the numerator. We will apply **Theorem 1.3** where appropriate:

$$\lim_{x\to-\infty}\left(\frac{1}{x^4}-\frac{2}{x^5}+\frac{13}{x^7}\right)=\lim_{x\to-\infty}\left(\overset{\nearrow 0}{\cancel{\frac{1}{x^4}}}-\overset{\nearrow 0}{\cancel{\frac{2}{x^5}}}+\overset{\nearrow 0}{\cancel{\frac{13}{x^7}}}\right)$$
$$=0$$

Next, we will determine the behavior of the denominator:

$$\lim_{x\to-\infty}\left(\frac{7}{x^6}-\frac{1}{x^2}-4\right)=\lim_{x\to-\infty}\left(\overset{\nearrow 0}{\cancel{\frac{7}{x^6}}}-\overset{\nearrow 0}{\cancel{\frac{1}{x^2}}}-\overset{\nearrow 4}{\cancel{4}}\right)$$
$$=-4$$

We see both limits exist and the limit of the function in the denominator is nonzero. Now, we can finish finding the limit using the Properties of Limits and dividing the limits of the numerator and denominator:

$$\lim_{x\to-\infty}\frac{\frac{1}{x^4}-\frac{2}{x^5}+\frac{13}{x^7}}{\frac{7}{x^6}+\frac{1}{x^2}-4}=\frac{\lim\limits_{x\to-\infty}\left(\frac{1}{x^4}-\frac{2}{x^5}+\frac{13}{x^7}\right)}{\lim\limits_{x\to-\infty}\left(\frac{7}{x^6}+\frac{1}{x^2}-4\right)}$$
$$=\frac{0}{-4}$$
$$=0$$

Hence,

$$\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\frac{-2x^7+3x^6-1}{x^6-3x^4+122}=\lim_{x\to-\infty}\frac{\frac{1}{x^4}-\frac{2}{x^5}+\frac{13}{x^7}}{\frac{7}{x^6}+\frac{1}{x^2}-4}=0$$

In summary, $\lim\limits_{x\to\infty}f(x)=0$ and $\lim\limits_{x\to-\infty}f(x)=0$.

N *In part **a** above, both the limit as $x\to\infty$ and limit as $x\to-\infty$ were the same (both equaled 0). This is not a coincidence! If a rational function has a finite limit at one infinity, then it has the same finite limit at the other infinity.*

b. Recall that we must find $\lim\limits_{x\to\infty}f(x)$ and $\lim\limits_{x\to-\infty}f(x)$ for $f(x)=\dfrac{-2x^7+3x^6-1}{x^6-3x^4+122}$. Again, regardless of whether we are looking for $\lim\limits_{x\to\infty}f(x)$ or $\lim\limits_{x\to-\infty}f(x)$, the strategy is the same: divide both the numerator and denominator by the highest power of x in the denominator.

Let's start by finding $\lim\limits_{x\to\infty}f(x)$. The highest power of x in the denominator is x^6, so we start by dividing both the numerator and denominator by x^6:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{-2x^7+3x^6-1}{x^6-3x^4+122}$$

$$= \lim_{x\to\infty} \frac{\dfrac{-2x^7+3x^6-1}{x^6}}{\dfrac{x^6-3x^4+122}{x^6}}$$

$$= \lim_{x\to\infty} \frac{\dfrac{-2x^7}{x^6}+\dfrac{3x^6}{x^6}-\dfrac{1}{x^6}}{\dfrac{x^6}{x^6}-\dfrac{3x^4}{x^6}+\dfrac{122}{x^6}}$$

$$= \lim_{x\to\infty} \frac{-2x+3-\dfrac{1}{x^6}}{1-\dfrac{3}{x^2}+\dfrac{122}{x^6}}$$

Again, we have to be careful when finding the limit of this resulting quotient. We must investigate the behavior of the numerator and denominator separately.

Let's start by finding the behavior of the numerator. We will apply **Theorem 1.3** where appropriate as well as the fact that $\lim\limits_{x\to\infty}(-2x) \to -\infty$:

$$\lim_{x\to\infty}\left(-2x+3-\frac{1}{x^6}\right) = \lim_{x\to\infty}\left(\overset{-\infty}{\cancel{-2x}} + \overset{3}{\cancel{3}} - \overset{0}{\cancel{\frac{1}{x^6}}}\right)$$

$$\to -\infty$$

The first term in this function approaches negative infinity because $\lim\limits_{x\to\infty}(-2x) \to -\infty$. Adding 3 to a number that is getting "more and more negative", or decreasing without bound, will still result in a number that is getting "more and more negative". So the function in the numerator is approaching negative infinity. Thus, this limit does not exist.

Next, we will determine the behavior of the denominator:

$$\lim_{x\to\infty}\left(1-\frac{3}{x^2}+\frac{122}{x^6}\right) = \lim_{x\to\infty}\left(1-\overset{0}{\cancel{\frac{3}{x^2}}}+\overset{0}{\cancel{\frac{122}{x^6}}}\right)$$

$$= 1$$

Therefore, the function in the denominator is approaching 1. However, because the limit of the function in the numerator does not exist (because the function approaches negative infinity), we cannot use the Properties of Limits to divide the limits of the numerator and denominator. So we must carefully analyze the overall behavior of the resulting quotient.

Due to the fact that the function in the numerator is getting "more and more negative", or decreasing without bound, and the function in the denominator is approaching a positive finite number, the quotient itself is approaching negative infinity:

$$\lim_{x\to\infty} \frac{-2x+3-\dfrac{1}{x^6}}{1-\dfrac{3}{x^2}+\dfrac{122}{x^6}} \to -\infty$$

Hence,

$$\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} \frac{-2x^7+3x^6-1}{x^6-3x^4+122} = \lim_{x\to\infty} \frac{-2x+3-\dfrac{1}{x^6}}{1-\dfrac{3}{x^2}+\dfrac{122}{x^6}} \to -\infty$$

Remember, technically this limit does not exist because it is infinite, but our goal in this section is to use limit notation to describe the infinite behavior.

Similarly, we will find $\lim\limits_{x\to-\infty} f(x)$ by dividing both the numerator and denominator by x^6:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{-2x^7+3x^6-1}{x^6-3x^4+122}$$

$$= \lim_{x\to-\infty} \frac{\dfrac{-2x^7+3x^6-1}{x^6}}{\dfrac{x^6-3x^4+122}{x^6}}$$

$$= \lim_{x\to-\infty} \frac{\dfrac{-2x^7}{x^6}+\dfrac{3x^6}{x^6}-\dfrac{1}{x^6}}{\dfrac{x^6}{x^6}-\dfrac{3x^4}{x^6}+\dfrac{122}{x^6}}$$

$$= \lim_{x\to-\infty} \frac{-2x+3-\dfrac{1}{x^6}}{1-\dfrac{3}{x^2}+\dfrac{122}{x^6}}$$

Checking the behavior of the numerator gives

$$\lim_{x\to-\infty}\left(-2x+3-\frac{1}{x^6}\right) = \lim_{x\to-\infty}\left(\overset{\infty}{\cancel{-2x}} + \overset{3}{\cancel{3}} - \overset{0}{\cancel{\frac{1}{x^6}}}\right)$$
$$\to \infty$$

The first term in this function approaches positive infinity because $\lim\limits_{x\to-\infty}(-2x) \to \infty$. Adding 3 to a number that is getting "more and more positive", or increasing without bound, will still result in a number that is getting "more and more positive". So the function in the numerator is approaching positive infinity. Thus, this limit does not exist.

Next, we will determine the behavior of the denominator:

$$\lim_{x\to-\infty}\left(1-\frac{3}{x^2}+\frac{122}{x^6}\right)=\lim_{x\to-\infty}\left(\overset{1}{\cancel{1}}-\overset{0}{\cancel{\frac{3}{x^2}}}+\overset{0}{\cancel{\frac{122}{x^6}}}\right)$$
$$=1$$

Therefore, the function in the denominator is approaching 1. Because the limit of the function in the numerator does not exist (because the function approaches positive infinity), we cannot use the Properties of Limits to divide the limits. So we must carefully analyze the overall behavior of the resulting quotient.

Due to the fact that the function in the numerator is getting "more and more positive", or increasing without bound, and the function in the denominator is approaching a positive finite number, the quotient itself is approaching positive infinity:

$$\lim_{x\to-\infty}\frac{-2x+3-\frac{1}{x^6}}{1-\frac{3}{x^2}+\frac{122}{x^6}}\to\infty$$

Hence,

$$\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\frac{-2x^7+3x^6-1}{x^6-3x^4+122}=\lim_{x\to-\infty}\frac{-2x+3-\frac{1}{x^6}}{1-\frac{3}{x^2}+\frac{122}{x^6}}\to\infty$$

In summary, $\lim_{x\to\infty}f(x)\to-\infty$ and $\lim_{x\to-\infty}f(x)\to\infty$.

▪

Try It # 3:
Find $\lim_{x\to\infty}f(x)$ and $\lim_{x\to-\infty}f(x)$ for each of the following functions. If a limit does not exist because the function has infinite behavior, use limit notation to describe the infinite behavior.

a. $f(x)=\dfrac{-4x^3-12x^2+15}{16x^3+14x-17}$

b. $f(x)=\dfrac{23x^3-15x^2-6x}{188x^2+155x+413}$

Functions Involving Exponential Functions

To determine the end behavior, or limits at infinity, of functions involving exponential functions, we will use an approach similar to that for rational functions. To help develop our approach, let's consider the end behavior of $y = e^x$ whose graph is shown in **Figure 1.3.4**.

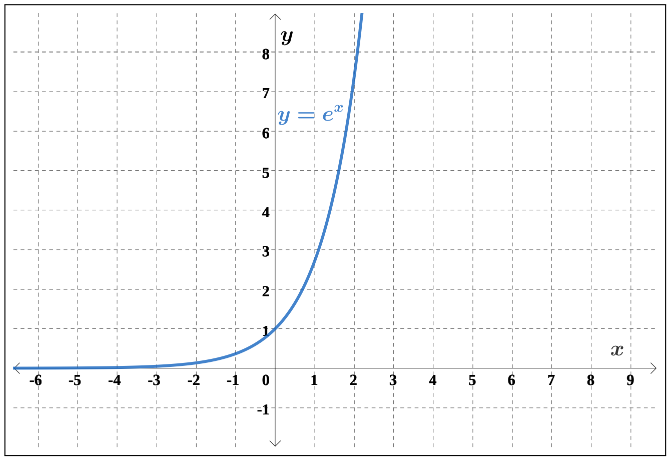


Figure 1.3.4: Graph of $y = e^x$

The graph shows $\lim_{x\to\infty} e^x \to \infty$ and $\lim_{x\to-\infty} e^x = 0$. This is true more generally:

Theorem 1.4 If n is a positive integer, then $\lim_{x\to\infty} e^{nx} \to \infty$ and $\lim_{x\to-\infty} e^{nx} = 0$.

▪ **Example 6** Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for each of the following functions. If a limit does not exist because the function has infinite behavior, use limit notation to describe the infinite behavior.

a. $f(x) = e^{5x}$

b. $f(x) = e^{-3x}$

Solution:

a. According to **Theorem 1.4**,

$$\lim_{x\to\infty} e^{5x} \to \infty \quad \text{and} \quad \lim_{x\to-\infty} e^{5x} = 0$$

b. Recall that we must find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for $f(x) = e^{-3x}$. We have to be a bit more careful here because of the negative exponent, but we can still compute the limits. Accounting for the negative sign of the coefficient in the exponent and using **Theorem 1.4** gives

$$\lim_{x\to\infty} e^{-3x} = 0 \quad \text{and} \quad \lim_{x\to-\infty} e^{-3x} \to \infty$$

▪

Try It # 4:
Find $\lim_{x\to\infty} \left(-e^{-2x}\right)$ and $\lim_{x\to-\infty} \left(-e^{-2x}\right)$. If a limit does not exist because the function has infinite behavior, use limit notation to describe the infinite behavior.

Let's expand upon these ideas to try and determine the end behavior of a function in which there is division of exponential functions or functions consisting of exponential functions.

For instance, consider $f(x) = \frac{e^{5x}}{e^{-3x}}$. To find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$, we can use laws of exponents to algebraically manipulate the function so it is of the form $f(x) = e^{8x}$. Now, we *can* find the limits at infinity using **Theorem 1.4**: $\lim_{x\to\infty} e^{8x} \to \infty$ and $\lim_{x\to-\infty} e^{8x} = 0$.

However, we will not always be so lucky. Consider finding $\lim_{x\to\infty} \frac{4e^{5x}+2}{e^{2x}+e^{-3x}+8}$. We cannot algebraically manipulate this function to arrive at a single exponential function like we did previously. To find the limits at infinity, or end behavior, of this function, we might attempt to observe the behavior of the numerator and denominator separately like we have done previously.

Let's look at the limit of the numerator. Accounting for the sign of the coefficient in the exponent and then applying **Theorem 1.4** gives

$$\begin{aligned}\lim_{x\to\infty} 4e^{5x}+2 &= \lim_{x\to\infty} \cancelto{\infty}{4e^{5x}} + \cancelto{2}{2} \\ &\to \infty\end{aligned}$$

Likewise, let's observe the limit, or behavior, of the denominator:

$$\begin{aligned}\lim_{x\to\infty} e^{2x}+e^{-3x}+8 &= \lim_{x\to\infty} \cancelto{\infty}{e^{2x}} + \cancelto{0}{e^{-3x}} + \cancelto{8}{8} \\ &\to \infty\end{aligned}$$

Unfortunately, this means we cannot determine the overall behavior of the given function as it is currently because the limit is of the indeterminate form $\frac{\infty}{\infty}$. We will have to algebraically manipulate the function to find the limit, if it exists.

Following our approach for rational functions, we will strategically select the exponential function *in the denominator* that is "getting the largest" and divide both the numerator and denominator by that function. For these types of functions, the function we select to divide by depends on whether $x \to \infty$ or $x \to -\infty$. In other words, if $x \to \infty$, we divide by the exponential function with the *most positive power* in the denominator. If $x \to -\infty$, we divide by the exponential function with the *most negative power* in the denominator.

Working with this function, $f(x) = \frac{4e^{5x}+2}{e^{2x}+e^{-3x}+8}$, in part **a** of the next example, you will see that we divide by e^{2x} to find the limit as $x \to \infty$ and divide by e^{-3x} to find the limit as $x \to -\infty$.

Similar to rational functions, applying this strategy of division will still result in a quotient, or ratio, of two functions. However, it will also ensure the limit of the resulting denominator exists and is nonzero (although the limit of the resulting numerator may or may not exist). Remember we can only use the Properties of Limits to divide limits if both exist and the limit of the function in the denominator is nonzero.

Example 7 Find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for each of the following functions. If a limit does not exist because the function has infinite behavior, use limit notation to describe the infinite behavior.

a. $f(x) = \dfrac{4e^{5x}+2}{e^{2x}+e^{-3x}+8}$

b. $f(x) = \dfrac{3e^{3x}-4e^{-4x}}{e^{-4x}+3e^{2x}-6e^{3x}}$

Solution:

a. To calculate $\lim_{x\to\infty} f(x)$, we must divide the numerator and denominator by e^{2x} because it is the exponential function in the denominator that is "getting the largest" when $x\to\infty$. In other words, it is the exponential function in the denominator with the most positive power. After dividing every term in the numerator and denominator by e^{2x}, we will use laws of exponents to algebraically manipulate each term:

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{4e^{5x}+2}{e^{2x}+e^{-3x}+8}$$

$$= \lim_{x\to\infty} \frac{\dfrac{4e^{5x}+2}{e^{2x}}}{\dfrac{e^{2x}+e^{-3x}+8}{e^{2x}}}$$

$$= \lim_{x\to\infty} \frac{\dfrac{4e^{5x}}{e^{2x}}+\dfrac{2}{e^{2x}}}{\dfrac{e^{2x}}{e^{2x}}+\dfrac{e^{-3x}}{e^{2x}}+\dfrac{8}{e^{2x}}}$$

$$= \lim_{x\to\infty} \frac{4e^{3x}+2e^{-2x}}{1+e^{-5x}+8e^{-2x}}$$

Now, to compute the limit of this resulting quotient, we must be careful and observe the behavior of the numerator and denominator separately due to the fact that the limit of the function in the numerator may or may not exist.

Starting with the numerator of the quotient and applying **Theorem 1.4** to determine its behavior as $x\to\infty$ gives

$$\lim_{x\to\infty}\left(4e^{3x}+2e^{-2x}\right) = \lim_{x\to\infty}\left(\overset{\infty}{\cancel{4e^{3x}}}+\overset{0}{\cancel{2e^{-2x}}}\right)$$

$$\to \infty$$

Thus, the function in the numerator is approaching positive infinity, so this limit does not exist. Looking at the limit of the denominator of the quotient, we see

$$\lim_{x\to\infty}\left(1+e^{-5x}+8e^{-2x}\right) = \lim_{x\to\infty}\left(\overset{1}{\cancel{1}}+\overset{0}{\cancel{e^{-5x}}}+\overset{0}{\cancel{e^{-2x}}}\right)$$

$$= 1$$

Therefore, the function in the denominator is approaching 1. Because the limit of the function in the numerator does not exist, we cannot use the Properties of Limits to divide the limits. So we must carefully analyze the overall behavior of the resulting quotient.

Due to the fact that the function in the numerator is increasing without bound and the function in the denominator is approaching a positive finite number, the quotient is approaching positive infinity:

$$\lim_{x\to\infty} \frac{4e^{3x}+2e^{-2x}}{1+e^{-5x}+8e^{-2x}} \to \infty$$

Hence,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{4e^{5x}+2}{e^{2x}+e^{-3x}+8} = \lim_{x\to\infty} \frac{4e^{3x}+2e^{-2x}}{1+e^{-5x}+8e^{-2x}} \to \infty$$

Remember, technically this limit does not exist because it is infinite, but our goal in this section is to use limit notation to describe the infinite behavior.

Next, we need to calculate $\lim_{x\to-\infty} f(x)$.

We cannot assume that $\lim_{x\to-\infty} f(x) \to \infty$ also. We must calculate the limit to see. The only way we know for certain that $\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(x)$ is if f is a rational function and one of the limits at infinity results in a finite number.

To calculate $\lim_{x\to-\infty} f(x)$, we must divide every term in the numerator and denominator by e^{-3x} because it is the exponential function in the denominator that is "getting the largest" when $x \to -\infty$. In other words, it is the exponential function in the denominator with the most negative power. After dividing every term in the numerator and denominator by e^{-3x}, we will use laws of exponents to algebraically manipulate each term:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{4e^{5x}+2}{e^{2x}+e^{-3x}+8}$$

$$= \lim_{x\to-\infty} \frac{\dfrac{4e^{5x}+2}{e^{-3x}}}{\dfrac{e^{2x}+e^{-3x}+8}{e^{-3}}}$$

$$= \lim_{x\to-\infty} \frac{\dfrac{4e^{5x}}{e^{-3x}}+\dfrac{2}{e^{-3x}}}{\dfrac{e^{2x}}{e^{-3x}}+\dfrac{e^{-3x}}{e^{-3x}}+\dfrac{8}{e^{-3x}}}$$

$$= \lim_{x\to-\infty} \frac{4e^{8x}+2e^{3x}}{e^{5x}+1+8e^{3x}}$$

Now, to compute the limit of this resulting quotient, we must be careful and observe the behavior of the numerator and denominator separately due to the fact that the limit of the function in the numerator may or may not exist.

Starting with the numerator of the quotient and looking at its behavior as $x \to \infty$ gives

$$\lim_{x\to-\infty}\left(4e^{8x}+2e^{3x}\right) = \lim_{x\to-\infty}\left(\cancelto{0}{4e^{8x}}+\cancelto{0}{2e^{3x}}\right)$$
$$= 0$$

Thus, the limit of the function in the numerator exists and equals 0. Looking at the limit of the denominator of the quotient, we see

$$\lim_{x\to-\infty}\left(e^{5x}+1+8e^{3x}\right) = \lim_{x\to-\infty}\left(\cancelto{0}{e^{5x}}+\cancelto{1}{1}+\cancelto{0}{8e^{3x}}\right)$$
$$= 1$$

Therefore, the function in the denominator is approaching 1. Because both the limit of the function in the numerator and the limit of the function in the denominator exist and the limit of the function in the denominator is nonzero, we can finish finding the limit using the Properties of Limits:

$$\lim_{x\to-\infty}\frac{4e^{8x}+2e^{3x}}{e^{5x}+1+8e^{3x}} = \frac{\lim_{x\to-\infty}\left(4e^{8x}+2e^{3x}\right)}{\lim_{x\to-\infty}\left(e^{5x}+1+8e^{3x}\right)}$$

$$= \frac{0}{1}$$

$$= 0$$

Therefore,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\frac{4e^{5x}+2}{e^{2x}+e^{-3x}+8} = \lim_{x\to-\infty}\frac{4e^{8x}+2e^{3x}}{e^{5x}+1+8e^{3x}} = 0$$

In summary, $\lim_{x\to\infty} f(x) \to \infty$ and $\lim_{x\to-\infty} f(x) = 0$.

We can check both answers by looking at the graph of the function $f(x) = \dfrac{4e^{5x}+2}{e^{2x}+e^{-3x}+8}$ shown in **Figure 1.3.5**.

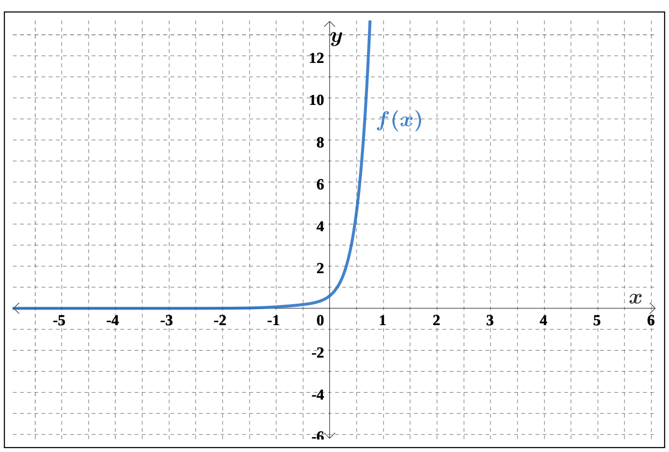


Figure 1.3.5: Graph of $f(x) = \dfrac{4e^{5x}+2}{e^{2x}+e^{-3x}+8}$

b. Recall that we must find $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ for $f(x) = \dfrac{3e^{3x} - 4e^{-4x}}{e^{-4x} + 3e^{2x} - 6e^{3x}}$.

To calculate $\lim_{x\to\infty} f(x)$, we need to divide every term in the numerator and denominator by e^{3x} because it is the exponential function in the denominator that is "getting the largest" when $x \to \infty$. In other words, it is the exponential function in the denominator with the most positive power. After dividing every term in the numerator and denominator by e^{3x}, we will use laws of exponents to algebraically manipulate each term:

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{3e^{3x} - 4e^{-4x}}{e^{-4x} + 3e^{2x} - 6e^{3x}} \\
&= \lim_{x\to\infty} \frac{\dfrac{3e^{3x} - 4e^{-4x}}{e^{3x}}}{\dfrac{e^{-4x} + 3e^{2x} - 6e^{3x}}{e^{3x}}} \\
&= \lim_{x\to\infty} \frac{\dfrac{3e^{3x}}{e^{3x}} - \dfrac{4e^{-4x}}{e^{3x}}}{\dfrac{e^{-4x}}{e^{3x}} + \dfrac{3e^{2x}}{e^{3x}} - \dfrac{6e^{3x}}{e^{3x}}} \\
&= \lim_{x\to\infty} \frac{3 - 4e^{-7x}}{e^{-7x} + 3e^{-x} - 6}
\end{aligned}$$

Again, to compute the limit of this resulting quotient, we must be careful and observe the behavior of the numerator and denominator separately.

Starting with the numerator of the quotient and looking at its behavior as $x \to \infty$ gives

$$\begin{aligned}
\lim_{x\to\infty} \left(3 - 4e^{-7x}\right) &= \lim_{x\to\infty} \left(\cancelto{3}{3} - \cancelto{0}{4e^{-7x}}\right) \\
&= 3
\end{aligned}$$

Thus, the limit of the function in the numerator exists and equals 3. Looking at the limit of the denominator of the quotient, we see

$$\begin{aligned}
\lim_{x\to-\infty} \left(e^{-7x} + 1 + 3e^{-x} - 6\right) &= \lim_{x\to-\infty} \left(\cancelto{0}{e^{-7x}} + \cancelto{0}{3e^{-x}} - \cancelto{6}{6}\right) \\
&= -6
\end{aligned}$$

Therefore, the function in the denominator is approaching -6. Because both the limit of the function in the numerator and the limit of the function in the denominator exist and the limit of the function in the denominator is nonzero, we can finish finding the limit using the Properties of Limits:

$$\lim_{x\to\infty} \frac{3-4e^{-7x}}{e^{-7x}+3e^{-x}-6} = \frac{\lim\limits_{x\to\infty}\left(3-4e^{-7x}\right)}{\lim\limits_{x\to\infty}\left(e^{-7x}+1+3e^{-x}-6\right)}$$

$$= \frac{3}{-6}$$

$$= -\frac{1}{2}$$

Hence,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3e^{3x}-4e^{-4x}}{e^{-4x}+3e^{2x}-6e^{3x}} = \lim_{x\to\infty} \frac{3-4e^{-7x}}{e^{-7x}+3e^{-x}-6} = -\frac{1}{2}$$

To calculate $\lim\limits_{x\to-\infty} f(x)$, we must divide every term in the numerator and denominator by e^{-4x} because it is the exponential function in the denominator that is "getting the largest" when $x \to -\infty$. In other words, it is the exponential function in the denominator with the most negative power. After dividing every term in the numerator and denominator by e^{-4x}, we will algebraically manipulate each term using laws of exponents:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{3e^{3x}-4e^{-4x}}{e^{-4x}+3e^{2x}-6e^{3x}}$$

$$= \lim_{x\to-\infty} \frac{\dfrac{3e^{3x}-4e^{-4x}}{e^{-4x}}}{\dfrac{e^{-4x}+3e^{2x}-6e^{3x}}{e^{-4x}}}$$

$$= \lim_{x\to-\infty} \frac{\dfrac{3e^{3x}}{e^{-4x}}-\dfrac{4e^{-4x}}{e^{-4x}}}{\dfrac{e^{-4x}}{e^{-4x}}+\dfrac{3e^{2x}}{e^{-4x}}-\dfrac{6e^{3x}}{e^{-4x}}}$$

$$= \lim_{x\to-\infty} \frac{3e^{7x}-4}{1+3e^{6x}-6e^{7x}}$$

Again, to compute the limit of this resulting quotient, we must be careful and observe the behavior of the numerator and denominator separately.

Starting with the numerator of the quotient and looking at its behavior as $x \to -\infty$ gives

$$\lim_{x\to-\infty}\left(3e^{7x}-4\right) = \lim_{x\to-\infty}\left(\cancelto{0}{3e^{7x}} - \cancelto{4}{4}\right)$$
$$= -4$$

Thus, the limit of the function in the numerator exists and equals -4.

Looking at the limit of the denominator of the quotient, we see

$$\lim_{x\to-\infty}\left(1+3e^{6x}-6e^{7x}\right) = \lim_{x\to-\infty}\left(\overset{1}{\cancel{1}}+\overset{0}{\cancel{3e^{6x}}}-\overset{0}{\cancel{6e^{7x}}}\right)$$
$$= 1$$

Therefore, the function in the denominator is approaching 1. Because both the limit of the function in the numerator and the limit of the function in the denominator exist and the limit of the function in the denominator is nonzero, we can finish finding the limit using Properties of Limits:

$$\lim_{x\to-\infty}\frac{3e^{7x}-4}{1+3e^{6x}-6e^{7x}} = \frac{\lim\limits_{x\to-\infty}\left(3e^{7x}-4\right)}{\lim\limits_{x\to-\infty}\left(1+3e^{6x}-6e^{7x}\right)}$$

$$= \frac{-4}{1}$$

$$= -4$$

Hence,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\frac{3e^{3x}-4e^{-4x}}{e^{-4x}+3e^{2x}-6e^{3x}} = \lim_{x\to-\infty}\frac{3e^{7x}-4}{1+3e^{6x}-6e^{7x}} = -4$$

In summary, $\lim\limits_{x\to\infty} f(x) = -\frac{1}{2}$ and $\lim\limits_{x\to-\infty} f(x) = -4$. Remember, you can always graph the function to verify your answers!

■

Try It # 5:
Given $f(x) = \dfrac{4e^{2x}-6e^{-3x}}{e^{2x}+e^{-6x}+2e^{-8x}}$, find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$. If a limit does not exist because the function has infinite behavior, use limit notation to describe the infinite behavior.

Now, you may be wondering what to do if the function you are given does not have an exponential function in the denominator with the necessary positive or negative power needed to perform division. For example, consider $f(x) = \dfrac{3e^{2x}-7e^{-5x}}{10-e^{-2x}}$. To find the $\lim\limits_{x\to-\infty} f(x)$, we divide every term in the function by e^{-2x}. But, what if we need to calculate $\lim\limits_{x\to\infty} f(x)$? There is no exponential function in the denominator with a positive exponent.

In this situation, when there is no exponential function by which to divide, we simply observe the behavior of the function as $x\to\infty$ or $x\to-\infty$ depending on which limit at infinity we are trying to find. Also, we must still account for the sign of the coefficient in each exponent. Let's look at an example to demonstrate this.

■ **Example 8** Given $f(x) = \dfrac{8e^{-5x}+2e^{4x}}{e^{9x}-7e^{2x}+3}$, find $\lim\limits_{x\to-\infty} f(x)$. If the limit does not exist because the function has infinite behavior, use limit notation to describe the infinite behavior.

Solution:
Because we are trying to find $\lim\limits_{x\to-\infty} f(x)$, we need to divide every term in the numerator and denominator by the exponential function with the most negative power in the denominator. However, there is no exponential function in the denominator with a negative power by which to divide. Thus, to determine the behavior of the function as

$x \to -\infty$, we must observe the behavior of the functions in the numerator and denominator as $x \to -\infty$ separately because the given function is a quotient, or ratio, of two functions (and we have to be careful about existence of the limits and whether or not the limit of the function in the denominator is zero).

We will start by observing the behavior of the function in the numerator. Accounting for the sign of the coefficient in each exponent and applying **Theorem 1.4** gives

$$\lim_{x\to-\infty}\left(8e^{-5x}+2e^{4x}\right) = \lim_{x\to-\infty}\left(\overset{\infty}{8e^{-5x}}+\overset{0}{2e^{4x}}\right)$$
$$\to \infty$$

Thus, the function in the numerator is approaching positive infinity, so this limit does not exist. Looking at the limit of the denominator, we see

$$\lim_{x\to-\infty}\left(e^{9x}-7e^{2x}+3\right) = \lim_{x\to-\infty}\left(\overset{0}{e^{9x}}-\overset{0}{7e^{2x}}+\overset{3}{3}\right)$$
$$= 3$$

Therefore, the function in the denominator is approaching 3. Because the limit of the function in the numerator does not exist, we cannot use the Properties of Limits to divide the limits. So we must carefully analyze the overall behavior of the quotient.

Due to the fact that the function in the numerator is increasing without bound and the function in the denominator is approaching a positive finite number, the quotient is approaching positive infinity:

$$\lim_{x\to-\infty}\frac{8e^{-5x}+2e^{4x}}{e^{9x}-7e^{2x}+3} \to \infty$$

Thus,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\frac{8e^{-5x}+2e^{4x}}{e^{9x}-7e^{2x}+3} \to \infty$$

■

Try It # 6:

Given $f(x) = \dfrac{e^{6x}-5e^{-3x}}{7-10e^{4x}-e^{2x}}$, find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$. If a limit does not exist because the function has infinite behavior, use limit notation to describe the infinite behavior.

Logarithmic Functions

We end this subsection with a very important class of functions: logarithmic functions of the form $y = \log_b(x)$. The calculations for the limits we will state can be found by recalling $y = \log_b(x)$ is the inverse function of $y = b^x$, but we will satisfy our curiosity by looking at the graph of $y = \ln(x)$ shown in **Figure 1.3.6**.

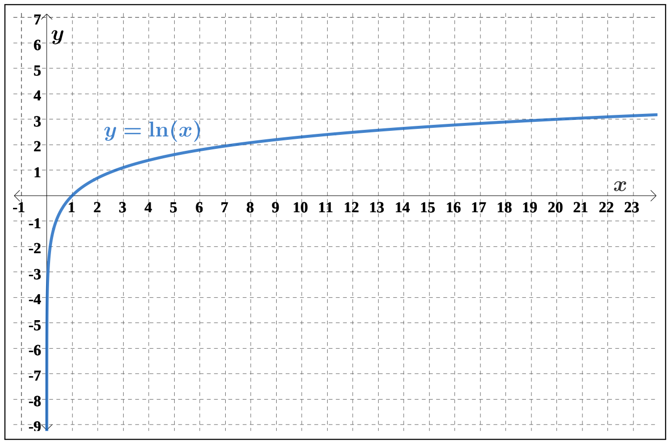


Figure 1.3.6: Graph of $y = \ln(x)$

Remember that the domain of $y = \log_b(x)$ is $(0, \infty)$ for any base b, where $b > 0$ and $b \neq 1$. Previously to find the end behavior of functions, we described it as finding both the limits as $x \to -\infty$ and $x \to \infty$. But in this case, we look at $x \to 0^+$ and $x \to \infty$ because of the domain of the logarithmic function.

From the graph, we see $\lim\limits_{x\to 0^+} (\ln(x)) \to -\infty$, which shows there is a vertical asymptote at $x = 0$. However, it is difficult to see $\lim\limits_{x\to\infty} (\ln(x))$. Because the function is increasing slowly, we may be tempted to think there is a horizontal asymptote. However, it turns out that $\lim\limits_{x\to\infty} (\ln(x)) \to \infty$. This means there is no horizontal asymptote of the function.

This is true for logarithms of any base, and we will now formalize this result:

Theorem 1.5 For any base $b > 0$ and $b \neq 1$,

$$\lim_{x\to 0^+} (\log_b(x)) \to -\infty \text{ and } \lim_{x\to\infty} (\log_b(x)) \to \infty$$

N *While we have discussed the end behavior of logarithmic functions above, it is beyond the scope of this textbook to work examples involving limits at infinity of these functions.*

Horizontal Asymptotes

We discussed the definition of **horizontal asymptotes** earlier in this section. Now, we will focus on how to find horizontal asymptotes given a function. Recall the definition of a horizontal asymptote:

Definition
If $\lim\limits_{x\to\infty} f(x) = L$ or $\lim\limits_{x\to-\infty} f(x) = L$, where L is a finite number, we say the line $y = L$ is a **horizontal asymptote** of the graph of f. ▪

Thus, to find horizontal asymptotes of a function, we need to find the end behavior of the function. Specifically, we must find both limits at infinity.

N *Because horizontal asymptotes occur when these limits exist, it is only possible for a function to have zero, one, or two horizontal asymptotes.*

▪ **Example 9** Find any horizontal asymptotes of the following functions. If there are none, describe the end behavior using limit notation.

a. $f(x) = \dfrac{4x^3 - 12x + 2}{-3x^3 + 3x - 4}$

b. $f(x) = \dfrac{9x^{15} - 7x^8 - 22}{8x^6 + 20x^4 + 19x}$

c. $f(x) = \dfrac{8e^{3x} + 12e^{-5x}}{4e^{-5x} + 1}$

d. $f(x) = \dfrac{4e^{2x} - 12}{3e^{2x} + 24}$

Solution:

a. To find horizontal asymptotes, we need to find both $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$. Remember that for rational functions, if the limit as either $x \to \infty$ or $x \to -\infty$ is a finite number, then the other limit at infinity will equal the same finite number.

Recall that regardless of which limit at infinity we are finding, for rational functions we divide both the numerator and the denominator by the highest power of x in the denominator, which in this case is x^3.

Let's start by finding $\lim\limits_{x\to\infty} f(x)$. Dividing both the numerator and denominator by x^3 gives

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{4x^3 - 12x + 2x}{-3x^3 + 3x - 4}$$

$$= \lim_{x\to\infty} \frac{\dfrac{4x^3 - 12x + 2x}{x^3}}{\dfrac{-3x^3 + 3x - 4}{x^3}}$$

$$= \lim_{x\to\infty} \frac{\dfrac{4x^3}{x^3} - \dfrac{12x}{x^3} + \dfrac{2}{x^3}}{\dfrac{-3x^3}{x^3} + \dfrac{3x}{x^3} - \dfrac{4}{x^3}}$$

$$= \lim_{x\to\infty} \frac{4 - \dfrac{12}{x^2} + \dfrac{2}{x^3}}{-3 + \dfrac{3}{x^2} - \dfrac{4}{x^3}}$$

Again, we have to be careful when finding the limit of this resulting quotient. We must investigate the behavior of the numerator and denominator separately.

Let's start by finding the behavior of the numerator. We will apply **Theorem 1.3** where appropriate:

$$\lim_{x\to\infty}\left(4-\frac{12}{x^2}+\frac{2}{x^3}\right)=\lim_{x\to\infty}\left(\overset{4}{\cancel{4}}-\overset{0}{\cancel{\frac{12}{x^2}}}+\overset{0}{\cancel{\frac{2}{x^3}}}\right)$$
$$=4$$

Next, we will determine the behavior of the denominator:

$$\lim_{x\to\infty}\left(-3+\frac{3}{x^2}-\frac{4}{x^3}\right)=\lim_{x\to\infty}\left(\overset{-3}{\cancel{-3}}+\overset{0}{\cancel{\frac{3}{x^2}}}-\overset{0}{\cancel{\frac{4}{x^3}}}\right)$$
$$=-3$$

We see both limits exist and the limit of the function in the denominator is nonzero. Now, we can finish finding the limit using the Properties of Limits:

$$\lim_{x\to\infty}\frac{4-\frac{12}{x^2}+\frac{2}{x^3}}{-3+\frac{3}{x^2}-\frac{4}{x^3}}=\frac{\lim_{x\to\infty}\left(4-\frac{12}{x^2}+\frac{2}{x^3}\right)}{\lim_{x\to\infty}\left(-3+\frac{3}{x^2}-\frac{4}{x^3}\right)}$$

$$=\frac{4}{-3}$$

Hence,

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{4x^3-12x+2}{-3x^3+3x-4}=\lim_{x\to\infty}\frac{4-\frac{12}{x^2}+\frac{2}{x^3}}{-3+\frac{3}{x^2}-\frac{4}{x^3}}=-\frac{4}{3}$$

Therefore, as $x\to\infty$, the function has a horizontal asymptote: $y=-\frac{4}{3}$. Due to the fact that f is a rational function and the limit as $x\to\infty$ equals a finite number, $-\frac{4}{3}$, we know the limit as $x\to-\infty$ is also $-\frac{4}{3}$. In other words, $\lim_{x\to\infty}f(x)=-\frac{4}{3}$ and $\lim_{x\to-\infty}f(x)=-\frac{4}{3}$. So the function has one horizontal asymptote: $y=-\frac{4}{3}$.

We can check our work by looking at the graph of $f(x) = \dfrac{4x^3 - 12x + 2}{-3x^3 + 3x - 4}$ shown in **Figure 1.3.7**.

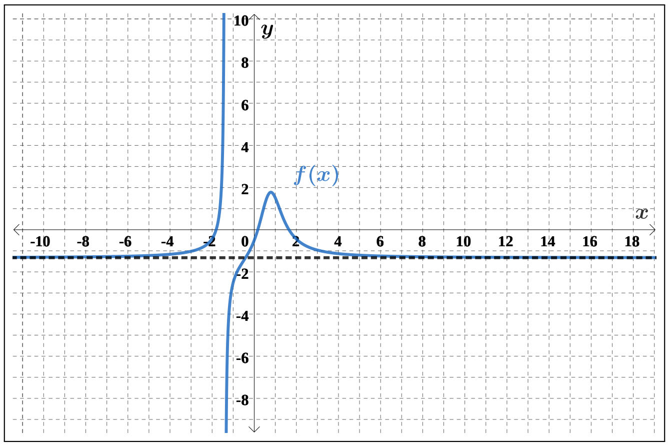


Figure 1.3.7: Graph of $f(x) = \dfrac{4x^3 - 12x + 2}{-3x^3 + 3x - 4}$

b. Recall that we must find any horizontal asymptotes of $f(x) = \dfrac{9x^{15} - 7x^8 - 22}{8x^6 + 20x^4 + 19x}$.

We will take the same approach as part **a** and divide both the numerator and denominator by x^6 because it is the highest power of x in the denominator.

Let's start by finding $\lim\limits_{x\to\infty} f(x)$. Dividing both the numerator and denominator by x^6 gives

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{9x^{15} - 7x^8 - 22}{8x^6 + 20x^4 + 19x} \\
&= \lim_{x\to\infty} \frac{\dfrac{9x^{15} - 7x^8 - 22}{x^6}}{\dfrac{8x^6 + 20x^4 + 19x}{x^6}} \\
&= \lim_{x\to\infty} \frac{\dfrac{9x^{15}}{x^6} - \dfrac{7x^8}{x^6} - \dfrac{22}{x^6}}{\dfrac{8x^6}{x^6} + \dfrac{20x^4}{x^6} + \dfrac{19x}{x^6}} \\
&= \lim_{x\to\infty} \frac{9x^9 - 7x^2 - \dfrac{22}{x^6}}{8 + \dfrac{20}{x^2} + \dfrac{19}{x^5}}
\end{aligned}$$

Again, we have to be careful when finding the limit of this resulting quotient. We must investigate the behavior of the numerator and denominator separately.

We will start by finding the behavior of the numerator. We have to be extra careful with this function (we will see why soon!), but first let's apply **Theorem 1.3** where appropriate:

$$\lim_{x\to\infty}\left(9x^9 - 7x^2 - \frac{22}{x^6}\right) = \lim_{x\to\infty}\left(9x^9 - 7x^2 - \cancelto{0}{\frac{22}{x^6}}\right)$$

The last term in this function tends to zero as $x \to \infty$, and we can view the first two terms as behaving like a polynomial. We learned previously that to find a limit at infinity of a polynomial, we only need to consider the degree and sign of the leading coefficient. In other words, we only need to consider $\lim\limits_{x\to\infty}\left(9x^9\right)$. Because $\lim\limits_{x\to\infty}\left(9x^9\right) \to \infty$, we have

$$\lim_{x\to\infty}\left(9x^9 - 7x^2 - \frac{22}{x^6}\right) \to \infty$$

So the function in the numerator is approaching positive infinity. Thus, this limit does not exist.

Next, we will determine the behavior of the denominator:

$$\begin{aligned}\lim_{x\to\infty}\left(8 + \frac{20}{x^2} + \frac{19}{x^5}\right) &= \lim_{x\to\infty}\left(\cancelto{8}{8} + \cancelto{0}{\frac{20}{x^2}} + \cancelto{0}{\frac{19}{x^5}}\right)\\ &= 8\end{aligned}$$

Therefore, the function in the denominator is approaching 8. Because the limit of the function in the numerator does not exist, we cannot use the Properties of Limits to divide the limits. So we must carefully analyze the overall behavior of the resulting quotient.

Due to the fact that the function in the numerator is increasing without bound and the function in the denominator is approaching a positive finite number, the quotient itself is approaching positive infinity:

$$\lim_{x\to\infty}\frac{9x^9 - 7x^2 - \frac{22}{x^6}}{8 + \frac{20}{x^2} + \frac{19}{x^5}} \to \infty$$

Hence,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{9x^{15} - 7x^8 - 22}{8x^6 + 20x^4 + 19x} = \lim_{x\to\infty}\frac{9x^9 - 7x^2 - \frac{22}{x^6}}{8 + \frac{20}{x^2} + \frac{19}{x^5}} \to \infty$$

So $f(x)$ does not have a horizontal asymptote as $x \to \infty$.

Recall that if the behavior of one end of a rational function is finite, then the other must be as well. The same holds for infinite behavior! If one end of a rational function is infinite (either positive or negative), then the other end must be infinite (either positive or negative). Because $\lim\limits_{x\to\infty} f(x) \to \infty$ in this example, $\lim\limits_{x\to-\infty} f(x)$ must tend toward either positive or negative infinity. Either way, this means the function does not have any horizontal asymptotes!

However, we still need to describe the end behavior of the function. We know $\lim_{x\to\infty} f(x) \to \infty$, so now we need to determine the type of infinite behavior (positive or negative) the function exhibits as $x \to -\infty$.

To find $\lim_{x\to-\infty} f(x)$, we will again divide both the numerator and denominator by x^6 because it is the highest power of x in the denominator:

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{9x^{15} - 7x^8 - 22}{8x^6 + 20x^4 + 19x}$$

$$= \lim_{x\to-\infty} \frac{\dfrac{9x^{15} - 7x^8 - 22}{x^6}}{\dfrac{8x^6 + 20x^4 + 19x}{x^6}}$$

$$= \lim_{x\to-\infty} \frac{\dfrac{9x^{15}}{x^6} - \dfrac{7x^8}{x^6} - \dfrac{22}{x^6}}{\dfrac{8x^6}{x^6} + \dfrac{20x^4}{x^6} + \dfrac{19x}{x^6}}$$

$$= \lim_{x\to-\infty} \frac{9x^9 - 7x^2 - \dfrac{22}{x^6}}{8 + \dfrac{20}{x^2} + \dfrac{19}{x^5}}$$

Again, we have to be careful when finding the limit of this resulting quotient. We must investigate the behavior of the numerator and denominator separately.

We will start by finding the behavior of the numerator and apply **Theorem 1.3** where appropriate:

$$\lim_{x\to-\infty} \left(9x^9 - 7x^2 - \frac{22}{x^6}\right) = \lim_{x\to-\infty} \left(9x^9 - 7x^2 - \overset{0}{\cancel{\frac{22}{x^6}}}\right)$$

The last term in this function tends to zero as $x \to -\infty$, and we can view the first two terms as behaving like a polynomial. Recall that to find a limit at infinity of a polynomial, we only need to consider the degree and sign of the leading coefficient. In other words, we only need to consider $\lim_{x\to-\infty} \left(9x^9\right)$. Because $\lim_{x\to-\infty} \left(9x^9\right) \to -\infty$, we have

$$\lim_{x\to-\infty} \left(9x^9 - 7x^2 - \frac{22}{x^6}\right) \to -\infty$$

So the function in the numerator is approaching negative infinity. Thus, this limit does not exist.

Next, we will determine the behavior of the denominator:

$$\lim_{x\to-\infty} \left(8 + \frac{20}{x^2} + \frac{19}{x^5}\right) = \lim_{x\to-\infty} \left(\overset{8}{\cancel{8}} + \overset{0}{\cancel{\frac{20}{x^2}}} + \overset{0}{\cancel{\frac{19}{x^5}}}\right)$$

$$= 8$$

Therefore, the function in the denominator is approaching 8. Because the limit of the function in the numerator does not exist, we cannot use the Properties of Limits to divide the limits. So we must carefully analyze the overall behavior of the resulting quotient.

Due to the fact that the function in the numerator is decreasing without bound and the function in the denominator is approaching a positive finite number, the quotient itself is approaching negative infinity:

$$\lim_{x\to-\infty} \frac{9x^9 - 7x^2 - \dfrac{22}{x^6}}{8 + \dfrac{20}{x^2} + \dfrac{19}{x^5}} \to -\infty$$

Hence,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{9x^{15} - 7x^8 - 22}{8x^6 + 20x^4 + 19x} = \lim_{x\to-\infty} \frac{9x^9 - 7x^2 - \dfrac{22}{x^6}}{8 + \dfrac{20}{x^2} + \dfrac{19}{x^5}} \to -\infty$$

This calculation confirms our earlier statement that the graph of f does not have any horizontal asymptotes. However, we can still describe the end behavior of the function:

$$\lim_{x\to-\infty} f(x) \to -\infty \text{ and } \lim_{x\to\infty} f(x) \to \infty$$

We can check our work by looking at the graph of $f(x) = \dfrac{9x^{15} - 7x^8 - 22}{8x^6 + 20x^4 + 19x}$ shown in **Figure 1.3.8**.

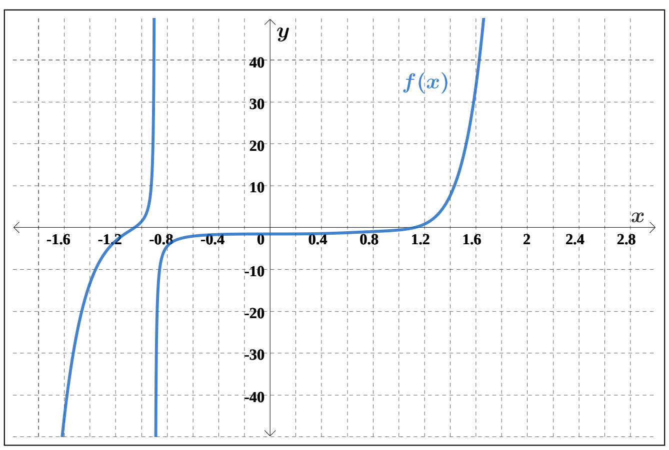


Figure 1.3.8: Graph of $f(x) = \dfrac{9x^{15} - 7x^8 - 22}{8x^6 + 20x^4 + 19x}$

c. Recall that we must find any horizontal asymptotes of $f(x) = \dfrac{8e^{3x} + 12e^{-5x}}{4e^{-5x} + 1}$, so we need to calculate both limits at infinity. Let's start with $\lim\limits_{x\to\infty} f(x)$.

Notice that there is no exponential function in the denominator with a positive power, so we cannot perform division. Thus, we will attempt to find the limit directly by considering the behavior of the function as $x \to \infty$.

To determine the behavior of the function as $x \to \infty$, we must observe the behavior of the functions in the numerator and denominator as $x \to \infty$ separately because the given function is a quotient, or ratio, of two functions. We will start by observing the behavior of the function in the numerator. Accounting for the sign of the coefficient in each exponent and applying **Theorem 1.4** gives

$$\lim_{x\to\infty}\left(8e^{3x}+12e^{-5x}\right)=\lim_{x\to\infty}\left(\overset{\infty}{\cancel{8e^{3x}}}+\overset{0}{\cancel{12e^{-5x}}}\right)$$
$$\to\infty$$

Thus, the function in the numerator is approaching positive infinity, so this limit does not exist. Looking at the limit of the denominator, we see

$$\lim_{x\to\infty}\left(4e^{-5x}+1\right)=\lim_{x\to\infty}\left(\overset{0}{\cancel{4e^{-5x}}}+\overset{1}{\cancel{1}}\right)$$
$$=1$$

Therefore, the function in the denominator is approaching 1. Because the limit of the function in the numerator does not exist, we cannot use the Properties of Limits to divide the limits. So we must carefully analyze the overall behavior of the quotient.

Due to the fact that the function in the numerator is increasing without bound and the function in the denominator is approaching a positive finite number, the quotient is approaching positive infinity:

$$\lim_{x\to\infty}\frac{8e^{3x}+12e^{-5x}}{4e^{-5x}+1}\to\infty$$

Thus,

$$\lim_{x\to\infty}f(x)=\lim_{x\to\infty}\frac{8e^{3x}+12e^{-5x}}{4e^{-5x}+1}\to\infty$$

This shows there is no horizontal asymptote as $x\to\infty$. However, unlike rational functions, *there may still be a horizontal asymptote on the other end of the function as $x\to-\infty$!* It is absolutely necessary to check the other limit at infinity.

To compute $\lim\limits_{x\to-\infty}f(x)$, we divide the numerator and the denominator by the exponential function in the denominator with the most negative power. In this case, that is e^{-5x}:

$$\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\frac{8e^{3x}+12e^{-5x}}{4e^{-5x}+1}$$

$$=\lim_{x\to-\infty}\frac{\dfrac{8e^{3x}+12e^{-5x}}{e^{-5x}}}{\dfrac{4e^{-5x}+1}{e^{-5x}}}$$

$$=\lim_{x\to-\infty}\frac{\dfrac{8e^{3x}}{e^{-5x}}+\dfrac{12e^{-5x}}{e^{-5x}}}{\dfrac{4e^{-5x}}{e^{-5x}}+\dfrac{1}{e^{-5x}}}$$

$$=\lim_{x\to-\infty}\frac{8e^{8x}+12}{4+e^{5x}}$$

Again, to compute the limit of this resulting quotient, we must be careful and observe the behavior of the numerator and denominator separately.

Starting with the numerator of the quotient and looking at its behavior as $x \to -\infty$ gives

$$\lim_{x\to-\infty}\left(8e^{8x}+12\right) = \lim_{x\to-\infty}\left(\overset{0}{\cancel{8e^{8x}}}+\overset{12}{\cancel{12}}\right)$$
$$= 12$$

Thus, the limit of the function in the numerator exists and equals 12. Looking at the limit of the denominator of the quotient, we see

$$\lim_{x\to-\infty}\left(4+e^{5x}\right) = \lim_{x\to-\infty}\left(\overset{4}{\cancel{4}}+\overset{0}{\cancel{e^{5x}}}\right)$$
$$= 4$$

Therefore, the function in the denominator is approaching 4. Because both the limit of the function in the numerator and the limit of the function in the denominator exist and the limit of the function in the denominator is nonzero, we can finish finding the limit using Properties of Limits:

$$\lim_{x\to-\infty}\frac{8e^{8x}+12}{4+e^{5x}} = \frac{\lim\limits_{x\to-\infty}\left(8e^{8x}+12\right)}{\lim\limits_{x\to-\infty}\left(4+e^{5x}\right)}$$

$$= \frac{12}{4}$$

$$= 3$$

Hence,

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\frac{8e^{3x}+12e^{-5x}}{4e^{-5x}+1} = \lim_{x\to-\infty}\frac{8e^{8x}+12}{4+e^{5x}} = 3$$

Even though $\lim\limits_{x\to\infty}\dfrac{8e^{3x}+12e^{-5x}}{4e^{-5x}+1} \to \infty$, because $\lim\limits_{x\to-\infty}\dfrac{8e^{3x}+12e^{-5x}}{4e^{-5x}+1} = 3$, the function has a horizontal asymptote at $y = 3$.

In summary, the graph of f has a horizontal asymptote at $y = 3$ as $x \to -\infty$, and we can describe the end behavior as $x \to \infty$ using limit notation: $\lim\limits_{x\to\infty} f(x) \to \infty$.

Again, we can check our work by looking at the graph of $f(x) = \dfrac{8e^{3x}+12e^{-5x}}{4e^{-5x}+1}$ shown in **Figure 1.3.9**.

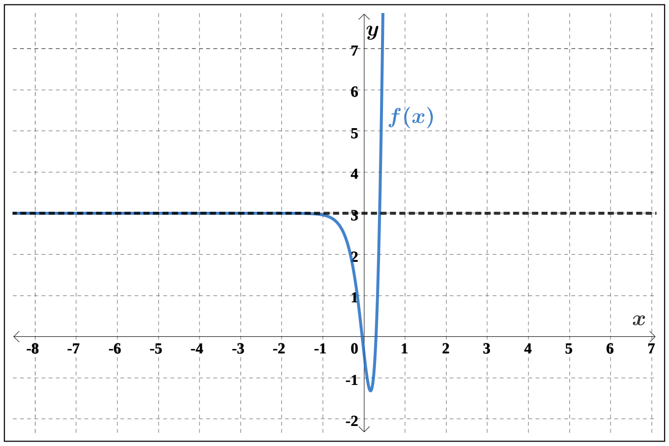


Figure 1.3.9: Graph of $f(x) = \dfrac{8e^{3x}+12e^{-5x}}{4e^{-5x}+1}$

d. Recall that we must find any horizontal asymptotes of $f(x) = \dfrac{4e^{2x}-12}{3e^{2x}+24}$, so we need to calculate both limits at infinity. Let's start with $\lim\limits_{x\to\infty} f(x)$.

To find the limit as $x \to \infty$, we divide both the numerator and denominator by e^{2x}:

$$\begin{aligned}
\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{4e^{2x}-12}{3e^{2x}+24} \\
&= \lim_{x\to\infty} \frac{\dfrac{4e^{2x}-12}{e^{2x}}}{\dfrac{3e^{2x}+24}{e^{2x}}} \\
&= \lim_{x\to\infty} \frac{\dfrac{4e^{2x}}{e^{2x}} - \dfrac{12}{e^{2x}}}{\dfrac{3e^{2x}}{e^{2x}} + \dfrac{24}{e^{2x}}} \\
&= \lim_{x\to\infty} \frac{4-12e^{-2x}}{3+24e^{-2x}}
\end{aligned}$$

Again, to compute the limit of this resulting quotient, we must be careful and observe the behavior of the numerator and denominator separately.

Starting with the numerator of the quotient and looking at its behavior as $x \to \infty$ gives

$$\begin{aligned}
\lim_{x\to\infty}\left(4-12e^{-2x}\right) &= \lim_{x\to\infty}\left(\overset{\nearrow 4}{4} - \overset{\nearrow 0}{12e^{-2x}}\right) \\
&= 4
\end{aligned}$$

Thus, the limit of the function in the numerator exists and equals 4. Looking at the limit of the denominator of the quotient, we see

$$\lim_{x\to-\infty}\left(3+24e^{-2x}\right) = \lim_{x\to-\infty}\left(\cancelto{3}{3}+\cancelto{0}{24e^{-2x}}\right)$$
$$= 3$$

Therefore, the function in the denominator is approaching 3. Because both the limit of the function in the numerator and the limit of the function in the denominator exist and the limit of the function in the denominator is nonzero, we can finish finding the limit using Properties of Limits:

$$\lim_{x\to\infty}\frac{4-12e^{-2x}}{3+24e^{-2x}} = \frac{\lim\limits_{x\to\infty}\left(4-12e^{-2x}\right)}{\lim\limits_{x\to\infty}\left(3+24e^{-2x}\right)}$$

$$= \frac{4}{3}$$

Hence,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{4e^{2x}-12}{3e^{2x}+24} = \lim_{x\to\infty}\frac{4-12e^{-2x}}{3+24e^{-2x}} = \frac{4}{3}$$

This tells us there is *at least* one horizontal asymptote, $y = \frac{4}{3}$, but we still need to check the limit as $x \to -\infty$ for a possible second. Because there is no exponential function in the denominator with a negative power, we find the limit directly by considering the behavior of the function as $x \to -\infty$.

To determine the behavior of the function as $x \to -\infty$, we must observe the behavior of the functions in the numerator and denominator as $x \to -\infty$ separately because the given function is a quotient, or ratio, of two functions.

We will start by observing the behavior of the function in the numerator. Accounting for the sign of the coefficient in the exponent and applying **Theorem 1.4** gives

$$\lim_{x\to-\infty}\left(4e^{2x}-12\right) = \lim_{x\to-\infty}\left(\cancelto{0}{4e^{2x}}-\cancelto{12}{12}\right)$$
$$= -12$$

Thus, the limit of the function in the numerator exists and equals -12. Looking at the limit of the denominator of the quotient, we see

$$\lim_{x\to-\infty}\left(3e^{2x}+24\right) = \lim_{x\to-\infty}\left(\cancelto{0}{3e^{2x}}+\cancelto{24}{24}\right)$$

$$= 24$$

Therefore, the function in the denominator is approaching 24. Because both the limit of the function in the numerator and the limit of the function in the denominator exist and the limit of the function in the denominator is nonzero, we can finish finding the limit using Properties of Limits:

$$\lim_{x\to-\infty}\frac{4e^{2x}-12}{3e^{2x}+24}=\frac{\lim\limits_{x\to-\infty}\left(4e^{2x}-12\right)}{\lim\limits_{x\to-\infty}\left(3e^{2x}+24\right)}$$

$$=\frac{-12}{24}$$

$$=-\frac{1}{2}$$

Hence,

$$\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\frac{4e^{2x}-12}{3e^{2x}+24}=-\frac{1}{2}$$

and we see there *is* a second horizontal asymptote!

Thus, the graph of f has two horizontal asymptotes: $y=\frac{4}{3}$ and $y=-\frac{1}{2}$.

Once more, we check our work by looking at the graph of $f(x)=\frac{4e^{2x}-12}{3e^{2x}+24}$ shown in **Figure 1.3.10**.

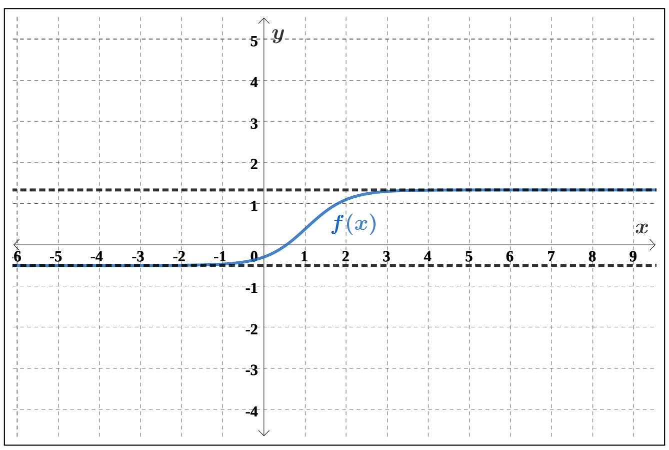


Figure 1.3.10: Graph of $f(x)=\frac{4e^{2x}-12}{3e^{2x}+24}$

*The functions in the previous example demonstrate all the possibilities a function can have: no horizontal asymptotes, the same horizontal asymptote for both ends, a horizontal asymptote on one end and an end without a horizontal asymptote, and two distinct horizontal asymptotes. The example shows the importance of checking **both** limits at infinity.* ▪

Try It # 7:
Find any horizontal asymptotes of the following functions. If there are none, describe the end behavior using limit notation.

a. $f(x)=\frac{8x^8+22x^5-9x^3}{15x^8-1200x^4-1}$.

b. $f(x)=\frac{2e^{4x}-9}{7e^{2x}+8e^{-6x}}$

Infinite Limits

We will now turn our attention from limits at infinity to **infinite limits**. We discussed infinite limits in **Section 1.1** when we computed limits and found a function was not heading toward a finite number. Rather, the function was heading toward positive (or negative) infinity. We see this behavior when a function has a **vertical asymptote**.

Vertical Asymptotes and Holes

Recall the definition of vertical asymptote from **Section 1.1**:

Definition
Let f be a function. If any of the following conditions hold for some real number c, then the line $x = c$ is a **vertical asymptote** of the graph of f.

$$\lim_{x\to c^-} f(x) \to \infty \quad \text{or} \quad \lim_{x\to c^-} f(x) \to -\infty$$

$$\lim_{x\to c^+} f(x) \to \infty \quad \text{or} \quad \lim_{x\to c^+} f(x) \to -\infty$$

$$\lim_{x\to c} f(x) \to \infty \quad \text{or} \quad \lim_{x\to c} f(x) \to -\infty$$

■

Now we ask the question: Given a function f, how do we find the vertical asymptotes? To find horizontal asymptotes, we know we have to calculate $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. However, for vertical asymptotes, we need to find the value x is tending toward.

We will mainly restrict our focus to rational functions. Let's start with $f(x) = \dfrac{p(x)}{q(x)}$, where p and q are polynomials. Because rational functions are nicely defined on their domains, we only need to consider the x-values where $q(x) = 0$ (i.e., where f is undefined). This gives us candidates to check.

■ **Example 10** Find any vertical asymptotes of $f(x) = \dfrac{x-1}{x^2-1}$. For each vertical asymptote, describe the infinite behavior using limit notation.

Solution:
First, we find where $x^2 - 1 = 0$. Factoring this expression gives $(x+1)(x-1) = 0$, and this occurs when $x = -1$ and $x = 1$. This means we need to check the behavior of the function at $x = -1$ and $x = 1$. We can do this by finding $\lim_{x\to-1} f(x)$ and $\lim_{x\to1} f(x)$.

Using direct substitution to find $\lim_{x\to-1} f(x)$ leads us to a nonzero number, -2, in the numerator and 0 in the denominator. Recall from the previous section that this means the limit does not exist and that there is a vertical asymptote at $x = -1$ (this was Case 2).

Now, let's now turn to our other candidate, $x = 1$. Direct substitution leads us to the indeterminate form $\frac{0}{0}$, so we work to algebraically manipulate the function:

$$\begin{aligned}\lim_{x\to 1}\frac{x-1}{x^2-1} &= \lim_{x\to 1}\frac{x-1}{(x+1)(x-1)}\\ &= \lim_{x\to 1}\frac{\cancel{x-1}}{(x+1)\cancel{(x-1)}}\\ &= \lim_{x\to 1}\frac{1}{x+1}\\ &= \frac{1}{1+1} = \frac{1}{2}\end{aligned}$$

It does not particularly matter that the numeric value of the limit is $\dfrac{1}{2}$. The important thing is that the limit equals *a finite number*. This means $x = 1$ is *not* a vertical asymptote.

Thus, the only vertical asymptote of the graph of f is $x = -1$.

Recall that because the $\lim_{x\to 1} f(x) = \dfrac{1}{2}$, *the graph of* f *has a hole at the point* $\left(1, \dfrac{1}{2}\right)$. *This corresponds to Case 3 in the previous section.*

Not every x-value where the denominator equals zero will be a vertical asymptote!

Because there is a vertical asymptote at $x = -1$, we must also describe the infinite behavior near $x = -1$ using limit notation. We can investigate the infinite behavior near $x = -1$ using either graphical or numerical methods. We will observe the graph of f near $x = -1$ to determine the infinite behavior. See **Figure 1.3.11**.

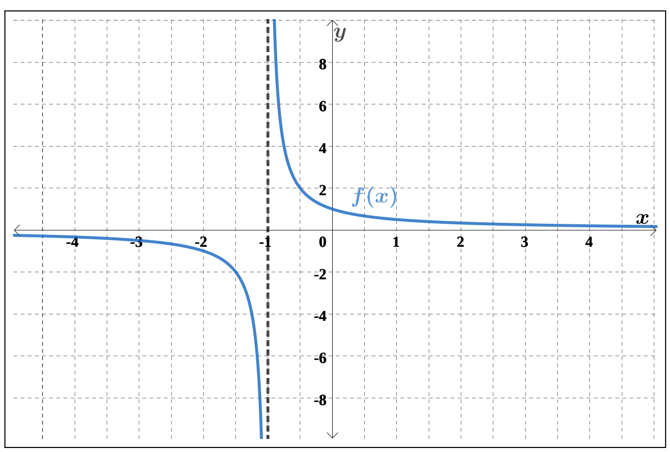


Figure 1.3.11: Graph of $f(x) = \dfrac{x-1}{x^2-1}$

Looking at the graph, we see that the infinite behavior near the vertical asymptote at $x = -1$ can be described by the following limits:

$$\lim_{x\to -1^-} f(x) \to -\infty \quad \text{and} \quad \lim_{x\to -1^+} f(x) \to \infty$$

■

Try It # 8:
Find any vertical asymptotes of the following functions. For each vertical asymptote, describe the infinite behavior using limit notation.

a. $f(x) = \dfrac{x(x+3)}{x^2(x+3)(x+5)}$

b. $f(x) = \dfrac{(x+7)(x+2)}{2x^3+10x^2-28x}$

In the previous example, $x = -1$ and $x = 1$ were not in the domain of f. We concluded that the graph of f had a vertical asymptote at $x = -1$ and a hole at $x = 1$. Thus, at the values of x that are not in the domain of a rational function, there will either be a hole or a vertical asymptote. We summarize below how to find the holes and vertical asymptotes of rational functions:

Method for Determining Holes and Vertical Asymptotes of Rational Functions

1. Factor the numerator and denominator. Divide any common factors.
2. The factors in the denominator that *divide completely* will determine the holes. Set each factor in the denominator that divides completely equal to zero to find the x-value of the **hole**.
3. The factors that *remain in the denominator* (and denominator only!) will determine the vertical asymptotes. Again, set each factor that remains in the denominator equal to zero to find the x-value of the **vertical asymptote**.

N *For each vertical asymptote we find using the method above, we will describe the infinite behavior near the vertical asymptote using limit notation.*

Example 11 Find the location of any holes and vertical asymptotes of the following functions. For each vertical asymptote, describe the infinite behavior using limit notation.

a. $f(x) = \dfrac{(x+3)(x-4)}{(x-2)(x-3)}$

b. $f(x) = \dfrac{(2x-9)x^2}{8x(2x-9)^2}$

Solution:

a. This function has the numerator and denominator factored for us already. Furthermore, there are no common factors to divide! This means there are no holes and that the factors in the denominator will determine the vertical asymptotes: $x-2=0$ tells us there is a vertical asymptote at $x=2$, and $x-3=0$ shows there is a vertical asymptote at $x=3$.

To describe the infinite behavior near the vertical asymptotes, we can use either numerical or graphical methods. We will observe the graph of f near $x=2$ and $x=3$ to determine the infinite behavior. See **Figure 1.3.12**.

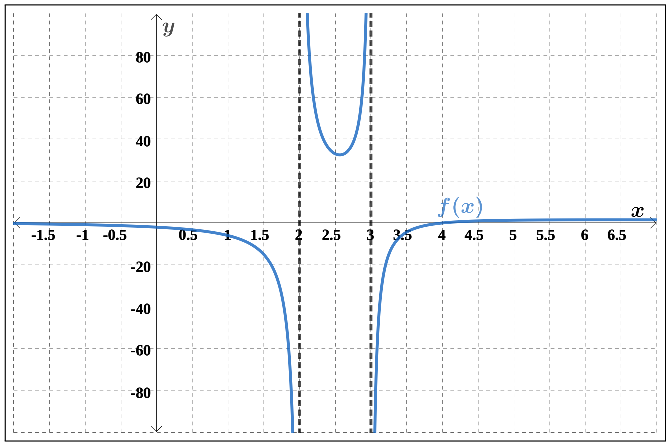


Figure 1.3.12: Graph of $f(x) = \dfrac{(x+3)(x-4)}{(x-2)(x-3)}$

Looking at the infinite behavior near the vertical asymptote at $x = 2$, we see

$$\lim_{x\to 2^-} f(x) \to -\infty \quad \text{and} \quad \lim_{x\to 2^+} f(x) \to \infty$$

Looking at the infinite behavior near the vertical asymptote at $x = 3$, we see

$$\lim_{x\to 3^-} f(x) \to \infty \quad \text{and} \quad \lim_{x\to 3^+} f(x) \to -\infty$$

b. Recall that we must find the location of any holes and vertical asymptotes of $f(x) = \dfrac{(2x-9)x^2}{8x(2x-9)^2}$, and again, the function is already factored in both the numerator and denominator. If we divide the common factors we have

$$\frac{(2x-9)x^2}{8x(2x-9)^2} \to \frac{\cancel{(2x-9)}x^{\cancel{2}}}{8\cancel{x}(2x-9)^{\cancel{2}}}$$

Notice that we were able to divide the factor x in the denominator completely even though there is still a factor of x in the numerator! This means there is a hole at $x = 0$.

However, the factor $2x-9$ did *not* divide completely from the denominator. This means that $2x-9 = 0$ determines a vertical asymptote. In other words, there is a vertical asymptote at $x = \dfrac{9}{2}$.

We will observe the infinite behavior near the vertical asymptote at $x = \dfrac{9}{2}$ graphically. See **Figure 1.3.13**.

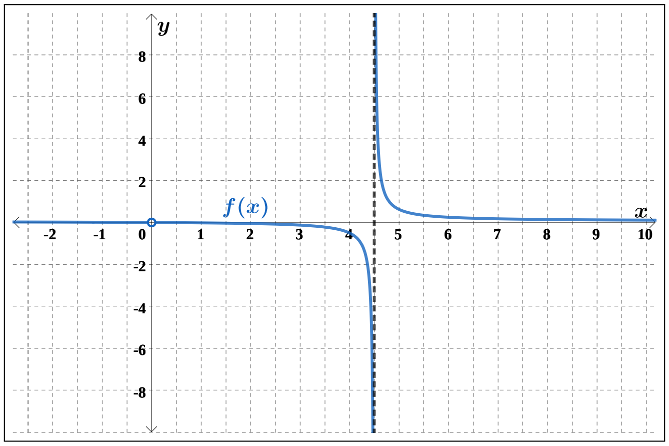


Figure 1.3.13: Graph of $f(x) = \dfrac{(2x-9)x^2}{8x(2x-9)^2}$

Looking at the infinite behavior near the vertical asymptote at $x = \dfrac{9}{2}$, we see

$$\lim_{x \to \frac{9}{2}^-} f(x) \to -\infty \quad \text{and} \quad \lim_{x \to \frac{9}{2}^+} f(x) \to \infty$$

To summarize, the graph of f has a hole at $x = 0$ and a vertical asymptote at $x = \dfrac{9}{2}$ in which $\displaystyle\lim_{x \to \frac{9}{2}^-} f(x) \to -\infty$ and $\displaystyle\lim_{x \to \frac{9}{2}^+} f(x) \to \infty$.

■

Try It # 9:

Find the location of any holes and vertical asymptotes of $f(x) = \dfrac{(x-8)(x+9)(x+4)^2}{x(x+4)(x+9)^3}$. For each vertical asymptote, describe the infinite behavior using limit notation.

Now, let's try applying what we have learned in this section to find any holes and asymptotes, both horizontal and vertical, of a rational function!

■ **Example 12** Find any horizontal asymptotes, holes, and vertical asymptotes of the function $f(x) = \dfrac{x^2-8x+12}{x^3-2x^2-24x}$. If there are no horizontal asymptotes, describe the end behavior using limit notation. For each vertical asymptote, describe the infinite behavior using limit notation.

Solution:

We should start with finding horizontal asymptotes because, unlike when finding vertical asymptotes and holes, we do not want the function in factored form. After we find any horizontal asymptotes, we can factor the function to find any holes and vertical asymptotes.

To find horizontal asymptotes, we must find $\displaystyle\lim_{x \to \infty} f(x)$ and $\displaystyle\lim_{x \to -\infty} f(x)$. Recall that regardless of which limit at infinity we are finding, for rational functions we divide both the numerator and denominator by the term in the denominator with the highest power of x, which in this case is x^3.

Let's start by finding $\lim_{x\to\infty} f(x)$. Dividing both the numerator and denominator by x^3 gives

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2-8x+12}{x^3-2x^2-24x}$$

$$= \lim_{x\to\infty} \frac{\dfrac{x^2-8x+12}{x^3}}{\dfrac{x^3-2x^2-24x}{x^3}}$$

$$= \lim_{x\to\infty} \frac{\dfrac{x^2}{x^3}-\dfrac{8x}{x^3}+\dfrac{12}{x^3}}{\dfrac{x^3}{x^3}-\dfrac{2x^2}{x^3}-\dfrac{24x}{x^3}}$$

$$= \lim_{x\to\infty} \frac{\dfrac{1}{x}-\dfrac{8}{x^2}+\dfrac{12}{x^3}}{1-\dfrac{2}{x}-\dfrac{24}{x^2}}$$

Again, we have to be careful when finding the limit of this resulting quotient. We must investigate the behavior of the numerator and denominator separately.

Let's start by finding the behavior of the numerator. We will apply **Theorem 1.3** where appropriate:

$$\lim_{x\to\infty}\left(\frac{1}{x}-\frac{8}{x^2}+\frac{12}{x^3}\right) = \lim_{x\to\infty}\left(\cancelto{0}{\frac{1}{x}}-\cancelto{0}{\frac{8}{x^2}}+\cancelto{0}{\frac{12}{x^3}}\right)$$

$$= 0$$

Next, we will determine the behavior of the denominator:

$$\lim_{x\to\infty}\left(1-\frac{2}{x}-\frac{24}{x^2}\right) = \lim_{x\to\infty}\left(1-\cancelto{0}{\frac{2}{x}}-\cancelto{0}{\frac{24}{x^2}}\right)$$

$$= 1$$

We see both limits exist and the limit of the function in the denominator is nonzero. Now, we can finish finding the limit using the Properties of Limits:

$$\lim_{x\to\infty} \frac{\dfrac{1}{x}-\dfrac{8}{x^2}+\dfrac{12}{x^3}}{1-\dfrac{2}{x}-\dfrac{24}{x^2}} = \frac{\lim_{x\to\infty}\left(\dfrac{1}{x}-\dfrac{8}{x^2}+\dfrac{12}{x^3}\right)}{\lim_{x\to\infty}\left(1-\dfrac{2}{x}-\dfrac{24}{x^2}\right)}$$

$$= \frac{0}{1}$$

$$= 0$$

Hence,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2-8x+12}{x^3-2x^2-24x} = \lim_{x\to\infty} \frac{\frac{1}{x}-\frac{8}{x^2}+\frac{12}{x^3}}{1-\frac{2}{x}-\frac{24}{x^2}} = 0$$

Therefore, as $x\to\infty$, the graph of the function has a horizontal asymptote: $y=0$.

Due to the fact that f is a rational function and the limit as $x\to\infty$ equals a finite number, 0, we know the limit as $x\to-\infty$ is also 0. In other words, $\lim\limits_{x\to\infty} f(x)=0$ and $\lim\limits_{x\to\infty} f(x)=0$. So the graph of the function has one horizontal asymptote: $y=0$.

Next, let's find any holes and vertical asymptotes. We need to factor the numerator and denominator of the rational function so we can divide any common factors:

$$f(x) = \frac{x^2-8x+12}{x^3-2x^2-24x}$$

$$= \frac{(x-6)(x-2)}{x(x-6)(x+4)}$$

Dividing common factors we have

$$\frac{(x-6)(x-2)}{x(x-6)(x+4)} \to \frac{\cancel{(x-6)}(x-2)}{x\cancel{(x-6)}(x+4)}$$

We see that the factor $x-6$ divides completely from the denominator, meaning there is a hole where $x-6=0$, or at $x=6$. To find the point where the hole is, we need to find the corresponding y-value. This is the value of the limit!

$$\lim_{x\to6} \frac{x^2-8x+12}{x^3-2x^2-24x} = \lim_{x\to6} \frac{(x-6)(x-2)}{x(x-6)(x+4)}$$

$$= \lim_{x\to6} \frac{\cancel{(x-6)}(x-2)}{x\cancel{(x-6)}(x+4)}$$

$$= \lim_{x\to6} \frac{x-2}{x(x+4)}$$

$$= \frac{6-2}{6(6+4)}$$

$$= \frac{1}{15}$$

Therefore, the hole is at the point $\left(6, \frac{1}{15}\right)$.

To find any vertical asymptotes, we look at the factors remaining in the denominator after dividing: namely, x and $x+4$. We set each of these factors equal to zero and find that the graph of f has vertical asymptotes at $x=0$ and $x=-4$.

To describe the infinite behavior near the vertical asymptotes, we can use either numerical or graphical methods. We will observe the graph of f near $x=0$ and $x=-4$ to determine the infinite behavior. See **Figure 1.3.14**.

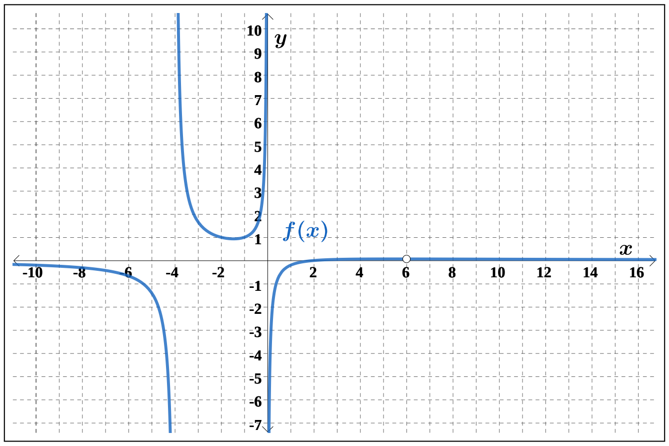


Figure 1.3.14: Graph of $f(x) = \dfrac{x^2 - 8x + 12}{x^3 - 2x^2 - 24x}$

Looking at the infinite behavior near the vertical asymptote at $x = 0$, we see

$$\lim_{x \to 0^-} f(x) \to \infty \quad \text{and} \quad \lim_{x \to 0^+} f(x) \to -\infty$$

Looking at the infinite behavior near the vertical asymptote at $x = -4$, we see

$$\lim_{x \to -4^-} f(x) \to -\infty \quad \text{and} \quad \lim_{x \to -4^+} f(x) \to \infty$$

To summarize, the graph of f has one horizontal asymptote at $y = 0$ (occurring at both ends), a hole at the point $\left(6, \frac{1}{15}\right)$, a vertical asymptote at $x = -4$ in which $\lim\limits_{x \to -4^-} f(x) \to -\infty$ and $\lim\limits_{x \to -4^+} f(x) \to \infty$, and a vertical asymptote at $x = 0$ in which $\lim\limits_{x \to 0^-} f(x) \to \infty$ and $\lim\limits_{x \to 0^+} f(x) \to -\infty$.

■

Try It # 10:
Find any horizontal asymptotes, holes, and vertical asymptotes of the following functions. If there are no horizontal asymptotes, describe the end behavior using limit notation. For each vertical asymptote, describe the infinite behavior using limit notation.

a. $f(x) = \dfrac{3x^2 + 12x + 12}{2x^3 - 8x}$

b. $f(x) = \dfrac{-2x^4 + 2x^3 + 12x^2}{x^3 - 7x^2 - 18x}$

Try It Answers

1. **a.** $\lim\limits_{x \to \infty} \left(x^5 - 3x^2 + 12\right) \to \infty$

   b. $\lim\limits_{x \to -\infty} \left(\left(-2x^4\right)\left(x^2 - x^{12}\right)\right) \to \infty$

2. 2

3. **a.** $\lim_{x\to\infty} f(x) = -1/4$; $\lim_{x\to-\infty} f(x) = -1/4$

   b. $\lim_{x\to\infty} f(x) \to \infty$; $\lim_{x\to-\infty} f(x) \to -\infty$

4. $\lim_{x\to\infty} -e^{-2x} = 0$; $\lim_{x\to-\infty} -e^{-2x} \to -\infty$

5. $\lim_{x\to\infty} f(x) = 4$; $\lim_{x\to-\infty} f(x) = 0$

6. $\lim_{x\to\infty} f(x) \to -\infty$; $\lim_{x\to-\infty} f(x) \to -\infty$

7. **a.** $y = 8/15$

   b. $y = 0$ when $x \to -\infty$; None when $x \to \infty$: $\lim_{x\to\infty} f(x) \to \infty$

8. **a.** $x = -5$, where $\lim_{x\to-5^-} f(x) \to \infty$ and $\lim_{x\to-5^+} f(x) \to -\infty$; $x = 0$, where $\lim_{x\to0^-} f(x) \to -\infty$ and $\lim_{x\to0^+} f(x) \to \infty$

   b. $x = 0$, where $\lim_{x\to0^-} f(x) \to \infty$ and $\lim_{x\to0^+} f(x) \to -\infty$; $x = 2$, where $\lim_{x\to2^-} f(x) \to -\infty$ and $\lim_{x\to2^+} f(x) \to \infty$

9. Hole at $x = -4$; Vertical Asympotes: $x = -9$, where $\lim_{x\to-9} f(x) \to -\infty$, and $x = 0$, where $\lim_{x\to0^-} f(x) \to \infty$ and $\lim_{x\to0^+} f(x) \to -\infty$

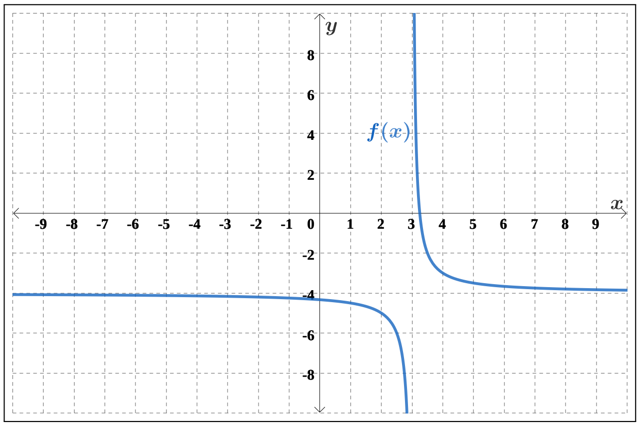
10. **a.** Horizontal Asymptote: $y = 0$; Hole: $(-2, 0)$; Vertical Asymptotes: $x = 0$, where $\lim_{x\to0^-} f(x) \to \infty$ and $\lim_{x\to0^+} f(x) \to -\infty$, and $x = 2$, where $\lim_{x\to2^-} f(x) \to -\infty$ and $\lim_{x\to2^+} f(x) \to \infty$

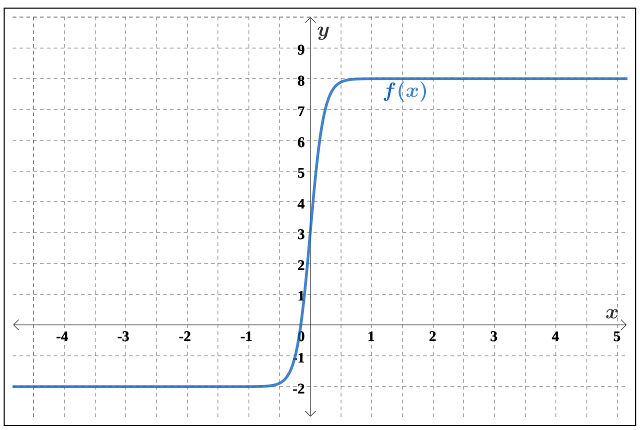
    b. No Horizontal Asymptote: $\lim_{x\to-\infty} f(x) \to \infty$ and $\lim_{x\to\infty} f(x) \to -\infty$; Holes: $(-2, 20/11)$ and $(0, 0)$; Vertical Asymptote: $x = 9$, where $\lim_{x\to9^-} f(x) \to \infty$ and $\lim_{x\to9^+} f(x) \to -\infty$

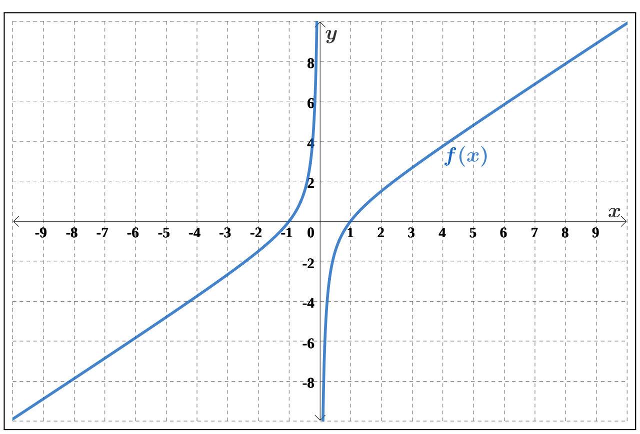
EXERCISES

Basic Skills Practice

For Exercises 1 - 3, use the graph of f to estimate (a) $\lim_{x\to\infty} f(x)$ and (b) $\lim_{x\to-\infty} f(x)$ (i.e., find the end behavior of the function). If a limit does not exist because the function has infinite behavior, use limit notation to describe the infinite behavior.

1. 

2. 

3. 

For Exercises 4 - 15, find the limit. If the limit does not exist because the function has infinite behavior, use limit notation to describe the infinite behavior.

4. $\lim_{x\to\infty} -3x^6$

5. $\lim_{x\to-\infty} -7x^5$

6. $\lim_{x\to\infty} \left(4x^2+7x-5\right)$

7. $\lim_{x\to\infty} \left(-4x^7+2x-9\right)$

8. $\lim_{x\to-\infty} \dfrac{-8x^2+4}{2x^3-9x}$

9. $\lim_{x\to-\infty} \dfrac{2x^3-3}{-4x+7}$

10. $\lim_{x\to\infty} \dfrac{4x^2+7}{5x-8}$

11. $\lim_{x\to-\infty} \dfrac{-x^2-x}{4x^2+1}$

12. $\lim_{x\to\infty} 5e^{-4x}$

13. $\lim_{x\to\infty} 2e^{3x}$

14. $\lim_{x\to-\infty} -3e^{-x}$

15. $\lim_{x\to-\infty} 4e^{7x}$

For Exercises 16 - 19, find the x-values of any (a) holes and (b) vertical asymptotes of the function.

16. $f(x)=\dfrac{(x+2)(x-4)}{(x-4)(2x+5)}$

17. $f(x)=\dfrac{(x+7)(x-9)}{(x-9)(4x-5)}$

18. $f(x)=\dfrac{(2x-3)(x+6)}{(2x+3)(x-6)}$

19. $f(x)=\dfrac{(3x-8)(x+4)}{(2x-9)(x+1)}$

For Exercises 20 - 27, find any horizontal asymptotes of the function.

20. $f(x) = -2x^6 + 5x$

21. $f(x) = -5x^3 - x^2 - 9$

22. $f(x) = \dfrac{2x-7}{4x^2+3}$

23. $f(x) = \dfrac{3x^3-7}{4x+3}$

24. $f(x) = \dfrac{-9x^2+3x}{2x^2-5}$

25. $f(x) = \dfrac{-2x^3+4}{3x^2+7x}$

26. $f(x) = 4e^{-3x}$

27. $f(x) = -3e^{6x}$

Intermediate Skills Practice

For Exercises 28 - 37, find the limit. If the limit does not exist because the function has infinite behavior, use limit notation to describe the infinite behavior.

28. $\lim\limits_{x\to-\infty} \left(2x^3 + x^8 - 4x^2\right)$

29. $\lim\limits_{x\to-\infty} \left(32 - 3x^3 + 10x^{11}\right)$

30. $\lim\limits_{x\to\infty} \dfrac{4x^3 - 8x^{10} + 4x^6}{8x^2 - 7x + 5}$

31. $\lim\limits_{x\to-\infty} \dfrac{12 + 4x^7 + x}{-19x^6 - 4x^2}$

32. $\lim\limits_{x\to\infty} \dfrac{4x^2 - 5x + 3x^3}{15x - 3x^5 - x^4}$

33. $\lim\limits_{x\to-\infty} \dfrac{4x^2 + 7x^3 - 9}{-x^3 - 7x + 8}$

34. $\lim\limits_{x\to\infty} \dfrac{5}{8 + e^{2x}}$

35. $\lim_{x\to-\infty} \frac{6}{e^{3x}-12}$

36. $\lim_{x\to\infty} \frac{5e^{7x}}{e^{-2x}+9}$

37. $\lim_{x\to-\infty} \frac{e^{x}}{e^{-x}+1}$

For Exercises 38 - 43, find any (a) holes and (b) vertical asymptotes of the function. For each vertical asymptote, describe the infinite behavior using limit notation.

38. $f(x) = \frac{(2x+7)^2(x-4)}{(x+3)(2x+7)}$

39. $f(x) = \frac{(5x-17)(x+4)}{(3x-8)(x+4)^2}$

40. $f(x) = \frac{x^2+3x-28}{x^2-x-12}$

41. $f(x) = \frac{2x^2+5x-3}{2x^2-9x+4}$

42. $f(x) = \frac{3x^3+11x^2-4x}{2x^3-3x^2-2x}$

43. $f(x) = \frac{x^3-3x^2-28x}{x^4-4x^3+3x^2}$

For Exercises 44 - 51, find any horizontal asymptotes of the function. If there are none, describe the end behavior using limit notation.

44. $f(x) = \frac{5x^2-7x^4+3x}{4x-x^4}$

45. $f(x) = \frac{10x^5-4x-x^3}{3x^2+7x^5}$

46. $f(x) = \frac{-14x^{17}-4x-5}{3x^2-8x}$

47. $f(x) = \frac{25x^4-x^9}{8x^{11}-4x^2+19}$

48. $f(x) = \frac{5}{1-2e^{-2x}}$

49. $f(x) = \dfrac{10}{e^{4x}+5}$

50. $f(x) = \dfrac{7+3e^{3x}}{4-8e^{3x}}$

51. $f(x) = \dfrac{5-e^{-4x}}{3e^{-4x}-1}$

For Exercises 52 - 55, find any (a) horizontal asymptotes, (b) holes, and (c) vertical asymptotes of the function.

52. $f(x) = \dfrac{2x^2-x-21}{3x^2+14x-5}$

53. $f(x) = \dfrac{4x^2-9}{2x^2+5x-12}$

54. $f(x) = \dfrac{x^3+x^2-12x}{x^2+3x-4}$

55. $f(x) = \dfrac{x^3-x^2-6x}{x^4-5x^3-14x^2}$

Mastery Practice

For Exercises 56 - 66, find the limit. If the limit does not exist because the function has infinite behavior, use limit notation to describe the infinite behavior.

56. $\lim\limits_{x\to\infty} ((x+2)(-7+2x))$

57. $\lim\limits_{x\to-\infty} \dfrac{9x^2-x^6+3x}{31x^7-42x^9+\sqrt{5}x^3}$

58. $\lim\limits_{x\to\infty} \dfrac{-4e^{2x}-5e^{-2x}}{10+e^{-3x}+7e^{2x}}$

59. $\lim\limits_{x\to-\infty} \left((-2x+3)\left(5+4x^2\right)\right)$

60. $\lim\limits_{x\to\infty} \dfrac{-7x^{10}+x^{25}}{-x^4+3x^{10}}$

61. $\lim\limits_{x\to\infty} \dfrac{16x^7+9x^8-\pi}{-2x^8-9x^7+3}$

62. $\lim\limits_{x\to-\infty} \left(\left(2x^2-3\right)(10+8x)\right)$

63. $\lim_{x\to-\infty} \dfrac{6e^{4x}+5e^{-2x}}{-2e^{5x}+e^{9x}-13}$

64. $\lim_{x\to-\infty} \dfrac{-4e^{2x}-5e^{-2x}}{10+e^{-3x}+7e^{4x}}$

65. $\lim_{x\to\infty} \left(\left(4x^3-5\right)\left(-3x^3+10x\right)\right)$

66. $\lim_{x\to\infty} \dfrac{e^{4x}+7e^{-8x}}{e^{-3x}-5e^{2x}}$

For Exercises 67 - 73, find any horizontal asymptotes of the function. If there are none, describe the end behavior using limit notation.

67. $f(x) = \dfrac{3e^{4x}+5e^{-2x}}{-e^{-3x}+4e^{8x}}$

68. $f(x) = \dfrac{-41x^5-3x^2+\pi x^8}{x^5+2x^8-6x^3}$

69. $f(x) = \dfrac{4e^{-x}+e^{2x}}{1+6e^{-x}+2e^{-3x}}$

70. $f(x) = \dfrac{\sqrt{2}x^{40}-\frac{7}{3}x^{20}+5}{-6x^{38}+2x^9-\frac{8}{9}}$

71. $f(x) = \dfrac{2e^{-4x}-e^{5x}}{e^{2x}-6e^{x}+8}$

72. $f(x) = \dfrac{2e^{7x}-4e^{-5x}}{100+3e^{-5x}+5e^{8x}}$

73. $f(x) = \dfrac{7x^2-4x^9+1}{\frac{4}{5}x^5-x^8+2x^{10}}$

For Exercises 74 - 76, find any (a) horizontal asymptotes, (b) holes, and (c) vertical asymptotes of the function. If there are no horizontal asymptotes, describe the end behavior using limit notation. For each vertical asymptote, describe the infinite behavior using limit notation.

74. $f(x) = \dfrac{2x^3+13x^2-45x}{x+9}$

75. $f(x) = \dfrac{8x^2+28x-60}{5x^2-8x-21}$

76. $f(x) = \dfrac{3x^3-15x^2-18x}{4x^4-4}$

For Exercises 77 - 80, find (a) $\lim_{x\to\infty} f(x)$ and (b) $\lim_{x\to-\infty} f(x)$. If a limit does not exist because the function has infinite behavior, use limit notation to describe the infinite behavior.

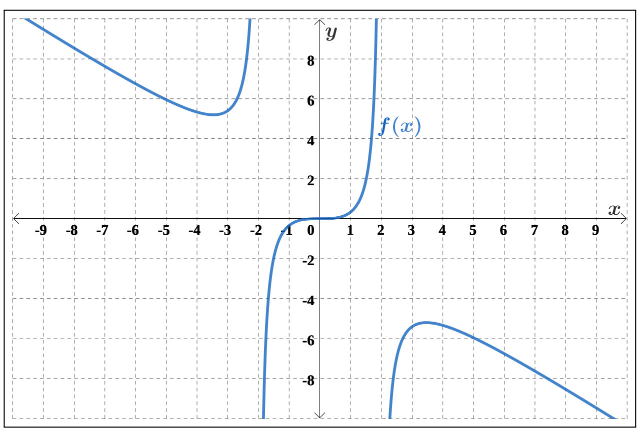
77. $f(x) = \begin{cases} 4x^2 - 7 & x \le 3 \\ \dfrac{5x^7 - 4x}{2x - x^2} & x > 3 \end{cases}$

78. $f(x) = \begin{cases} \dfrac{9x^2 - 4x - x^6}{10 - 8x^6} & x \le -10 \\ \dfrac{5x^2 - 9x}{-6x^8 + 25} & x > 5 \end{cases}$

79. $f(x) = \begin{cases} \dfrac{e^{-10x} - e^{3x}}{4e^{2x} - 5e^{-x}} & x \le -2 \\ \dfrac{x^2 - x}{x^4 - x^2} & x > -2 \end{cases}$

80. $f(x) = \begin{cases} \dfrac{91x^8 + 4}{10x - x^8} & x \le -4 \\ \dfrac{5e^{3x} - 4e^{-2x}}{7e^{-x} - 10e^{5x}} & x > 10 \end{cases}$

81. Use the graph of f shown below to find (a) $\lim_{x\to\infty} f(x)$ and (b) $\lim_{x\to-\infty} f(x)$. If a limit does not exist because the function has infinite behavior, use limit notation to describe the infinite behavior.



For Exercises 82 - 85, find the limit. If the limit does not exist because the function has infinite behavior, use limit notation to describe the infinite behavior. Note that a represents a nonzero real number.

82. $\displaystyle\lim_{x\to\infty} \frac{ax^2 - 7x + 4}{4x^7 - 3x^2 - 4}$

83. $\displaystyle\lim_{x\to-\infty} \frac{7x^3 - 4x^6 + 2x^8}{ax^8 - 4x^2}$

84. $\lim_{x\to\infty} \dfrac{2x^4 - 7x^2 - ax^5}{3x^4 - 4x^2}$

85. $\lim_{x\to-\infty} \dfrac{a - x^6 + 2x^4}{3x^6 + 4x^3}$

For Exercises 86 - 89, find any (a) holes and (b) vertical asymptotes of the function. Note that a represents a real number.

86. $f(x) = \dfrac{(x-a)(x+2)^3}{x^2 + 2x}$

87. $f(x) = \dfrac{2x^2 + x - 1}{(x-a)(2x-1)^2}$

88. $f(x) = \dfrac{(3x+1)(x-a)}{x^2 - ax}$

89. $f(x) = \dfrac{(a-x)(2x-3)}{(x-a)(x+1)}$

90. Find a function f whose graph has a vertical asymptote at $x = 5$ and a horizontal asymptote at $y = 5$.

91. Find a function f whose graph has a vertical asymptote at $x = 1$, a hole at $x = -1$, and a horizontal asymptote at $y = -2$.

For Exercises 92 and 93, find the limit. If the limit does not exist because the function has infinite behavior, use limit notation to describe the infinite behavior.

92. $\lim_{x\to-\infty} \dfrac{4e^{-6x} + 2}{7e^{2x} + 5e^{3x}}$

93. $\lim_{x\to\infty} \dfrac{9e^{4x} - 2e^{x}}{8e^{-3x} - e^{-5x}}$

94. Find any horizontal asymptote(s) of $y = \dfrac{ax^{46} - bx^2 - c}{bx^{46} - a}$, where a, b, and c are real numbers with $a > 0$, $b < 0$, and $c < 0$.

95. Find $\lim_{x\to-\infty} \dfrac{ax^3 + bx - c}{4x + 2}$, where a, b, and c are real numbers with $a < 0$, $b > 0$, and $c > 0$. If the limit does not exist because the function has infinite behavior, use limit notation to describe the infinite behavior.

96. Determine the end behavior of the function $g(x) = ax^2 + bx^4 - cx^3 - dx + 6$, where a, b, c, and d are real numbers with $a > 0$, $b < 0$, $c < 0$, and $d > 0$.

Communication Practice

97. Explain, in general, how to determine the end behavior of a function f.

98. Can a function have no horizontal asymptotes? Explain.

99. Can a function have three horizontal asymptotes? Explain.

100. If $\lim_{x\to 1^-} f(x) \to -\infty$, does $\lim_{x\to 1^+} f(x) \to \infty$? Explain.

101. If $\lim_{x\to 5} f(x) \to \infty$, does the graph of f have a vertical asymptote at $x = 5$? Explain.

102. Both $f(x) = \dfrac{1}{x-1}$ and $g(x) = \dfrac{1}{(x-1)^2}$ have a vertical asymptote at $x = 1$. What is the difference in the behavior of these two functions as x approaches 1?

103. Explain how to find the horizontal asymptotes of a rational function.

104. Compare and contrast horizontal and vertical asymptotes.

105. Explain what leads you to conclude that the graph of a function has a vertical asymptote instead of a hole (or vice-versa) at $x = a$.

106. Explain why the graph of $f(x) = x^2 + 2x - 4$ does not have any holes or asymptotes.

1.4 Continuity from a Calculus Perspective

Many functions have the property that their graphs can be traced with a pencil without lifting the pencil from the page. Such functions are called **continuous**. Other functions have points at which a break in the graph occurs, but satisfy this property over intervals contained in their domains. They are continuous on these intervals and are said to have a discontinuity at a point where a break occurs.

We begin our investigation of continuity by exploring what it means for a function to have continuity at a point. Intuitively, a function is continuous at a particular point if there is no break in its graph at that point.

Learning Objectives:

In this section, you will learn how to determine where a function is continuous graphically and algebraically using the definition of continuity at a point. Upon completion you will be able to:

- List, using mathematical notation, the three conditions which must be satisfied for a function to be continuous at a point where $x = c$.
- Determine where a piecewise-defined function is continuous graphically.
- Justify, mathematically, whether a function is continuous at a particular x-value using the definition of continuity given the graph of the function.
- Justify, mathematically, whether a function is continuous at a particular x-value using the definition of continuity at a point given the rule of the function.
- Determine where a function is continuous algebraically, including piecewise-defined functions.
- Calculate an unknown value to ensure continuity of a function.

The Calculus Definition of Continuity

Before we look at the formal definition of what it means for a function to be continuous at a point, let's consider various functions that fail to meet our intuitive notion of what it means to be continuous at a point. We then create a list of conditions that will ensure continuity at a point.

Our first function of interest is shown in **Figure 1.4.1**. We see that the graph of f has a hole at $x = c$. In fact, $f(c)$ is undefined. At the very least, for f to be continuous at $x = c$, we need the following condition:

I. $f(c)$ is defined.

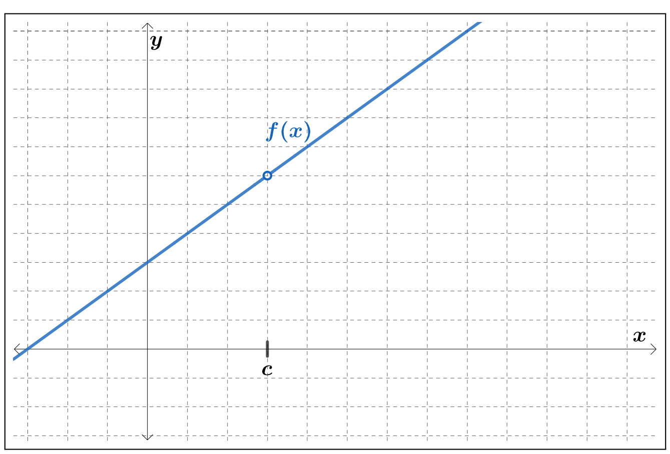


Figure 1.4.1: The function f is not continuous at $x = c$ because $f(c)$ is undefined.

However, as we see in **Figure 1.4.2**, this condition alone is insufficient to guarantee continuity at $x = c$. Although $f(c)$ is defined, the function has a "jump" at $x = c$. In this example, the "jump" exists because $\lim_{x\to c} f(x)$ does not exist. We must add another condition for continuity at $x = c$. Namely,

$$\text{II. } \lim_{x\to c} f(x) \text{ exists.}$$

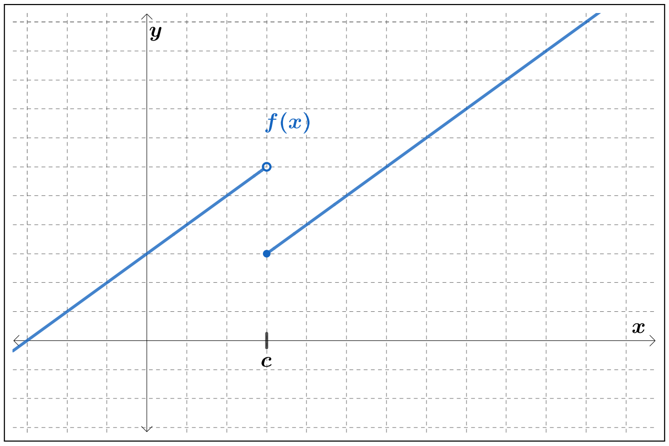


Figure 1.4.2: The function f is not continuous at $x = c$ because $\lim_{x\to c} f(x)$ does not exist.

However, as we see in **Figure 1.4.3** below, these two conditions by themselves do not guarantee continuity at a point. The function in this figure satisfies both of our first two conditions, but it is still not continuous at $x = c$. We must add a third condition to our list:

$$\text{III. } \lim_{x\to c} f(x) = f(c).$$

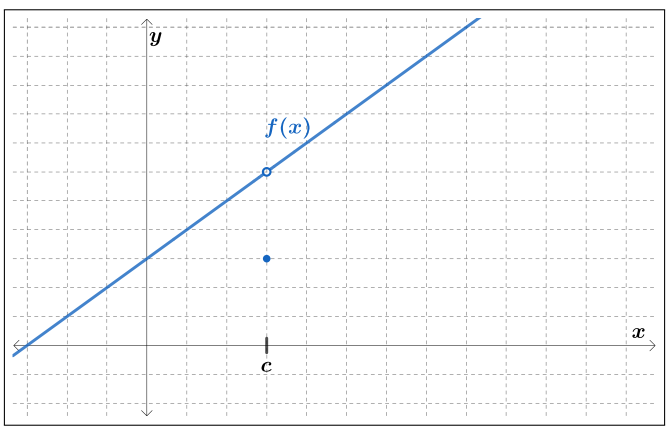


Figure 1.4.3: The function f is not continuous at $x = c$ because $\lim_{x\to c} f(x) \neq f(c)$.

Now, we put our list of conditions together and form a definition of continuity at a point:

Definition
A function f is **continuous at a point** where $x = c$ if and only if the following three conditions are satisfied:

I. $f(c)$ is defined

II. $\lim_{x \to c} f(x)$ exists

III. $\lim_{x \to c} f(x) = f(c)$

*In **Section 1.2**, we were vague regarding the functions for which the Direct Substitution Property yields an answer. This definition tells us that those functions are exactly those that are continuous at the point!*

Related to the definition of continuity, we say that a function is **discontinuous at a point** where $x = c$ if it fails to be continuous at $x = c$. We also say that a function is **continuous on an interval** if it is continuous at every point in the interval.

Notice that the limits we use in the definition are two-sided limits. There is a whole study of left-hand and right-hand continuity that is beyond the scope of this textbook.

The following procedure can be used to analyze the continuity of a function at a point using this definition.

Determining Continuity at a Point

1. Check to see if $f(c)$ is defined. If $f(c)$ is undefined, we do not need to go any further! The function is not continuous at $x = c$. If $f(c)$ is defined, continue to step 2.

2. Compute $\lim_{x \to c} f(x)$. In some cases, we may need to do this by first computing $\lim_{x \to c^-} f(x)$ and $\lim_{x \to c^+} f(x)$. If $\lim_{x \to c} f(x)$ does not exist (either because the left- and right-hand limits do not equal or because the function has a vertical asymptote), then the function is not continuous at $x = c$ and the problem is solved. If $\lim_{x \to c} f(x)$ exists, then continue to step 3.

3. Compare $\lim_{x \to c} f(x)$ and $f(c)$ (steps 1 and 2). If $\lim_{x \to c} f(x) \neq f(c)$, then the function is not continuous at $x = c$. If $\lim_{x \to c} f(x) = f(c)$, then the function is continuous at $x = c$.

Determining Continuity Graphically

First, we will practice applying the method above to determine if a function is continuous at a point by observing the graph of the function. Then, we will determine where a function is continuous algebraically.

Example 1 Determine the x-value(s) where f, shown in **Figure 1.4.4**, is not continuous (i.e., is discontinuous), and state the condition in the definition of continuity at a point that fails first at each x-value mathematically.

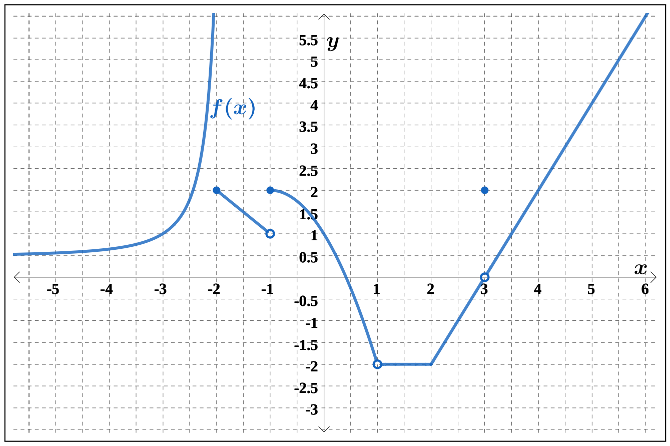


Figure 1.4.4: Graph of a piecewise-defined function f

Solution:
f is not continuous (i.e., f is discontinuous) at $x = -2$, $x = -1$, $x = 1$, and $x = 3$.

At $x = -2$, the function is defined because $f(-2) = 2$. Next, we look at $\lim_{x\to -2} f(x)$. Notice that because this is a piecewise-defined function, we have to look at both the left- and right-hand limits. Observing the behavior of the function as x approaches -2 from the right, we see $\lim_{x\to -2^+} f(x) = 2$. However, since $\lim_{x\to -2^-} f(x) \to \infty$, technically this left-hand limit does not exist. In any case, the left- and right-hand limits are not heading toward the same finite y-value. Thus, the second condition in the definition fails first because $\lim_{x\to -2} f(x)$ does not exist.

N *Note that the third condition in the definition of continuity at a point also fails in part* **a** *above. In fact, if the first condition and/or second condition fail(s), then the third condition must also fail.*

At $x = -1$, the function is defined because $f(-1) = 2$. However, looking at the left- and right-hand limits, we see $\lim_{x\to -1^-} f(x) = 1$ and $\lim_{x\to -1^+} f(x) = 2$. Thus, the second condition in the definition of continuity at a point is the first to fail because $\lim_{x\to -1} f(x)$ does not exist.

At $x = 1$, we see the function is undefined. Thus, the first condition in the definition of continuity at a point is the first to fail: $f(1)$ is undefined.

At $x = 3$, the function is defined because $f(3) = 2$. Next, we look at $\lim_{x\to 3} f(x)$. We see $\lim_{x\to 3^-} f(x) = 0$ and $\lim_{x\to 3^+} f(x) = 0$. Because the left- and right-hand limits are equal, the limit exists. Specifically, $\lim_{x\to 3} f(x) = 0$. Moving on to the last (third) condition in the definition of continuity at a point, we check to see if the value of the limit equals the function value. Because $2 \neq 0$, we conclude that this condition is the first to fail because $\lim_{x\to 3} f(x) \neq f(3)$.

■

Try It # 1:
Determine the x-value(s) where g, shown in **Figure 1.4.5**, is not continuous (i.e., is discontinuous), and state the condition in the definition of continuity at a point that fails first at each x-value mathematically.

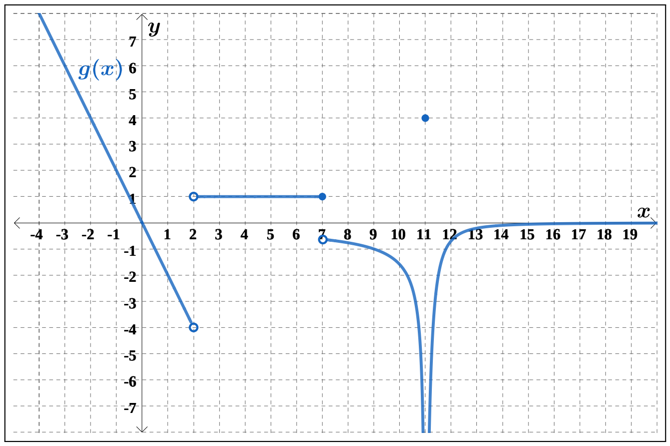


Figure 1.4.5: Graph of a piecewise-defined function g

Determining Continuity Algebraically

We will now learn how to determine where a function is continuous without looking at the graph of the function (i.e., algebraically). To do this, we need an important theorem:

Theorem 1.6 Polynomials, rational functions, power functions, exponential functions, logarithmic functions, and combinations of these are continuous on their domain.

Thus, in order to find where these types of functions, as well as combinations of these functions, are continuous, we need to find their domain. We list the three main domain restrictions below.

Domain Restrictions

1. The denominator must be nonzero.
2. The argument of an even root must be nonnegative.
3. The argument of a logarithm (of any base) must be positive.

Let's practice determining continuity algebraically!

▪ **Example 2** Find where each of the following functions is continuous algebraically. Write your answer using interval notation.

a. $y = \dfrac{x^2-4}{x-2}$

b. $g(x) = \ln(4-x) - \log_8(7-3x)$

c. $k(x) = \dfrac{1}{\sqrt{x+2}}$

d. $f(x) = \dfrac{e^{\frac{2x-4}{x}}}{\ln(-4x+9)}$

e. $h(x) = \dfrac{\sqrt{6x-12}}{8^{x-25}}$

Solution:

a. For this function, $y = \dfrac{x^2-4}{x-2}$, there are no even roots or logarithms, so we only need to make sure there is no division by zero:

$$x-2 \neq 0 \implies$$
$$x \neq 2$$

Therefore, the domain, and where the function is continuous, is $(-\infty,2)\cup(2,\infty)$.

! *Although you may be tempted to factor the numerator so the function looks like*

$$y = \frac{(x+2)(x-2)}{x-2},$$

you ***should not do this!*** *If you write the function this way, you might divide the common factor $x-2$ and conclude the domain is all real numbers. However, the function has a hole at $x=2$ because the factor $x-2$ will divide completely from the denominator. Thus, $x=2$ is not in the domain of the function.*

b. For this function, $g(x) = \ln(4-x) - \log_8(7-3x)$, we do not have any even roots or division, so we only have to worry about the logarithms. We need to make sure that the quantities we are taking the logarithm of are positive:

$$4-x > 0$$
$$4 > x$$

Or, $x < 4$.

$$7-3x > 0$$
$$7 > 3x$$
$$\frac{7}{3} > x$$

Or, $x < \dfrac{7}{3}$.

N *Remember when multiplying or dividing an inequality by a number, it is possible that the inequality may change direction. That occurs if we multiply or divide by a negative number. Because we divided by 3, the the direction of the inequality remained the same.*

Now, we need to find a solution that satisfies both of our requirements: $x < \dfrac{7}{3}$ and $x < 4$. When you have multiple inequality conditions, it may be helpful to graph each on a number line and see where their solutions intersect. See **Figure 1.4.6**.

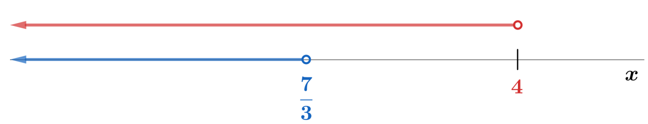


Figure 1.4.6: Number line to help determine the interval(s) where the function $y = \dfrac{x^2-4}{x-2}$ is continuous

Based on our number line, both conditions will be satisfied when $x < \dfrac{7}{3}$. Thus, the function is continuous on $\left(-\infty, \dfrac{7}{3}\right)$.

c. Recall that $k(x) = \dfrac{1}{\sqrt{x+2}}$. In a case like this where there is an even root in the denominator, and it is the *only term* in the denominator, you can combine the first and second domain restrictions in one step of work by setting the quantity you are taking the even root of strictly greater than zero. This will ensure the denominator does not equal zero:

$$\begin{aligned} x+2 &> 0 \\ x &> -2 \end{aligned}$$

Therefore, the function is continuous on $(-2, \infty)$.

d. This function, $f(x) = \dfrac{e^{\frac{2x-4}{x}}}{\ln(-4x+9)}$, has two different fractions as well as a logarithm to consider for the domain. Let's look at the denominators of the fractions. From the fraction in the exponent of the numerator, we have

$$x \neq 0$$

From the main fraction we have

$$\begin{aligned} \ln(-4x+9) &\neq 0 \\ e^{\ln(-4x+9)} &\neq e^0 \\ -4x+9 &\neq 1 \\ -4x &\neq -8 \\ x &\neq 2 \end{aligned}$$

Because the function contains a logarithm, we also have to consider the third domain restriction, regardless of the base of the logarithm:

$$\begin{aligned} -4x+9 &> 0 \\ -4x &> -9 \\ x &< \frac{-9}{-4} \\ x &< \frac{9}{4} \end{aligned}$$

The domain, in words, is all x-values less than (but not equal to) $\frac{9}{4}$ with the exception of 0 and 2. Using interval notation, we have that the function is continuous on $(-\infty, 0) \cup (0, 2) \cup \left(2, \frac{9}{4}\right)$.

N *Remember, you can always draw a number line, if necessary, to help you determine on what interval(s) the requirements for continuity intersect as shown in the solution for part* **a**.

e. For this function, $h(x) = \dfrac{\sqrt{6x-12}}{8^{x-25}}$, there is division and an even root. First, notice the denominator contains only an exponential function. Because an exponential function is always positive, it will never equal zero. Thus, we obtain no domain restrictions from the denominator.

We now look at the even root. The quantity we are taking the even root of must be nonnegative:

$$\begin{aligned} 6x-12 &\geq 0 \\ 6x &\geq 12 \\ x &\geq 2 \end{aligned}$$

Thus, the function is continuous on $[2, \infty)$.

N *In the answer above, by having the bracket at $x = 2$, we are implying that the function is continuous from the right at $x = 2$. This involves one-sided continuity, which is a concept beyond the scope of this textbook. For our purposes, it is sufficient to understand that a non piecewise-defined function is continuous on its domain.*

! *You could try to graph the functions in parts* **a** *through* **e** *to verify your answers, but be aware that you may have trouble seeing the exact points of discontinuity on the calculator.*

■

Try It # 2:
Find where each of the following functions is continuous algebraically. Write your answer using interval notation.

a. $f(x) = \dfrac{4x^3 - 2x^2 + 7}{x^2 - 15}$

b. $y = \dfrac{5x - 8}{\sqrt[3]{x^2 - 16}}$

c. $g(x) = \dfrac{\log_9(100 - 2x)}{x^2 - x - 6}$

d. $y = \dfrac{\sqrt{x + 7}}{\ln(20 - 3x)}$

Continuity of Piecewise-defined Functions

We will now turn our attention to determining continuity algebraically of piecewise-defined functions. A piecewise-defined function may or may not be continuous on its domain. Thus, unlike the functions in our previous examples, the process for finding where a piecewise-defined function is continuous will involve more than finding its domain; it will also involve using the definition of continuity at a point!

If we are given a piecewise-defined function and only need to determine whether or not the function is continuous at a specific x-value, usually a cutoff number, we use the definition of continuity at a point. In other words, we check to see if the function is defined, if the limit exists, and if the function value equals the value of the limit at the x-value in question.

■ **Example 3** The function f is given below. Using the definition of continuity at a point, determine whether f is continuous at $x = 3$.

$$f(x) = \begin{cases} -x^2 + 4 & x \leq 3 \\ 4x - 8 & x > 3 \end{cases}$$

Solution:
Note that $x = 3$ is the cutoff number in the piecewise-defined function (and it is the only cutoff number). Recall the cutoff numbers are the x-values where the rules of the piecewise-defined function change. So we need to apply the three conditions in the definition of continuity at a point to determine if the function is continuous at $x = 3$.

First, we attempt to find $f(3)$. Because the first rule of the function, $-x^2 + 4$, includes $x = 3$, we substitute $x = 3$ into this rule to get the function value. Thus,

$$\begin{aligned} f(3) &= -(3)^2 + 4 \\ &= -5 \end{aligned}$$

Next, we find $\lim_{x\to 3} f(x)$, if it exists. Because f is a piecewise-defined function and the x-value in question is a cutoff number, we have to calculate the left- and right-hand limits. Using the first rule which corresponds to x approaching 3 from the left, we have

$$\begin{aligned}\lim_{x\to 3^-} f(x) &= \lim_{x\to 3^-} \left(-x^2+4\right)\\ &= -(3)^2+4\\ &= -9+4\\ &= -5\end{aligned}$$

Using the second rule for the right-hand limit, we have

$$\begin{aligned}\lim_{x\to 3^+} f(x) &= \lim_{x\to 3^+} (4x-8)\\ &= 4(3)-8\\ &= 12-8\\ &= 4\end{aligned}$$

Thus, we see $\lim_{x\to 3} f(x)$ does not exist because the left- and right-hand limits are not equal.

Because this limit does not exist, the second condition in the definition of continuity at a point fails, and we do not need to continue any further. The function is not continuous at $x = 3$.

We can verify our answer by looking at the graph of f in **Figure 1.4.7**:

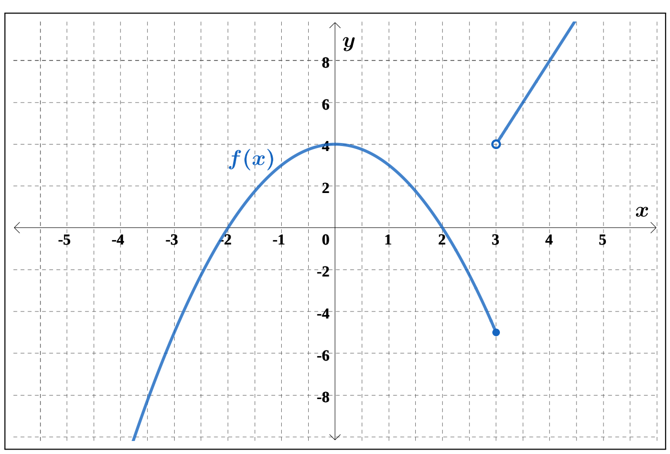


Figure 1.4.7: Graph of the piecewise-defined function f

■

Try It # 3:
The function f is given below. Using the definition of continuity at a point, determine whether f is continuous at $x = 1$.

$$f(x) = \begin{cases} 2x+1 & x < 1 \\ 2 & x = 1 \\ -x+4 & x > 1 \end{cases}$$

Until now, we have only discussed how to find the interval(s) where a function that is not piecewise-defined is continuous. Although we will use the domain rules as part of the process to determine where a piecewise-defined function is continuous, a piecewise-defined function is not necessarily continuous on its domain. Thus, we will have to do a little more work to determine where a piecewise-defined function is continuous (as stated previously).

First, we will observe for which values of x we are given a rule. The function cannot be continuous at any values of x that are not in the domain of the function.

Then, we will find the domain of each rule (remember, these are the "pieces" of the function). When doing so, we will also need to check that the resulting domain of each rule is relevant to the set of x-values defined for the rule. For example, if one of the rules requires $x \neq 3$, but that rule is defined for $x > 5$, then we do not need the restriction $x \neq 3$ because it is not relevant to the x-values defined for that rule.

Lastly, we will investigate the behavior of the function at each cutoff number. To investigate the behavior at the cutoff number(s), we will test the three conditions in the definition of continuity at a point.

The process we will use is summarized below.

Finding Where a Piecewise-defined Function is Continuous

1. Observe for which values of x we are given a rule. The answer cannot include any values of x for which we do not have a rule.
2. Find any domain restrictions of each rule and compare them to the x-values defined for that rule to see which, if any, are relevant.
3. Apply the three conditions in the definition of continuity at a point to determine if the function is continuous at each cutoff number (i.e., check to see if the function is defined, if the limit exists, and if the value of the limit equals the function value at each cutoff number).
4. Determine a solution that satisfies all the requirements found above, and write your answer using interval notation.

Example 4 The function f is given below. Determine where f is continuous algebraically.

$$f(x) = \begin{cases} \dfrac{4x^3-2x^2-9}{x^2-9} & x<2 \\ 8 & x=2 \\ -3+\log_6(3-x) & x>2 \end{cases}$$

Solution:
We first observe that there is a rule for all values of x because if we combine $x<2$, $x=2$, and $x>2$, we get all real numbers.

Next, we find any domain restrictions for each rule and check whether those are relevant to the x-values defined for the respective rule:

Rule 1: $\dfrac{4x^3-2x^2-9}{x^2-9}$. The only domain issue with this rule we need to worry about is division by zero. Thus, we need to ensure

$$\begin{aligned} x^2-9 &\neq 0 \\ x^2 &\neq 9 \\ \implies x &\neq -3,\ x\neq 3 \end{aligned}$$

Now, we check the x-values for which the first rule is defined: $x<2$. This means the only restriction relevant to rule 1 that we need to include in our final answer is $x\neq -3$. Why? Because rule 1 is defined for $x<2$, the restriction $x\neq 3$ will not be an issue due to the fact that we are only allowed to substitute x-values into the rule that are less than 2.

Rule 2: 8. The constant function $y=8$ has a domain of all real numbers, so there are no restrictions for this rule. Likewise, there is no need to check the x-values where the rule is defined.

Rule 3: $-3+\log_6(3-x)$. The only domain issue here is the logarithm. Regardless of the base of the logarithm, the argument must be positive:

$$\begin{aligned} 3-x &> 0 \\ 3 &> x \end{aligned}$$

Or, $x<3$. Now, we compare this restriction with the x-values defined for this rule: $x>2$. Because 3 is greater than 2, we need to be sure that we include the restriction $x<3$ as part of our answer. Note this means 3 will be an "upper bound" for x in our final answer.

To summarize, for the rules of the function we have the following restrictions: $x\neq -3$ and $x<3$.

Now, we need to determine if the function is continuous at the cutoff number, $x=2$, using the definition of continuity at a point. Recall that we only have to check the conditions until one fails, if one does:

I. Is $f(2)$ defined? According to the function, we use rule 2 and find $f(2)=8$. Thus, f is defined at $x=2$, and we move to the second condition.

II. Does $\lim_{x\to 2} f(x)$ exist? Because f is a piecewise-defined function and $x=2$ is the cutoff number, we have to calculate the left -and right-hand limits and see if they exist and are equal.

We must use rule 1 to calculate the left-hand limit: $\lim\limits_{x\to 2^-} \dfrac{4x^3-2x^2-9}{x^2-9}$. But first, notice rule 1 consists of a ratio of two functions. So we need to make sure the limits of the two functions exist and that the limit of the function in the denominator is nonzero. We can find the limits of the functions in the numerator and denominator using direct substitution because there are no domain issues:

$$\begin{aligned}\lim_{x\to 2^-}\left(4x^3-2x^2-9\right) &= 4(2)^3-2(2)^2-9\\ &= 15\end{aligned}$$

and

$$\begin{aligned}\lim_{x\to 2^-}\left(x^2-9\right) &= (2)^2-9\\ &= -5\end{aligned}$$

Now, we know the limits of the functions in the numerator and denominator exist and that the limit of the function in the denominator is nonzero. Thus, we can use the Properties of Limits and divide the limits:

$$\begin{aligned}\lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-}\frac{4x^3-2x^2-9}{x^2-9}\\ &= \frac{\lim\limits_{x\to 2^-}\left(4x^3-2x^2-9\right)}{\lim\limits_{x\to 2^-}(x^2-9)}\\ &= \frac{15}{-5}\\ &= -3\end{aligned}$$

Next, we use rule 3 to calculate the right-hand limit: $\lim\limits_{x\to 2^+}(-3+\log_6(3-x))$. Notice part of this rule contains a logarithmic function, $\log_6(3-x)$. Remember we can only use direct substitution to calculate a limit if there are no domain issues. Because $x=2$ is in the domain of this logarithmic function, we can proceed with direct substitution to calculate the right-hand limit:

$$\begin{aligned}\lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+}(-3+\log_6(3-x))\\ &= -3+\log_6(3-2)\\ &= -3+\log_6(1)\\ &= -3+0\\ &= -3\end{aligned}$$

Thus, the left- and right-hand limits both equal -3, so $\lim\limits_{x\to 2} f(x) = -3$. We move to the last condition.

III. Does $\lim\limits_{x\to 2} f(x) = f(2)$? Because $\lim\limits_{x\to 2} f(x) = -3$ and $f(2) = 8$, $\lim\limits_{x\to 2} f(x) \neq f(2)$. Thus, the function is *not* continuous at $x = 2$, and we need to exclude it from the x-values where f is continuous (our final answer).

Combining this restriction ($x \neq 2$), as well as the restrictions we calculated when looking at the domain of each rule of the function ($x \neq -3$ and $x < 3$), we have that f is continuous on

$$(-\infty,-3)\cup(-3,2)\cup(2,3)$$

Notice this answer differs from the domain of f! f is *defined* at $x = 2$, so the domain is $(-\infty,-3)\cup(-3,3)$. But, the function is not continuous at $x = 2$, and that is why we excluded it from our answer.

Let's check both the domain of f and the intervals where f is continuous by looking at the graph of the function. See **Figure 1.4.8**.

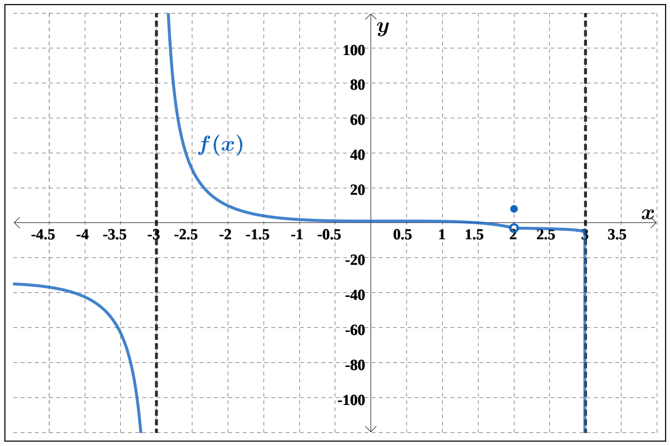


Figure 1.4.8: Graph of the piecewise-defined function f

■

N *Although it may appear that the function is touching the asymptote, it is just very close given the viewing window.*

■ **Example 5** The function f is given below. Determine where f is continuous algebraically.

$$f(x) = \begin{cases} \dfrac{x+6}{(x+6)(3x-2)} & x < -3 \\[2ex] \dfrac{\sqrt[3]{x+2}}{11^{x+4}} & -3 < x < 1 \\[2ex] \dfrac{e^{2x}}{900\ln(x+1)} & x \geq 1 \end{cases}$$

Solution:
We first observe that we have a rule for all values of x except -3. We need to keep this in mind when we are writing our final answer.

Next, we find any domain restrictions for each rule and check whether those are relevant to the x-values defined for the respective rule:

Rule 1: $\dfrac{x+6}{(x+6)(3x-2)}$. The only domain issue with this rule we need to worry about is division by zero. Thus, we need to ensure

$$(x+6)(3x-2) \neq 0 \implies$$
$$x+6 \neq 0 \text{ and } 3x-2 \neq 0$$
$$\implies x \neq -6 \text{ and } x \neq \frac{2}{3}$$

Now, we check the x-values for which the first rule is defined: $x < -3$. This means the only restriction we need to include in our final answer for rule 1 is $x \neq -6$. Why? Because rule 1 is defined for $x < -3$, the restriction $x \neq \dfrac{2}{3}$ will not be an issue.

> ⚠ *Although you may be tempted to divide the common factor $x+6$ from the numerator and denominator before finding the domain of this rule, don't! It may lead you to ignore the fact that $x \neq -6$! Recall from the previous section that because the factor $x+6$ divides completely from the denominator, there is a hole in the function where $x+6=0$ (i.e., at $x=-6$).*

Rule 2: $\dfrac{\sqrt[3]{x+2}}{11^{x+4}}$. Because the numerator contains an odd root, we only have to be concerned about making sure the denominator does not equal zero:

$$11^{x+4} \neq 0$$

However, 11^{x+4} is an exponential function. Therefore, it will never equal zero! Thus, we do not have any restrictions for rule 2.

Rule 3: $\dfrac{e^{2x}}{900\ln(x+1)}$. We have two restrictions to calculate here: division by zero and the argument of the logarithm. Starting with the denominator, we have

$$\begin{aligned} \ln(x+1) &\neq 0 \\ e^{\ln(x+1)} &\neq e^0 \\ x+1 &\neq 1 \\ x &\neq 0 \end{aligned}$$

Now, we need to compare this restriction with the x-values where rule 3 is defined to see if it is relevant. Rule 3 is defined for $x \geq 1$. Because 0 is less than 1, this restriction is irrelevant because we would only attempt to substitute x-values into rule 3 such that $x \geq 1$.

Next, we must ensure the argument of the logarithm is positive:

$$\begin{aligned} x+1 &> 0 \\ x &> -1 \end{aligned}$$

Now, we compare this restriction with the x-values defined for this rule: $x \geq 1$. Because our restriction says $x > -1$, this condition will be satisfied because this rule is defined for $x \geq 1$. You can think about it this way: If the function is defined for x-values greater than or equal to 1, then the requirement that x is greater than -1 will be satisfied. Thus, we do not have any restrictions resulting from the argument of the logarithm.

So we do not have any restrictions to include in our final answer for rule 3 (or rule 2). The only restriction thus far comes from rule 1: $x \neq -6$.

Now, we check to see if the function is continuous at each of the cutoff numbers, $x=-3$ and $x=1$, by applying the definition of continuity at a point to each. Recall we only have to check the conditions until one fails, if one does. We start with $x=-3$:

First cutoff number: $x=-3$

I. Is $f(-3)$ defined? According to the function, $f(-3)$ is undefined because there is no rule for which $x=-3$ is defined (i.e., there is no equality corresponding to $x=-3$). Thus, we can stop here because we know f is *not* continuous at $x=-3$. Thus, we must include the restriction $x \neq -3$ in our final answer. Recall that we stated this at the beginning of our solution when we observed the function had a rule for all values of x except -3.

Now, we must apply the three conditions in the definition of continuity at a point to the remaining cutoff number, $x = 1$:

Second cutoff number: $x = 1$

I. Is $f(1)$ defined? According to the function, we use rule 3 to find $f(1)$:

$$f(1) = \frac{e^{2(1)}}{900\ln(1+1)}$$

$$= \frac{e^2}{900\ln(2)}$$

$$\approx 0.012$$

Thus, f is defined at $x = 1$ and $f(1) = \dfrac{e^2}{900\ln(2)}$, and we move to the second condition.

II. Does $\lim\limits_{x\to 1} f(x)$ exist? Because f is a piecewise-defined function and $x = 1$ is a cutoff number, we have to calculate the left- and right-hand limits.

We must use rule 2 to calculate the left-hand limit: $\lim\limits_{x\to 1^-} \dfrac{\sqrt[3]{x+2}}{11^{x+4}}$. But first, notice rule 2 consists of a ratio of two functions. So we need to make sure the limits of the two functions exist and that the limit of the function in the denominator is nonzero. We can find the limits of the functions in the numerator and denominator using direct substitution because there are no domain issues:

$$\lim_{x\to 1^-} \sqrt[3]{x+2} = \sqrt[3]{1+2}$$
$$= \sqrt[3]{3}$$

and

$$\lim_{x\to 1^-} 11^{x+4} = 11^{1+4}$$
$$= 11^5$$
$$= 161{,}051$$

Now, we know the limits of the functions in the numerator and denominator exist and that the limit of the function in the denominator is nonzero. Thus, we can use the Properties of Limits and divide the limits:

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{\sqrt[3]{x+2}}{11^{x+4}}$$

$$= \frac{\lim\limits_{x\to 1^-} \sqrt[3]{x+2}}{\lim\limits_{x\to 1^-} 11^{x+4}}$$

$$= \frac{\sqrt[3]{3}}{161{,}051}$$

$$\approx 0.00000896$$

This number is very small, but it is still a number. Thus, the limit from the left exists.

We use rule 3 to calculate the right-hand limit: $\lim\limits_{x\to 1^+} \dfrac{e^{2x}}{900\ln(x+1)}$. Even though this rule is a ratio of two functions, because we already found its value at $x = 1$ while working with condition I above (and there were no domain issues), we can proceed with direct substitution to find the limit (and we will get the same value):

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{e^{2x}}{900\ln(x+1)}$$

$$= \frac{e^{2(1)}}{900\ln(1+1)}$$

$$= \frac{e^2}{900\ln(2)}$$

$$\approx 0.012$$

Thus, we see $\lim\limits_{x\to 1} f(x)$ does not exist because the left- and right-hand limits are not equal.

Because this limit does not exist, the second condition in the definition of continuity at a point fails, and we do not need to continue any further. The function is not continuous at $x = 1$.

Combining this restriction ($x \neq 1$) with our other restrictions ($x \neq -6$ and $x \neq -3$), we have that f is continuous on

$$(-\infty,-6)\cup(-6,-3)\cup(-3,1)\cup(1,\infty)$$

N *Note that in a textbook where one-sided continuity is discussed, the authors would claim that the answer to the example above is* $(-\infty,-6)\cup(-6,-3)\cup(-3,1)\cup[1,\infty)$, *which is oh so very slightly different from ours (notice the square bracket on the* 1 *in the last interval). Because one-sided continuity is not discussed in this textbook, we will always give our answers as open intervals (i.e., using parentheses) when finding where a piecewise-defined function is continuous because the answer will be equally true.*

Let's verify our answer by looking at the graph of f shown in **Figure 1.4.9**.

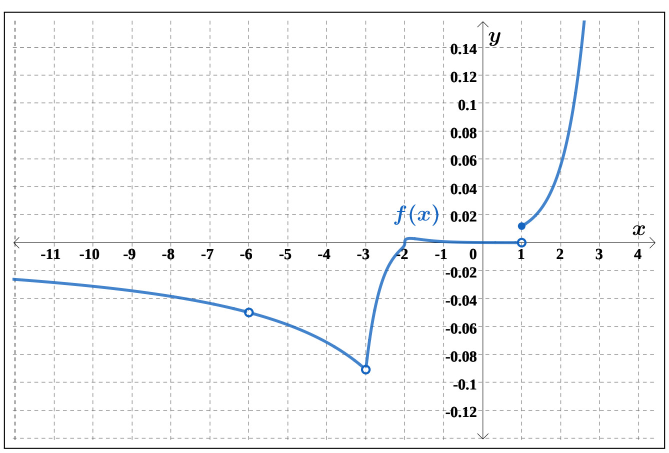


Figure 1.4.9: Graph of the piecewise-defined function f

■

Try It # 4:
The function f is given below. Determine where f is continuous algebraically.

$$f(x) = \begin{cases} 2x^2 - 4x - 8 & x < -1 \\ \dfrac{\sqrt{37+x}}{x-2} & -1 \leq x < 8 \\ \dfrac{x-9}{\sqrt[5]{x^2-8x-9}} & x \geq 8 \end{cases}$$

■ **Example 6** Find the value(s) of a that make(s) the following piecewise-defined function continuous for all real numbers. If there is no such value of a, explain why.

$$f(x) = \begin{cases} x^2 + ax & x \leq 6 \\ x + a & x > 6 \end{cases}$$

Solution:
To find the value of a that makes the function continuous for all real numbers, we follow a procedure that is similar, yet slightly different, than the procedure we used in the previous two examples. First, we observe that $f(x)$ has a rule for all values of x. If the function did not have a rule for all values of x, then we would stop and conclude there is no value of a that would make the function continuous for all real numbers.

Now, we need to make certain there are no domain restrictions for each rule. If there are domain restrictions that are applicable, then we can stop and conclude that there is no value of a that would make the function continuous for all real numbers. Notice both rules of the function are polynomials, which have a domain of all real numbers. Thus, there are no domain restrictions. This means that the only value of x where the function has a chance of not being continuous is at the cutoff number $x = 6$.

Therefore, all we have left to do is find the value of a that "forces" the function to be continuous at $x = 6$. Thus, we need all three conditions in the definition of continuity at a point to be satisfied when $x = 6$:

I. First, we see that the function is defined at $x = 6$. Because $x = 6$ corresponds to the first rule, $x^2 + ax$, we have

$$\begin{aligned} f(6) &= 6^2 + a(6) \\ &= 36 + 6a \end{aligned}$$

II. Next, we need $\lim_{x \to 6} f(x)$ to exist. Just like before, we must calculate the left- and right-hand limits. Using the first rule for the left-hand limit (note we just calculated the same value in the previous step), we have

$$\begin{aligned} \lim_{x \to 6^-} f(x) &= \lim_{x \to 6^-} \left(x^2 + ax\right) \\ &= (6)^2 + a(6) \\ &= 36 + 6a \end{aligned}$$

Using the second rule, $x + a$, to calculate the right-hand limit gives

$$\begin{aligned} \lim_{x \to 6^+} f(x) &= \lim_{x \to 6^+} (x + a) \\ &= 6 + a \end{aligned}$$

Now, for $\lim_{x\to 6} f(x)$ to exist, the left- and right-hand limits must be equal:

$$\begin{aligned} 36+6a &= 6+a \\ 5a &= -30 \\ a &= \frac{-30}{5} \\ &= -6 \end{aligned}$$

Thus, for the left- and right-hand limits to be equal, $a = -6$.

III. We normally want to make sure $\lim_{x\to 6} f(x) = f(6)$ in this last step, but for this particular example we have essentially already completed this step while working with condition II. Because $f(6) = \lim_{x\to 6^-} f(x) = 36+6a$ and we set the left- and right-hand limits equal to each other, we have ensured that the function value equals the value of the limit.

Thus, the function is continuous for all real numbers when $a = -6$.

The resulting function with $a = -6$ is given by

$$f(x) = \begin{cases} x^2 - 6x & x \leq 6 \\ x - 6 & x > 6 \end{cases}$$

The graph of this function is shown in **Figure 1.4.10**.

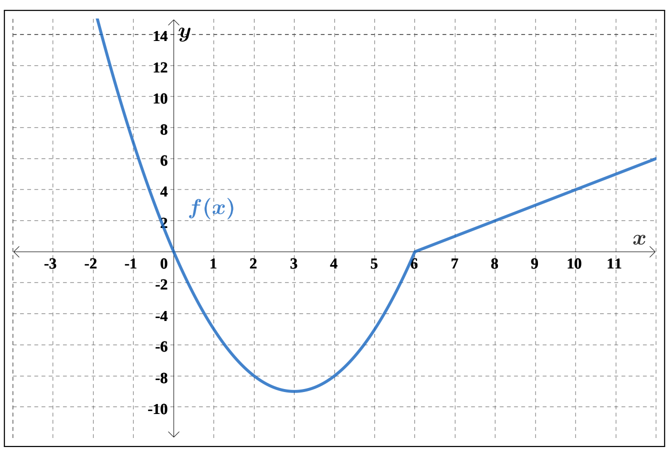


Figure 1.4.10: Graph of the piecewise-defined function f in which $a = -6$

▪

Try It # 5:
Find the value(s) of b that make(s) the following piecewise-defined function continuous for all real numbers. If there is no such value of b, explain why.

$$f(x) = \begin{cases} |2x-13| & x \leq -5 \\ \dfrac{bx}{x+7} & x > -5 \end{cases}$$

■ **Example 7** Find the value(s) of k that make(s) the following piecewise-defined function continuous at $x = -4$. If there is no such value of k, explain why.

$$f(x) = \begin{cases} \dfrac{x^2-16}{x+4} & x < -4 \\ kx^2 - \ln(5+x) & x \geq -4 \end{cases}$$

Solution:
Notice this example is slightly different than the previous example. We are not concerned with ensuring the entire function is continuous for all real numbers. We only have to find the value of k that "forces" the function to be continuous at $x = -4$. Thus, we do not have to find the domain of each rule of the function. We just need to make sure all three conditions in the definition of continuity at a point are satisfied when $x = -4$:

I. First, we see that the function is defined at $x = -4$. Because $x = -4$ corresponds to the second rule, we have

$$\begin{aligned} f(-4) &= k(-4)^2 - \ln(5-4) \\ &= 16k - \ln(1) \\ &= 16k - 0 \\ &= 16k \end{aligned}$$

II. Next, we need $\lim\limits_{x\to-4} f(x)$ to exist. Just like before, we must calculate the left- and right-hand limits.

The left-hand limit corresponds to the first rule, $\dfrac{x^2-16}{x+4}$. Notice this rule consists of a ratio of two functions, so we need to make sure the limits of the two functions exist and that the limit of the function in the denominator is nonzero. We can find the limits of the functions in the numerator and denominator using direct substitution because there are no domain issues:

$$\begin{aligned} \lim_{x\to-4^-} \left(x^2-16\right) &= (-4)^2 - 16 \\ &= 16 - 16 \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} \lim_{x\to-4^-} (x+4) &= -4+4 \\ &= 0 \end{aligned}$$

Because the limits of the two functions equal zero, the limit is of the indeterminate form $\frac{0}{0}$. Thus, we must algebraically manipulate the function. Namely, we try factoring the numerator, dividing common factors, and substituting $x = -4$ again:

$$\begin{aligned} \lim_{x\to-4^-} f(x) &= \lim_{x\to-4^-} \frac{(x+4)(x-4)}{x+4} \\ &= \lim_{x\to-4^-} \frac{\cancel{(x+4)}(x-4)}{\cancel{x+4}} \\ &= \lim_{x\to-4^-} (x-4) \\ &= -4-4 \\ &= -8 \end{aligned}$$

So the left-hand limit equals -8. Now, we use the second rule, $kx^2 - \ln(5+x)$, to calculate the right-hand limit (note we calculated this value while working with condition I when we found $f(-4)$):

$$\begin{aligned}\lim_{x\to -4^+} f(x) &= \lim_{x\to -4^+}\left(kx^2 - \ln(5+x)\right)\\ &= k(-4)^2 - \ln(5-4)\\ &= 16k\end{aligned}$$

We must set the left- and right-hand limits equal to each other so that $\lim\limits_{x\to -4} f(x)$ exists:

$$\begin{aligned}-8 &= 16k\\ \frac{-8}{16} &= k\\ -\frac{1}{2} &= k\end{aligned}$$

Thus, $\lim\limits_{x\to -4} f(x)$ will exist when $k = -\frac{1}{2}$.

III. We normally want to make sure $\lim\limits_{x\to -4} f(x) = f(-4)$ in this last step, but again, we have essentially already completed this step while working with condition II. Because $f(-4) = \lim\limits_{x\to -4^+} f(x) = 16k$ and we set the left- and right-hand limits equal to each other, we have ensured that the function value equals the value of the limit.

Thus, the function is continuous for all real numbers when $k = -\frac{1}{2}$.

▪

! *For certain types of functions, we may not be able to find a value that will make the function continuous at a particular value of x or for all real numbers!*

Try It # 6:
Show that no value of k can make the following piecewise-defined function continuous at $x = -4$.

$$f(x) = \begin{cases} \dfrac{x^2-16}{x+4} & x < -4 \\ 0 & x = -4 \\ kx^2 - \ln(5+x) & x > -4 \end{cases}$$

Try It Answers

1. $x = 2$, $g(2)$ is undefined; $x = 7$, $\lim\limits_{x\to 7} g(x)$ does not exist; $x = 11$, $\lim\limits_{x\to 11} g(x)$ does not exist

2. **a.** $\left(-\infty, -\sqrt{15}\right) \cup \left(-\sqrt{15}, \sqrt{15}\right) \cup \left(\sqrt{15}, \infty\right)$
   b. $(-\infty, -4) \cup (-4, 4) \cup (4, \infty)$
   c. $(-\infty, -2) \cup (-2, 3) \cup (3, 50)$
   d. $[-7, 19/3) \cup (19/3, 20/3)$

3. f is not continuous at $x = 1$ because the third condition in the definition of continuity at a point fails: $\lim_{x\to 1} f(x) \neq f(1)$.

4. $(-\infty, 2) \cup (2, 8) \cup (8, 9) \cup (9, \infty)$

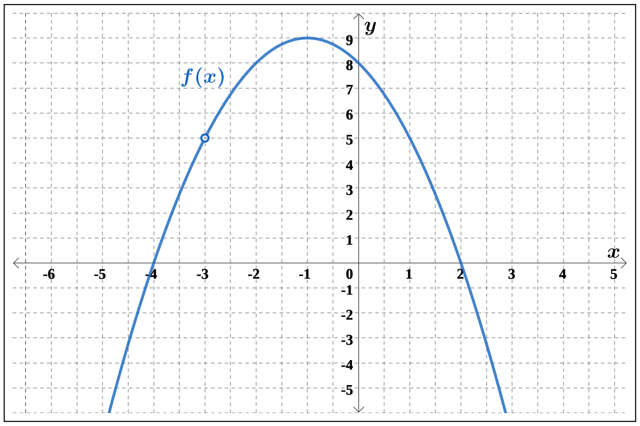
5. $b = -46/5$

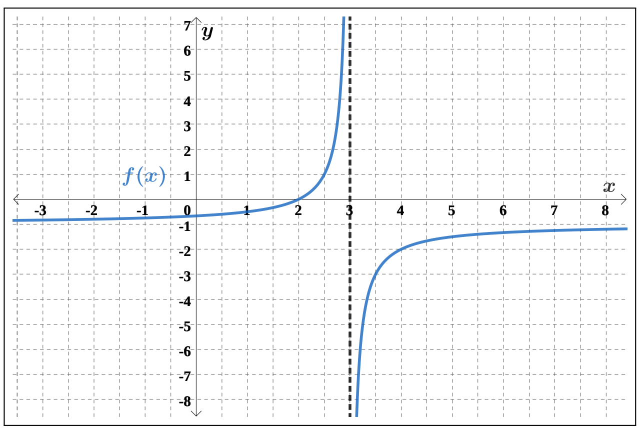
6. The first and second conditions in the definition of continuity at a point are satisfied: $f(-4) = 0$ and $\lim_{x\to -4} f(x) = -8$ when $k = -1/2$. However, the third condition cannot be satisfied because no value of k will enable $\lim_{x\to -4} f(x) = f(-4)$. In other words, -8 will never equal 0.

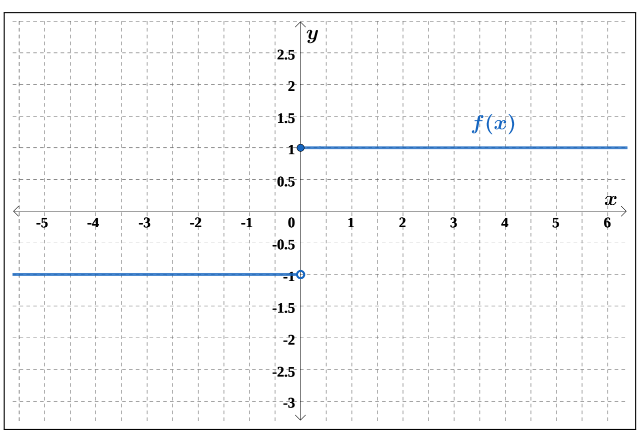
EXERCISES

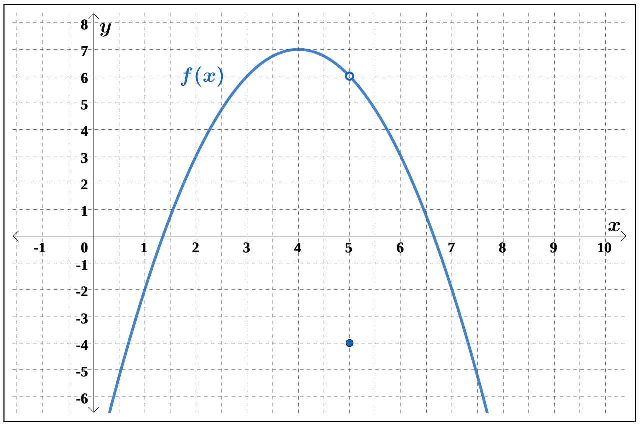
Basic Skills Practice

For Exercises 1 - 4, use the graph of f to determine the x-value(s) where f is discontinuous. State the condition in the definition of continuity at a point that fails first at each x-value mathematically.

1. 

2. 

3. 

4. 

For Exercises 5 - 10, determine if the function is continuous at the given value of c. If the function is not continuous at $x = c$, also state the condition in the definition of continuity at a point that fails first mathematically.

5. $f(x) = 3x^2 - 4x + 5, \quad c = 0$

6. $f(x) = -2x^3 - x^2 - 10, \quad c = 2$

7. $f(x) = \dfrac{x-2}{x-10}, \quad c = 2$

8. $f(x) = \dfrac{x+3}{x-5}, \quad c = 5$

9. $f(x) = \dfrac{(x+3)(x-7)}{x-7}, \quad c = 7$

10. $f(x) = \dfrac{2x+3}{(x+2)(x-4)}, \quad c = 2$

For Exercises 11 - 22, determine where the function is continuous algebraically. Write your answer using interval notation.

11. $f(x) = 4x^2 - 7x + 3$

12. $f(x) = -7x^3 - x - 1$

13. $f(x) = \dfrac{x+1}{x-5}$

14. $f(x) = \dfrac{2x+1}{x+3}$

15. $f(x) = \dfrac{(2x-7)(x-3)}{(x-3)(2x+3)}$

16. $f(x) = \dfrac{(x+1)(3x-5)}{(3x-5)(7x-1)}$

17. $f(x) = \sqrt[3]{x-7}$

18. $f(x) = \dfrac{1}{\sqrt{x+1}}$

19. $f(x) = 2^{3-x}$

20. $f(x) = e^{7x-10}$

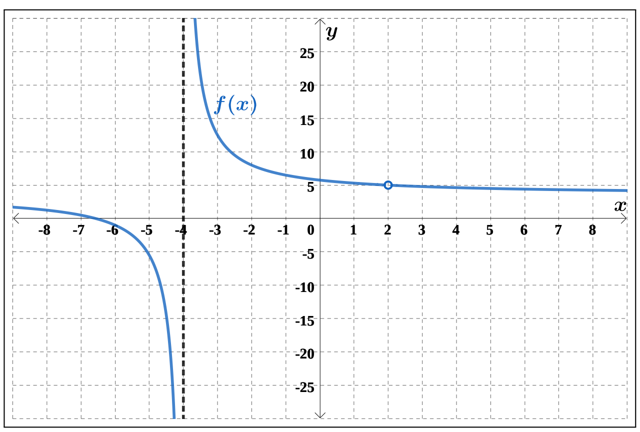
21. $f(x) = \log_4(2x-5)$

22. $f(x) = \ln(8-3x)$

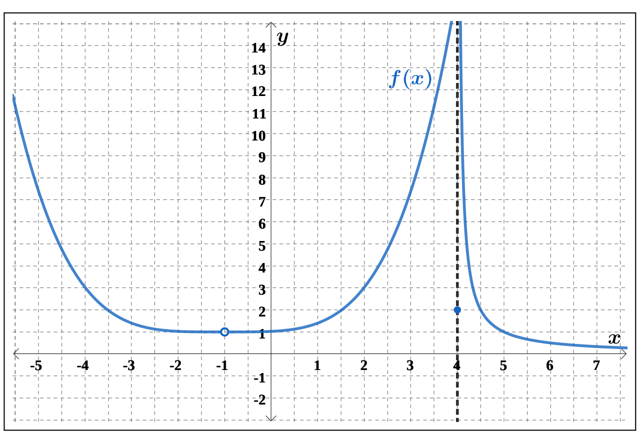
Intermediate Skills Practice

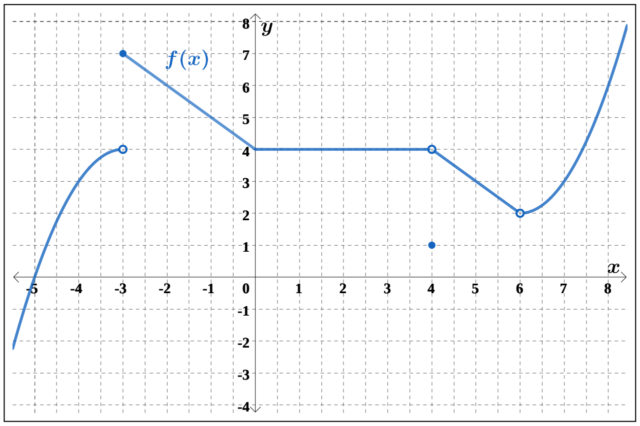
For Exercises 23 - 26, use the graph of f to determine the x-value(s) where f is discontinuous. State the condition in the definition of continuity at a point that fails first at each x-value mathematically.

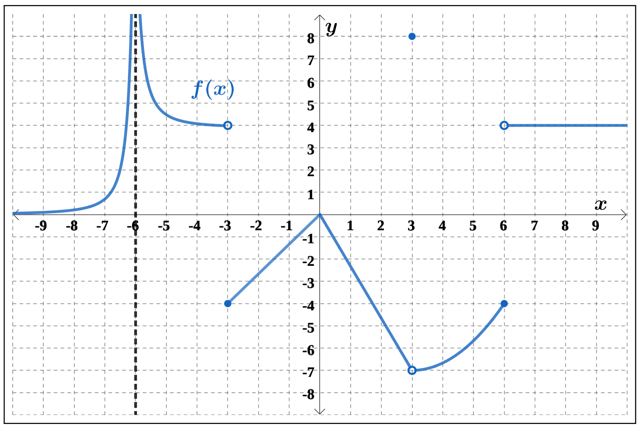
23.



24.



25. 

26. 

For Exercises 27 - 34, determine if the function is continuous at the given value of c. If the function is not continuous at $x = c$, also state the condition in the definition of continuity at a point that fails first mathematically.

27. $f(x) = \dfrac{x^2 + x - 2}{x^2 - x - 12}$, $c = -2$

28. $f(x) = \dfrac{2x^2 - x - 3}{x^2 - 6x - 7}$, $c = 7$

29. $f(x) = \dfrac{3x^2 + 17x - 28}{2x^2 - 17x + 8}$, $c = 8$

30. $f(x) = \dfrac{2x^2 - x - 15}{3x^2 - 2x - 16}$, $c = 10$

31. $f(x) = \begin{cases} 4x + 7 & x \leq 2 \\ 3x - 5 & x > 2 \end{cases}$, $c = 2$

32. $f(x) = \begin{cases} 3x^2 + 9 & x \leq -5 \\ (x-3)^2 + 20 & x > -5 \end{cases}$, $c = -5$

33. $f(x) = \begin{cases} 4x^3 - 11 & x < 8 \\ 39x^2 - 459 & x > 8 \end{cases}, \quad c = 8$

34. $f(x) = \begin{cases} 17x^3 + 4x - 10 & x < -2 \\ 14x^2 - 25 & x \geq -2 \end{cases}, \quad c = -2$

For Exercises 35 - 54, determine where the function is continuous algebraically. Write your answer using interval notation.

35. $f(x) = 2x^3 - \sqrt{x} + \ln(2x - 8)$

36. $f(x) = \sqrt{1 - x} + \sqrt{x + 1}$

37. $f(x) = \dfrac{x^2 - 3x - 10}{\sqrt[6]{2x - 7}}$

38. $f(x) = \dfrac{20 - \ln(9x - 1)}{x + 5}$

39. $f(x) = \dfrac{\sqrt[3]{x^3 - 4x^2 + 8}}{e^{x-5}}$

40. $f(x) = \dfrac{4^{x-3} + 2x^2}{\sqrt[5]{x^2 - 2x - 24}}$

41. $f(x) = \dfrac{\sqrt[4]{9 - x}}{x^2 - 12x}$

42. $f(x) = \dfrac{4\ln(x - 8)}{\sqrt{5x - 3}}$

43. $f(x) = \begin{cases} x^2 - 4x + 7 & x \leq 5 \\ 4x - 8 & x > 5 \end{cases}$

44. $f(x) = \begin{cases} 2x^2 - 4x + 7 & x < -3 \\ x^3 + 2x - 4 & x \geq -3 \end{cases}$

45. $f(x) = \begin{cases} x^2 - 3x + 2 & x < 1 \\ 3x^3 - 4x^2 + 1 & x \geq 1 \end{cases}$

46. $f(x) = \begin{cases} \dfrac{x^2 - 64}{x^2 - 11x + 24} & x \neq 8 \\ 0 & x = 8 \end{cases}$

47. $f(x) = \begin{cases} 2^{x+4} - 19.5 & x < 0 \\ \dfrac{5x^2 - 4x + 7}{2x^2 + 3x - 2} & x \geq 0 \end{cases}$

48. $f(x) = \begin{cases} \dfrac{4x - 7}{x + 5} & x \leq 3 \\ \sqrt{3x - 7} & x > 3 \end{cases}$

49. $f(x) = \begin{cases} \dfrac{x}{x - 7} & x < -4 \\ \ln(x + 6) & x > -4 \end{cases}$

50. $f(x) = \begin{cases} \dfrac{5x^2 + 20x + 20}{x + 2} & x < -1.6 \\ \sqrt[5]{24 - 5x} & x \geq -1.6 \end{cases}$

51. $f(x) = \begin{cases} 3x^2 - 41e^{x-6} & x < 6 \\ -39 & x = 6 \\ 2x^2 - 5 & x > 6 \end{cases}$

52. $f(x) = \begin{cases} -x - 6 & x \leq 0 \\ 2x^2 - 4x - 6 & 0 < x < 1 \\ -x + 5 & x > 1 \end{cases}$

53. $f(x) = \begin{cases} \dfrac{x^2 - 4}{x^2 + 6x + 8} & x < -6 \\ -3x^3 - 12x^2 + 26x - 56 & -6 \leq x < 1 \\ \dfrac{-(x + 20)^2 - 8x - 1}{x + 9} & x \geq 1 \end{cases}$

54. $f(x) = \begin{cases} -3x^4 + 2x - 1 & x < -7 \\ 5e^{x-2} + 16 & -7 < x \leq 2 \\ 8x^2 - 11 & x > 2 \end{cases}$

For Exercises 55 - 61, find the value(s) of k that make(s) the function continuous for all real numbers. If there is no such value of k, explain why.

55. $f(x) = \begin{cases} kx & x < 3 \\ 3x^2 & x \geq 3 \end{cases}$

56. $f(x) = \begin{cases} 2x^2 + k & x < 2 \\ 3x - 8 & x \geq 2 \end{cases}$

57. $f(x) = \begin{cases} 2x + k & x \leq 1 \\ -5x^2 + 9 & x > 1 \end{cases}$

58. $f(x) = \begin{cases} 3^{x-4} & x \leq 5 \\ 2x^2 - kx + 4 & x > 5 \end{cases}$

59. $f(x) = \begin{cases} 5x & x \leq k \\ 2x + 1 & x > k \end{cases}$

60. $f(x) = \begin{cases} 4x - 3 & x < k \\ 6 - x & x \geq k \end{cases}$

61. $f(x) = \begin{cases} 2x^2 + 3x - 4 & x \leq k \\ x^2 - x - 8 & x > k \end{cases}$

62. Find the value(s) of k that make(s) the following function continuous at $x = -1$. If there is no such value of k, explain why.

$$f(x) = \begin{cases} 12 - kx & x \leq -1 \\ x^2 - 3x & x > -1 \end{cases}$$

63. Find the value(s) of k that make(s) the following function continuous at $x = 4$. If there is no such value of k, explain why.

$$f(x) = \begin{cases} \dfrac{1}{x-8} & x < 4 \\ kx - 7 & x \geq 4 \end{cases}$$

64. Find the value(s) of k that make(s) the following function continuous at $x = 8$. If there is no such value of k, explain why.

$$f(x) = \begin{cases} 13e^{8-x} - 2 & x \leq 8 \\ 1 - k & x > 8 \end{cases}$$

Mastery Practice

For Exercises 65 - 83, determine where the function is continuous algebraically. Write your answer using interval notation.

65. $f(x) = \dfrac{\ln(7x-14)}{\sqrt{2x+10}}$

66. $f(x) = \dfrac{\ln(x) - \sqrt[7]{21x+3}}{6x^2+23x-18}$

67. $f(x) = \dfrac{\sqrt[5]{x^2-49}}{x^2+36}$

68. $f(x) = \dfrac{e^{x/(x+3)}}{\sqrt[3]{x^2-16}}$

69. $f(x) = \dfrac{\sqrt{x+5}}{\ln(2-x)}$

70. $f(x) = \dfrac{\sqrt{x+5}}{5^{\sqrt{2x+3}}}$

71. $f(x) = \dfrac{4^{\sqrt{x+2}}}{5\log(7x+3)}$

72. $f(x) = \dfrac{-10 \cdot \left(\frac{2}{3}\right)^{x^2-3x}}{x^6 - 25x^4}$

73. $f(x) = \dfrac{\ln(13-x)}{\sqrt{x+4}-3}$

74. $f(x) = \dfrac{\sqrt[5]{7x-11}}{\log_5(x-3)-4}$

75. $f(x) = \begin{cases} \log(5x+20) & x < 0 \\ \dfrac{3x^2-5x+2}{x^2+5x-24} & 0 < x \leq 5 \\ 3x^2-13x-8 & x > 5 \end{cases}$

76. $f(x) = \begin{cases} 4x^2-7 & x \leq -4 \\ \dfrac{x-2}{\sqrt[4]{2x+8}} & -4 < x \leq -1 \\ \dfrac{5x^7-4x}{6x+x^2} & x > -1 \end{cases}$

77. $f(x) = \begin{cases} \dfrac{\sqrt[5]{x+4}}{\ln(3x+24)} & -8 < x \leq 0 \\ 3^{x-4}+8 & 0 < x < 6 \\ x^2-4x+5 & x \geq 6 \end{cases}$

78. $f(x) = \begin{cases} \dfrac{x^2+6x-27}{x^2+8x-9} & x < -2 \\ \frac{1}{4}x^2-16\cdot 5^{x-8} & -2 < x \leq 8 \\ \ln(9-x) & x > 8 \end{cases}$

79. $f(x) = \begin{cases} \dfrac{x^2-1}{x^2-12x+35} & x \leq 4 \\ -2\cdot 3^{4-x}+7 & 4 < x < 6 \\ \dfrac{205-4x^2}{9e^{2x-12}} & x \geq 6 \end{cases}$

80. $f(x) = \begin{cases} \dfrac{x+2}{x^2+5x+6} & x \leq -1 \\ 2x^2+3x+\frac{3}{2} & -1 < x \leq 12 \\ \dfrac{4x^2+5x+15}{2e^{12-x}} & x > 12 \end{cases}$

81. $f(x) = \begin{cases} \dfrac{6}{\log_2(-x-1)} & x < -3 \\ \sqrt{x+4}+5 & -3 \leq x < 2 \\ \dfrac{2x}{x+18} & x > 2 \end{cases}$

82. $f(x) = \begin{cases} \dfrac{x^2+2x}{x^2-4} & x < 1 \\ -1 & x = 1 \\ 3-4e^{(x-1)/(5-x)} & x > 1 \end{cases}$

83. $f(x) = \begin{cases} \dfrac{\sqrt[7]{2x-7}}{2x^2-18x+40} & x < 3 \\ \ln(21-3x) & 3 < x < 5 \\ \dfrac{x^2+7x}{4^{x+1}-20} & 5 < x < 7 \end{cases}$

For Exercises 84 - 91, find the value(s) of k that make(s) the function continuous for all real numbers. If there is no such value of k, explain why.

84. $f(x) = \begin{cases} 2x^2-4 & x < 0 \\ 10 & x = 0 \\ 4k-x & x > 0 \end{cases}$

85. $f(x) = \begin{cases} -x^2+3 & x < -2 \\ x+1 & -2 \leq x \leq 4 \\ kx^2-4 & x > 4 \end{cases}$

86. $f(x) = \begin{cases} \dfrac{x^2-9}{x-3} & x < 3 \\ kx^2-21 & x \geq 3 \end{cases}$

87. $f(x) = \begin{cases} \dfrac{x+k}{\sqrt[4]{2-x}} & x < 1 \\ \dfrac{2^x-k}{x^2+6x+8} & x \geq 1 \end{cases}$

88. $f(x) = \begin{cases} \ln(2x+7) & x \leq -1 \\ -9x+k & x > -1 \end{cases}$

89. $f(x) = \begin{cases} \dfrac{k-x}{\sqrt[8]{1-x}} & x \leq 0 \\ \dfrac{x^2+k}{x^7+1} & x > 0 \end{cases}$

90. $f(x) = \begin{cases} x+1 & x \leq k \\ \sqrt{x+7} & x > k \end{cases}$

91. $f(x) = \begin{cases} \dfrac{10-x}{3^x} & x < k \\ 3^{-x-2} & x \geq k \end{cases}$

92. Find the value(s) of k that make(s) the following function continuous at $x = 5$. If there is no such value of k, explain why.

$$f(x) = \begin{cases} \sqrt{30-x} & x < 5 \\ \dfrac{e^{2k}}{x-4} & x \geq 5 \end{cases}$$

93. Find the value(s) of k that make(s) the following function continuous at $x = 6$. If there is no such value of k, explain why.

$$f(x) = \begin{cases} \dfrac{x^2-36}{x-6} & x < 6 \\ kx^2-7 & x \geq 6 \end{cases}$$

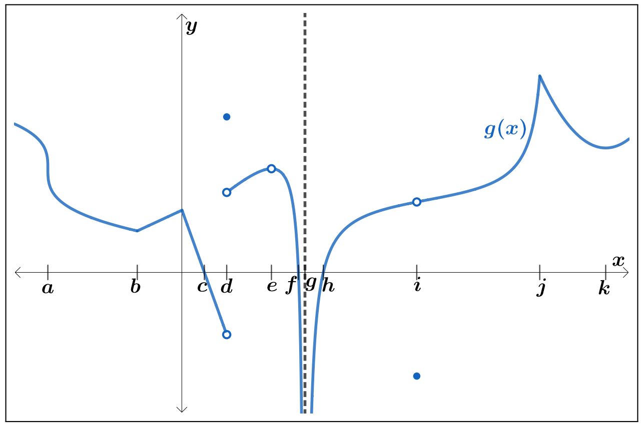
94. Find the value(s) of k that make(s) the following function continuous at $x = 3$. If there is no such value of k, explain why.

$$f(x) = \begin{cases} 4x-k & x \leq 3 \\ \ln(8x-1) & x > 3 \end{cases}$$

95. Find the value(s) of k that make(s) the following function continuous at $x = 2$. If there is no such value of k, explain why.

$$f(x) = \begin{cases} 3^{x-1} & x < 2 \\ \sqrt{kx-4} & x \geq 2 \end{cases}$$

96. Use the graph of g shown below to determine the x-value(s) where g is discontinuous. State the condition in the definition of continuity at a point that fails first at each x-value mathematically.



Communication Practice

97. State, in words, the conditions for a function f to be continuous at $x = 5$.

98. If f is continuous at $x = c$, does $\lim_{x \to c} f(x)$ exist? Explain.

99. If f is continuous at $x = c$, does $\lim_{x \to c^+} f(x) = f(c)$? Explain.

100. Explain how to find where a function that is not piecewise-defined is continuous algebraically.

101. Explain how to find where a piecewise-defined function is continuous algebraically.

Chapter 2

2. The Derivative

Suppose a company has determined that its profit from making and selling 10,000 items is \$1,990,000. If the company makes and sells additional items, will its profit increase or decrease?

If we know the rate of change (or slope) of the company's profit function, we can determine whether or not the company should increase the number of items it makes and sells. For instance, if the rate of change of the company's profit is positive when it sells 10,000 items, making and selling more items will increase profit.

This is far from the only application of rates of change. Many of them are seen every day: the speedometer in a car, an accelerometer in a video game motion control sensor, the rate of change of a business function such as revenue, cost, or profit, the pressure of fluid in a container, air traffic flight patterns and schedules, and many more.

In this chapter, we will explore average and instantaneous rates of change and their applications. We will also investigate differentiation, the process for finding a function, called the **derivative**, which represents the rate of change of a given function. We will also learn techniques for finding the derivative including the Product, Quotient, and Chain Rules.

Finally, we will learn how to find the derivative implicitly of an equation with two variables in which one variable cannot be defined explicitly in terms of the other. We will then apply this technique, called implicit differentiation, to solve real-world problems where several variables, or quantities, are related to each other and all but one of the variables are changing at a known rate, so we can determine how rapidly the other variable must be changing. These are known as related-rates problems.

2.1 Average and Instantaneous Rates of Change

A common amusement park ride lifts riders to a particular height and then allows them to free-fall a certain distance before safely stopping. Suppose such a ride drops riders from a height of 150 feet. Students of physics may recall that the height, in feet, of the riders t seconds after free-fall (ignoring air resistance, etc.) can be accurately modeled by $f(t) = -16t^2 + 150$. Using this function, which is called a position function, we can verify that the riders will hit the ground at $t \approx 3.06$ seconds.

Suppose the designers of the ride decide to begin slowing the riders' falls after two seconds, which corresponds to a height of 86 feet. How fast will the riders be traveling at that time?

We have the position function, $f(t) = -16t^2 + 150$, but what we wish to compute now is the velocity of the riders at a specific point in time (at exactly two seconds after beginning free-fall). In other words, we want to calculate an **instantaneous velocity**. We currently do not know how to calculate this precisely, but we can use a somewhat intuitive technique to arrive at a good estimate.

For example, if we traveled 90 miles in 3 hours, we know we had an **average velocity** of 30 miles per hour. We can take the same approach to calculate the average velocity over any time interval:

$$\text{average velocity} = \frac{\text{change in position}}{\text{change in time}}$$

Thus, we can approximate the riders' instantaneous velocity at $t = 2$ seconds by considering their average velocity over some time period containing $t = 2$ seconds. If we make the time interval small, we will get a good approximation of their instantaneous velocity.

N *This fact is commonly used. For instance, high speed cameras are used to track fast moving objects. Distances are measured over a fixed number of frames to generate an accurate approximation of their instantaneous velocity.*

Consider finding the average velocity of the riders on the interval $[2,3]$, which is just before they hit the ground. We can calculate the change in the riders' height (position) on this interval by finding the difference in their position at $t = 2$ and $t = 3$ seconds using the position function $f(t) = -16t^2 + 150$ feet. The change in time would be the difference in time. Thus, on the interval $[2,3]$, the average velocity is

$$\frac{f(3) - f(2)}{3 - 2} = \frac{6 - 86}{1} = -80 \text{ feet per second}$$

where the negative sign indicates the riders are moving downward. This gives us an approximation for the instantaneous velocity at $t = 2$. In order to find the exact value for the instantaneous velocity, or the **instantaneous rate of change** of position, we need to build upon this idea of average velocity, or **average rate of change** of position, further.

Learning Objectives:

In this section, you will learn how to calculate the average rate of change and the instantaneous rate of change of a function and use these ideas to solve problems involving real-world applications. Upon completion you will be able to:

- Calculate the slope of a secant line given a function, table of function values, or graph of a function.
- Describe the slope of the secant line as the average rate of change, difference quotient, and average velocity.
- Calculate the average rate of change of a function involving a real-world scenario, including cost, revenue, profit, and position.

- Interpret the meaning of the average rate of change of a function involving a real-world scenario, including cost, revenue, and profit.
- Interpret the slope of the tangent line as the limit of slopes of secant lines.
- Describe the slope of a tangent line as the instantaneous rate of change (rate of change), limit of the difference quotient, and instantaneous velocity (velocity).
- Estimate the slope of a tangent line by computing the limit of slopes of secant lines given a function, table of function values, or graph of a function.
- Calculate the slope of a tangent line using the limit definition of the instantaneous rate of change.
- Find the equation of a tangent line using the limit definition of the instantaneous rate of change.
- Calculate the instantaneous rate of change of a function involving a real-world scenario, including cost, revenue, profit, and position, using the limit definition of the instantaneous rate of change.
- Interpret the meaning of the instantaneous rate of change of a function involving a real-world scenario, including cost, revenue, and profit.

Average Rate of Change

In the motivating example, we used the function $f(t) = -16t^2 + 150$ to calculate the average velocity of the riders on the interval from $t = 2$ to $t = 3$ seconds. To find the average velocity, we found the slope of the line containing the points $(2, 86)$ and $(3, 6)$. This line is known as the **secant line** because it passes through two points on the graph of f. This leads to the formal definition of the **slope of the secant line**, which we also discovered is the average rate of change of f from $t = 2$ to $t = 3$ seconds.

Definition

A line through two points $(a, f(a))$ and $(b, f(b))$ on the graph of f is called a **secant line**. The **slope of the secant line** is given by

$$m = \frac{f(b) - f(a)}{b - a}$$

This is also known as the **average rate of change** of f on the interval $[a, b]$.

In the motivating example, we were dealing with a position function. When a function represents the position of an object, the average rate of change of that function represents the average velocity.

The graphs of the position function f and the secant line on the interval $[2, 3]$ are shown in **Figure 2.1.1**.

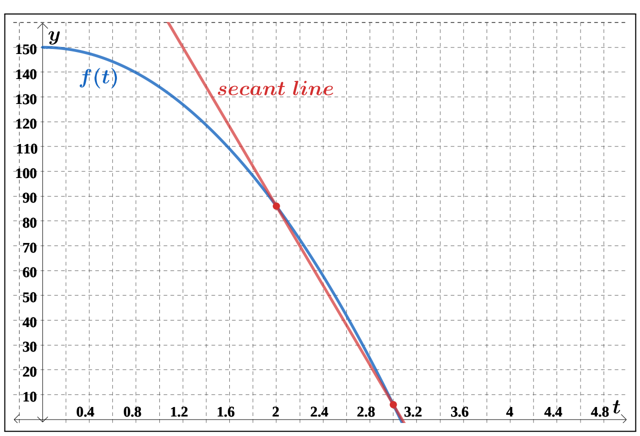


Figure 2.1.1: Graphs of $f(t) = -16t^2 + 150$ and a secant line

▪ **Example 1** Find the slope of the secant line passing through the points on the graph of $f(t) = -16t^2 + 150$ corresponding to $t = 2$ and $t = 2.5$.

Solution:
To find the y-value of the point on the graph of f corresponding to $t = 2$, we must find $f(2)$:

$$f(2) = -16(2)^2 + 150 = 86$$

Likewise, to find the y-value of the point on the graph of f corresponding to $t = 2.5$, we must find $f(2.5)$:

$$f(2.5) = -16(2.5)^2 + 150 = 50$$

Thus, the secant line passes through the points $(2, 86)$ and $(2.5, 50)$ and has a slope of

$$\begin{aligned} m &= \frac{50 - 86}{2.5 - 2} \\ &= \frac{-36}{0.5} \\ &= -72 \end{aligned}$$

▪

In the previous example, we were using the function from our motivating example, $f(t) = -16t^2 + 150$, which had input units of *seconds* and output units of *feet*. Thus, the slope of the secant line, which is equivalent to the average rate of change, has units of *feet per second*.

This is always true for any rate of change (i.e., slope). For example, if the units of x are *years* and the units of $f(x)$ are *people*, then the units of both the average rate of change and instantaneous rate of change are *people per year*. In general, the units for any rate of change (both average and instantaneous) of f are given by

$$\textbf{Units for a Rate of Change of } f = \frac{\text{units of } f(x)}{\text{units of } x} = \frac{\text{output units}}{\text{input units}}$$

▪ **Example 2** Given the units of x and the units of $f(x)$ in **Table 2.1**, find the units for the average rate of change of f.

	Units of x	**Units of $f(x)$**
a.	hours	miles
b.	people	automobiles
c.	dollars	pancakes
d.	seconds	miles per second

Table 2.1: Units of independent and dependent variables

Solution:

a. When calculating the average rate of change, the units for the numerator are the units of $f(x)$, which in this case is miles. The units for the denominator are the units of x, which in this case is hours. Thus, our units for the average rate of change are miles per hour.

b. Again, the units for the average rate of change are the units of $f(x)$ per unit of x. So here we have automobiles per person.

c. Again, the units for the average rate of change are the units of $f(x)$ per unit of x. Thus, the units are pancakes per dollar.

d. The units for the average rate of change are miles per second per second, or we may write this as miles per second squared $\left(\frac{mi}{s^2}\right)$. These are the units for acceleration! The rate of change of position is velocity, and the rate of change of velocity is acceleration.

N *Acceleration can be thought of as **the rate of change of the rate of change of the position**. In Chapter 3, we will further discuss the rate of change of a rate of change function.*

■

Try It # 1:
Given the units of x and the units of $f(x)$ in **Table 2.2**, find the units for the average rate of change of f.

	Units of x	**Units of $f(x)$**
a.	days	trout
b.	seconds	gallons
c.	gallons	seconds
d.	study hours	test points

Table 2.2: Units of independent and dependent variables

■ **Example 3** A manufacturer produces pocket-sized monster plushes. The cost of making x plushes is given by the function $C(x) = -x^2 + 1000x + 50,000$ dollars. What is the average rate of change of cost when the first 600 plushes are made? Interpret your answer.

Solution:
We must calculate the average rate of change of cost on the interval [0, 600]:

$$\begin{aligned}\frac{C(600) - C(0) \text{ dollars}}{600 - 0 \text{ plushes}} &= \frac{\$290,000 - \$50,000}{600 \text{ plushes}} \\ &= \frac{\$240,000}{600 \text{ plushes}} \\ &= \$400 \text{ per plush}\end{aligned}$$

Thus, the average rate of change of cost is \$400 per plush. To interpret our answer, we write a sentence:

"When the first 600 plushes are made, cost is increasing at an average rate of \$400 per plush."

These better be very well made plushes!

■

Try It # 2:
Using the cost function from the previous example, what is the average rate of change of cost when the number of plushes made is between 600 and 1200? Interpret your answer.

■ **Example 4** Jenny is a police officer in Viridian City. Her radar gun measures the position of an oncoming car, first when she presses the button, and then every half second after that while she holds down the button. The radar gun measures the position of a car as 1150 feet away, and half a second later it measures the position of the car as 1225 feet away. What is the average velocity, in miles per hour, that Officer Jenny's radar gun shows for this car? Round your answer to one decimal place.

Solution:
We will start our interval at $t = 0$ seconds indicating the exact moment Officer Jenny pressed the trigger. A half second later (the time of the second reading) corresponds to $t = 0.5$ second. The output of the radar gun is 1150 feet and 1225 feet, respectively. Using our formula above for average rate of change, or in this case, average velocity, we have

$$\begin{aligned}\frac{1225-1150 \text{ feet}}{0.5-0 \text{ seconds}} &= \frac{75 \text{ feet}}{0.5 \text{ second}}\\ &= 150 \text{ feet per second}\end{aligned}$$

Be careful! This is not the answer we are looking for! The problem asks for our answer in miles per hour. We need to convert the units:

$$\frac{150 \cancel{\text{ feet}}}{1 \cancel{\text{ second}}} \cdot \frac{1 \text{ mile}}{5280 \cancel{\text{ feet}}} \cdot \frac{60 \cancel{\text{ seconds}}}{1 \cancel{\text{ minute}}} \cdot \frac{60 \cancel{\text{ minutes}}}{1 \text{ hour}} = 102.3 \text{ miles per hour}$$

Thus, the average velocity of the car is 102.3 miles per hour. While Officer Jenny writes this driver a ticket, we will move on to our next example.

■

■ **Example 5** Nurse Joy measures a patient's heart rate, in beats per minute, every thirty seconds for five minutes using a machine. The results are shown in **Table 2.3**.

Time (in seconds)	0	30	60	90	120	150	180	210	240	270	300
Heart Rate (in BPM)	85	87	90	100	89	105	110	90	85	89	87

Table 2.3: A patient's heart rate, in BPM, over a five minute period

Find the average rate of change of the heart rate

a. over the entire 5 minute period.
b. over the first 3 minutes.
c. between 150 seconds and 300 seconds.

Solution:

a. The entire 5 minute period corresponds to the interval $[0, 300]$. The average rate of change of the heart rate on the interval $[0, 300]$ is given by

$$\frac{87-85 \text{ BPM}}{300-0 \text{ seconds}} = \frac{2 \text{ BPM}}{300 \text{ s}} = \frac{1}{150} \text{ BPM per second}$$

b. The first 3 minutes corresponds to the interval $[0, 180]$. The average rate of change of the heart rate on $[0, 180]$ is given by

$$\frac{110-85 \text{ BPM}}{180-0 \text{ seconds}} = \frac{25 \text{ BPM}}{180 \text{ s}} = \frac{5}{36} \text{ BPM per second}$$

c. The average rate of change of the heart rate on the interval $[150, 300]$ is given by

$$\frac{87-105 \text{ BPM}}{300-150 \text{ seconds}} = \frac{-18 \text{ BPM}}{150 \text{ s}} = -\frac{3}{25} \text{ BPM per second}$$

■

Try It # 3:
Using **Table 2.3** from the previous example, find the average rate of change of the heart rate

a. over the last two minute period.
b. between 120 seconds and 240 seconds.

Instantaneous Rate of Change

Let's revisit our motivating example concerning the amusement park ride. Recall that the height, in feet, of the riders t seconds after free-fall was modeled by $f(t) = -16t^2 + 150$. We were interested in obtaining the instantaneous velocity at $t = 2$ seconds. We approximated the instantaneous velocity at $t = 2$ by finding the average velocity on the interval $[2,3]$, which we found was -80 feet per second. In Example 1, we found the slope of the secant line on the graph of f on the interval $[2,2.5]$ to be -72. As we discussed earlier, this leads us to conclude that the average velocity, or average rate of change of the position function, is -72 feet per second on that interval.

By continuing to narrow the interval we consider, we can improve our approximation for the instantaneous velocity at $t = 2$ seconds. For instance, on the interval $[2,2.1]$, we see the average velocity is

$$\frac{f(2.1)-f(2)}{2.1-2} = \frac{79.44-86}{0.1} = -65.6 \text{ feet per second}$$

On the interval $[2,2.01]$, the average velocity is

$$\frac{f(2.01)-f(2)}{2.01-2} = \frac{85.3584-86}{0.01} = -64.16 \text{ feet per second}$$

On the interval $[2,2.001]$, the average velocity is

$$\frac{f(2.001)-f(2)}{2.001-2} = \frac{85.935984-86}{0.001} = -64.016 \text{ feet per second}$$

Based on these calculations, it appears that a good estimate for the instantaneous velocity at $t = 2$ seconds is -64 feet per second. Thus, we know at exactly two seconds after the start of free-fall, the riders are moving in the downward direction at an approximate speed of 64 feet per second.

N *We could repeat the process above with 2 being the right endpoint of the interval, and we would reach a similar conclusion.*

■ **Example 6** The profit for Singing Samuel's TVs when x TVs are manufactured and sold is $P(x)$ dollars. **Table 2.4** gives some values of $P(x)$.

x	497	498	499	500	501	502	503
$P(x)$	\$145,629.91	\$145,919.96	\$146,209.99	\$146,500	\$146,789.99	\$147,079.96	\$147,369.91

Table 2.4: Profit values when x TVs are sold

a. Find the average rate of change of profit on the following intervals.

i. $[497,500]$

ii. $[498,500]$

iii. $[499,500]$

iv. $[500,503]$

v. $[500,502]$

vi. $[500,501]$

b. Use part **a** to approximate the instantaneous rate of change of profit when 500 of Singing Samuel's TVs are manufactured and sold, and interpret your answer.

Solution:

a. In general, the average rate of change of the profit function, P, on the interval $[a,b]$ is given by

$$\frac{P(b)-P(a)\text{ dollars}}{b-a\text{ TVs}}$$

i. Thus, the average rate of change of profit on the interval $[497,500]$ is

$$\begin{aligned}\frac{P(500)-P(497)\text{ dollars}}{500-497\text{ TVs}} &= \frac{\$146,500-\$145,629.91}{3\text{ TVs}}\\ &= \frac{\$870.09}{3\text{ TVs}}\\ &= \$290.03\text{ per TV}\end{aligned}$$

ii. The average rate of change of profit on the interval $[498,500]$ is

$$\begin{aligned}\frac{P(500)-P(498)\text{ dollars}}{500-498\text{ TVs}} &= \frac{\$146,500-\$145,629.91}{2\text{ TVs}}\\ &= \frac{\$580.04}{2\text{ TVs}}\\ &= \$290.02\text{ per TV}\end{aligned}$$

iii. The average rate of change of profit on the interval $[499,500]$ is

$$\begin{aligned}\frac{P(500)-P(499)\text{ dollars}}{500-499\text{ TVs}} &= \frac{\$146,500-\$146,209.99}{1\text{ TVs}}\\ &= \$290.01\text{ per TV}\end{aligned}$$

iv. The average rate of change of profit on the interval $[500,503]$ is

$$\begin{aligned}\frac{P(503)-P(500)\text{ dollars}}{503-500\text{ TVs}} &= \frac{\$147,369.91-\$146,500}{3\text{ TVs}}\\ &= \frac{\$869.91}{3\text{ TVs}}\\ &= \$289.97\text{ per TV}\end{aligned}$$

v. The average rate of change of profit on the interval $[500, 502]$ is

$$\begin{aligned}\frac{P(502)-P(500) \text{ dollars}}{502-500 \text{ TVs}} &= \frac{\$147,079.96-\$146,500}{2 \text{ TVs}}\\ &= \frac{\$579.96}{2 \text{ TVs}}\\ &= \$289.98 \text{ per TV}\end{aligned}$$

vi. The average rate of change of profit on the interval $[500, 501]$ is

$$\begin{aligned}\frac{P(501)-P(500) \text{ dollars}}{501-500 \text{ TVs}} &= \frac{\$146,789.99-\$146,500}{1 \text{ TVs}}\\ &= \$289.99 \text{ per TV}\end{aligned}$$

b. Each average rate of change in part **a** is an estimate for the instantaneous rate of change when 500 TVs are produced and sold, but if we look more closely at the values of these average rates of change, we may be able to obtain a better estimate.

Notice that the first three intervals have 500 as the right endpoint and the last three intervals have 500 as the left endpoint. In each set of three, the intervals are getting smaller and smaller. Also, each set of approximations is approaching $290 per TV.

Thus, it appears that a good approximation for the instantaneous rate of change of profit when 500 TVs are manufactured and sold is $290 per TV. To interpret our answer, we write a sentence:

"When 500 TVs are manufactured and sold, profit is increasing at a rate of $290 per TV."

▪

Try It # 4:
The revenue for Ugly Mug brand coffee cups when x coffee cups are sold is $R(x)$ dollars. **Table 2.5** gives some values of $R(x)$.

x	527	528	529	530	531	532	533
$R(x)$	$1,965.71	$1,964.16	$1,962.59	$1,961	$1,959.39	$1,957.76	$1,956.11

Table 2.5: Revenue values when x coffee cups are sold

a. Find the average rate of change of revenue on the following intervals.

i. $[527, 530]$

ii. $[528, 530]$

iii. $[529, 530]$

iv. $[530, 533]$

v. $[530, 532]$

vi. $[530, 531]$

b. Use part **a** to approximate the instantaneous rate of change of revenue when 530 coffee cups are sold, and interpret your answer.

In the previous example, we approximated the instantaneous rate of change by calculating the average rate of change on smaller and smaller intervals. This concept can also be applied to slopes of secant lines.

Example 7 Given $f(x) = 10x^2 - 30x + 23.5$,

a. find the slope of the secant line passing through the points on the graph of f at $x = 1$ and $x = 1.5$.

b. find the slope of the secant line passing through the points on the graph of f at $x = 1$ and $x = 1.2$.

c. find the slope of the secant line passing through the points on the graph of f at $x = 1$ and $x = 1.01$.

d. estimate the limit of the slopes of these secant lines as the right endpoint of the intervals approaches 1.

Solution:

a. To find the y-value of the point on the graph of f corresponding to $x = 1$, we must find $f(1)$:

$$f(1) = 10(1)^2 - 30(1) + 23.5 = 3.5$$

Likewise, to find the y-value of the point on the graph of f corresponding to $x = 1.5$, we must find $f(1.5)$:

$$f(1.5) = 10(1.5)^2 - 30(1.5) + 23.5 = 1$$

Thus, the secant line passes through the points $(1, 3.5)$ and $(1.5, 1)$ and has slope

$$m = \frac{1 - 3.5}{1.5 - 1}$$

$$= \frac{-2.5}{0.5}$$

$$= -5$$

b. Once again, we need to calculate $f(1)$ and $f(1.2)$ to find the y-values of the points on the graph of f corresponding to $x = 1$ and $x = 1.2$, respectively. From part **a**, we know $f(1) = 3.5$. We now find $f(1.2)$:

$$f(1.2) = 10(1.2)^2 - 30(1.2) + 23.5 = 1.9$$

Thus, the secant line passes through the points $(1, 3.5)$ and $(1.2, 1.9)$ and has slope

$$m = \frac{1.9 - 3.5}{1.2 - 1}$$

$$= \frac{-1.6}{0.2}$$

$$= -8$$

c. Once again, we already have $f(1)$, so we only need to find $f(1.01)$:

$$f(1.01) = 10(1.01)^2 - 30(1.01) + 23.5 = 3.401$$

Thus, the secant line passes through the points $(1, 3.5)$ and $(1.01, 3.401)$ and has slope

$$m = \frac{3.401 - 3.5}{1.01 - 1}$$

$$= \frac{-0.099}{0.01}$$

$$= -9.9$$

d. Recall that we need to estimate the limit of the slopes of these secant lines as the right endpoint of the intervals approaches 1. As the intervals get smaller and smaller in parts **a** through **c**, the slopes of the secant lines go from -5 to -8 and then from -8 to -9.9. Thus, it appears that these values are approaching -10.

▪

Try It # 5:
Given $f(x) = -x^3 + 4$,

a. find the slope of the secant line passing through the points on the graph of f at $x = 4.5$ and $x = 5$.

b. find the slope of the secant line passing through the points on the graph of f at $x = 4.9$ and $x = 5$.

c. find the slope of the secant line passing through the points on the graph of f at $x = 4.99$ and $x = 5$.

d. estimate the limit of the slopes of these secant lines as the left endpoint of the intervals approaches 5.

Let's consider the previous example graphically to investigate what happens when we find slopes of secant lines on smaller and smaller intervals. The function from the previous example, $f(x) = 10x^2 - 30x + 23.5$, along with three different secant lines whose slopes we found in parts **a** through **c**, are shown in **Figures 2.1.2**, **2.1.3**, and **2.1.4**, respectively.

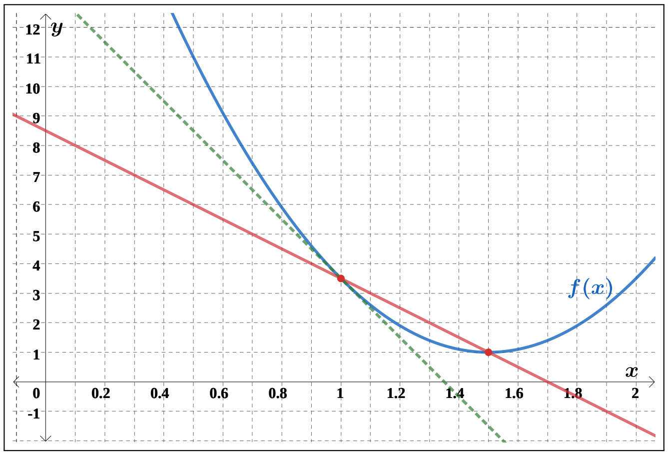


Figure 2.1.2: Graphs of $f(x) = 10x^2 - 30x + 23.5$, the line tangent to the graph of f at $x = 1$, and the secant line that passes through the points at $x = 1$ and $x = 1.5$

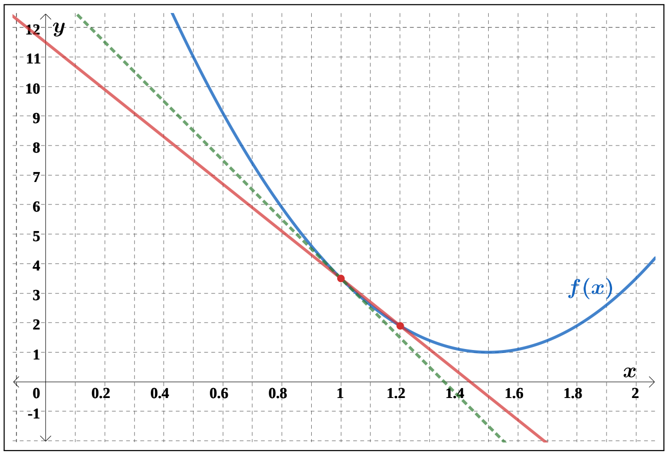


Figure 2.1.3: Graphs of $f(x) = 10x^2 - 30x + 23.5$, the line tangent to the graph of f at $x = 1$, and the secant line that passes through the points at $x = 1$ and $x = 1.2$

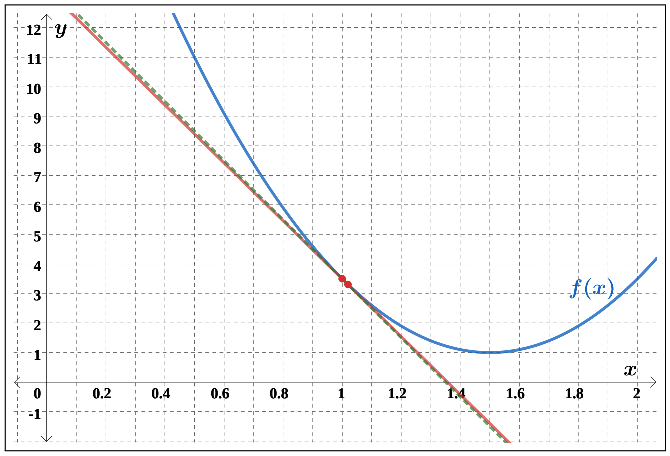


Figure 2.1.4: Graphs of $f(x) = 10x^2 - 30x + 23.5$, the line tangent to the graph of f at $x = 1$, and the secant line that passes through the points at $x = 1$ and $x = 1.01$

Notice that as the intervals get smaller and smaller, the secant lines approach the dotted line. This dotted line appears to touch the graph at only $x = 1$. We refer to this line as the **tangent line**, and its slope can be approximated by finding slopes of secant lines on smaller and smaller intervals. We define the tangent line below and then discuss how to find the exact value of the **slope of the tangent line**.

Definition

The **tangent line** of a function is the line that touches the graph of the function at a specific point and approximates the function around that point. ▪

Note that this definition is not mathematically precise. We will now work to formally define the slope of the tangent line at the point $(a, f(a))$. We start by generalizing the ideas from **Figures 2.1.2**, **2.1.3**, and **2.1.4**.

To find the slope of the tangent line at the point $(a, f(a))$, we will start by finding slopes of secant lines around the point $(a, f(a))$. The slope of the secant line passing through the points $(a, f(a))$ and $(x, f(x))$ is given by

$$m = \frac{f(x) - f(a)}{x - a}$$

Because we want smaller intervals, we consider secant lines in which the x-values are getting closer to a. See **Figure 2.1.5**, **Figure 2.1.6**, and **Figure 2.1.7**. The dotted line in the figures is the line tangent to the graph of f at $x = a$:

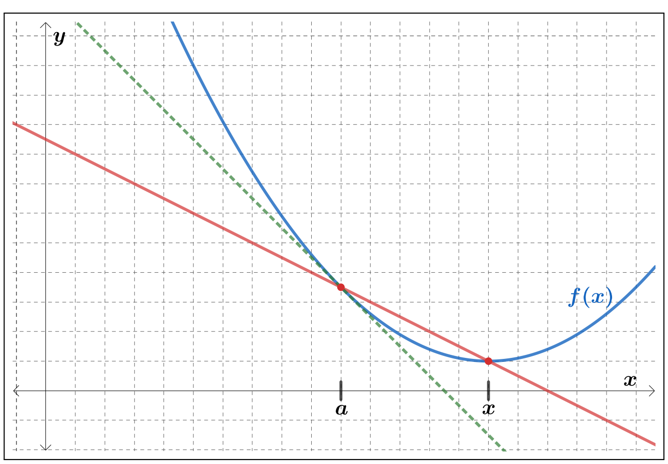


Figure 2.1.5: Graphs of f, the line tangent to the graph of f at $x = a$, and the secant line that passes through the points $(a, f(a))$ and $(x, f(x))$

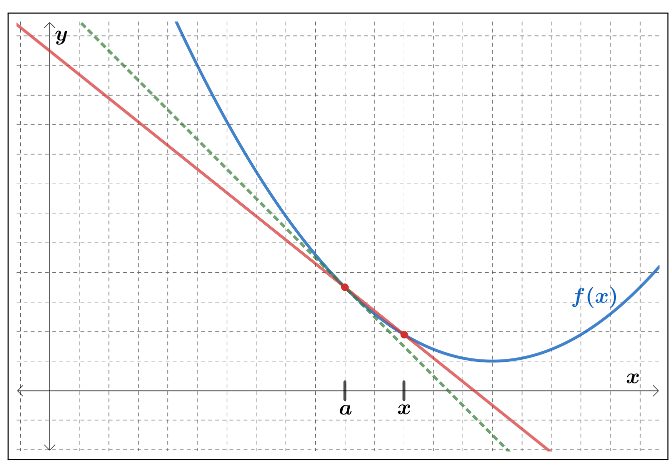


Figure 2.1.6: Graphs of f, the line tangent to the graph of f at $x = a$, and the secant line that passes through the points $(a, f(a))$ and $(x, f(x))$

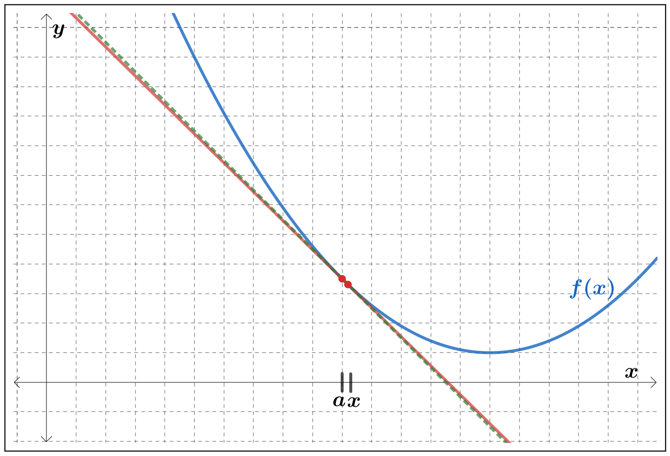


Figure 2.1.7: Graphs of f, the line tangent to the graph of f at $x = a$, and the secant line that passes through the points $(a, f(a))$ and $(x, f(x))$

Mathematically, when we talk about x "getting close" to a particular value, we are talking about a limit! This allows us to formally define the slope of the tangent line, which is equivalent to the instantaneous rate of change:

Definition

The **instantaneous rate of change** of a function f at $x = a$ is the **slope of the tangent line** that passes through the point $(a, f(a))$, if it exists, and is given by

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

if this limit exists. ▪

N *Even though the graphs in **Figure 2.1.5**, **Figure 2.1.6**, and **Figure 2.1.7** showed the secant lines as x approached a from the right, for the above limit to exist, the slopes of the secant lines from both the left and right of x = a must approach the same number (a good fact to remember from Sections 1.1 and 1.2!).*

This definition is very useful conceptually because it shows that the slope of the tangent line is the limit of the slopes of secant lines, but it is difficult to compute. Let's revisit the amusement park ride example to help lead us to a definition that is easier to work with.

Recall that we found five estimates for the instantaneous velocity, or instantaneous rate of change, at $t = 2$ seconds by finding the average velocity on smaller and smaller intervals. These computations can be viewed as finding the average velocity, or average rate of change, on the interval $[2, 2+h]$ for small values of h. That is, we computed

$$\frac{f(2+h) - f(2)}{(2+h) - 2} = \frac{f(2+h) - f(2)}{h}$$

for small values of h ($h \neq 0$).

Our previous calculations using this new notation are summarized in **Table 2.6**.

h	Average Velocity on $[2,2+h] = \dfrac{f(2+h)-f(2)}{h}$
1	-80 feet per second
0.5	-72 feet per second
0.1	-65.6 feet per second
0.01	-64.16 feet per second
0.001	-64.016 feet per second

Table 2.6: Approximating the instantaneous velocity

Because we want h to become small, we are really wanting to find the limit as h approaches 0. Thus, the instantaneous velocity, or instantaneous rate of change, at $t = 2$ seconds can be found exactly by finding the following limit:

$$\lim_{h\to 0} \frac{f(2+h)-f(2)}{h}$$

This leads us to a definition that is easier to work with than our previous definition:

Definition

The **instantaneous rate of change** of a function f at $x = a$ is the **slope of the tangent line** that passes through the point $(a, f(a))$, if it exists, and is given by

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

if this limit exists. ▪

To visualize this definition, look at **Figures 2.1.8**, **2.1.9**, and **2.1.10**, which are identical to **Figures 2.1.5**, **2.1.6**, and **2.1.7** except we replace x with $a+h$:

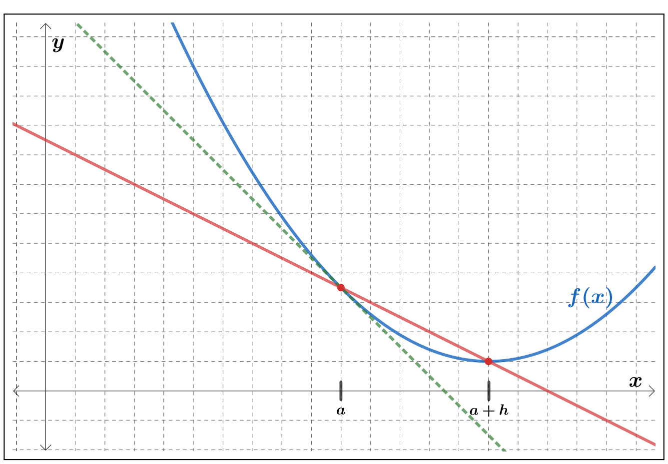


Figure 2.1.8: Graphs of f, the line tangent to the graph of f at $x = a$, and the secant line that passes through the points at $x = a$ and $x = a+h$

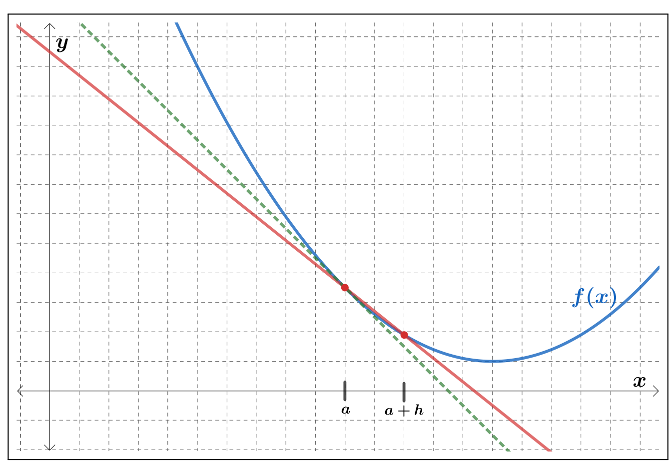


Figure 2.1.9: Graphs of f, the line tangent to the graph of f at $x = a$, and the secant line that passes through the points at $x = a$ and $x = a+h$

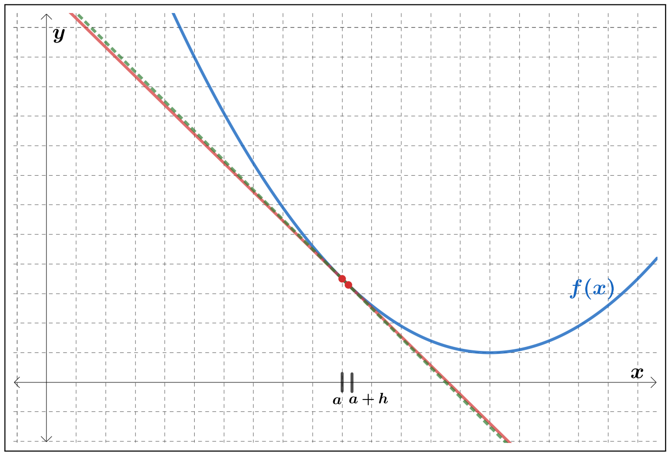


Figure 2.1.10: Graphs of f, the line tangent to the graph of f at $x = a$, and the secant line that passes through the points at $x = a$ and $x = a + h$

Note that in this definition of the instantaneous rate of change, the quantity $\dfrac{f(a+h)-f(a)}{h}$, where $h \neq 0$, is sometimes referred to as the **difference quotient**. Because this quantity represents the slope of a secant line, it can also be referred to as the average rate of change.

Now, let's focus on computing this limit algebraically! Recall from Section 1.2 that we should first attempt to find $\lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ using direct substitution. However, because the difference quotient consists of a ratio of two functions, we need to make sure the limits of the numerator and denominator exist and that the limit of the denominator is nonzero. Let's start by checking the numerator:

$$\begin{aligned}\lim_{h\to 0} f(a+h) - f(a) &= f(a+0) - f(a) \\ &= f(a) - f(a) \\ &= 0\end{aligned}$$

Finding the limit of the denominator we get

$$\lim_{h\to 0} h = 0$$

These two limits exist, but because the limits of the two functions equal zero, the limit is of the indeterminate form $\frac{0}{0}$. This will always be the case when finding the limit of the difference quotient as $h \to 0$. Thus, we must always use algebraic techniques to further investigate this type of limit. The ultimate goal of our algebraic manipulations is to *divide the h from the denominator* of the difference quotient. Doing so will ensure the limit of the denominator is nonzero.

Let's look at this process for a specific function in the example below.

■ **Example 8** Find the exact value of the instantaneous rate of change of $f(x) = x^2 - 3$ at $x = 4$.

Solution:
To find the exact value of the instantaneous rate of change, we must use the limit definition of the instantaneous rate of change. We defined two different limit definitions, but in this textbook, we will always use the second definition, $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$, in our computations.

As seen previously, direct substitution with the difference quotient leads to the indeterminate form $\frac{0}{0}$. Thus, we must use algebraic techniques to further investigate the limit. Remember, the goal is to eventually divide the h from the denominator so we can use direct substitution to find the limit. Because we are trying to find the instantaneous rate of change at $x = 4$, we let $a = 4$ in the limit:

$$\begin{aligned}
\lim_{h \to 0} \frac{f(4+h) - f(4)}{h} &= \lim_{h \to 0} \frac{((4+h)^2 - 3) - (4^2 - 3)}{h} \\
&= \lim_{h \to 0} \frac{((4+h)(4+h) - 3) - 13}{h} \\
&= \lim_{h \to 0} \frac{\left(\left(16 + 4h + 4h + h^2\right) - 3\right) - 13}{h} \\
&= \lim_{h \to 0} \frac{\cancel{16} + 8h + h^2 - \cancel{16}}{h} \\
&= \lim_{h \to 0} \frac{8h + h^2}{h} \\
&= \lim_{h \to 0} \frac{\cancel{h}(8+h)}{\cancel{h}} \\
&= \lim_{h \to 0} \frac{8+h}{1} \\
&= \lim_{h \to 0} (8+h)
\end{aligned}$$

We were able to divide the h from the denominator as we hoped. We now attempt direct substitution again:

$$\begin{aligned}
\lim_{h \to 0} \frac{f(4+h) - f(4)}{h} &= \lim_{h \to 0} (8+h) \\
&= 8 + 0 \\
&= 8
\end{aligned}$$

Thus, the instantaneous rate of change, or slope of the tangent line, of $f(x) = x^2 - 3$ at $x = 4$ is 8.

■

Some students may be content with finding this type of limit in one step. But for many, the process may feel overwhelming. Also, computing the limit all at once makes it much more likely that algebraic mistakes will be made!

For the remainder of this section, we will use the Suggested Four-Step Process outlined below to evaluate this type of limit using a step-by-step process.

Suggested Four-Step Process (to find the exact value of the instantaneous rate of change at $x = a$)

1. Calculate $f(a+h)$, and algebraically manipulate the resulting expression, if possible.
2. Calculate the numerator of the difference quotient, and algebraically manipulate the resulting expression, if possible:
$$f(a+h)-f(a)$$
3. Calculate the difference quotient (i.e., the slope of the secant line), and algebraically manipulate the resulting expression:
$$\frac{f(a+h)-f(a)}{h}, \quad h \neq 0$$
4. Take the limit of the difference quotient as h tends to zero (i.e., find the slope of the tangent line):
$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Let's rework the previous example using the Suggested Four-Step Process.

Example 9 Find the exact value of the instantaneous rate of change of $f(x) = x^2 - 3$ at $x = 4$.

Solution:
We still need to find $\lim_{h\to 0}\frac{f(4+h)-f(4)}{h}$ as in the previous example, but instead of algebraically manipulating the limit in one single (long) step, we will use the Suggested Four-Step Process:

1. Calculate $f(4+h)$, and algebraically manipulate the resulting expression, if possible:

$$\begin{aligned} f(4+h) &= (4+h)^2 - 3 \\ &= (4+h)(4+h) - 3 \end{aligned}$$

Using FOIL gives

$$\begin{aligned} &= 16 + 4h + 4h + h^2 - 3 \\ &= 13 + 8h + h^2 \end{aligned}$$

2. Calculate the numerator of the difference quotient, $f(4+h) - f(4)$, and algebraically manipulate the resulting expression, if possible:

We start with $f(4)$:

$$f(4) = (4)^2 - 3 = 13$$

So in the numerator of the difference quotient, we have

$$\begin{aligned} f(4+h)-f(4) &= 13+8h+h^2-13 \\ &= 8h+h^2 \end{aligned}$$

3. Calculate the difference quotient (i.e., divide the result from the previous step by h, where $h \neq 0$), and algebraically manipulate the resulting expression:

$$\frac{f(4+h)-f(4)}{h} = \frac{8h+h^2}{h}$$

Factoring an h from the numerator gives

$$= \frac{h(8+h)}{h}$$

Dividing the h's gives

$$\begin{aligned} &= \frac{\cancel{h}(8+h)}{\cancel{h}} \\ &= 8+h \end{aligned}$$

4. Take the limit as $h \to 0$:

$$\begin{aligned} \lim_{h\to 0}\frac{f(4+h)-f(4)}{h} &= \lim_{h\to 0}(8+h) \\ &= 8+0 \\ &= 8 \end{aligned}$$

Thus, the instantaneous rate of change, or slope of the tangent line, of $f(x) = x^2 - 3$ at $x = 4$ is 8. If you look at the solution to the previous example, you will see that we reached the same conclusion. Either technique is correct, but we will continue to use the Suggested Four-Step Process for the remainder of this section.

■

N *We were able to use direct substitution to find the limit in step 4 when using the Suggested Four-Step Process in the previous example. In other words, we were able to substitute* 0 *for h to find* $\lim_{h\to 0}(8+h)$. *Recall that we are able to use direct substitution if there are no domain issues.*

Try It # 6:
Find the exact value of the instantaneous rate of change of $f(x) = 7 + x^2$ at $x = 2$.

■ **Example 10** Find the slope of the line tangent to the graph of $f(x) = \sqrt{6x+2}$ at $x = 1$.

Solution:
The slope of the tangent line is equivalent to the instantaneous rate of change, which is what we found in the previous two examples. Because we are asked to find the slope of the tangent line at $x = 1$, we let $a = 1$ in the limit:

$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

Again, to simplify the process of finding this limit, we will use the Suggested Four-Step Process:

1. Calculate $f(1+h)$, and algebraically manipulate the resulting expression, if possible:

$$\begin{aligned} f(1+h) &= \sqrt{6(1+h)+2} \\ &= \sqrt{6+6h+2} \\ &= \sqrt{8+6h} \end{aligned}$$

2. Calculate the numerator of the difference quotient, $f(1+h)-f(1)$, and algebraically manipulate the resulting expression, if possible:

We start with $f(1)$:

$$f(1) = \sqrt{6(1)+2} = \sqrt{8}$$

So in the numerator of the difference quotient, we have:

$$f(1+h)-f(1) = \sqrt{8+6h} - \sqrt{8}$$

Notice there is no algebraic manipulation to perform at this step. When working with functions containing square roots, the algebra will mostly be performed in step 3.

3. Calculate the difference quotient (i.e., divide the result from the previous step by h, where $h \neq 0$), and algebraically manipulate the resulting expression:

$$\frac{f(1+h)-f(1)}{h} = \frac{\sqrt{8+6h}-\sqrt{8}}{h}$$

To algebraically manipulate this expression, we rationalize the numerator by multiplying by the conjugate:

$$\begin{aligned} \frac{\sqrt{8+6h}-\sqrt{8}}{h} &= \left(\frac{\sqrt{8+6h}-\sqrt{8}}{h}\right)\left(\frac{\sqrt{8+6h}+\sqrt{8}}{\sqrt{8+6h}+\sqrt{8}}\right) \\ &= \frac{(8+6h)-8}{h\left(\sqrt{8+6h}+\sqrt{8}\right)} \\ &= \frac{6h}{h\left(\sqrt{8+6h}+\sqrt{8}\right)} \end{aligned}$$

N *Remember that multiplying by the conjugate is extremely useful for calculations where there is a square root in the function because it eliminates the square roots where necessary. It is not without cost, however. The process helps us simplify the numerator, but the denominator becomes more complicated. That's okay for us, though, because we can calculate the limit with this slightly more complicated denominator.*

Dividing the h's gives

$$\frac{6\not{h}}{\not{h}\left(\sqrt{8+6h}+\sqrt{8}\right)} = \frac{6}{\sqrt{8+6h}+\sqrt{8}}$$

4. Take the limit as $h \to 0$:

$$\lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{h\to 0}\frac{6}{\sqrt{8+6h}+\sqrt{8}}$$

We have to be careful before proceeding. Unlike the previous example, we should not immediately substitute 0 for h to find this limit. Why? You guessed it! The expression we are taking the limit of consists of a ratio of two functions, so we must check that the limits of both the numerator and denominator exist and that the limit of the function in the denominator is nonzero.

Let's start by checking the limit of the numerator:

$$\lim_{h\to 0} 6 = 6$$

Next, we check the limit of the denominator (note we can use direct substitution because there will not be any domain issues):

$$\begin{aligned}\lim_{h\to 0}\left(\sqrt{8+6h}+\sqrt{8}\right) &= \sqrt{8+6(0)}+\sqrt{8} \\ &= \sqrt{8}+\sqrt{8} \\ &= 2\sqrt{8}\end{aligned}$$

We see both limits exist and the limit of the function in the denominator is nonzero. Thus, we can find the limit using the Properties of Limits:

$$\begin{aligned}\lim_{h\to 0}\frac{6}{\sqrt{8+6h}+\sqrt{8}} &= \frac{\lim_{h\to 0} 6}{\lim_{h\to 0}\left(\sqrt{8+6h}+\sqrt{8}\right)} \\ &= \frac{6}{2\sqrt{8}} \\ &= \frac{3}{\sqrt{8}}\end{aligned}$$

Thus, the slope of the line tangent to the graph of f at $x = 1$ is $\frac{3}{\sqrt{8}}$.

▪

For the instantaneous rates of change we will find in this textbook using the limit definition, algebraically manipulating the difference quotient will ensure that we can substitute 0 for h to find the limit in step 4 of the Suggested Four-Step Process, even if the resulting expression in step 3 consists of a ratio of two functions (as seen in the previous example). The limits of the resulting numerator and denominator will exist and the limit of the resulting denominator will be nonzero (due to dividing h from the denominator in step 3). Thus, from this point forward, we will use direct substitution and substitute 0 for h when calculating the limit in step 4 of the Suggested Four-Step Process because this will lead us to the same answer as using the Properties of Limits to divide the limits of the resulting numerator and denominator.

N *We will discuss the x-values where the limit definition of the instantaneous rate of change does not exist in* **Section 2.2**.

Try It # 7:
Find the slope of the line tangent to the graph of $f(x) = \sqrt{x-18}+3$ at $x = 27$.

■ **Example 11** Find the exact value of the instantaneous rate of change of $f(x) = \dfrac{7}{5-9x}$ at $x = -3$.

Solution:
To find the instantaneous rate of change of f at $x = -3$, we let $a = -3$ in the limit definition of the instantaneous rate of change:

$$\lim_{h \to 0} \frac{f(-3+h) - f(-3)}{h}$$

Again, to break down the steps for finding this limit, we will use the Suggested Four-Step Process:

1. Calculate $f(-3+h)$, and algebraically manipulate the resulting expression, if possible:

$$f(-3+h) = \frac{7}{5-9(-3+h)}$$

$$= \frac{7}{5+27-9h}$$

$$= \frac{7}{32-9h}$$

2. Calculate the numerator of the difference quotient, $f(-3+h) - f(-3)$, and algebraically manipulate the resulting expression, if possible:

We start with $f(-3)$:

$$f(-3) = \frac{7}{5-9(-3)} = \frac{7}{32}$$

So in the numerator of the difference quotient, we have

$$f(-3+h) - f(-3) = \frac{7}{32-9h} - \frac{7}{32}$$

To algebraically manipulate this expression further, we will get a common denominator:

$$\frac{7}{32-9h} - \frac{7}{32} = \left(\frac{32}{32}\right)\left(\frac{7}{32-9h}\right) - \left(\frac{7}{32}\right)\left(\frac{32-9h}{32-9h}\right)$$

$$= \frac{224}{32(32-9h)} - \frac{224-63h}{32(32-9h)}$$

$$= \frac{224-(224-63h)}{32(32-9h)}$$

$$= \frac{224-224+63h}{32(32-9h)}$$

$$= \frac{63h}{32(32-9h)}$$

3. Calculate the difference quotient (i.e., divide the result from the previous step by h, where $h \neq 0$), and algebraically manipulate the resulting expression:

$$\frac{f(-3+h)-f(-3)}{h} = \frac{\frac{63h}{32(32-9h)}}{h}$$

At first glance, it may not be obvious how to proceed because we have a complex fraction. Anytime we are presented with a complex fraction such as this one, we will rewrite the division by h as multiplication by $\frac{1}{h}$. Recall that we can do this because dividing by a fraction is equivalent to multiplying by its reciprocal. Thus, we continue algebraically manipulating the difference quotient:

$$\frac{\frac{63h}{32(32-9h)}}{h} = \frac{\frac{63h}{32(32-9h)}}{\frac{h}{1}}$$

$$= \left(\frac{63h}{32(32-9h)}\right)\left(\frac{1}{h}\right)$$

$$= \left(\frac{63\cancel{h}}{32(32-9h)}\right)\left(\frac{1}{\cancel{h}}\right)$$

$$= \frac{63}{32(32-9h)}$$

4. Take the limit as $h \to 0$ (i.e., substitute 0 for h):

$$\begin{aligned}\lim_{h\to 0}\frac{f(-3+h)-f(-3)}{h} &= \lim_{h\to 0}\frac{63}{32(32-9h)}\\ &= \frac{63}{32(32-9(0))}\\ &= \frac{63}{1024}\end{aligned}$$

Thus, the instantaneous rate of change of f at $x=-3$ is $\frac{63}{1024}$.

■

Try It # 8:

Find the exact value of the instantaneous rate of change of $f(x) = \frac{1}{2x+4}$ at $x = 0$.

■ **Example 12** Find the exact value of the instantaneous velocity at $t = 2$ seconds for the amusement park ride example. Recall that the height, in feet, of the riders t seconds after free-fall was given by $f(t) = -16t^2 + 150$.

Solution:

Remember, if the function f represents the position of an object, the instantaneous rate of change of f represents the instantaneous velocity of the object. To find the instantaneous velocity at $t = 2$, we let $a = 2$ in the limit:

$$\lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

Again, we will use the Suggested Four-Step Process:

1. Calculate $f(2+h)$, and algebraically manipulate the resulting expression, if possible:

$$f(2+h) = -16(2+h)^2 + 150$$

Using FOIL and distributing gives

$$\begin{aligned} &= -16\left(4+4h+h^2\right)+150 \\ &= -64-64h-16h^2+150 \\ &= 86-64h-16h^2 \end{aligned}$$

2. Calculate the numerator of the difference quotient, $f(2+h)-f(2)$, and algebraically manipulate the resulting expression, if possible:

We start with $f(2)$:

$$f(2) = -16(2)^2 + 150 = 86$$

So in the numerator of the difference quotient, we have

$$\begin{aligned} f(2+h)-f(2) &= 86-64h-16h^2-86 \\ &= -64h-16h^2 \end{aligned}$$

3. Calculate the difference quotient (i.e., divide the result from the previous step by h, where $h \neq 0$), and algebraically manipulate the resulting expression:

$$\frac{f(2+h)-f(2)}{h} = \frac{-64h-16h^2}{h}$$

Factoring an h from the numerator gives

$$= \frac{h(-64-16h)}{h}$$

Dividing the h's gives

$$\begin{aligned} &= \frac{\not{h}(-64-16h)}{\not{h}} \\ &= -64-16h \end{aligned}$$

4. Take the limit as $h \to 0$ (i.e., substitute 0 for h):

$$\begin{aligned} \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0}(-64-16h) \\ &= -64-16(0) \\ &= -64 \text{ feet per second} \end{aligned}$$

Thus, instantaneous velocity at $t = 2$ seconds is -64 feet per second. Remember that the negative sign indicates the ride is moving downward!

Recall that we approximated this value earlier in this section and actually guessed that the value was -64 feet per second. However, we can be more confident in our answer when we use the limit definition to find the exact value!

▪

Try It # 9:
Find the exact value of the instantaneous velocity at $t = 1$ second for the amusement park ride in the previous example.

■ **Example 13** Ninvento, a toy company, can sell x Swap gaming consoles at a price of $p(x) = -0.01x + 400$ dollars per Swap. The cost of manufacturing x of these consoles is given by $C(x) = 100x + 10{,}000$ dollars. Find the instantaneous rate of change of profit when $10{,}000$ Swaps are produced and sold, and interpret your answer.

Solution:
First, we need the profit function. Recall profit is equal to revenue minus cost. We have the cost function, so we need to find the revenue function. The revenue function, R, is given by the number of items sold, x, times the price of each item, p:

$$\begin{aligned} R(x) &= x \cdot p(x) \\ &= x(-0.01x + 400) \\ &= -0.01x^2 + 400x \end{aligned}$$

Thus, the profit function is

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= \left(-0.01x^2 + 400x\right) - (100x + 10{,}000) \\ &= -0.01x^2 + 400x - 100x - 10{,}000 \\ &= -0.01x^2 + 300x - 10{,}000 \end{aligned}$$

Now that we have the profit function, $P(x) = -0.01x^2 + 300x - 10{,}000$ dollars, where x is the number of Swap gaming consoles made and sold, we can find the instantaneous rate of change of the function P at $x = 10{,}000$ by letting $a = 10{,}000$ in the limit definition:

$$\lim_{h \to 0} \frac{P(10{,}000 + h) - P(10{,}000)}{h}$$

Again, we will use the Suggested Four-Step Process:

1. Calculate $P(10{,}000 + h)$, and algebraically manipulate the resulting expression, if possible:

$$P(10{,}000 + h) = -0.01(10{,}000 + h)^2 + 300(10{,}000 + h) - 10{,}000$$

Using FOIL and distributing gives

$$\begin{aligned} &= -0.01\left(100{,}000{,}000 + 20{,}000h + h^2\right) + 3{,}000{,}000 + 300h - 10{,}000 \\ &= -1{,}000{,}000 - 200h - 0.01h^2 + 3{,}000{,}000 + 300h - 10{,}000 \\ &= 100h - 0.01h^2 + 1{,}990{,}000 \end{aligned}$$

2. Calculate the numerator of the difference quotient, $P(10{,}000 + h) - P(10{,}000)$, and algebraically manipulate the resulting expression, if possible:

We start with $P(10{,}000)$:

$$P(10{,}000) = -0.01(10{,}000)^2 + 300(10{,}000) - 10{,}000 = 1{,}990{,}000.$$

So in the numerator of the difference quotient, we have

$$\begin{aligned} P(10{,}000 + h) - P(10{,}000) &= 100h - 0.01h^2 + 1{,}990{,}000 - 1{,}990{,}000 \\ &= 100h - 0.01h^2 \end{aligned}$$

3. Calculate the difference quotient (i.e., divide the result from the previous step by h, where $h \neq 0$), and algebraically manipulate the resulting expression:

$$\frac{P(10,000+h)-P(10,000)}{h} = \frac{100h-0.01h^2}{h}$$

Factoring an h from the numerator gives

$$= \frac{h(100-0.01h)}{h}$$

Dividing the h's gives

$$= \frac{\cancel{h}(100-0.01h)}{\cancel{h}}$$
$$= 100-0.01h$$

4. Take the limit as $h \to 0$ (i.e., substitute 0 for h):

$$\begin{aligned}\lim_{h\to 0}\frac{P(10,000+h)-P(10,000)}{h} &= \lim_{h\to 0}(100-0.01h)\\ &= 100-0.01(0)\\ &= \$100 \text{ per Swap}\end{aligned}$$

Thus, profit is increasing at a rate of \$100 per Ninvento Swap. To interpret our answer, we write a sentence:

"When $10,000$ Swaps are produced and sold, profit is increasing at a rate of \$100 per Swap."

Notice we did not include the word "average" in our interpretation because we calculated an instantaneous rate of change at one point (when $x = 10,000$), not an average rate of change between two points. When we interpret an average rate of change, we should always include the word "average" to distinguish it from an instantaneous rate of change.

■

Try It # 10:
Find the instantaneous rate of change of cost and revenue in the previous example when 10,000 Swaps are produced and sold, and interpret your answers.

After the previous example and Try It, you should see that the instantaneous rate of change of profit at $10,000$ units is the instantaneous rate of change of revenue at $10,000$ units minus the instantaneous rate of change of cost at $10,000$ units.

■ **Example 14** An office super store sells a laptop stylus pen. The price-demand function for the stylus pen is $p(x) = -0.2x+55$, where $p(x)$ is the price, in dollars, per stylus pen when x stylus pens are purchased. Find the instantaneous rate of change of revenue when the super store sells 225 stylus pens, and interpret your answer.

Solution:
First, we need to find the revenue function:

$$\begin{aligned}R(x) &= x\cdot p(x)\\ &= x(-0.2x+55)\\ &= -0.2x^2+55x\end{aligned}$$

Now that we have the revenue function, $R(x) = -0.2x^2 + 55x$ dollars, where x is the number of stylus pens sold, we can find the instantaneous rate of change of the function R at $x = 225$ by letting $a = 225$ in the limit definition:

$$\lim_{h\to 0} \frac{R(225+h) - R(225)}{h}$$

Again, we will use the Suggested Four-Step Process:

1. Calculate $R(225+h)$, and algebraically manipulate the resulting expression, if possible:

$$R(225+h) = -0.2(225+h)^2 + 55(225+h)$$

Using FOIL and distributing gives

$$\begin{aligned}
&= -.02\left(50,625 + 450h + h^2\right) + 12,375 + 55h \\
&= -10,125 - 90h - 0.2h^2 + 12,375 + 55h \\
&= 2250 - 35h - 0.2h^2
\end{aligned}$$

2. Calculate the numerator of the difference quotient, $R(225+h) - R(225)$, and algebraically manipulate the resulting expression, if possible:

We start with $R(225)$:

$$R(225) = -0.2(225)^2 + 55(225) = 2250$$

So in the numerator of the difference quotient, we have

$$\begin{aligned}
R(225+h) - R(225) &= 2250 - 35h - 0.2h^2 - 2250 \\
&= -35h - 0.2h^2
\end{aligned}$$

3. Calculate the difference quotient (i.e., divide the result from the previous step by h, where $h \neq 0$), and algebraically manipulate the resulting expression:

$$\frac{R(225+h) - R(225)}{h} = \frac{-35h - 0.2h^2}{h}$$

Factoring an h from the numerator gives

$$= \frac{h(-35 - 0.2h)}{h}$$

Dividing the h's gives

$$\begin{aligned}
&= \frac{\not{h}(-35 - 0.2h)}{\not{h}} \\
&= -35 - 0.2h
\end{aligned}$$

4. Take the limit as $h \to 0$ (i.e., substitute 0 for h):

$$\begin{aligned}
\lim_{h\to 0} \frac{R(225+h) - R(225)}{h} &= \lim_{h\to 0}(-35 - 0.2h) \\
&= -35 - 0.2(0) \\
&= -\$35 \text{ per stylus pen}
\end{aligned}$$

Thus, revenue is decreasing at a rate of \$35 per stylus pen. To interpret our answer, we write a sentence:

"When 225 stylus pens are sold, revenue is decreasing at a rate of \$35 per stylus pen."

▪

Try It # 11:
If the cost function for the stylus pens in the previous example is given by $C(x) = 15x + 60$ dollars, where x is the number of stylus pens produced, find the instantaneous rate of change of profit when 225 stylus pens are produced and sold. Interpret your answer.

Finding the Equation of the Tangent Line

Recall that to find the equation of a line, we need two pieces of information: a point the line passes through and the slope of the line. We saw earlier in this section that we can use the limit definition of the instantaneous rate of change to find the slope of the line tangent to the graph of f at $x = a$. Thus, in order to find the equation of the line tangent to the graph of f at $x = a$, the only question we have left to answer is, "Where do we get the y-value of the point?"

We know that the tangent line passes through the graph of f at $x = a$. Thus, the y-value of the point can be found by calculating $f(a)$. In other words, the tangent line passes through the point $(a, f(a))$. Once we have the point and the slope of the tangent line, we can find the equation of the tangent line using either the point-slope form or the slope-intercept form of the equation of a line.

▪ **Example 15** For each of the following functions, find the equation of the line tangent to the graph of the function at the indicated value of x, and graph the function and its tangent line on the same axes.

a. $f(x) = x^2$ at $x = 2$

b. $f(x) = \sqrt{x-6} + 3$ at $x = 15$

c. $f(x) = \dfrac{2}{3x-3}$ at $x = 4$

Solution:

a. We start by calculating $f(2)$ to find the y-value of the point:

$$f(2) = 2^2 = 4$$

Thus, the tangent line passes through the point $(2, 4)$.

Now, we use the limit definition of the instantaneous rate of change at $x = 2$, $\lim_{h \to 0} \dfrac{f(2+h) - f(2)}{h}$, to find the slope of the tangent line at $x = 2$. Once again, we use the Suggested Four-Step Process:

1. Calculate $f(2+h)$, and algebraically manipulate the resulting expression, if possible:

$$\begin{aligned} f(2+h) &= (2+h)^2 \\ &= 4 + 4h + h^2 \end{aligned}$$

2. Calculate the numerator of the difference quotient, $f(2+h) - f(2)$, and algebraically manipulate the resulting expression, if possible:

We found above that $f(2) = 4$, so

$$\begin{aligned} f(2+h) - f(2) &= 4 + 4h + h^2 - 4 \\ &= 4h + h^2 \end{aligned}$$

3. Calculate the difference quotient (i.e., divide the result from the previous step by h, where $h \neq 0$), and algebraically manipulate the resulting expression:

$$\frac{f(2+h)-f(2)}{h} = \frac{4h+h^2}{h}$$

Factoring an h from the numerator gives

$$= \frac{h(4+h)}{h}$$

Dividing the h's gives

$$\begin{aligned} &= \frac{\not{h}(4+h)}{\not{h}} \\ &= 4+h \end{aligned}$$

4. Take the limit as $h \to 0$ (i.e., substitute 0 for h):

$$\begin{aligned} \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} &= \lim_{h\to 0}(4+h) \\ &= 4+0 \\ &= 4 \end{aligned}$$

Now, we know the tangent line passes through the point $(2,4)$ and has a slope of 4 (i.e., $m = 4$). To find the equation of the tangent line, we can either use the point-slope form of the equation of a line or the slope-intercept form of the equation of a line. In this example, we will demonstrate both.

The point-slope form of the equation of a line that passes through the point (x_1, y_1) and has a slope of m is given by

$$y - y_1 = m(x - x_1)$$

Substituting the point $(2,4)$ and slope of the tangent line $m = 4$ gives

$$y - 4 = 4(x-2)$$

This is an equation for the tangent line at $x = 2$. Oftentimes, we need the equation of the line written in slope-intercept form: $y = mx + b$, where m is the slope and b is the y-intercept. We can achieve this by solving the above equation for y:

$$\begin{aligned} y - 4 &= 4(x-2) \\ y &= 4x - 8 + 4 \\ &= 4x - 4 \end{aligned}$$

Thus, the equation of the line tangent to the graph of f at $x = 2$ is given by $y = 4x - 4$.

Instead, we could choose to use the slope-intercept form to find the equation of the tangent line from the beginning. Substituting the point (for x and y) and slope, we can solve for b:

$$\begin{aligned} y &= mx + b \implies \\ 4 &= 4(2) + b \\ -4 &= b \end{aligned}$$

Thus, because the slope is 4 and the y-intercept is -4, the equation of the tangent line is

$$y = 4x - 4$$

Notice that both methods lead to the same tangent line! For the remainder of this section, we will use the point-slope form to find the equation of a line and then solve for y so it is ultimately in slope-intercept form. You should use whichever method you feel more comfortable with.

The graphs of $f(x) = x^2$ and the line tangent to the graph of f at $x = 2$ (i.e., $y = 4x - 4$) are shown in **Figure 2.1.11**.

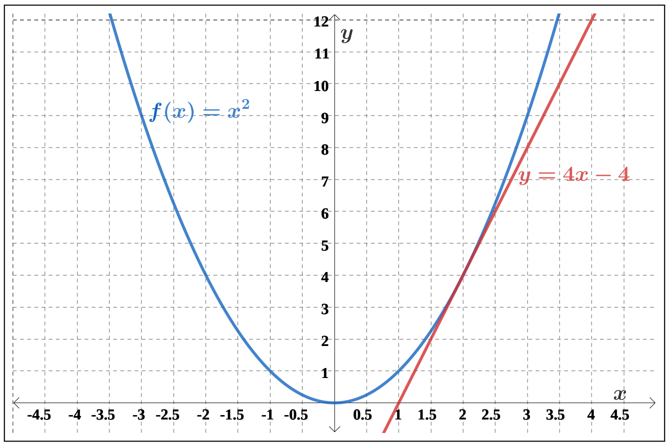


Figure 2.1.11: Graphs of $f(x) = x^2$ and $y = 4x - 4$

b. Recall that we must find the equation of the line tangent to the graph of $f(x) = \sqrt{x-6} + 3$ at $x = 15$. We will start by evaluating the function, $f(x) = \sqrt{x-6} + 3$, at $x = 15$ to find the y-value of the point:

$$\begin{aligned} f(15) &= \sqrt{15-6} + 3 \\ &= 3 + 3 \\ &= 6 \end{aligned}$$

Thus, the tangent line passes through the point $(15, 6)$.

Now, we use the limit definition of the instantaneous rate of change at $x = 15$, $\lim\limits_{h \to 0} \dfrac{f(15+h) - f(15)}{h}$, to find the slope of the tangent line. Once again, we use the Suggested Four-Step Process:

1. Calculate $f(15+h)$, and algebraically manipulate the resulting expression, if possible:

$$\begin{aligned} f(15+h) &= \sqrt{(15+h) - 6} + 3 \\ &= \sqrt{9+h} + 3 \end{aligned}$$

2. Calculate the numerator of the difference quotient, $f(15+h) - f(15)$, and algebraically manipulate the resulting expression, if possible:

Again, we already have $f(15) = 6$:

$$\begin{aligned} f(15+h) - f(15) &= \sqrt{9+h} + 3 - 6 \\ &= \sqrt{9+h} - 3 \end{aligned}$$

3. Calculate the difference quotient (i.e., divide the result from the previous step by h, where $h \neq 0$), and algebraically manipulate the resulting expression:

$$\frac{f(15+h)-f(15)}{h} = \frac{\sqrt{9+h}-3}{h}$$

Multiplying by the conjugate gives

$$\begin{aligned}\frac{\sqrt{9+h}-3}{h} &= \left(\frac{\sqrt{9+h}-3}{h}\right)\left(\frac{\sqrt{9+h}+3}{\sqrt{9+h}+3}\right)\\ &= \frac{(9+h)-9}{h\left(\sqrt{9+h}+3\right)}\\ &= \frac{h}{h\left(\sqrt{9+h}+3\right)}\end{aligned}$$

Dividing the h's gives

$$\begin{aligned}&= \frac{\cancel{h}}{\cancel{h}\left(\sqrt{9+h}+3\right)}\\ &= \frac{1}{\sqrt{9+h}+3}\end{aligned}$$

4. Take the limit as $h \to 0$ (i.e., substitute 0 for h):

$$\begin{aligned}\lim_{h\to 0}\frac{f(15+h)-f(15)}{h} &= \lim_{h\to 0}\frac{1}{\sqrt{9+h}+3}\\ &= \frac{1}{\sqrt{9+0}+3}\\ &= \frac{1}{3+3}\\ &= \frac{1}{6}\end{aligned}$$

Now, we know that the tangent line passes through the point $(15,6)$ and has slope $m = \frac{1}{6}$. To find the equation of the line, we will use point-slope form:

$$\begin{aligned}y-y_1 = m(x-x_1) &\implies\\ y-6 = \frac{1}{6}(x-15)&\end{aligned}$$

Solving for y to get the equation in a more useful form (i.e., slope-intercept form), we have

$$\begin{aligned}y &= \frac{1}{6}x-\frac{5}{2}+6\\ &= \frac{1}{6}x+\frac{7}{2}\end{aligned}$$

Thus, the equation of the line tangent to the graph of f at $x = 15$ is given by $y = \frac{1}{6}x+\frac{7}{2}$.

The graphs of $f(x) = \sqrt{x-6}+3$ and the line tangent to the graph of f at $x = 15$ (i.e., $y = \frac{1}{6}x+\frac{7}{2}$) are shown in **Figure 2.1.12**.

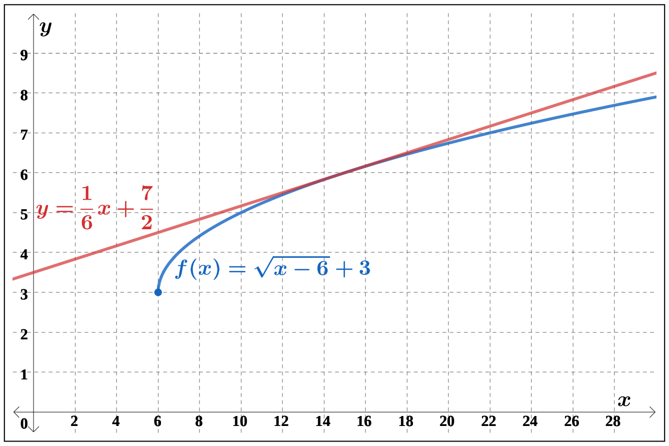


Figure 2.1.12: Graphs of $f(x) = \sqrt{x-6}+3$ and $y = \frac{1}{6}x + \frac{7}{2}$

c. Recall that we must find the equation of the line tangent to the graph of $f(x) = \dfrac{2}{3x-3}$ at $x = 4$. We will start by evaluating the function, $f(x) = \dfrac{2}{3x-3}$, at $x = 4$ to find the y-value of the point:

$$\begin{aligned} f(4) &= \frac{2}{3(4)-3} \\ &= \frac{2}{9} \end{aligned}$$

Thus, the tangent line passes through the point $\left(4, \frac{2}{9}\right)$.

Now, we use the limit definition of the instantaneous rate of change at $x = 4$, $\lim\limits_{h \to 0} \dfrac{f(4+h) - f(4)}{h}$, to find the slope of the tangent line.

As you can probably guess, we will use the Suggested Four-Step Process:

1. Calculate $f(4+h)$, and algebraically manipulate the resulting expression, if possible:

$$\begin{aligned} f(4+h) &= \frac{2}{3(4+h)-3} \\ &= \frac{2}{12+3h-3} \\ &= \frac{2}{9+3h} \end{aligned}$$

2. Calculate the numerator of the difference quotient, $f(4+h) - f(4)$, and algebraically manipulate the resulting expression, if possible:

$$f(4+h) - f(4) = \frac{2}{9+3h} - \frac{2}{9}$$

To algebraically manipulate this expression further, we will get a common denominator:

$$\frac{2}{9+3h}-\frac{2}{9}=\left(\frac{9}{9}\right)\left(\frac{2}{9+3h}\right)-\left(\frac{2}{9}\right)\left(\frac{9+3h}{9+3h}\right)$$

$$=\frac{18}{9(9+3h)}-\frac{2(9+3h)}{9(9+3h)}$$

$$=\frac{18-2(9+3h)}{9(9+3h)}$$

$$=\frac{18-18-6h}{9(9+3h)}$$

$$=\frac{-6h}{9(9+3h)}$$

3. Calculate the difference quotient (i.e., divide the result from the previous step by h, where $h \neq 0$), and algebraically manipulate the resulting expression:

$$\frac{f(4+h)-f(4)}{h}=\frac{\frac{-6h}{9(9+3h)}}{h}$$

$$=\frac{\frac{-6h}{9(9+3h)}}{\frac{h}{1}}$$

$$=\left(\frac{-6h}{9(9+3h)}\right)\left(\frac{1}{h}\right)$$

$$=\left(\frac{-6\cancel{h}}{9(9+3h)}\right)\left(\frac{1}{\cancel{h}}\right)$$

$$=\frac{-6}{9(9+3h)}$$

N *Notice when we divided by h in this step, we created a complex fraction because the numerator contained a fraction. We saw this in a previous example and stated that anytime we have this situation, we will rewrite the division by h as multiplication by* $\frac{1}{h}$.

4. Take the limit as $h \to 0$ (i.e., substitute 0 for h):

$$\begin{aligned}\lim_{h\to 0}\frac{f(4+h)-f(4)}{h} &= \lim_{h\to 0}\frac{-6}{9(9+3h)}\\ &= \frac{-6}{9(9+3(0))}\\ &= \frac{-6}{81}\\ &= -\frac{2}{27}\end{aligned}$$

Now, we know that the tangent line passes through the point $\left(4, \frac{2}{9}\right)$ and has slope $m = -\frac{2}{27}$. Using the point-slope form to find the equation of the line gives

$$y - \frac{2}{9} = -\frac{2}{27}(x-4)$$

Or, in slope-intercept form, we have

$$y = -\frac{2}{27}x - \frac{8}{27} + \frac{2}{9}$$

$$= -\frac{2}{27}x + \frac{14}{27}$$

Thus, the equation of the line tangent to the graph of f at $x = 4$ is given by $y = -\frac{2}{27}x + \frac{14}{27}$.

In **Figure 2.1.13**, the function and the tangent line are graphed on the same set of axes.

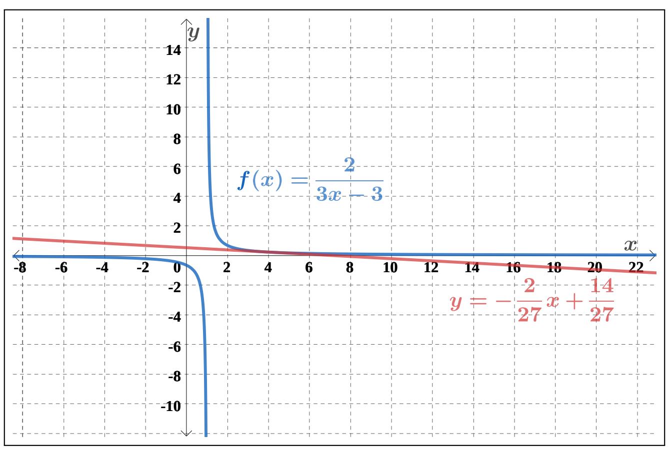


Figure 2.1.13: Graphs of $f(x) = \frac{2}{3x-3}$ and $y = -\frac{2}{27}x + \frac{14}{27}$

■

Try It # 12:
Find the equation of the line tangent to the graph of $f(x) = x^2 + 12x - 22$ at the point $(-2, -42)$.

We learned (and used) a lot of new terms in this section! To help summarize the important terminology (thus far), we have included the following for your reference:

The following are equivalent:	**The following are equivalent:**
Slope of the Secant Line	Slope of the Tangent Line
Average Rate of Change	Limit of the Slopes of Secant Lines
Difference Quotient	Limit of the Difference Quotient
Average Velocity	Instantaneous Rate of Change (or Rate of Change)
	Instantaneous Velocity (or Velocity)

Try It Answers

1. **a.** trout per day
   b. gallons per second
   c. seconds per gallons
   d. test points per study hour
2. −\$800 per plush; When the number of plushes made is between 600 and 1200 plushes, cost is decreasing at an average rate of \$800 per plush.
3. **a.** $-23/120$ BPM per second
   b. $-1/30$ BPM per second
4. **a.** **i.** −\$1.57 per coffee cup
   ii. −\$1.58 per coffee cup
   iii. −\$1.59 per coffee cup
   iv. −\$1.63 per coffee cup
   v. −\$1.62 per coffee cup
   vi. −\$1.61 per coffee cup
   b. −\$1.60 per coffee cup; When 530 coffee cups are sold, revenue is decreasing at a rate of \$1.60 per coffee cup.
5. **a.** -67.75
   b. -73.51
   c. -74.8501
   d. -75
6. 4

7. $1/6$

8. $-1/8$

9. -32 feet per second

10. Instantaneous Rate of Change of Cost: \$100 per Swap; When 10,000 Swaps are produced, cost is increasing at a rate of \$100 per Swap.

    Instantaneous Rate of Change of Revenue: \$200 per Swap; When 10,000 Swaps are sold, revenue is increasing at a rate of \$200 per Swap.

11. −\$50 per stylus pen; When 225 stylus pens are produced and sold, profit is decreasing at a rate of \$50 per stylus pen.

12. $y = 8x - 26$

EXERCISES

Basic Skills Practice

1. Use the table of function values below to answer each of the following.

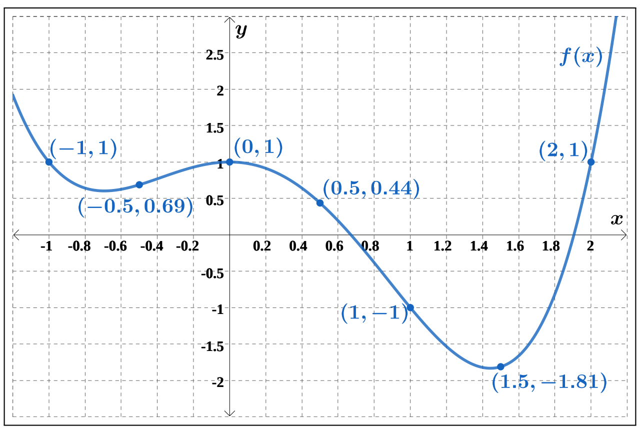
x	-8	-6	-4	-2	0	2	4	6	8	10	12
$f(x)$	-6	-17	-18	16	-17	12	22	3	0	3	26

   (a) Find the average rate of change of f on the interval $[-8, 12]$.

   (b) Find the average rate of change of f on the interval $[-6, 0]$.

   (c) Find the slope of the secant line passing through the points at $x = 4$ and $x = 10$.

2. Use the graph of f below to answer each of the following. Round your answers to two decimal places, if necessary.



   (a) Find the slope of the secant line passing through the points at $x = 0$ and $x = 2$.

   (b) Find the average rate of change of g on the interval $[-0.5, 0.5]$.

   (c) Find the slope of the secant line passing through the points at $x = 1.5$ and $x = 2$.

For Exercises 3 - 5, find the average rate of change of the function on the given interval. Round your answer to three decimal places, if necessary.

3. $y = 4x + 16$ on the interval $[-4, 1]$

4. $g(x) = -x^2 + 3x - 1$ on the interval $[5, 7]$

5. $f(t) = \dfrac{e^t}{t^2}$ on the interval $[-2, 2]$

For Exercises 6 - 8, find the instantaneous rate of change of the function at the given x-value.

6. $y = -8x - 12$ at $x = 2$

7. $y = x^2 - 4$ at $x = 4$

8. $f(x) = -x^2 + x + 5$ at $x = 0$

For Exercises 9 and 10, find the slope of the line tangent to the graph of f at the given x-value.

9. $f(x) = 7x$ at $x = 6$

10. $f(x) = x^2 - 2x + 4$ at $x = 0$

For Exercises 11 - 13, the instantaneous rate of change and the value of $f(x)$ at a particular x-value are given. Use this information to find the equation of the line tangent to the graph of f at the given x-value.

11. The instantaneous rate of change at $x = 2$ is 3, and $f(2) = 7$.

12. The instantaneous rate of change at $x = 14$ is -7, and $f(14) = 0$.

13. The instantaneous rate of change at $x = -5$ is 0, and $f(-5) = 9$.

14. Given the units for x and the units for $f(x)$ in the table below, find the units for the average rate of change of f.

	Units for x	**Units for $f(x)$**
a.	books	dollars
b.	seconds	hit points
c.	gallons	miles
d.	dollars	donuts

15. A small fast-food restaurant specializing in selling chicken sandwiches has a weekly price-demand function given by $p(x) = 12.5 - 0.01x$, where x is the number of chicken sandwiches sold at a price of $p(x)$ dollars each.
    (a) Find the average rate of change of price when the number of chicken sandwiches sold each week is between 800 and 900.
    (b) Find the instantaneous rate of change of price when 700 chicken sandwiches are sold each week.

16. The restaurant referred to in the previous exercise has a weekly revenue function given by $R(x) = 12.5x - 0.01x^2$, where $R(x)$ is the revenue, in dollars, from selling x chicken sandwiches.
    (a) Find the average rate of change of revenue when the number of chicken sandwiches sold each week is between 900 and 1000.
    (b) Find the instantaneous rate of change of revenue when 800 chicken sandwiches are sold each week.

17. Brown & Co. sells flux capacitors. They have determined their weekly cost function to be $C(x) = 75x + 50,000$ dollars, where x is the number of flux capacitors produced each week.

    (a) Find the average rate of change of cost when the number of flux capacitors produced each week is between 125 and 175.

    (b) Find the instantaneous rate of change of cost when 150 flux capacitors are produced each week.

18. The position of an object moving along a horizontal line after t seconds is given by $s(t) = 4t + 2$ feet.

    (a) Find the average velocity of the object between 2 and 6 seconds.

    (b) Find the instantaneous velocity of the object after 8 seconds.

Intermediate Skills Practice

19. When x of Magnus's hand carved wooden ducks are sold, it costs him $C(x) = 0.008x^2 + 0.261x + 159.2$ dollars to make them.

    (a) What is the average rate of change of cost when the first 350 hand carved ducks are made? Interpret your answer.

    (b) What is the average rate of change of cost when the number of hand carved ducks made is between 500 and 700? Interpret your answer.

20. The position of an experimental supersonic car driving down a straight track is shown in the table below, where t is measured in seconds since starting the engine and $f(t)$ is measured in feet from the starting position.

t	0	10	20	30	40	50	60
$f(t)$	0	7130	18,113	27,213	39,642	50,935	62,818

    (a) Find the average velocity of the car over the minute measured. Round your answer to two decimal places, if necessary.

    (b) Find the average velocity of the car from time $t = 20$ to $t = 40$ seconds. Round your answer to two decimal places, if necessary.

21. The table below gives function values of $g(x)$ for certain values of x.

x	-4	-3	-2	-1	0	1	2
$g(x)$	9	3	0	-1	0	7	26

    (a) Find the slope of the secant line on the following intervals.

        i. $[-4, -1]$
        ii. $[-3, -1]$
        iii. $[-2, -1]$
        iv. $[-1, 2]$
        v. $[-1, 1]$
        vi. $[-1, 0]$

    (b) Use part (a) to approximate the slope of the line tangent to the graph of g at $x = -1$.

For exercises 22 - 24, (a) find the average rate of change of the function on the given intervals, and (b) use that information to estimate the instantaneous rate of change of the function at the given x-value, if it exists. Round your answers to three decimal places, if necessary.

22. $f(x) = x^3 + 5x$
    (a) i. $[1,2]$
        ii. $[1.5,2]$
        iii. $[1.9,2]$
        iv. $[2,3]$
        v. $[2,2.5]$
        vi. $[2,2.1]$
    (b) $x = 2$

23. $f(x) = \sqrt[3]{x-2} + 7$
    (a) i. $[7,8]$
        ii. $[7.5,8]$
        iii. $[7.9,8]$
        iv. $[8,9]$
        v. $[8,8.5]$
        vi. $[8,8.1]$
    (b) $x = 8$

24. $f(x) = \dfrac{3x}{x^2+4}$
    (a) i. $[-0.15,0]$
        ii. $[-0.1,0]$
        iii. $[-0.05,0]$
        iv. $[0,0.15]$
        v. $[0,0.1]$
        vi. $[0,0.05]$
    (b) $x = 0$

25. The profit for Ugly Mug brand coffee cups when x coffee cups are manufactured and sold is $P(x)$ dollars. The table below gives some values of $P(x)$.

x	720	721	722	723	724	725	726
$P(x)$	\$815.89	\$812.34	\$810.60	\$810.18	\$809.57	\$807.94	\$806.29

(a) Find the average rate of change of profit on the following intervals.

i. $[720, 723]$

ii. $[721, 723]$

iii. $[722, 723]$

iv. $[723, 726]$

v. $[723, 725]$

vi. $[723, 724]$

(b) Use part (a) to approximate the instantaneous rate of change of profit when 723 coffee cups are sold.

26. The daily cost function for Stinking in the Rain wet active deodorant for rainy days is given by $C(x) = 10\sqrt{x} + 20$, where $C(x)$ is the cost, in dollars, of making x deodorant sticks.

(a) Find the average rate of change of cost on the following intervals. Round your answers to three decimal places, if necessary.

i. $[97, 100]$

ii. $[98, 100]$

iii. $[99, 100]$

iv. $[100, 103]$

v. $[100, 102]$

vi. $[100, 101]$

(b) Use part (a) to approximate the instantaneous rate of change of cost when 100 sticks of deodorant are made. Round your answer to two decimal places, if necessary.

27. Barnabus Noble's Bookstore has a weekly revenue function given by $R(x) = 50x \cdot (0.958)^x$ dollars when x books are sold.

(a) Find the average rate of change of the weekly revenue on the following intervals.

i. $[27, 30]$

ii. $[28, 30]$

iii. $[29, 30]$

iv. $[30, 33]$

v. $[30, 32]$

vi. $[30, 31]$

(b) Use part (a) to approximate the instantaneous rate of change of the weekly revenue when 30 books are sold.

28. A laboratory experiment measures a runner's velocity for 15 seconds. The scientists have determined their velocity to be $f(t) = 4t - \frac{4t^2}{15}$ meters per second, where t is time in seconds.
    (a) Find the average rate of change of the runner's velocity on the following intervals. Round your answers to three decimal places, if necessary.
        i. $[5, 6]$
        ii. $[5.5, 6]$
        iii. $[5.9, 6]$
        iv. $[6, 7]$
        v. $[6, 6.5]$
        vi. $[6, 6.1]$
    (b) Use part (a) to approximate the runner's instantaneous acceleration (i.e., instantaneous rate of change of velocity) after six seconds have passed. Round your answer to one decimal place, if necessary.

For Exercises 29 - 31 , compute the difference quotient for the function at the given x-value using the indicated value of h. Round your answer to four decimal places, if necessary.

29. $f(x) = 2x + 4$ at $x = 3$ with $h = 0.2$

30. $f(x) = 5 - x^2$ at $x = -2$ with $h = 0.1$

31. $f(x) = \sqrt{3 + x}$ at $x = 1$ with $h = 0.1$

For Exercises 32 - 34, find the instantaneous rate of change of the function at the given x-value.

32. $f(x) = \frac{4}{3x}$ at $x = 2$

33. $g(x) = \frac{11}{5x + 5}$ at $x = 1$

34. $h(x) = \sqrt{16x}$ at $x = 4$

For Exercises 35 - 37, find the slope of the line tangent to the graph of the function at the given x-value.

35. $g(x) = \frac{8x}{x - 2}$ at $x = 10$

36. $f(x) = 3\sqrt{x} + 5$ at $x = 9$

37. $h(x) = \sqrt{x + 13}$ at $x = 3$

For Exercises 38 - 40, find the equation of the line tangent to the graph of the function at the given x-value.

38. $g(x) = x^2 - 4$ at $x = 6$

39. $h(x) = -x^2 + 3x - 1$ at $x = -3$

40. $f(x) = \dfrac{7}{x-3}$ at $x = 2$

41. The amount of sales for Walter Green's pharmacy is given by $S(t) = 22t + 0.3t^2$ thousand dollars after t months. What is the instantaneous rate of change of sales after one year? Interpret your answer.

42. A car's value depreciates according to the function $V(t) = 20{,}000 - 400t^2$, where t represents the number of years since the car was purchased and $V(t)$ is its value in dollars. Find the instantaneous rate of change of the car's value seven years after it is purchased, and interpret your answer.

43. The profit function for a ghost busting business is given by $P(x) = -x^2 + 100x - 2000$ thousand dollars when x ghosts are busted. What is the instantaneous rate of change of profit when 40 ghosts are busted? Interpret your answer.

44. The position of a car relative to an observer t seconds after they notice the car is given by $f(t) = t^2 - 16t + 85$ feet. Find the instantaneous velocity of the car 12 seconds after the observer notices the car.

Mastery Practice

45. Given $g(x) = 3x^2 - 1$, find the slope of the secant line passing through the points at $x = 4$ and $x = 4 + h$.

46. Given the units for x and the units for $f(x)$ in the table below, find the units for the instantaneous rate of change of f.

	Units for x	**Units for $f(x)$**
a.	months	blizzards
b.	dollars	yo-yos
c.	hours	puzzles
d.	bottles of glue	slime balls

47. The position, in feet, of an object traveling along a horizontal line after t seconds is shown in the table below. Find the average velocity of the object between 3 and 9 seconds. Round your answer to two decimal places, if necessary.

t	0	1	3	4	6	9	12
$f(t)$	0	2.5	4.4	5.8	7.4	8.5	11.3

48. The table below gives function values of $g(x)$ for certain values of x. Approximate the slope of the line tangent to the graph of g at $x = -0.1$. Round your answer to one decimal place, if necessary.

x	−0.3	−0.2	−0.1	0	0.1	0.2	0.3	0.4
$g(x)$	0.09	0.04	0.01	0	0.01	0.04	0.09	0.16

49. A company has a revenue function given by $R(x) = 20x \cdot (0.997)^x$ dollars when x items are sold. Approximate the instantaneous rate of change of revenue when 100 items are sold, and interpret your answer.

50. The position, in meters, of an object after t seconds is given by $s(t) = 2\sqrt{t+1} - 2$. Approximate the object's velocity after 15 seconds.

For Exercises 51 - 53, compute the difference quotient for the function at the given x-value using the indicated value of h. Round your answer to four decimal places, if necessary.

51. $f(x) = -x^3 + x$ at $x = 1$ with $h = 0.1$

52. $f(x) = \dfrac{x}{x^2 + 1}$ at $x = 3$ with $h = 0.02$

53. $f(x) = x + \sqrt{8 - x}$ at $x = -1$ with $h = 0.01$

For Exercises 54 - 58, find the instantaneous rate of change of the function at the given value.

54. $f(x) = -7x + 8$ at $x = 14$

55. $f(x) = -2x^2 + 3x + 12$ at $x = -1$

56. $g(x) = x^3 - 2x^2 + 1$ at $x = -2$

57. $g(t) = 2\sqrt{5 - t}$ at $t = 4$

58. $h(x) = \dfrac{8 - x}{8 + x}$ at $x = -9$

For Exercises 59 and 60, find the equation of the line tangent to the graph of the function at the given value.

59. $f(t) = \dfrac{4 + 3t}{t^2 - 2t}$ at $t = 3$

60. $g(x) = \sqrt{8x + 9} - 2x$ at $x = 2$

61. A company has a revenue function given by $R(x) = 50xe^{-.002x}$ dollars, where x is the number of items sold. Find and interpret the slope of the secant line of the graph of this function passing through the points at $x = 600$ and $x = 675$.

62. Suppose that the profit, in dollars, obtained from the sale of x fish-fry dinners each week is given by $P(x) = -0.03x^2 + 6x - 50$. Find and interpret the slope of the line tangent to the graph of this function at
    (a) $x = 75$.
    (b) $x = 100$.
    (c) $x = 135$.

63. The height of a ball thrown upward is given by $s(t) = -18t^2 + 150t$ feet, where t is time in seconds. Find the slope of the line tangent to the graph of this function at $t = 15$ seconds.

64. The cost of producing x cameras each week is given by $C(x) = 600 + 100\sqrt{x}$ dollars.

    (a) Find and interpret the average rate of change of cost when the number of cameras produced each week is between 20 and 25.

    (b) Find and interpret the average rate of change of cost when the first 15 cameras are produced each week.

    (c) Find and interpret the instantaneous rate of change of cost when 17 cameras are produced each week.

65. The position of a car relative to an observer t seconds after they notice the car is given by $f(t) = t^2 - 16t + 80$ feet. Find the average velocity of the car between 2 and 5 seconds after the observer notices the car.

66. A company's profit when x gift baskets are made and sold is $P(x)$ dollars. If $\dfrac{P(35+h) - P(35)}{h} = 10 - h$, find the instantaneous rate of change of profit when 35 gift baskets are made and sold, and interpret your answer.

67. The equation of the line tangent to the graph of f at the point $(2, 7)$ is $y = -3x + 13$. Find $\lim\limits_{h \to 0} \dfrac{f(2+h) - 7}{h}$.

68. (a) Find the average rate of change of $f(x) = |4x + 8|$ on each of the following intervals. Round your answers to three decimal places, if necessary.

    i. $[-2.1, -2]$

    ii. $[-2.01, -2]$

    iii. $[-2.001, -2]$

    iv. $[-2, -1.9]$

    v. $[-2, -1.99]$

    vi. $[-2, -1.999]$

    (b) Use part (a) to approximate the instantaneous rate of change of f, if it exists, at $x = -2$.

69. Given $f(x) = x^2$, find (a) the slope of the line tangent to the graph of f at $x = a$, where a is any real number, and (b) the equation of the line tangent to the graph of f at the point $(a, f(a))$.

Communication Practice

70. Briefly explain the difference between slopes of secant and tangent lines.

71. Are the terms "average rate of change" and "slope of the tangent line" equivalent? Why or why not?

72. If $R(x)$ is the revenue of a company, in dollars, when it sells x items per week, interpret $\dfrac{R(500) - R(450)}{50} = 20{,}000$.

73. If $C(x)$ is the cost of a company, in dollars, when it makes x items per week, interpret $\lim_{h \to 0} \frac{C(800+h)-C(800)}{h} = 650$.

74. If the average rate of change of revenue when a company sells between 100 and 125 items each week is \$85/item, does that mean when the company sells 150 items the revenue will increase on average by \$85 if it sells one more item? Explain.

2.2 The Limit Definition of the Derivative

In **Section 2.1, Example 13**, we found the profit function, P, for producing and selling x Ninvento Swaps was $P(x) = -0.01x^2 + 300x - 10,000$ dollars. In that example, we found that when 10,000 Swaps were produced and sold, the instantaneous rate of change of profit was \$100 per Swap. To calculate this value, we used the limit definition of the instantaneous rate of change, which was quite a tedious process even using the Suggested Four-Step Process.

What if you were asked to repeat this calculation to find the instantaneous rate of change of profit when 12,000 Swaps are produced and sold? You would need to go through the entire tedious process again. What if you also needed this information for 15,000, 20,000, and 23,000 Swaps? We have summarized the results if you repeated this tedious process four more times in **Table 2.7**. **Figure 2.2.1** shows what a graph would look like if you also plotted the points in **Table 2.7**, where x is the number of Swaps and y is the instantaneous rate of change of profit (in \$ per Swap).

x	Instantaneous rate of change of $P(x)$ (in \$ per swap)
10,000	100
12,000	60
15,000	0
20,000	−100
23,000	−160

Table 2.7: Instantaneous rate of change of profit for five x-values

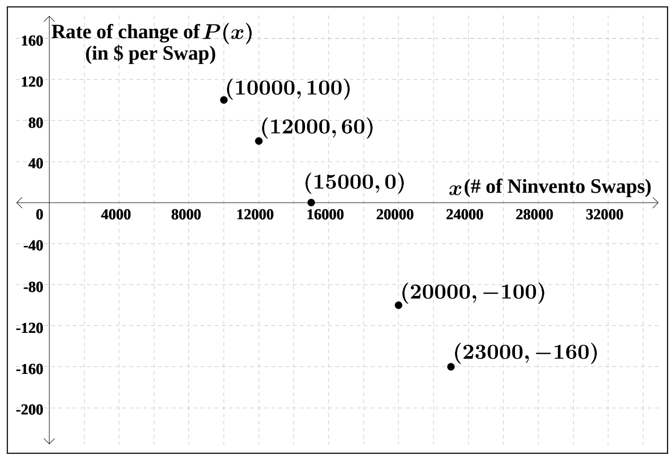


Figure 2.2.1: Graph showing the instantaneous rate of change of profit at five points

Notice in **Figure 2.2.1** that the graph of a function is starting to appear. If we keep plotting points, it looks as though they will all lie on the graph of a linear function. This function is known as the **derivative** of P, and its name is P' (read "P prime"). In this section, we will work to find a rule for the derivative, $P'(x)$ (read "P prime of x"), which will involve going through the Suggested Four-Step Process only once! Then, we can then easily substitute different values of x into $P'(x)$ to find their corresponding instantaneous rates of change, saving us a lot of energy.

Learning Objectives:

In this section, you will learn how to calculate the derivative using the limit definition, solve problems involving real-world applications using the derivative, determine where the derivative does not exist, and sketch a graph of the derivative function. Upon completion you will be able to:

- Interpret the derivative as a function with outputs being the slopes of tangent lines of the original function.
- Find the derivative function using the limit definition of the derivative.
- Calculate the slope of a tangent line using the limit definition of the derivative.
- Find the equation of a tangent line using the limit definition of the derivative.
- Determine the x-value(s) where the graph of a function has a horizontal tangent line using the limit definition of the derivative.

- Determine the x-value(s) where the graph of a function has a tangent line of a given slope using the limit definition of the derivative.
- Calculate the instantaneous rate of change of a function involving a real-world scenario, including cost, revenue, profit, and position, using the limit definition of the derivative.
- Interpret the meaning of the derivative of a function involving a real-world scenario, including cost, revenue, and profit.
- Determine the domain of a derivative function.
- Determine where a function is nondifferentiable given its graph.
- Justify where a function is nondifferentiable using calculus.
- Sketch a graph of the derivative of a function given a graph of the original function.

The Derivative as a Function

In general, for a function f, the **derivative function** is a function, which we call f' (read "f prime"), whose y-values (outputs) are the instantaneous rates of change (i.e., slopes of the tangent lines) of f at the x-value(s) we input.

In the previous section, we found the instantaneous rate of change, or slope of the tangent line, of a function f at a particular x-value, $x = a$. See **Figure 2.2.2**.

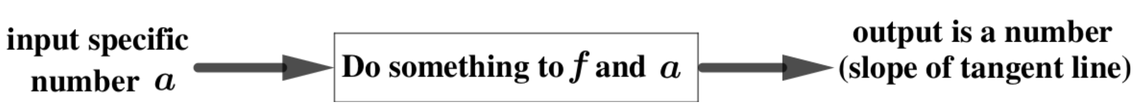


Figure 2.2.2: Process of finding the instantaneous rate of change of f at $x = a$

Our approach in this section will be to evaluate a similar limit definition. The only difference is that instead of having a specific number a in the limit definition, we will have the variable x. Thus, instead of our result being a specific number, our result will be a function of x. This is summarized in **Figure 2.2.3**.

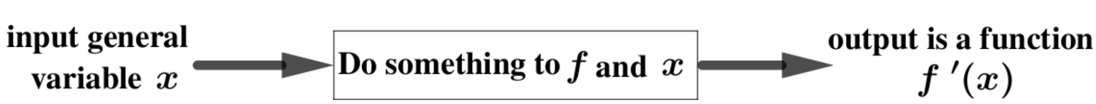


Figure 2.2.3: Process of finding the derivative function $f'(x)$

This approach is more efficient if we need multiple instantaneous rates of changes, or slopes of tangent lines. We front-load our work, and then we can substitute any specific value for x into the derivative function we created to get the corresponding instantaneous rates of change we need.

We formally define the derivative function as follows:

Definition
Let f be a function. The **derivative function**, denoted f', is the function whose domain consists of those values of x such that the following limit exists:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

The derivative function f' gives the instantaneous rate of change of the function f at each point in the domain of f in which the above limit exists (a concern we will address later in this section).

Example 1 Given $P(x) = -0.01x^2 + 300x - 10,000$, find $P'(x)$.

Solution:
Notice that in the limit definition of the derivative, the function is given the name of f. Here, the function is given the name of P. Thus, we replace every occurrence of f with P, and the limit definition of the derivative becomes

$$P'(x) = \lim_{h\to 0} \frac{P(x+h)-P(x)}{h}$$

As stated in **Section 2.1**, using direct substitution to find the limits of the numerator and denominator of the difference quotient as $h \to 0$ will lead to the indeterminate form $\frac{0}{0}$. Thus, we must work to algebraically manipulate the function. Again, our ultimate goal is to *divide the h from the denominator* of the difference quotient:

$$\begin{aligned}
P'(x) &= \lim_{h\to 0} \frac{P(x+h)-P(x)}{h} \\
&= \lim_{h\to 0} \frac{\left(-0.01(x+h)^2 + 300(x+h) - 10,000\right) - \left(-0.01x^2 + 300x - 10,000\right)}{h} \\
&= \lim_{h\to 0} \frac{\left(-0.01\left(x^2+2xh+h^2\right) + 300x + 300h - 10,000\right) - \left(-0.01x^2 + 300x - 10,000\right)}{h} \\
&= \lim_{h\to 0} \frac{-0.01x^2 - 0.02xh - 0.01h^2 + 300x + 300h - 10,000 + 0.01x^2 - 300x + 10,000}{h}
\end{aligned}$$

Combining like terms gives

$$\begin{aligned}
&= \lim_{h\to 0} \frac{\cancel{-0.01x^2} - 0.02xh - 0.01h^2 \cancel{+300x} + 300h \cancel{-10,000} \cancel{+0.01x^2} \cancel{-300x} \cancel{+10,000}}{h} \\
&= \lim_{h\to 0} \frac{-0.02xh - 0.01h^2 + 300h}{h} \\
&= \lim_{h\to 0} \frac{\cancel{h}(-0.02x - 0.01h + 300)}{\cancel{h}} \\
&= \lim_{h\to 0} (-0.02x - 0.01h + 300)
\end{aligned}$$

Again, as discussed in **Section 2.1**, we can substitute 0 for h after dividing h from the denominator (even if the resulting expression after dividing h from the denominator consists of a ratio of two functions). Substituting 0 for h (not x) gives

$$\begin{aligned}
&= -0.02x - 0.01(0) + 300 \\
&= -0.02x + 300
\end{aligned}$$

Thus, $P'(x) = -0.02x + 300$.

Similar to **Section 2.1**, *algebraically manipulating the difference quotient ensures we can substitute* 0 *for* h *to find* $f'(x)$. *Remember that there may be* x*-values where* $f'(x)$ *(as a limit) does not exist. Being able to use direct substitution for* h *does not mean* $f'(x)$ *exists for all* x*!*

In the previous example, we actually found a rule for the derivative of the function from the introductory example. If we substitute the x-values $10{,}000$, $12{,}000$, $15{,}000$, $20{,}000$, and $23{,}000$ into $P'(x)$, we get the same values for the instantaneous rates of change, in dollars per Swap, that are shown in **Table 2.7**:

$$\begin{aligned}
P'(10{,}000) &= -0.02(10{,}000) + 300 = 100\\
P'(12{,}000) &= -0.02(12{,}000) + 300 = 60\\
P'(15{,}000) &= -0.02(15{,}000) + 300 = 0\\
P'(20{,}000) &= -0.02(20{,}000) + 300 = -100\\
P'(23{,}000) &= -0.02(23{,}000) + 300 = -160
\end{aligned}$$

Some students may be content with finding this type of limit in one step. But for many, the process may feel overwhelming. Also, computing the limit all at once makes it much more likely that algebraic mistakes will be made!

For the remainder of this section, we will use an amended version of the Suggested Four-Step Process that was introduced in the previous section. The ultimate goal of this process is still to divide h from the denominator of the difference quotient so that we can use direct substitution.

Suggested Four-Step Process (to find $f'(x)$)

1. Calculate $f(x+h)$, and algebraically manipulate the resulting expression, if possible.
2. Calculate the numerator of the difference quotient, and algebraically manipulate the resulting expression, if possible:
$$f(x+h) - f(x)$$
3. Calculate the difference quotient (i.e., the slope of the secant line), and algebraically manipulate the resulting expression:
$$\frac{f(x+h) - f(x)}{h}, \quad h \neq 0$$
4. Take the limit of the difference quotient as h tends to zero (i.e., find the slope of the tangent line):
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

N *Remember to substitute* h *with zero,* ***not*** x***!*** *Our final answer will have* x *in it. Throughout steps 1, 2, and 3, we will (most likely) have* x *and* h*, but by the end of step 4,* h *should be gone!*

▪ **Example 2** Given $f(x) = 4x^2 + 2x + 12$, find

a. $f\,'(x)$.

b. the instantaneous rate of change of f at $x = -2$.

c. the slope of the line tangent to the graph of f at $x = 1$.

d. the value(s) of x for which the line tangent to the graph of f is horizontal.

Solution:

a. We find $f\,'(x)$ using the limit definition of the derivative:

$$f\,'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

To simplify the process of finding this limit, we will use the Suggested Four-Step Process:

1. Calculate $f(x+h)$, and algebraically manipulate the resulting expression, if possible:

$$\begin{aligned} f(x+h) &= 4(x+h)^2 + 2(x+h) + 12 \\ &= 4(x+h)(x+h) + 2(x+h) + 12 \end{aligned}$$

Using FOIL and distributing gives

$$\begin{aligned} &= 4\left(x^2 + 2xh + h^2\right) + 2x + 2h + 12 \\ &= 4x^2 + 8xh + 4h^2 + 2x + 2h + 12 \end{aligned}$$

2. Calculate the numerator of the difference quotient, $f(x+h) - f(x)$, and algebraically manipulate the resulting expression, if possible:

$$\begin{aligned} f(x+h) - f(x) &= \left(4x^2 + 8xh + 4h^2 + 2x + 2h + 12\right) - \left(4x^2 + 2x + 12\right) \\ &= \cancel{4x^2} + 8xh + 4h^2 \cancel{+2x} + 2h \cancel{+12} \cancel{-4x^2} \cancel{-2x} \cancel{-12} \\ &= 8xh + 4h^2 + 2h \end{aligned}$$

A common error is forgetting to distribute the negative when subtracting $f(x)$. This is necessary for proper algebra! With wrong algebra will come the wrong answer, even if you understand the concepts.

3. Calculate the difference quotient (i.e., divide the result from the previous step by h, where $h \neq 0$), and algebraically manipulate the resulting expression:

$$\frac{f(x+h) - f(x)}{h} = \frac{8xh + 4h^2 + 2h}{h}$$

Factoring an h from the numerator gives

$$= \frac{h(8x + 4h + 2)}{h}$$

Dividing the h's gives

$$\begin{aligned} &= \frac{\cancel{h}(8x + 4h + 2)}{\cancel{h}} \\ &= 8x + 4h + 2 \end{aligned}$$

4. Take the limit as $h \to 0$. Remember to substitute 0 for h, not x:

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}(8x+4h+2)\\ &= 8x+4(0)+2\\ &= 8x+2\end{aligned}$$

Thus, $f'(x) = 8x+2$.

b. The instantaneous rate of change of f at $x = -2$ can be found by calculating $f'(2)$:

$$\begin{aligned}f'(-2) &= 8(-2)+2\\ &= -14\end{aligned}$$

Thus, the instantaneous rate of change of f at $x = -2$ is -14.

c. Recall that the slope of the line tangent to the graph of f at $x = 1$ is equivalent to the instantaneous rate of change of f at $x = 1$. Thus, we must evaluate $f'(x)$ at $x = 1$:

$$\begin{aligned}f'(1) &= 8(1)+2\\ &= 10\end{aligned}$$

Hence, the slope of the line tangent to the graph of f at $x = 1$ is 10.

d. Recall that we must find the value(s) of x for which the line tangent to the graph of f is horizontal. Remember, the slope of a horizontal line is zero. Because $f'(x)$ gives us the slope of the line tangent to the graph of f at x, we need to find the value(s) of x such that

$$f'(x) = 0$$

Given that $f'(x) = 8x+2$, we have

$$8x+2 = 0$$

Solving for x gives

$$\begin{aligned}8x &= -2\\ x &= \frac{-2}{8}\\ x &= -\frac{1}{4}\end{aligned}$$

Thus, the graph of f has a horizontal tangent line at $x = -\frac{1}{4}$.

■

The graphs of f and f' from the previous example are shown in **Figure 2.2.4**.

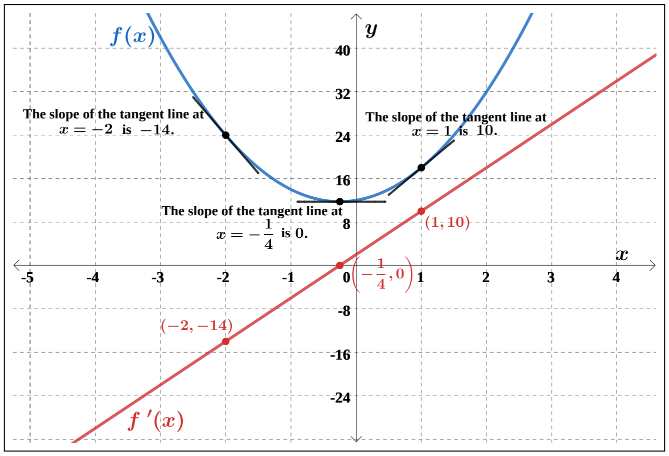


Figure 2.2.4: Graphs of $f(x) = 4x^2 + 2x + 12$ and $f'(x) = 8x + 2$

Notice that where the slopes of the lines tangent to the graph of f are negative (i.e., where f is decreasing), the graph of f' is below the x-axis (i.e., the y-values of $f'(x)$ are negative). We see in **Figure 2.2.4** that this occurs on the interval $\left(-\infty, -\frac{1}{4}\right)$.

On the other hand, where the slopes of the lines tangent to the graph of f are positive (i.e., where f is increasing), the graph of f' is above the x-axis (i.e., the y-values of $f'(x)$ are positive). We see in **Figure 2.2.4** that this occurs on the interval $\left(-\frac{1}{4}, \infty\right)$.

Lastly, notice that when the line tangent to the graph of f is horizontal, the graph of f' touches the x-axis (i.e., the y-value of $f'(x)$ is zero). We see in **Figure 2.2.4** that this occurs at $x = -\frac{1}{4}$.

Making these connections between the graphs of f and f' is incredibly important.

Try It # 1:
Given $f(x) = 3x^2 - 5x + 2$, find

a. $f'(x)$.
b. the instantaneous rate of change of f at $x = 0$.
c. the slope of the line tangent to the graph of f at $x = 2$.
d. the value(s) of x for which the line tangent to the graph of f is horizontal.

▪ **Example 3** Given $f(x) = \sqrt{x} - 3$, find $f'(x)$.

Solution:
We find $f'(x)$ by using the limit definition of the derivative:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

To simplify the process of finding this limit, we will use the Suggested Four-Step Process:

1. Calculate $f(x+h)$, and algebraically manipulate the resulting expression, if possible:

$$f(x+h) = \sqrt{x+h} - 3$$

Depending on the function f, you may or may not be able to algebraically manipulate the expression at this step. In the previous example, we had a quadratic function so we could FOIL and combine like terms. However, in this example, the expression cannot be algebraically manipulated further at this step.

2. Calculate the numerator of the difference quotient, $f(x+h)-f(x)$, and algebraically manipulate the resulting expression, if possible:

$$\begin{aligned} f(x+h)-f(x) &= \left(\sqrt{x+h}-3\right)-\left(\sqrt{x}-3\right) \\ &= \sqrt{x+h}\cancel{-3}-\sqrt{x}\cancel{+3} \\ &= \sqrt{x+h}-\sqrt{x} \end{aligned}$$

Remember, $\sqrt{x+h}-\sqrt{x} \neq \sqrt{x+h-x}$*! Incorrect simplification will lead to the wrong answer!*

3. Calculate the difference quotient (i.e., divide the result from the previous step by h, where $h \neq 0$), and algebraically manipulate the resulting expression:

$$\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

Multiplying the numerator and denominator by the conjugate of $\sqrt{x+h}-\sqrt{x}$ gives

$$\begin{aligned} &= \left(\frac{\sqrt{x+h}-\sqrt{x}}{h}\right)\left(\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\right) \\ &= \frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \\ &= \frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)} \end{aligned}$$

Dividing the h's gives

$$\begin{aligned} &= \frac{\cancel{h}}{\cancel{h}\left(\sqrt{x+h}+\sqrt{x}\right)} \\ &= \frac{1}{\sqrt{x+h}+\sqrt{x}} \end{aligned}$$

4. Take the limit as $h \to 0$. Remember to substitute 0 for h, not x:

$$\begin{aligned} \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{1}{\sqrt{x+h}+\sqrt{x}} \\ &= \frac{1}{\sqrt{x+0}+\sqrt{x}} \\ &= \frac{1}{\sqrt{x}+\sqrt{x}} \\ &= \frac{1}{2\sqrt{x}} \end{aligned}$$

Thus, $f'(x) = \dfrac{1}{2\sqrt{x}}$.

■

Try It # 2:
Find the derivative of each of the following functions.

a. $g(x) = \sqrt{x}$

b. $h(x) = \dfrac{2}{x-4}$

c. $f(x) = \sqrt{x-6}$

*In part **a** of the above Try It, you should get the same answer we got in the previous example. This is not a coincidence! These two functions, $g(x) = \sqrt{x}$ and $f(x) = \sqrt{x} - 3$, have the same graph, except shifted up/down from each other. This shift will change the equations of the tangent lines, but not the slopes of the tangent lines. Thus, the two functions have the same derivative function.*

Derivative Notation

There are many different ways to write the derivative:

Derivative Notation

Lagrange notation: $f'(x)$ and y' (pronounced as "f prime of x" and "y prime")

Leibniz notation: $\dfrac{df}{dx}, \dfrac{dy}{dx}$, and $\dfrac{d}{dx}(f(x))$ (pronounced as "$df\ dx$", "$dy\ dx$", and "d-dx of f of x")

Even though the Leibniz notation, $\dfrac{dy}{dx}$, is one symbol, it is not a fraction! It looks like a fraction because the derivative is a slope. It can be thought of as the change in y over the change in x.

Example 4 Given $y = \dfrac{7}{5-9x}$, find $\dfrac{dy}{dx}$.

Solution:
We find $\dfrac{dy}{dx}$ by using the limit definition of the derivative:

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Keep in mind here that y is the function. Thus, $y = f(x) = \dfrac{7}{5-9x}$. Again, to simplify the process of finding this limit, we will use the Suggested Four-Step Process:

1. Calculate $f(x+h)$, and algebraically manipulate the resulting expression, if possible:

$$\begin{aligned} f(x+h) &= \frac{7}{5-9(x+h)} \\ &= \frac{7}{5-9x-9h} \end{aligned}$$

2. Calculate the numerator of the difference quotient, $f(x+h) - f(x)$, and algebraically manipulate the resulting expression, if possible:

$$f(x+h) - f(x) = \frac{7}{5-9x-9h} - \frac{7}{5-9x}$$

Now, we get a common denominator:

$$= \left(\frac{5-9x}{5-9x}\right)\left(\frac{7}{5-9x-9h}\right) - \left(\frac{7}{5-9x}\right)\left(\frac{5-9x-9h}{5-9x-9h}\right)$$

$$= \frac{7(5-9x)}{(5-9x)(5-9x-9h)} - \frac{7(5-9x-9h)}{(5-9x)(5-9x-9h)}$$

$$= \frac{7(5-9x)-7(5-9x-9h)}{(5-9x)(5-9x-9h)}$$

$$= \frac{\cancel{35}\cancel{-63x}\cancel{-35}\cancel{+63x}+63h}{(5-9x)(5-9x-9h)}$$

$$= \frac{63h}{(5-9x)(5-9x-9h)}$$

3. Calculate the difference quotient (i.e., divide the result from the previous step by h, where $h \neq 0$), and algebraically manipulate the resulting expression:

$$\frac{f(x+h)-f(x)}{h} = \frac{\frac{63h}{(5-9x)(5-9x-9h)}}{h}$$

At first glance, it may not be obvious how to proceed because we have a complex fraction. As discussed in **Section 2.1**, anytime we are presented with a complex fraction such as this one, we will rewrite the division by h as multiplication by $\frac{1}{h}$. Recall that we can do this because dividing by a fraction is equivalent to multiplying by its reciprocal:

$$\frac{\frac{63h}{(5-9x)(5-9x-9h)}}{h} = \frac{\frac{63h}{(5-9x)(5-9x-9h)}}{\frac{h}{1}}$$

$$= \left(\frac{63h}{(5-9x)(5-9x-9h)}\right)\left(\frac{1}{h}\right)$$

$$= \left(\frac{63\cancel{h}}{(5-9x)(5-9x-9h)}\right)\left(\frac{1}{\cancel{h}}\right)$$

$$= \frac{63}{(5-9x)(5-9x-9h)}$$

4. Take the limit as $h \to 0$. Remember to substitute 0 for h, not x:

$$\begin{aligned}\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \frac{63}{(5-9x)(5-9x-9(0))}\\ &= \frac{63}{(5-9x)(5-9x)}\\ &= \frac{63}{(5-9x)^2}\end{aligned}$$

Thus,

$$f'(x) = \frac{63}{(5-9x)^2}$$

We could have continued using algebra to expand the denominator above to get $f'(x) = \dfrac{63}{25-90x+81x^2}$, *but doing so is unnecessary.*

We found the instantaneous rate of change of this function at $x=-3$ *to be* $\dfrac{63}{1024}$ *in the previous section. Notice here that if we calculate* $f'(-3)$, *we get* $f'(-3) = \dfrac{63}{(5-9(-3))^2} = \dfrac{63}{(32)^2} = \dfrac{63}{1024}$ *as well!*

■

We summarize all of the interpretations of the derivative below. Some of these interpretations we have seen before, but we include them here to have them all in one place for easier reference:

Interpretations of the Derivative f'

1. Slope of the tangent line (or "slope", even if f is not linear)
2. Instantaneous rate of change (or "rate of change")
3. Instantaneous velocity (or "velocity")
4. Limit of the difference quotient
5. Limit of the slopes of secant lines

■ **Example 5** Find the equation of the line tangent to the graph of $f(x) = -3x^2+5x-2$ at $x=-4$.

Solution:
Recall from the previous section that to find the equation of a line, we need two pieces of information: a point the line passes through and the slope of the line. We will start by calculating $f(-4)$ to find the y-value of the point:

$$\begin{aligned}f(-4) &= -3(-4)^2+5(-4)-2\\ &= -70\end{aligned}$$

Thus, the tangent line passes through the point $(-4,-70)$.

Because the slope of the tangent line at $x = -4$ is given by $f'(-4)$, we will use the limit definition of the derivative to find $f'(x)$ and then calculate $f'(-4)$. Recall $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$. Once again, we use the Suggested Four-Step Process to simplify the evaluation of this limit:

1. Calculate $f(x+h)$, and algebraically manipulate the resulting expression, if possible:

$$\begin{aligned} f(x+h) &= -3(x+h)^2 + 5(x+h) - 2 \\ &= -3(x+h)(x+h) + 5(x+h) - 2 \end{aligned}$$

Using FOIL and distributing gives

$$\begin{aligned} &= -3\left(x^2 + 2xh + h^2\right) + 5x + 5h - 2 \\ &= -3x^2 - 6xh - 3h^2 + 5x + 5h - 2 \end{aligned}$$

2. Calculate the numerator of the difference quotient, $f(x+h) - f(x)$, and algebraically manipulate the resulting expression, if possible:

$$\begin{aligned} f(x+h) - f(x) &= \left(-3x^2 - 6xh - 3h^2 + 5x + 5h - 2\right) - \left(-3x^2 + 5x - 2\right) \\ &= \cancel{-3x^2} - 6xh - 3h^2 \cancel{+5x} + 5h \cancel{-2} \cancel{+3x^2} \cancel{-5x} \cancel{+2} \\ &= -6xh - 3h^2 + 5h \end{aligned}$$

3. Calculate the difference quotient (i.e., divide the result from the previous step by h, where $h \neq 0$), and algebraically manipulate the resulting expression:

$$\frac{f(x+h) - f(x)}{h} = \frac{-6xh - 3h^2 + 5h}{h}$$

Factoring an h from the numerator gives

$$\begin{aligned} &= \frac{h(-6x - 3h + 5)}{h} \\ &= \frac{\cancel{h}(-6x - 3h + 5)}{\cancel{h}} \\ &= -6x - 3h + 5 \end{aligned}$$

4. Take the limit as $h \to 0$. Remember to substitute 0 for h, not x:

$$\begin{aligned} \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h\to 0}(-6x - 3h + 5) \\ &= -6x - 3(0) + 5 \\ &= -6x + 5 \end{aligned}$$

This tells us

$$f'(x) = -6x + 5$$

So to find the slope of the line tangent to the graph of f at $x = -4$, we calculate $f'(-4)$:

$$\begin{aligned} f'(-4) &= -6(-4) + 5 \\ &= 29 \end{aligned}$$

Thus, the tangent line has slope $m = 29$ and passes through the point $(-4, -70)$. Using the point-slope form of the equation of a line, we have

$$y-(-70) = 29(x-(-4))$$
$$y+70 = 29(x+4)$$
$$y+70 = 29x+116$$

Solving for y to write the equation in slope-intercept form gives

$$y = 29x+46$$

Figure 2.2.5 shows the graph of f along with the line tangent to the graph of f at $x = -4$:

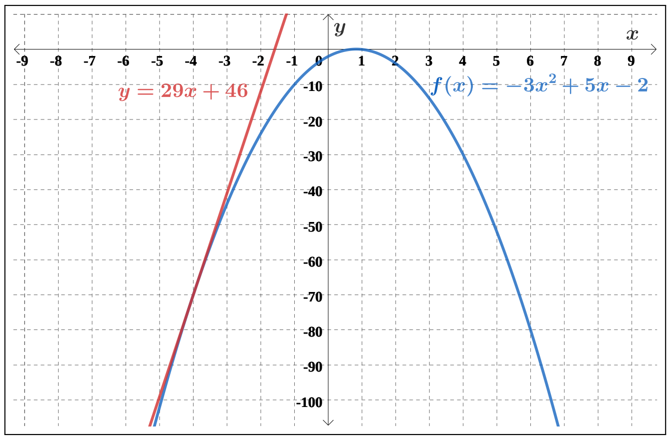


Figure 2.2.5: Graphs of $f(x) = -3x^2+5x-2$ and the line tangent to the graph of f at $x = -4$

▪

In the previous example, we were not explicitly asked to find $f'(x)$. Because we only needed to find $f'(-4)$, we could have used the method from the previous section to find the instantaneous rate of change specifically at $x = -4$ instead. However, to practice finding the derivative function in this section, we will always find the rule for the derivative, $f'(x)$, first before substituting a particular value of x. But, realize you always have the choice of using the process from the previous section if you are not explicitly asked to find $f'(x)$ nor needing to substitute multiple values of x into $f'(x)$.

Try It # 3:
Find the equation of the line tangent to the graph of $f(x) = 7x^2-3x+15$ at $x = 1$.

▪ **Example 6** Reid's Cookie Shop determines that its price-demand function for selling "I ♥ Math" cookies each day is given by $p(x) = -0.4x+20$, where $p(x)$ is the price, in dollars, per cookie when x cookies are sold. The cookie shop also has a daily cost function given by $C(x) = 0.1x+50$ dollars when x cookies are made. Find the rate of change of profit (i.e., instantaneous rate of change of profit) when Reid's Cookie Shop sells 30 cookies, and interpret your answer.

Solution:
To find the rate of change of profit when 30 cookies are sold, we need to find $P'(30)$, where P is the profit function. Thus, we first need to find the profit function and then find its derivative.

Recall that profit equals revenue minus cost. We are given the cost function, C, in the problem, but we need to find the revenue function, R, in order to find the profit function, P.

In the previous section, we were reminded that the revenue function, R, is given by the number of items sold, x, times the price of each item, p:

$$\begin{aligned} R(x) &= x \cdot p(x) \\ &= x(-0.4x+20) \\ &= -0.4x^2+20x \end{aligned}$$

Now, we can find the profit function:

$$\begin{aligned} P(x) &= R(x)-C(x) \\ &= \left(-0.4x^2+20x\right)-(0.1x+50) \\ &= -0.4x^2+19.9x-50 \end{aligned}$$

Thus, $P(x) = -0.4x^2+19.9x-50$ gives the profit, in dollars, when x cookies are made and sold.

Now, we can use the limit definition of the derivative to find $P'(x)$. Recall that $P'(x) = \lim_{h\to 0}\dfrac{P(x+h)-P(x)}{h}$. Once again, we use the Suggested Four-Step Process to simplify finding this limit:

1. Calculate $P(x+h)$, and algebraically manipulate the resulting expression, if possible:

$$\begin{aligned} P(x+h) &= -0.4(x+h)^2+19.9(x+h)-50 \\ &= -0.4(x+h)(x+h)+19.9(x+h)-50 \end{aligned}$$

Using FOIL and distributing gives

$$\begin{aligned} &= -0.4\left(x^2+2xh+h^2\right)+19.9x+19.9h-50 \\ &= -0.4x^2-0.8xh-0.4h^2+19.9x+19.9h-50 \end{aligned}$$

2. Calculate the numerator of the difference quotient, $P(x+h)-P(x)$, and algebraically manipulate the resulting expression, if possible:

$$\begin{aligned} P(x+h)-P(x) &= \left(-0.4x^2-0.8xh-0.4h^2+19.9x+19.9h-50\right)-\left(-0.4x^2+19.9x-50\right) \\ &= \cancel{-0.4x^2}-0.8xh-0.4h^2\cancel{+19.9x}+19.9h\cancel{-50}\cancel{+0.4x^2}\cancel{-19.9x}\cancel{+50} \\ &= -0.8xh-0.4h^2+19.9h \end{aligned}$$

3. Calculate the difference quotient (i.e., divide the result from the previous step by h, where $h \neq 0$), and algebraically manipulate the resulting expression:

$$\frac{P(x+h)-P(x)}{h} = \frac{-0.8xh-0.4h^2+19.9h}{h}$$

Factoring an h from the numerator gives

$$\begin{aligned} &= \frac{h(-0.8x-0.4h+19.9)}{h} \\ &= \frac{\cancel{h}(-0.8x-0.4h+19.9)}{\cancel{h}} \\ &= -0.8x-0.4h+19.9 \end{aligned}$$

4. Take the limit as $h \to 0$. Remember to substitute 0 for h, not x:

$$\begin{aligned} \lim_{h\to 0} \frac{P(x+h)-P(x)}{h} &= \lim_{h\to 0}(-0.8x-0.4h+19.9) \\ &= -0.8x-0.4(0)+19.9 \\ &= -0.8x+19.9 \end{aligned}$$

This tells us

$$P'(x) = -0.8x+19.9$$

Now, we calculate $P'(30)$:

$$\begin{aligned} P'(30) &= -0.8(30)+19.9 \\ &= -4.1 \end{aligned}$$

Rounding to the nearest cent because we are dealing with money gives

$$P'(30) = -\$4.10 \text{ per cookie}$$

Notice that the rate of change is negative, which indicates profit is decreasing when 30 cookies are made and sold. We write a sentence to interpret our answer:

"When 30 cookies are sold, profit is decreasing at a rate of \$4.10 per cookie."

■

N *Recall that because the derivative function, $f'(x)$, represents a rate of change, the units of $f'(x)$ are the same as we discussed in* **Section 2.1***:*

$$\textbf{\textit{Units of } } \boldsymbol{f'(x)} = \frac{\textit{units of } f(x)}{\textit{units of } x}$$

Try It # 4:
An office super store sells a laptop stylus pen. The price-demand function for the stylus pens is given by $p(x) = -0.2x+55$, where $p(x)$ is the price, in dollars, per stylus pen when x stylus pens are sold. Find the rate of change of revenue when the super store sells 120 stylus pens, and interpret your answer.

■ **Example 7** The height of a ball thrown upward is given by $s(t) = -16t^2+128t$ feet, where t is time in seconds. Find the velocity of the ball after 7 seconds.

Solution:
Recall that the rate of change of a position function gives us velocity. Thus, to find the velocity at 7 seconds, we need to calculate $s'(7)$. As mentioned previously, we could use the method from **Section 2.1** to find the velocity at this specific t-value, but in this section, we will find the velocity function, $s'(t) = \lim_{h\to 0} \frac{s(t+h)-s(t)}{h}$, first and then evaluate it at $t = 7$ seconds. We use the Suggested Four-Step Process to simplify finding this limit:

1. Calculate $s(t+h)$, and algebraically manipulate the expression, if possible:

$$\begin{aligned} s(t+h) &= -16(t+h)^2+128(t+h) \\ &= -16(t+h)(t+h)+128(t+h) \end{aligned}$$

Using FOIL and distributing gives

$$= -16\left(t^2 + 2th + h^2\right) + 128t + 128h$$
$$= -16t^2 - 32th - 16h^2 + 128t + 128h$$

2. Calculate the numerator of the difference quotient, $s(t+h) - s(t)$, and algebraically manipulate the expression, if possible:

$$\begin{aligned} s(t+h) - s(t) &= \left(-16t^2 - 32th - 16h^2 + 128t + 128h\right) - \left(-16t^2 + 128t\right) \\ &= \cancel{-16t^2} - 32th - 16h^2 \cancel{+128t} + 128h \cancel{+16t^2} \cancel{-128t} \\ &= -32th - 16h^2 + 128h \end{aligned}$$

3. Calculate the difference quotient (i.e., divide the result from the previous step by h, where $h \neq 0$), and algebraically manipulate the resulting expression:

$$\frac{s(t+h) - s(t)}{h} = \frac{-32th - 16h^2 + 128h}{h}$$

Factoring an h from the numerator gives

$$\begin{aligned} &= \frac{h(-32t - 16h + 128)}{h} \\ &= \frac{\cancel{h}(-32t - 16h + 128)}{\cancel{h}} \\ &= -32t - 16h + 128 \end{aligned}$$

4. Take the limit as $h \to 0$. Remember to substitute 0 for h, not t:

$$\begin{aligned} \lim_{h \to 0} \frac{s(t+h) - s(t)}{h} &= \lim_{h \to 0} (-32t - 16h + 128) \\ &= -32t - 16(0) + 128 \\ &= -32t + 128 \end{aligned}$$

This tells us

$$s\,'(t) = -32t + 128$$

Now, we calculate $s\,'(7)$:

$$\begin{aligned} s\,'(7) &= -32(7) + 128 \\ &= -96 \text{ feet per second} \end{aligned}$$

Thus, the velocity of the ball after 7 seconds is -96 feet per second. The negative sign indicates that the ball is moving in the downward direction (i.e., it is falling).

■

N *We could have named the velocity function we found in the previous example v instead of $s\,'$. Because the velocity function is the derivative of the position function, we can use v interchangeably with $s\,'$.*

■ **Example 8** A computer manufacturer's price-demand function is given by $p(x) = -0.3x + 540$, where $p(x)$ is the price, in dollars, of each computer when x computers are sold. Find the rate of change of revenue when 800 computers are sold, and interpret your answer.

Solution:
We want to find a rate of change, which means we need to find the derivative. But, we must be careful! We want the rate of change of *revenue*, not the rate of change of price. We must first find the revenue function:

$$\begin{aligned} R(x) &= x \cdot p(x) \\ &= x(-0.3x + 540) \\ &= -0.3x^2 + 540x \end{aligned}$$

Thus, $R(x) = -0.3x^2 + 540x$ gives the revenue, in dollars, when x computers are sold.

We need to calculate $R'(800)$, and so we will find $R'(x) = \lim_{h \to 0} \frac{R(x+h) - R(x)}{h}$ first and then evaluate it at $x = 800$. Again, we use the Suggested Four-Step Process to simplify finding this limit:

1. Calculate $R(x+h)$, and algebraically manipulate the resulting expression, if possible:

$$\begin{aligned} R(x+h) &= -0.3(x+h)^2 + 540(x+h) \\ &= -0.3(x+h)(x+h) + 540(x+h) \end{aligned}$$

Using FOIL and distributing gives

$$\begin{aligned} &= -0.3\left(x^2 + 2xh + h^2\right) + 540x + 540h \\ &= -0.3x^2 - 0.6xh - 0.3h^2 + 540x + 540h \end{aligned}$$

2. Calculate the numerator of the difference quotient, $R(x+h) - R(x)$, and algebraically manipulate the resulting expression, if possible:

$$\begin{aligned} R(x+h) - R(x) &= \left(-0.3x^2 - 0.6xh - 0.3h^2 + 540x + 540h\right) - \left(-0.3x^2 + 540x\right) \\ &= \cancel{-0.3x^2} - 0.6xh - 0.3h^2 \cancel{+540x} + 540h \cancel{+0.3x^2} \cancel{-540x} \\ &= -0.6xh - 0.3h^2 + 540h \end{aligned}$$

3. Calculate the difference quotient (i.e., divide the result from the previous step by h, where $h \neq 0$), and algebraically manipulate the resulting expression:

$$\frac{R(x+h) - R(x)}{h} = \frac{-0.6xh - 0.3h^2 + 540h}{h}$$

Factoring an h from the numerator gives

$$\begin{aligned} &= \frac{h(-0.6x - 0.3h + 540)}{h} \\ &= \frac{\cancel{h}(-0.6x - 0.3h + 540)}{\cancel{h}} \\ &= -0.6x - 0.3h + 540 \end{aligned}$$

4. Take the limit as $h \to 0$. Remember to substitute 0 for h, not x:

$$\begin{aligned}\lim_{h\to 0}\frac{R(x+h)-R(x)}{h} &= \lim_{h\to 0}(-0.6x-0.3h+540)\\ &= -0.6x-0.3(0)+540\\ &= -0.6x+540\end{aligned}$$

This tells us

$$R'(x) = -0.6x+540$$

Now, we calculate $R'(800)$:

$$\begin{aligned}R'(800) &= -0.6(800)+540\\ &= \$60 \text{ per computer}\end{aligned}$$

We can write a sentence to interpret our answer:

"When 800 computers are sold, revenue is increasing at a rate of \$60 per computer."

▪

Try It # 5:
The cost function for the computer manufacturer from the previous example is given by $C(x) = 122x+300$ dollars when x computers are manufactured. Find the rate of change of profit when 800 computers are manufactured and sold, and interpret your answer.

After the previous example and Try It, you will see that revenue is increasing, but profit is decreasing, when the manufacturer sells 800 computers. If the company wants to increase revenue, it should manufacture and sell more computers. Notice, however, that this would decrease profit! Manufacturing and selling less computers decreases revenue, but it increases profit because costs also decrease. Calculus is very handy for determining maximum and minimum values of a function. We'll explore this further in Chapter 3.

Differentiability

The limit definition of the derivative at a particular x-value, $x = a$, is given by

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

when the limit exists. We know that there are circumstances in which limits do not exist. If the limit representing $f'(a)$ does not exist, then $f'(a)$ does not exist.

Definition
If $f'(a)$ does not exist, we say f is **nondifferentiable** at $x = a$. If the limit does exist, we say f is **differentiable** at $x = a$. ▪

We know one way a limit does not exist is if the left- and right-hand limits are not equal. But, what does that look like on a graph when the limit is of a difference quotient? There are three cases in which $f'(a)$ does not exist:

Nonexistence of the Derivative

If the graph of f contains any of the following at $x = a$, then $f'(a)$ does not exist:

- Cusp or Corner
- Discontinuity
- Vertical Tangent Line

Let's look at each case individually and investigate why the derivative (i.e., limit) does not exist:

1. Cusp or Corner

It is difficult to verbally describe cusps and corners, but a basic description would be that they are points on the graph of a function where there is a sudden change in direction.

Recall that the slope of the tangent line only exists if the slopes of the secant lines from the left and right approach the same finite number. Looking at the cusp in **Figure 2.2.6** and **Figure 2.2.7**, we see the slopes of the secant lines approaching $x = 2$ from the right are getting "infinitely negative":

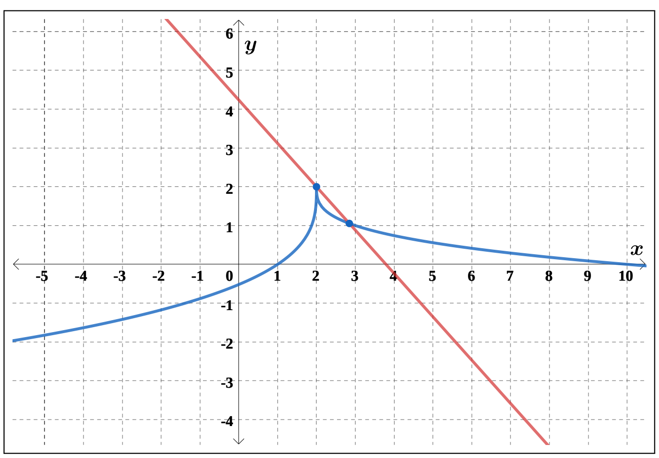


Figure 2.2.6: Graph of a function with a cusp at $x = 2$ and a secant line approaching from the right

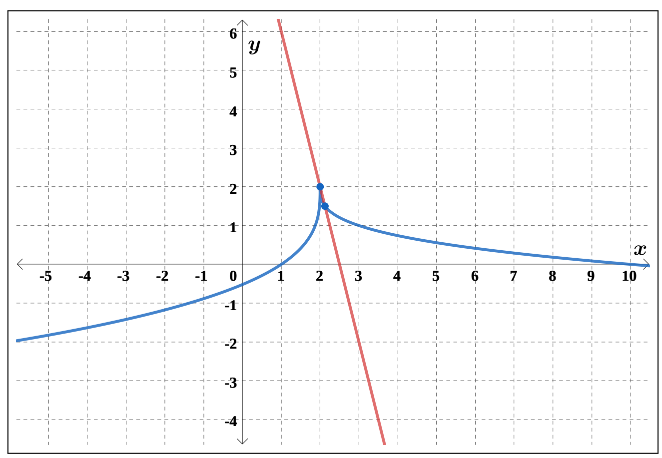


Figure 2.2.7: Graph of a function with a cusp at $x = 2$ and a secant line approaching from the right

In **Figure 2.2.8** and **Figure 2.2.9**, the slopes of the secant lines approaching $x = 2$ from the left are getting "infinitely positive":

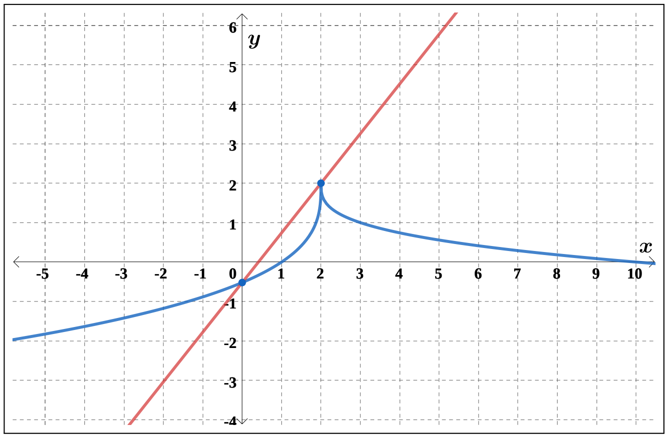


Figure 2.2.8: Graph of a function with a cusp at $x = 2$ and a secant line approaching from the left

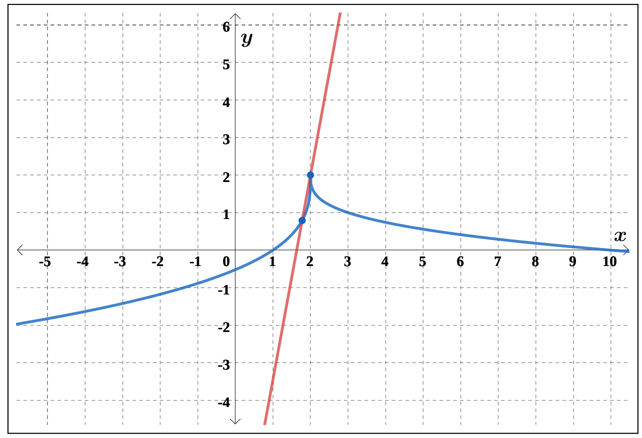


Figure 2.2.9: Graph of a function with a cusp at $x = 2$ and a secant line approaching from the left

Thus, the slopes of the secant lines from the left and the right of $x = 2$ do not approach a finite number, much less the same finite number (meaning the limit in the definition of $f'(a)$ does not exist). Therefore, there is no slope for a tangent line at $x = 2$, which in turn means there is no tangent line (and no derivative at $x = 2$).

The derivative of a function will also not exist at a corner for the same reason (i.e., no tangent line). Looking at the corner at $x = 1$ in **Figure 2.2.10**, we see the secant lines from the left will have positive slopes and the secant lines from the right will have a slope of zero. In **Figure 2.2.11**, the slopes of the secant lines from the left of $x = 1$ will be negative, and the slopes of the secant lines from the right will be positive:

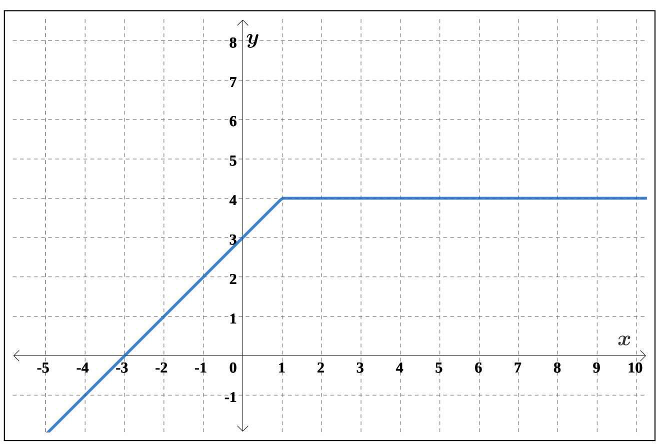


Figure 2.2.10: Graph of a piecewise-defined function that has a corner at $x = 1$

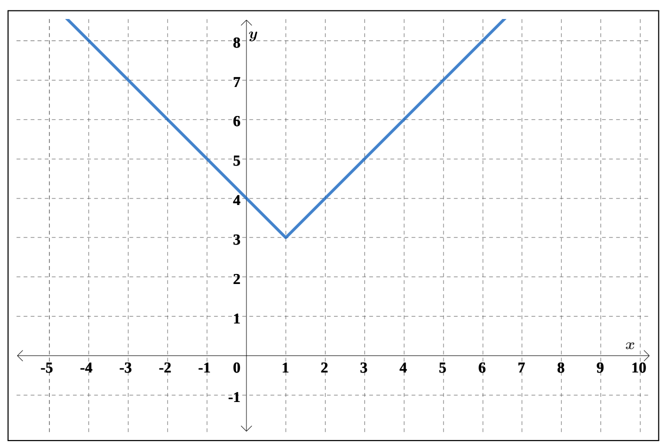


Figure 2.2.11: Graph of an absolute value function that has a corner at $x = 1$

In either case, the slopes of the secant lines from the left and right of $x = 1$ do approach the same number. Again, this means the limit in the definition of $f'(a)$ does not exist. And, as such, there is no slope for a tangent line. This again leads to the function not having a tangent line at $x = 1$ as well as the nonexistence of the derivative at $x = 1$. Therefore, we say f is nondifferentiable at $x = 1$, or $f'(1)$ does not exist.

2. Discontinuity

First of all, we cannot draw a secant line through a point that is undefined. So if $f(a)$ is not defined, we cannot even draw a secant line (much less a tangent line). However, even if the function is defined at $x = a$ and we can create secant lines, their slopes will not approach the same finite number from the left and right because one of the one-sided limits will approach positive or negative infinity!

Figure 2.2.12 and **Figure 2.2.13** show a function that is discontinuous, but defined, at $x = 2$. The slopes of the secant lines from the right are getting "infinitely positive". In other words, the limit of the slopes of the secant lines from the right tends to positive infinity and, hence, does not exist because it does not approach a finite number. If one of the one-sided limits does not exist, then there is no chance for the two-sided limit of the slopes of the secant lines (i.e., the slope of the tangent line) at $x = 2$ to exist:

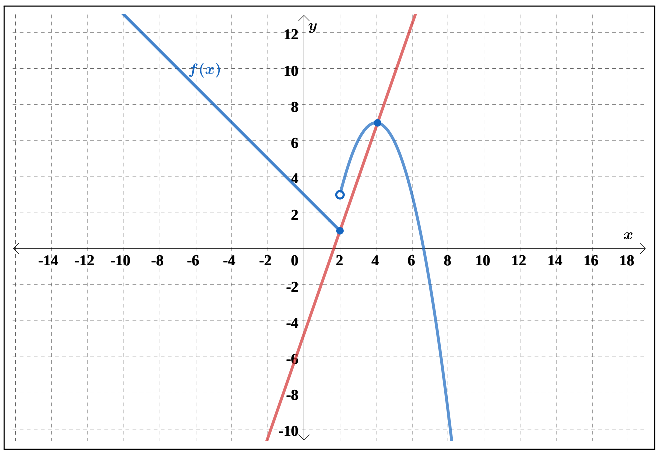


Figure 2.2.12: Graphs of a piecewise-defined function with a discontinuity at $x = 2$ and a secant line approaching from the right

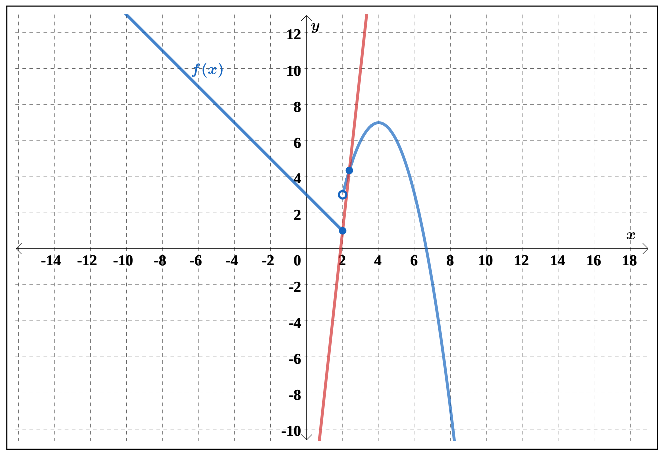


Figure 2.2.13: Graph of a piecewise-defined function with a discontinuity at $x = 2$ and a secant line approaching from the right

Thus, if a function has a discontinuity at $x = a$, there will be no slope of a tangent line, which again means there is no tangent line and $f'(a)$ does not exist.

This leads us to a very important theorem about the relationship between continuity and differentiability:

Theorem 2.1 If a function f is differentiable at $x = a$, then it must also be continuous at $x = a$.

*Although differentiability implies continuity, the converse is not true. A function can be continuous at $x = a$ without being differentiable at $x = a$. See **Figure 2.2.11** for an example. We will also see another way this can happen in the following discussion about vertical tangent lines!*

3. Vertical Tangent Line

Consider the slopes of the secant lines approaching $x = 3$ from the left shown in **Figure 2.2.14** and **Figure 2.2.15**:

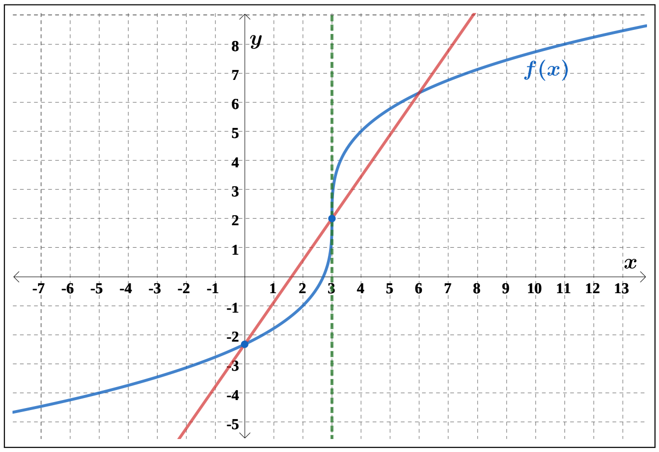


Figure 2.2.14: Graphs of a function that has a vertical tangent line at $x = 3$ and a secant line approaching $x = 3$ from the left

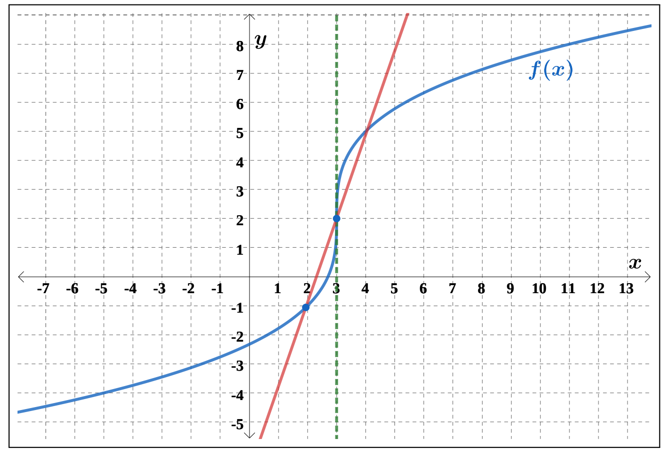


Figure 2.2.15: Graphs of a function that has a vertical tangent line at $x = 3$ and a secant line approaching $x = 3$ from the left

The slopes of the secant lines approaching $x = 3$ from the left are getting "infinitely positive". If we imagine the secant lines approaching $x = 3$ from the right, their slopes would be getting "infinitely positive" also. Thus,

$$f'(3) = \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} \to \infty$$

Remember that this notation describes the behavior of the limit; the limit itself does not exist because it does not equal a finite number. This is why f is nondifferentiable at $x = 3$.

Because the slopes of the secant lines from the left and right are approaching the same "infinitely positive slope", even though we cannot define a tangent line *function* at $x = 3$, there is a tangent line: $x = 3$. Due to the fact that this tangent line is vertical, it is not a function nor does it have a slope. Thus, the derivative (i.e., slope of this tangent line) does not exist. See **Figure 2.2.16**.

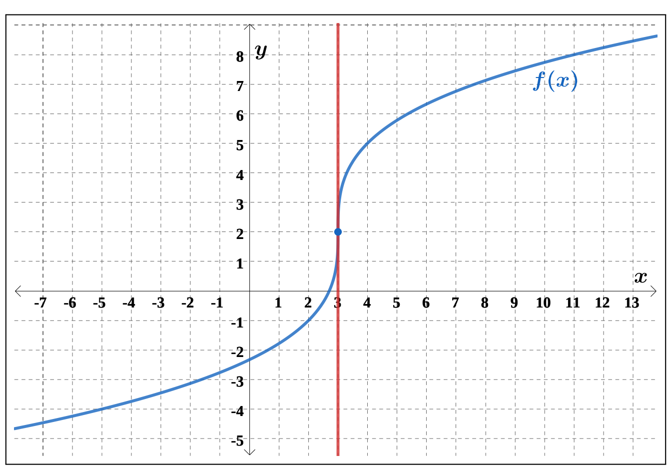


Figure 2.2.16: Graph of a function that has a vertical tangent line at $x = 3$

In summary, if a function has a vertical tangent line at $x = a$, $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ will always tend to positive or negative infinity. In either case, the limit does not exist because it is not approaching a finite number, and therefore, $f'(a)$ does not exist.

N *In cases 1 and 2, there is no tangent line, but in case 3, there is a tangent line. The problem is that the tangent line is vertical so its slope is undefined.*

! *Looking at the cusp in **Figure 2.2.6** through **Figure 2.2.9**, we may be tempted to say there is a vertical tangent line (like in case 3), but because the slopes from the left are positive and the slopes from the right are negative, there is no vertical tangent line. Cusps represent a different case than vertical tangent lines. They are more closely related to corners; hence, the reason they are included in case 1.*

■ **Example 9** Given the graph of the function f shown in **Figure 2.2.17**, determine the x-value(s) where the derivative of the function does not exist and explain why.

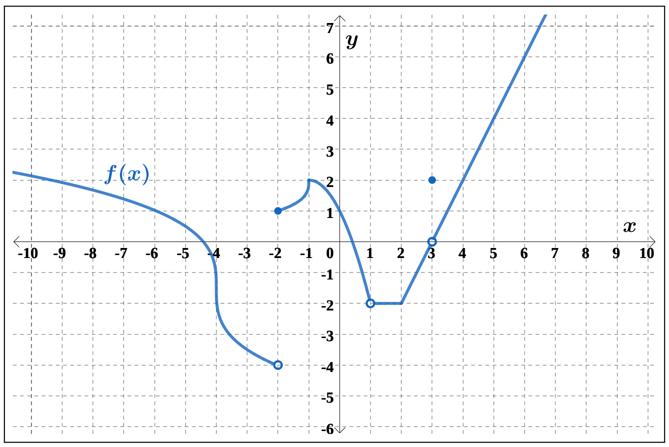


Figure 2.2.17: Graph of a piecewise-defined function f

Solution:
When determining where the derivative does not exist given a graph of f, the easiest x-values to locate are the ones where the graph of f is discontinuous (i.e., the x-values where the graph has "jumps" or is undefined). Looking at the graph, we see this occurs at $x = -2$, 1, and 3.

Next, we will look for corners and cusps. These are places where the graph changes direction suddenly. Looking at the graph of f, we see a cusp at $x = -1$ and a corner at $x = 2$. $f'(x)$ does not exist at either of these x-values because there is no tangent line.

! *You might be tempted to say there is a corner at $x = 1$. However, because there is no filled-in circle at $x = 1$, the function is undefined at $x = 1$. Thus, the graph of f cannot have a corner at $x = 1$.*

The "trickiest" x-values to see are the vertical tangent lines; places where the graph is approximately vertical at a single point. It appears that this occurs at $x = -4$.

To summarize, $f'(x)$ does not exist at $x = -4, -2, -1, 1, 2$, and 3. We can draw a (non-vertical) tangent line at all other points on the graph.

■

▪ **Example 10** Given the graph of f shown in **Figure 2.2.18**, determine where f is differentiable, and write your answer using interval notation.

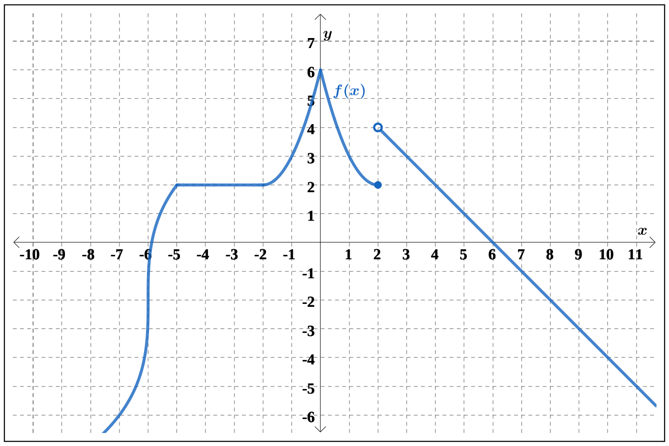


Figure 2.2.18: Graph of a piecewise-defined function f

Solution:
Once again, we start by looking for the x-values where f is discontinuous. This occurs at $x = 2$.

Next, we look for corners and cusps. These are places where the graph has a sharp turn. Looking at the graph of f, we see sharp turns at $x = -5$ and $x = 0$.

Finally, we look for where the graph has vertical tangent lines. It appears the graph has a vertical tangent line at $x = -6$.

To summarize, f is nondifferentiable at $x = -6$, -5, 0, and 2. Using interval notation, we say f is differentiable on the interval $(-\infty, -6) \cup (-6, -5) \cup (-5, 0) \cup (0, 2) \cup (2, \infty)$.

▪

Try It # 6:
Given the graph of f shown in **Figure 2.2.19**, determine where f is differentiable, and write your answer using interval notation.

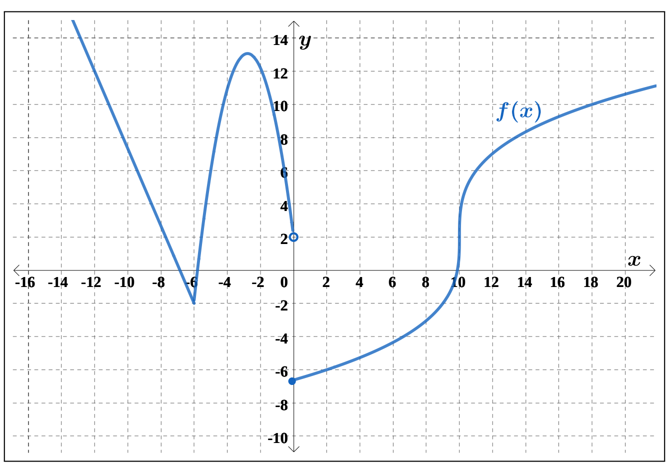


Figure 2.2.19: Graph of a piecewise-defined function f

Graphing the Derivative

Throughout this section, we were given the rule for a function f and asked to find the rule for its derivative function, f'. This is not the only way a function can be presented to us. If we are given the graph of a function, we can sketch the graph of its derivative. How could we do this?

Let's start by recalling the observations we made from the graphs of $f(x) = 4x^2 + 2x + 12$ and $f'(x) = 8x + 2$ shown again in **Figure 2.2.20**.

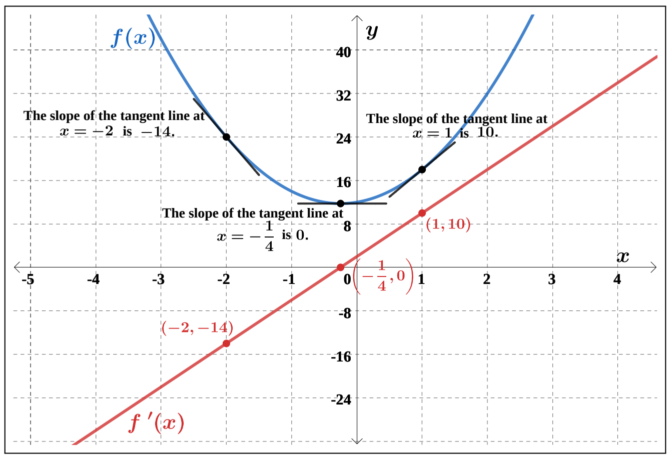


Figure 2.2.20: Graphs of $f(x) = 4x^2 + 2x + 12$ and $f'(x) = 8x + 2$

We noticed that the graph of f had a horizontal tangent line (i.e., the slope of the tangent line was zero) at $x = -\frac{1}{4}$, and as a result, the graph of f' touched the x-axis at $x = -\frac{1}{4}$.

We also noticed that the slopes of the lines tangent to the graph of f were negative on $\left(-\infty, -\frac{1}{4}\right)$, and as a result, the graph of f' was below the x-axis on that same interval: $\left(-\infty, -\frac{1}{4}\right)$.

Finally, we noticed that the slopes of the lines tangent to the graph of f were positive on $\left(-\frac{1}{4}, \infty\right)$, and as a result, the graph of f' was above the x-axis on that same interval: $\left(-\frac{1}{4}, \infty\right)$.

These ideas are generalized below:

- When the slopes of the lines tangent to the graph of f are **negative** (i.e., f is decreasing), the graph of f' is **below** the x-axis (i.e., the y-values of $f'(x)$ are negative).
- When the slopes of the lines tangent to the graph of f are **positive** (i.e., f is increasing), the graph of f' is **above** the x-axis (i.e., the y-values of $f'(x)$ are positive).
- When the lines tangent to the graph of f are **horizontal**, the graph of f' **touches** the x-axis (i.e., the y-values of $f'(x)$ equal zero).

We will use these ideas in the next example.

▪ **Example 11** Given the graph of f shown in **Figure 2.2.21**, sketch the graph of f'.

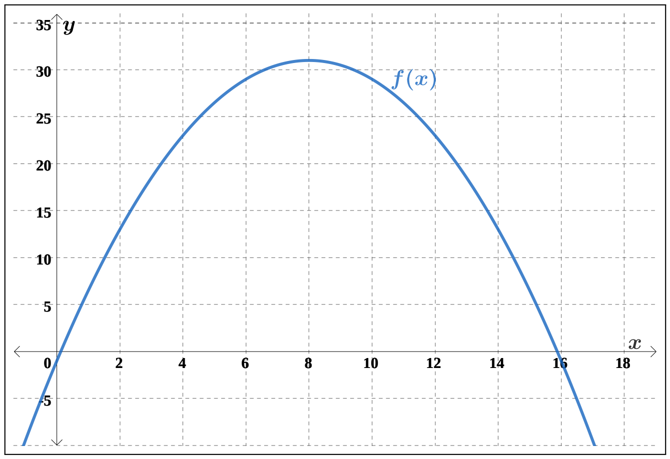


Figure 2.2.21: Graph of a function f

Solution:

Let's start by recalling that the graph of f' will consist of points of the form $(x, f'(x))$. In other words, for each x-coordinate, the y-coordinate on the graph of f' will be the slope of the line tangent to the graph of f at that x-coordinate. We start by drawing a few lines tangent to the graph of f in **Figure 2.2.22** to help us visualize this idea:

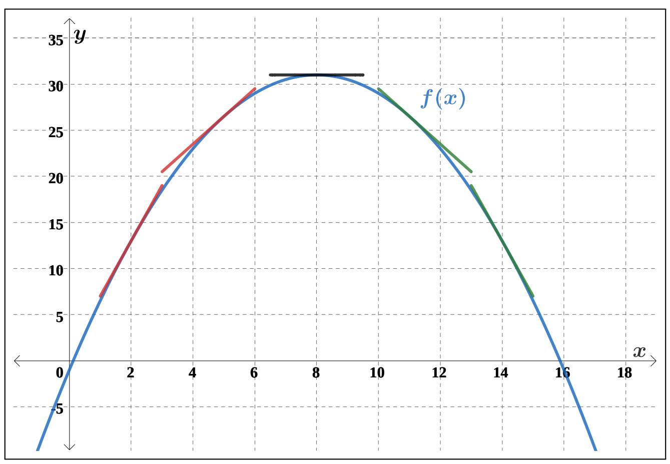


Figure 2.2.22: Graphs of f and five lines tangent to the graph of f

First, notice the line tangent to the graph of f at $x = 8$ is horizontal. Because $f'(x)$ represents the slope of the tangent line and the slope of a horizontal line is zero, we know that $f'(8) = 0$. Thus, when we graph f', the point $(8, 0)$ will be on the graph (meaning the graph will touch the x-axis at $x = 8$).

Next, notice that the red tangent lines drawn have positive slopes. This will be true for all tangent lines drawn to the left of $x = 8$ because f is increasing when $x < 8$. Thus, when we draw the graph of f', it will lie above the x-axis when $x < 8$.

Also, looking at the graph from left to right on the interval $(-\infty, 8)$, we see that the tangent lines are steep and then become flatter. The steep lines have a greater (or in this case, more positive) slope than the flatter lines. Therefore, as x approaches 8 from the left, the slopes of the tangent lines are getting smaller (or in this case, less positive). This means the values of $f'(x)$, which are the slopes of the tangent lines, will get less positive on the interval $(-\infty, 8)$. Thus, we have a starting point for our graph of f' as shown in **Figure 2.2.23**.

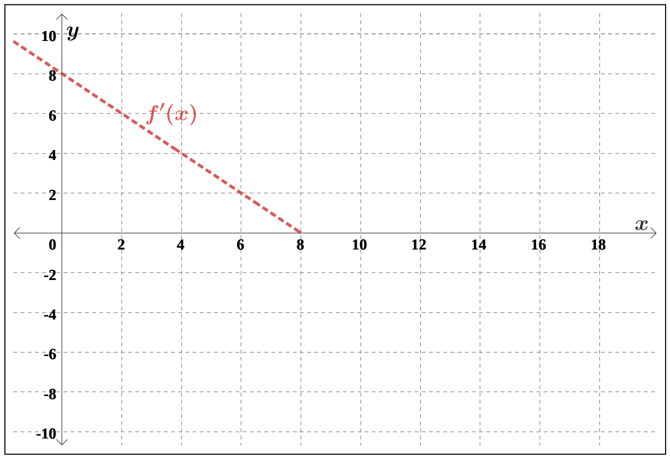


Figure 2.2.23: Sketch of f' corresponding to the interval $(-\infty, 8)$

Now, let's look at the x-values to the right of $x = 8$ in **Figure 2.2.22**. Both of the green tangent lines drawn have a negative slope. This will be true for all tangent lines drawn to the right of $x = 8$ because f is decreasing when $x > 8$. Thus, when we draw the graph of f', it will lie below the x-axis when $x > 8$.

Looking at the graph from left to right on the interval $(8, \infty)$, we see that the tangent lines start out fairly flat and then become steeper. Because the steep lines have a more negative slope than the flatter lines, we know that the values of $f'(x)$ will become more negative on the interval $(8, \infty)$.

Our final sketch of f' is shown in **Figure 2.2.25** along with the graph of f in **Figure 2.2.24**.

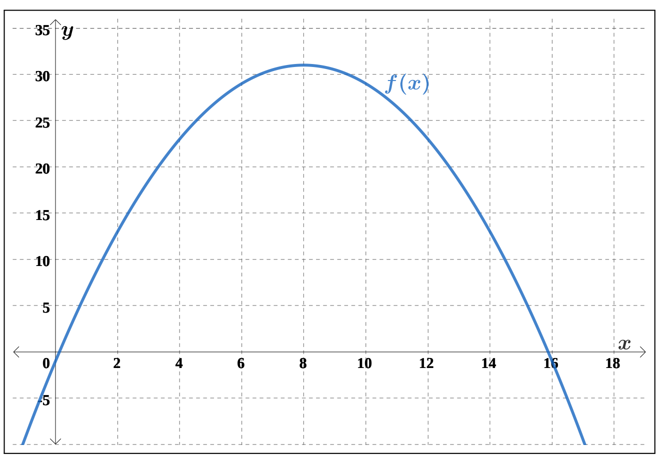


Figure 2.2.24: Graph of f

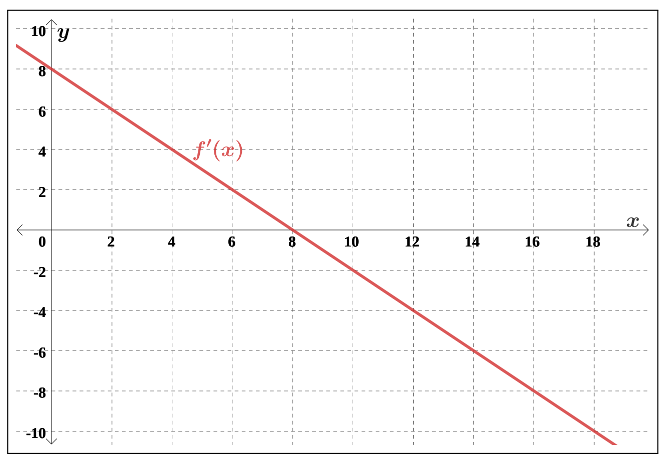


Figure 2.2.25: Sketch of f'

N *Keep in mind that this is a sketch of f'. Without actually computing the slopes of all the lines tangent to the graph of f, we are only able to approximate what the graph of f' looks like.*

■

Now, we give a step-by-step process to help you sketch the graph of the derivative of a function:

Graphing f'

Given a graph of f, find the following "important" x-values and write them on a number line:

1. x-values where the graph of f has a horizontal tangent line.
2. x-values where $f'(x)$ does not exist.

Determine and record the following information on the number line:

3. Intervals where the lines tangent to the graph of f have a positive/negative slope, as well as whether the slopes are getting "more" or "less" positive/negative on each interval (or are constant).

Use the information in the number line to sketch the graph of f'.

This technique is best seen through examples.

Example 12 Given the graph of f shown in **Figure 2.2.26**, sketch the graph of f'.

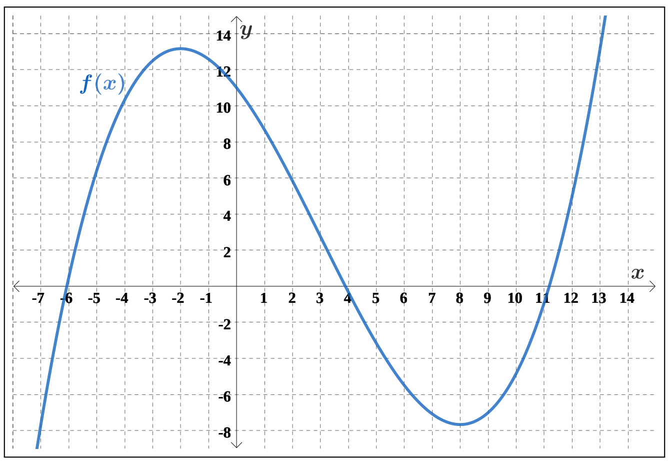


Figure 2.2.26: Graph of a function f

Solution:
Recall that one could approximate the value of $f'(x)$ at several x-values by drawing tangent lines and calculating their slopes. These points could then be plotted to sketch the graph of f'. In this textbook, we will only ask for a "rough" sketch of f'. Thus, you will not be expected to calculate the slopes of tangent lines to approximate the values of $f'(x)$ at multiple x-values (unless of course the function is linear and the tangent line is the line itself!).

Let's follow the formal process outlined previously to sketch the graph of f'. We start by creating a number line using the following "important" values of x discussed above:

1. x-values where the graph of f has a horizontal tangent line.

We see this happens at $x = -2$ and $x = 8$. These x-values are where $f'(x) = 0$, and so the graph of f' will touch the x-axis at these x-values.

2. x-values where $f'(x)$ does not exist.

Because f is a continuous function with no cusps, corners, or vertical tangent lines, it is differentiable everywhere. Thus, there are no values of x for which $f'(x)$ does not exist.

Next, we take the "important" x-values, $x = -2$ and $x = 8$, and put them on a number line with a label for tangent lines above the number line and a label for f' below the number line. We will also denote the values of $f'(x)$ at $x = -2$ and $x = 8$ below the number line because we know $f'(x) = 0$ at both x-values. See **Figure 2.2.27**.

Figure 2.2.27: First stage of creating a number line to aid in sketching the graph of f'

Now, we "fill in" our number line with the information specified in the last item listed:

3. Intervals where the lines tangent to the graph of f have a positive/negative slope.

For each interval, we will draw an example of what the tangent lines look like on the interval. This will help us determine if the slopes of the tangent lines are positive or negative (or constant) on the interval. We will draw a plus sign (+) below the number line corresponding to f' if the tangent lines have positive slopes on the interval, and we will draw a minus sign (−) below the number line if the tangent lines have negative slopes on the interval.

For each interval, we will also write below the number line whether the graph of f' is getting "more" or "less" positive/negative. This means we need to determine if the slopes of the tangent lines are getting more or less positive/negative. There is also another option: the slope of the tangent line is constant because the graph of f is linear. In this case, we will write "constant".

N *A helpful way to determine if the slopes of the tangent lines are getting "more" or "less" positive/negative on an interval is to ask yourself if the tangent lines are getting more or less steep (i.e., flatter).*

For our first interval, $(-\infty, -2)$, the slopes of all the tangent lines are positive. Thus, we will draw a sample tangent line above the number line with a positive slope and a plus sign below the number line corresponding to the fact that $f'(x)$ is positive on this interval. Also, below the plus sign, we need to write the word "more" or "less" depending on whether the slopes of the tangent lines are becoming more or less positive. If we look at the shape of the graph as x approaches -2 from the left, we see the graph is getting flatter (i.e., "less" steep). So we will write the word "less" below our plus sign because the slopes of the tangent lines are getting less positive. See **Figure 2.2.28**.

Figure 2.2.28: Creation of a number line to aid in sketching the graph of f'

Now, we continue to fill in the number line by analyzing the next interval, $(-2, 8)$. Because the slopes of all the tangent lines are negative on this interval, we draw an example of a tangent line with a negative slope above the number line and a minus sign below the number line.

Looking to see if the slopes of the tangent lines are getting more or less negative based on the shape of the graph, we conclude that because the graph is getting steeper (i.e., "more" steep) for part of the interval and then flatter for the latter part, the slopes are getting more negative and then less negative. It is hard to tell exactly where this change occurs, but it appears to occur somewhere near $x = 3$. We denote this by adding 3 to the number line, and we draw a tangent line with a negative slope above the number line and a minus sign below the number line for both the intervals $(-2,3)$ and $(3,8)$. However, because the tangent lines are getting steeper on $(-2,3)$ and then flatter on $(3,8)$, we write "more" below the number line for the interval $(-2,3)$ and "less" below the number line for the interval $(3,8)$. See **Figure 2.2.29**.

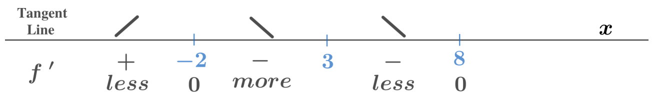


Figure 2.2.29: Creation of a number line to aid in sketching the graph of $f\,'$

For the last interval $(8,\infty)$, we see that all of the tangent lines have positive slopes. Thus, we draw a tangent line with a positive slope above the number line and a plus sign below the number line. Because the graph is getting steeper (i.e., the tangent lines are getting steeper), we write the word "more" below the plus sign to denote that the slopes are getting more positive. See **Figure 2.2.30**.

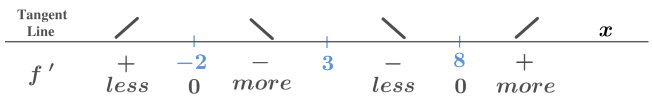


Figure 2.2.30: Number line to aid in sketching the graph of $f\,'$

Now, time for the fun part! We will translate our completed number line into a graph of $f\,'$. To do this, we will use the information corresponding to $f\,'$ on the number line to determine whether we start by drawing above or below the x-axis as well as in which direction.

Starting with the interval $(-\infty,-2)$, we begin drawing in Quadrant II because $f\,'(x)$ is positive (i.e., the y-values of $f\,'(x)$ are positive). We need to draw the graph of $f\,'$ with y-values that are getting less positive. That means the graph should not only be in Quadrant II, but it should also decrease. Furthermore, $f\,'(x)=0$ at $x=-2$, so the graph of $f\,'$ should touch the x-axis at $x=-2$. To summarize for this interval: the graph of $f\,'$ should start in Quadrant II, decrease, and continue downward until it touches the x-axis at $x=-2$. See **Figure 2.2.31**.

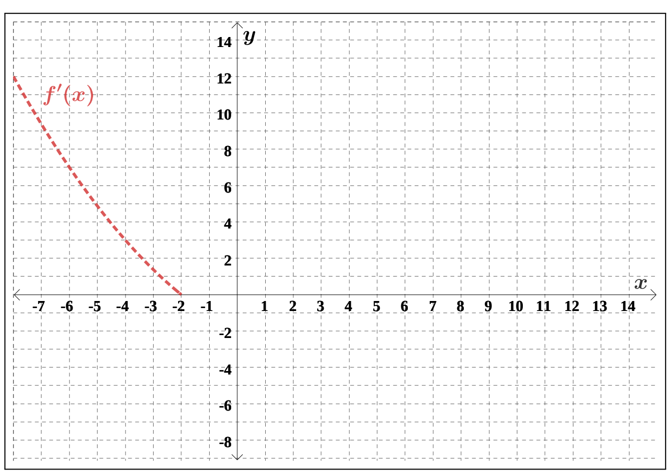


Figure 2.2.31: Sketching $f\,'$

Looking at the number line for the interval $(-2, 3)$, we see that we need to draw y-values of $f'(x)$ that are getting more negative. So we must continue drawing below the x-axis because this will correspond to y-values that are getting more negative. See **Figure 2.2.32**.

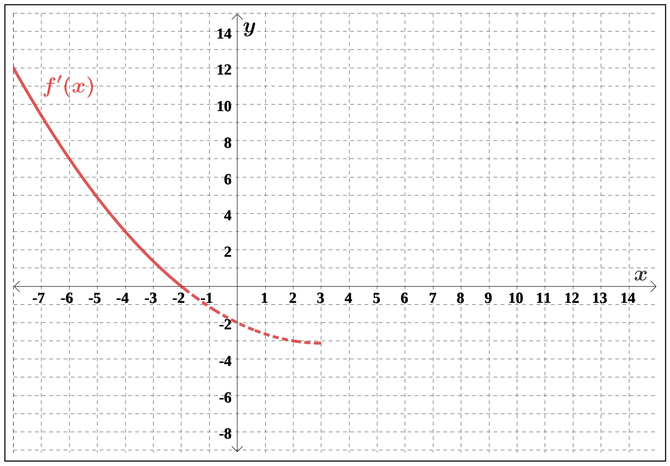


Figure 2.2.32: Sketching f'

For the interval $(3, 8)$, we need to draw y-values of $f'(x)$ that are getting less negative, which means we need to draw upward (but still in Quadrant IV). Note also that $f'(x) = 0$ at $x = 8$, so we will draw upward until the graph touches the x-axis. See **Figure 2.2.33**.

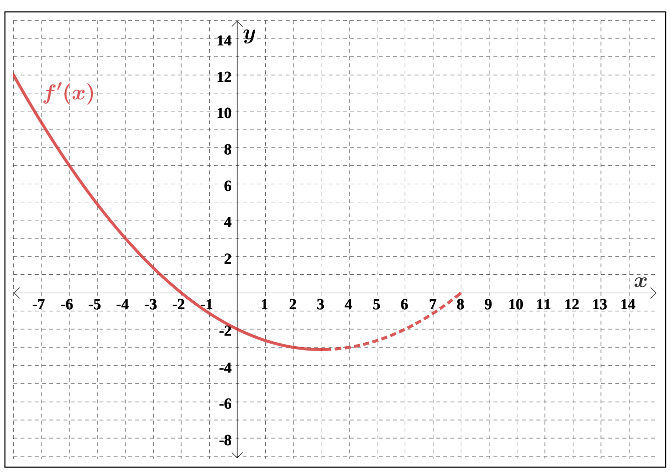


Figure 2.2.33: Sketching f'

We need to continue drawing upward (into Quadrant I) because the y-values of $f'(x)$ are getting more positive on the last interval, $(8,\infty)$. See **Figure 2.2.34**.

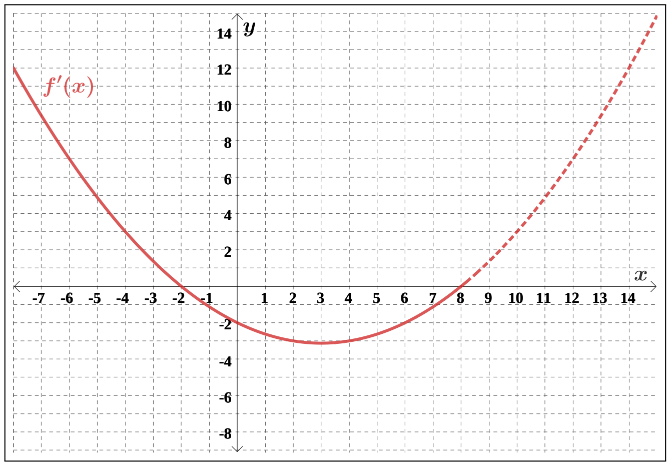


Figure 2.2.34: Sketch of f'

We now have the final sketch of f'! For reference, the original graph of f is shown in **Figure 2.2.35** along with our sketch of f' in **Figure 2.2.36**.

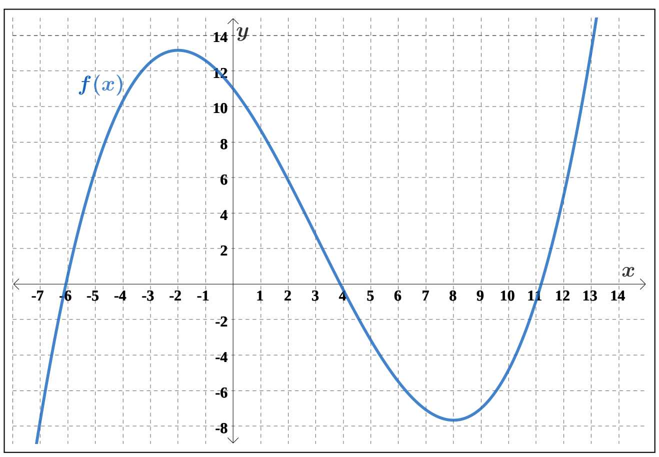


Figure 2.2.35: Graph of f

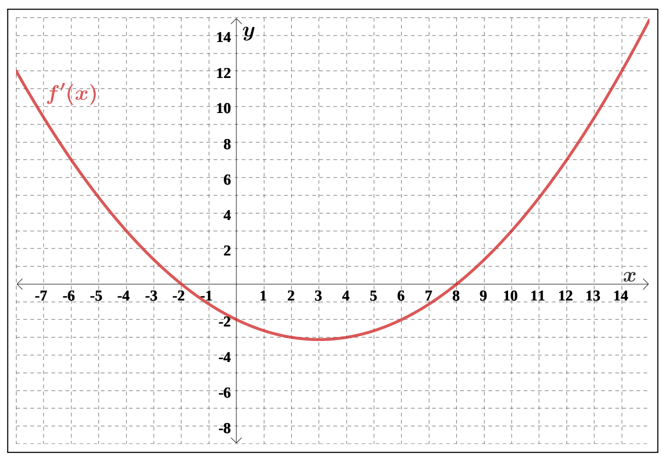


Figure 2.2.36: Sketch of f'

■

*The graph of f' shown in **Figure 2.2.36** was constructed using technology, so the y-values of $f'(x)$ are the exact values of the slopes of f at each x-value. The technique we demonstrated to sketch f' by first drawing a number line usually only approximates the graph of the derivative.*

Notice that at $x=3$, the graph of f' is at the bottom of a "valley". We will learn later that the point on the graph of f at $x=3$ has a special name. We will also learn that there is a relationship between points like this on the graph of a function and places on the graph of its derivative where the graph "switches" between increasing/decreasing. This relationship will be explored in detail in Chapter 3.

Example 13 Given the graph of f shown in **Figure 2.2.37**, sketch the graph of $f\,'$.

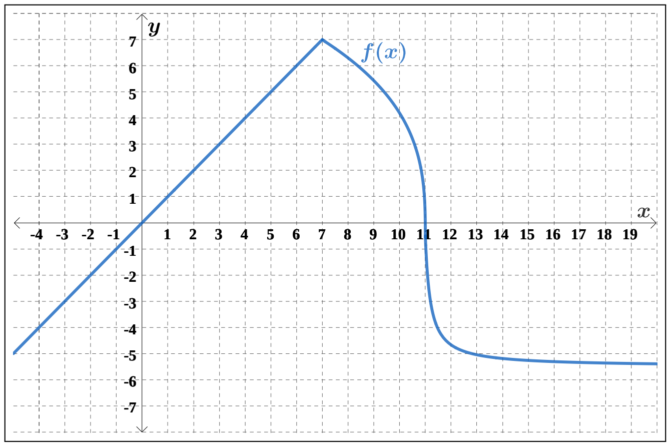


Figure 2.2.37: Graph of a function f

Solution:
Following the method demonstrated in the last example, we need to find the "important" x-values where the graph of f has horizontal tangent lines and where $f\,'(x)$ does not exist in order to partition our number line.

We see the graph of f does not have any horizontal tangent lines. However, because the graph of f has a sharp turn at $x = 7$ and a vertical tangent line at $x = 11$, $f\,'(x)$ does not exist at these x-values.

Now, we can construct a number line with these important x-values, $x = 7$ and $x = 11$, and their corresponding information about $f\,'$ (i.e., $f\,'(x)$ does not exist at these two values of x). See **Figure 2.2.38**.

Figure 2.2.38: First stage of creating a number line to aid in sketching the graph of $f\,'$

Next, we need to draw sample tangent lines and plus/minus signs for each interval as well as indicate if the slopes are getting "more" or "less" positive/negative (or are constant).

Due to the fact that f is linear on the interval $(-\infty, 7)$, the tangent lines lie on the actual graph. Because this line segment has a positive slope, the tangent lines have a positive slope. Thus, for the interval $(-\infty, 7)$, we will draw a sample tangent line with a positive slope above the number line and a plus sign below the number line. Because f is linear on this interval, the slopes of the tangent lines are constant. So we write "constant" below the number line. See **Figure 2.2.39**.

On the interval $(7, 11)$, the tangent lines have negative slopes. Thus, we draw a sample tangent line with a negative slope above the number line and a minus sign below the number line. Notice that the tangent lines are getting steeper on this interval, so we write "more" below the number line. See **Figure 2.2.39**.

On the interval $(11, \infty)$, the tangent lines have negative slopes, but they are now getting less steep. Thus, we draw a sample tangent line with a negative slope above the number line and a minus sign with the word "less" below the number line. See **Figure 2.2.39**.

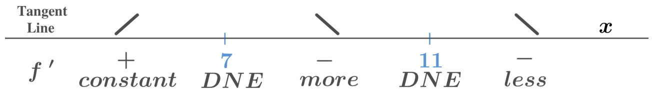


Figure 2.2.39: Number line to aid in sketching the graph of f'

Now, we can use the information on our number line to graph f'. On the interval $(-\infty, 7)$, f is linear, so its slope is constant. Thus, we can calculate the value of $f'(x)$ on this interval exactly. Looking at the graph of f, we see that the slope (rise over run) is 1. Hence, we can draw a horizontal line on the graph of f' at $y = 1$ on the interval $(-\infty, 7)$. Note that we draw a hole at $x = 7$ because $f'(x)$ does not exist at $x = 7$. See **Figure 2.2.42**.

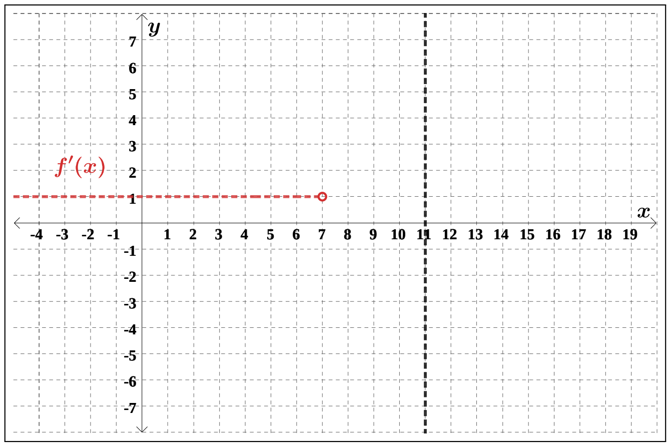


Figure 2.2.40: Sketching f'

On the interval $(7, 11)$, we need to move to Quadrant IV because the y-values of $f'(x)$ are negative. Note that we must also draw a hole at $x = 7$ for this part of the graph because $f'(7)$ does not exist. Also, because the slopes are getting more negative, we need to draw downward.

At $x = 11$, we see that $f'(x)$ does not exist. Due to the fact that the graph of f has a vertical tangent line at $x = 11$, the graph of f' will have a vertical asymptote at $x = 11$. This is true in general! The graph of f' will have a vertical asymptote if any part of the graph of f becomes vertical (note this also happens if the graph of f has a cusp). Why? If the graph of f becomes vertical, its slopes are getting infinitely positive or negative, and so the values (i.e., y-values) of $f'(x)$ are getting infinitely positive or negative resulting in a vertical asymptote. See **Figure 2.2.41**.

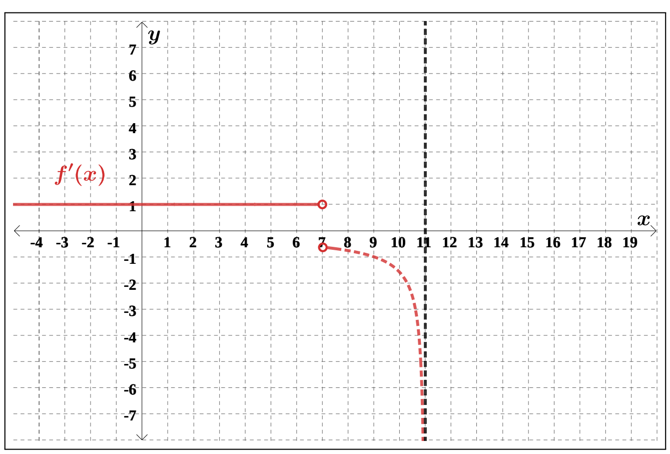


Figure 2.2.41: Sketching f'

Finally, we need to draw y-values of $f\,'(x)$ that are getting less negative on the right side of the vertical asymptote at $x = 11$. We see that this behavior continues as x approaches positive infinity. This means that the graph of $f\,'$ must remain in Quadrant IV and continue to get closer and closer to the x-axis without ever touching it. Therefore, the x-axis (i.e., $y = 0$) will be a horizontal asymptote for the graph of $f\,'$ in this direction. See **Figure 2.2.42**.

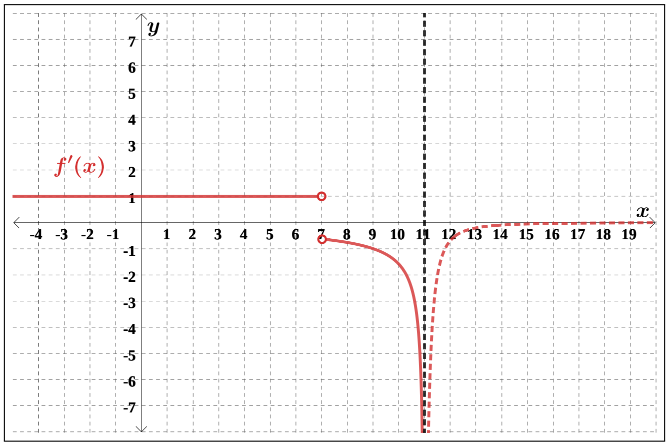


Figure 2.2.42: Sketching $f\,'$

We now have the final sketch of $f\,'$! For reference, the original graph of f is shown in **Figure 2.2.43** along with our sketch of $f\,'$ in **Figure 2.2.44**.

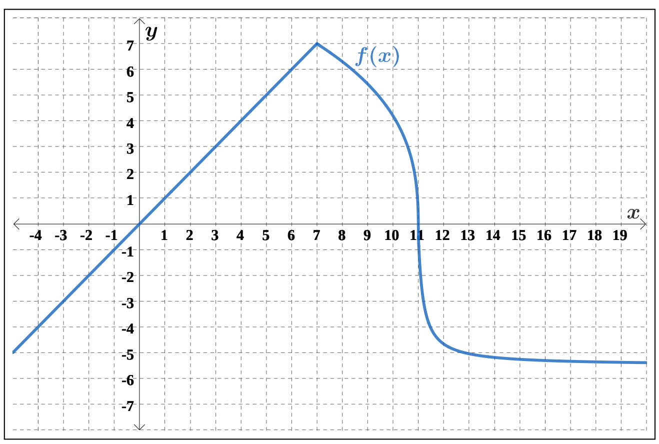


Figure 2.2.43: Graph of f

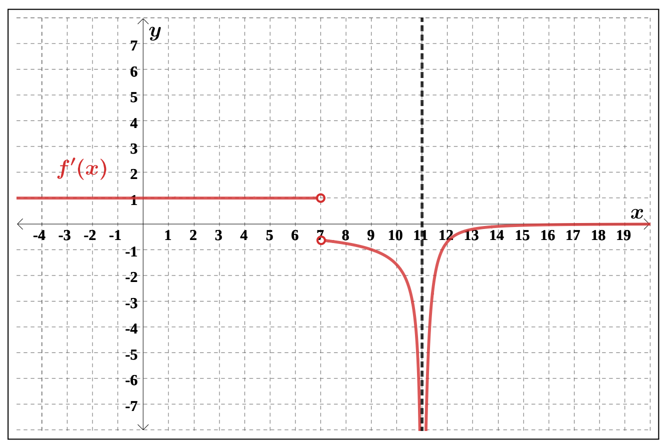


Figure 2.2.44: Sketch of $f\,'$

∎

As stated in the previous example, if $f\,'(x)$ does not exist at a particular x-value, the graph of $f\,'$ will either have a hole or vertical asymptote at that x-value. We must observe the behavior of the slopes of the tangent lines near such x-values to determine whether the graph of $f\,'$ will have a hole or vertical asymptote.

Try It # 7:
Given the graph of f shown in **Figure 2.2.45**, sketch the graph of $f\,'$.

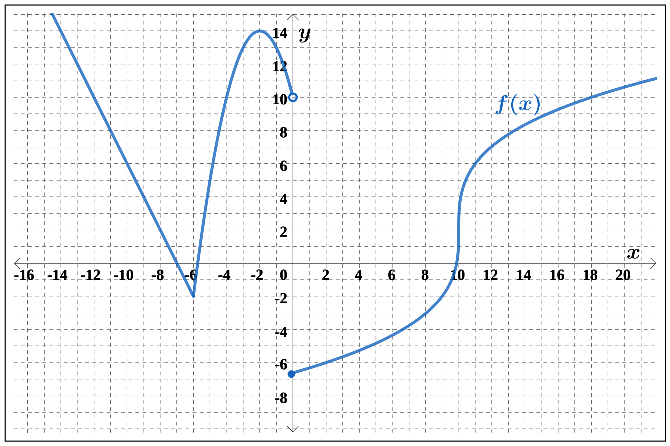


Figure 2.2.45: Graph of a function f

Enrichment: Weierstrass Function

We can create graphs of continuous functions that are nondifferentiable at several places just by putting corners at those places, but how many corners can a continuous function have? In other words, how "badly" can a continuous function fail to be differentiable?

In the mid-1800s, the German mathematician Karl Weierstrass surprised and even shocked the mathematical world by creating a function that was continuous everywhere but differentiable nowhere — a function whose graph was everywhere connected and everywhere corners! The creation of it is a multistep process, which shows just how rare these functions actually are. We will outline the idea of its creation without any rigorous proofs.

Let's start with a function f_1 that "zigzags" between the y-values -1 and 1 with a corner at each integer. This starting function, f_1, is continuous everywhere and is differentiable everywhere except at the integers. See **Figure 2.2.46**.

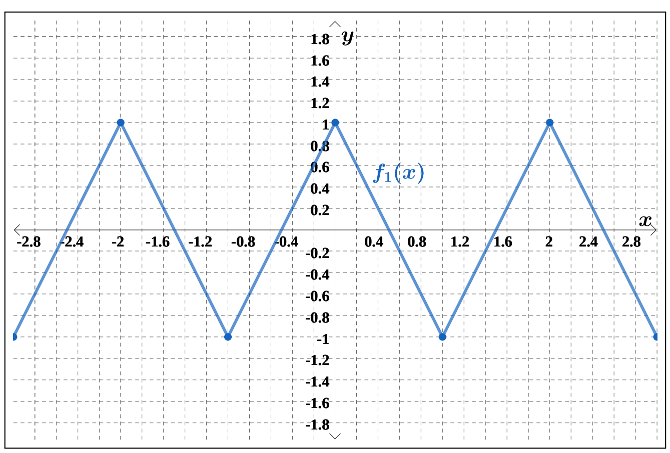


Figure 2.2.46: A function, f_1, that "zigzags" between $y = -1$ and $y = 1$

Next, we create a list of functions $f_2, f_3, f_4, \ldots$, each of which is "shorter" than the previous one but with many more corners than the previous one. For example, we might make f_2 "zigzag" between the y-values $-\frac{1}{4}$ and $\frac{1}{4}$ and have corners at $x = \pm\frac{1}{2}, \pm\frac{2}{2}$ (which equals ± 1), $\pm\frac{3}{2}, \pm\frac{4}{2}$ (which equals ± 2), $\pm\frac{5}{2}$, etc. See **Figure 2.2.47**.

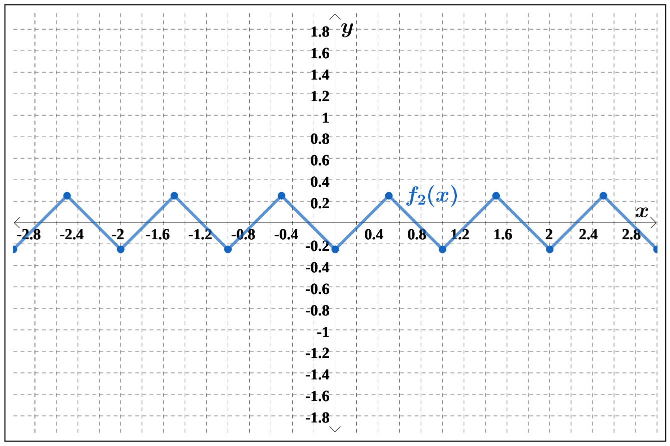


Figure 2.2.47: A function, f_2, that "zigzags" between $y = -1/4$ and $y = 1/4$

We make f_3 "zigzag" between the y-values $-\frac{1}{9}$ and $\frac{1}{9}$ with corners at $x = \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{3}{3}$ (which equals ± 1), etc. See **Figure 2.2.48**.

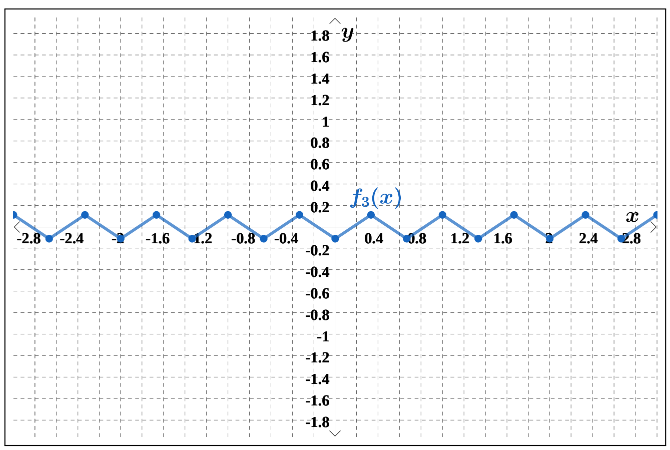


Figure 2.2.48: A function, f_3, that "zigzags" between $y = -1/9$ and $y = 1/9$

Now, if we add f_1 and f_2, we still have a continuous function. Adding two continuous functions results in a continuous function! (See if you can show why this is true using the Properties of Limits and the definition of continuity at a point from **Section 1.2** and **Section 1.4**, respectively.) Moreover, $f_1 + f_2$ has a corner at every integer and every half! If we then add f_3 to that, we still have a continuous function with even more corners.

We can continue adding the functions from our list together, and the result will be a function that has corners at almost every point. It is impossible to actually graph this function! You can attempt to imagine it and draw approximations, but it is a strange, totally "bent" function. Until Weierstrass created this function, most mathematicians thought a continuous function could only be nondifferentiable at a few places. This function was (and is) considered "pathological," a great (or not so great) example of how bad something can be. The authors wholeheartedly agree with the reaction of the famous mathematician Charles Hermite when confronted by the Weierstrass function: "I turn away with fright and horror from this lamentable evil of functions which do not have derivatives."

And so, all other functions in this textbook will be differentiable except at possibly a few points.

Try It Answers

1. **a.** $f'(x) = 6x - 5$

   b. -5

   c. 7

   d. $x = 5/6$

2. **a.** $g'(x) = \dfrac{1}{2\sqrt{x}}$

   b. $h'(x) = \dfrac{-2}{(x-4)^2}$

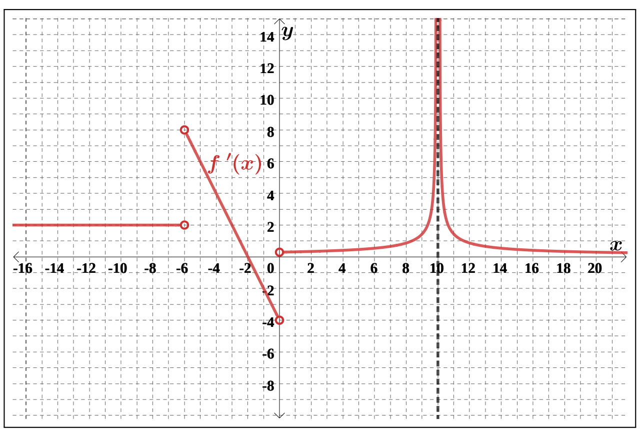
   c. $f'(x) = \dfrac{1}{2\sqrt{x-6}}$

3. $y = 11x + 8$

4. \$7 per stylus pen; When 120 stylus pens are sold, revenue is increasing at a rate of \$7 per stylus pen.

5. −\$62 per computer; When 800 computers are manufactured and sold, profit is decreasing at a rate of \$62 per computer.

6. $(-\infty, -6) \cup (-6, 0) \cup (0, 10) \cup (10, \infty)$

7. 

EXERCISES

Basic Skills Practice

For Exercises 1 - 3, find $f'(x)$.

1. $f(x) = 4x$

2. $f(x) = -3x + 1$

3. $f(x) = 2x - 9$

For Exercises 4 - 6, find $\dfrac{dy}{dx}$.

4. $y = x^2 - 7$

5. $y = 3x^2 - 6x + 1$

6. $y = -5x^2 + 2x - 18$

For Exercises 7 - 9, find the slope of the line tangent to the graph of the function at the given x-value.

7. $f(x) = 9x - 2$ at $x = 5$

8. $g(x) = -x^2 + 13$ at $x = -3$

9. $h(x) = 2x^2 + x - 4$ at $x = 1$

For Exercises 10 - 12, the derivative of a function is given. Use the derivative to find the equation of the line tangent to the graph of the function at the given point.

10. $f'(x) = 3x^2 + 2x$ at $(1, -8)$

11. $g'(x) = 16x^3 - 9x^2 + 4x$ at $(-1, 9)$

12. $h'(x) = -10x^4 + 21x^2 - 6$ at $(0, 7)$

13. Given $f(x) = x^2 + 5x + 2$ and $f'(x) = 2x + 5$, find the equation of the line tangent to the graph of f at the point $(-3, -4)$, and graph both f and the tangent line on the same axes.

For Exercises 14 - 16, the derivative function f' is given. Use the derivative to find the x-value(s) where the graph of f has a horizontal tangent line. In other words, determine where $f'(x) = 0$.

14. $f'(x) = x^2 + x - 6$

15. $f'(x) = 4x^3 - 16x^2 - 20x$

16. $f'(x) = 2x^2 + 4x - 3$

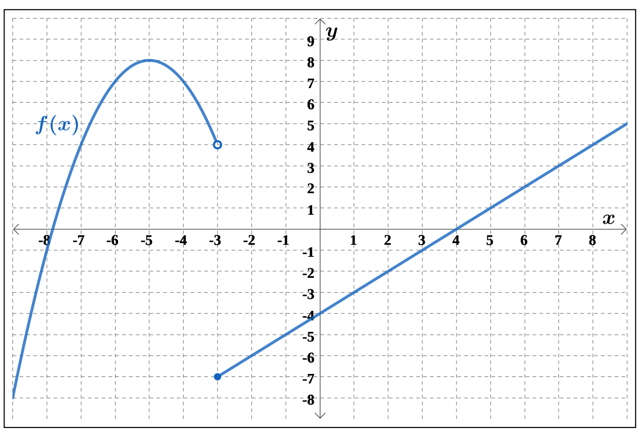
17. The price-demand function for kids' Bright Lights sneakers is given by $p(x) = 207 - 0.2x$, where $p(x)$ is the price, in dollars, when x pairs of sneakers are sold. Find the rate of change of price when 500 pairs of sneakers are sold.

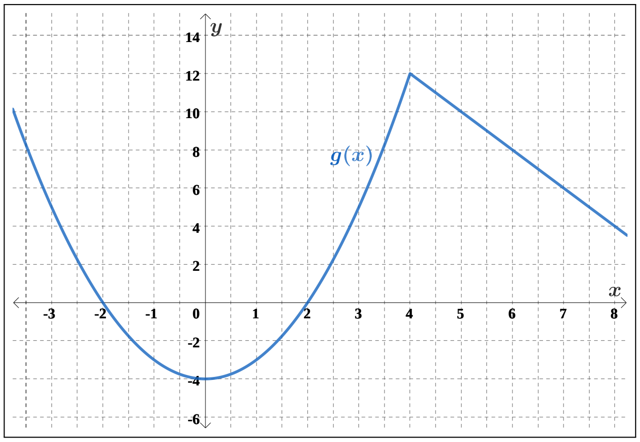
18. The revenue function for the manufacturer of Bright Lights sneakers referred to in the previous exercise is given by $R(x) = 207x - 0.2x^2$, where $R(x)$ is the revenue, in dollars, from selling x pairs of sneakers. Find the rate of change of revenue when 750 pairs of sneakers are sold.

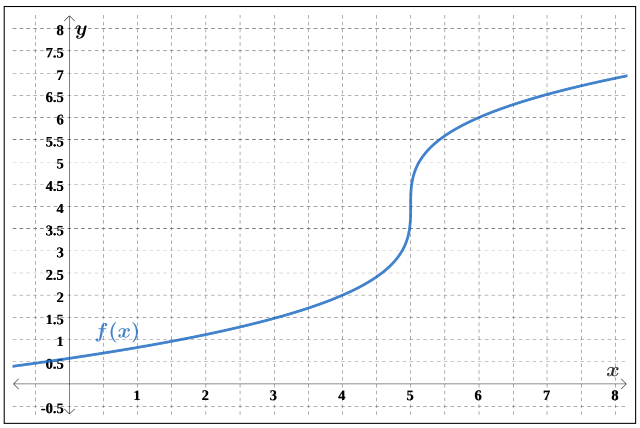
19. The profit function for a company that sells designer handbags is given by $P(x) = -0.2x^2 + 450x - 10{,}000$, where $P(x)$ is the profit, in dollars, when x handbags are sold. Find the rate of change of profit when 1000 handbags are sold.

20. The position of a particle after t seconds is given by $s(t) = 4t^2 - 5t$ feet. Find the velocity function, $v(t)$, of the particle.

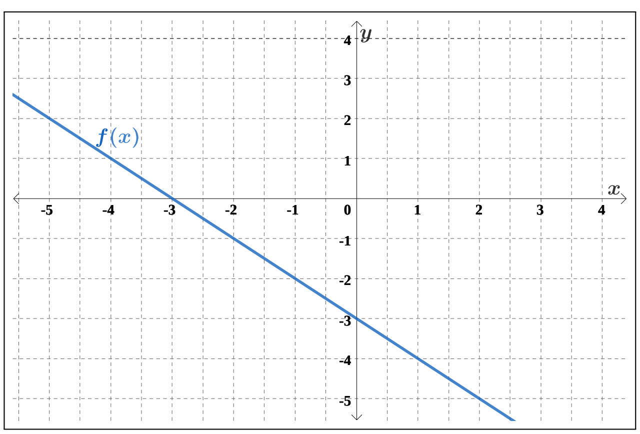
For Exercises 21 – 23, the graph of a function is shown. Use the graph to determine the x-value(s) where the derivative of the function does not exist.

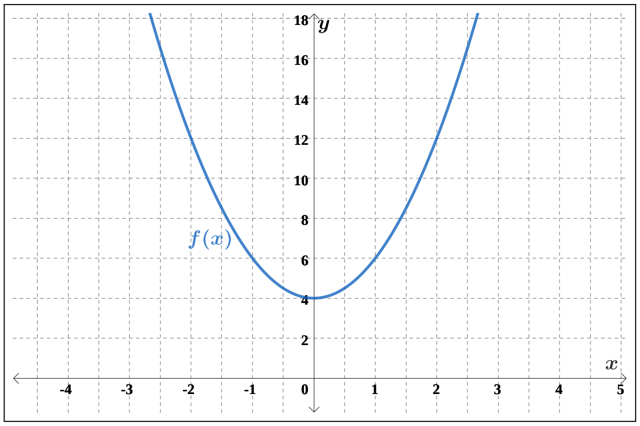
21. 

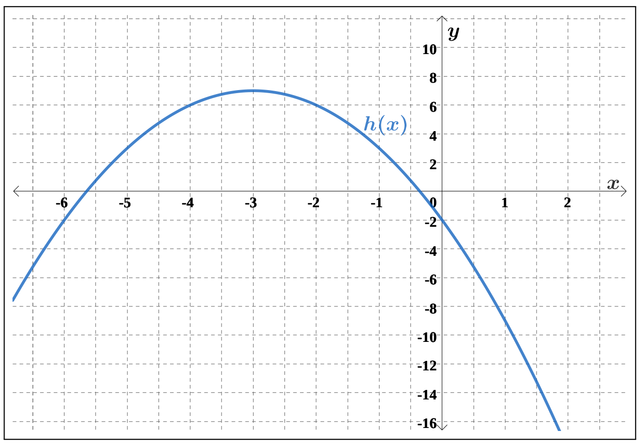
22. 

23. 

For Exercises 24 – 26, the graph of a function is shown. Use the graph to sketch the derivative of the function.

24. 

25. 

26. 

Intermediate Skills Practice

For Exercises 27 - 29, find $f'(x)$.

27. $f(x) = \dfrac{2}{x}$

28. $f(x) = \dfrac{4}{5x+3}$

29. $f(x) = \dfrac{x}{8-x}$

For Exercises 30 - 32, find $\dfrac{dy}{dx}$.

30. $y = \dfrac{3x}{x-7}$

31. $g(x) = \sqrt{5-x}$

32. $h(x) = \sqrt{2x+1}$

For Exercises 33 - 35, find the slope of the line tangent to the graph of the function at the given x-value.

33. $f(x) = \dfrac{-6}{12-x}$ at $x = 7$

34. $g(x) = \dfrac{2x}{3x+5}$ at $x = -5$

35. $h(x) = \sqrt{x+4}$ at $x = 12$

For Exercises 36 - 39, find the equation of the line tangent to the graph of the function at the given x-value.

36. $f(x) = 4x^2 - x + 1$ at $x = -3$

37. $g(x) = -x^2 + 5x - 10$ at $x = 2$

38. $h(x) = \dfrac{-x}{2x-7}$ at $x = 4$

39. $f(x) = \sqrt{9-x}$ at $x = 5$

40. Find the equation of the line tangent to the graph of $f(x) = 3x^2 - 2x - 5$ at $x = 1$, and graph both and the tangent line on the same axes.

For Exercises 41 - 43, find the x-value(s) where the graph of the function has a horizontal tangent line.

41. $f(x) = -4x^2 + 10$

42. $g(x) = 3x^2 - 5x + 2$

43. $h(x) = -2x^2 + 8x - 1$

For Exercises 44 - 46, find the x-value(s) where the graph of the function has the indicated slope.

44. $f(x) = 2x^2 - 7x$; slope is 25

45. $g(x) = -x^2 + 6x - 11$; slope is 18

46. $h(x) = 5x^2 + 8$; slope is 40

47. Professor Peacock and Mrs. Plum sell a board game called *Who Dunnit?*. They have a revenue function given by $R(x) = 28x - 0.01x^2$, where $R(x)$ is the revenue, in dollars, when x board games are sold. Find the rate of change of revenue when 300 board games are sold, and interpret your answer.

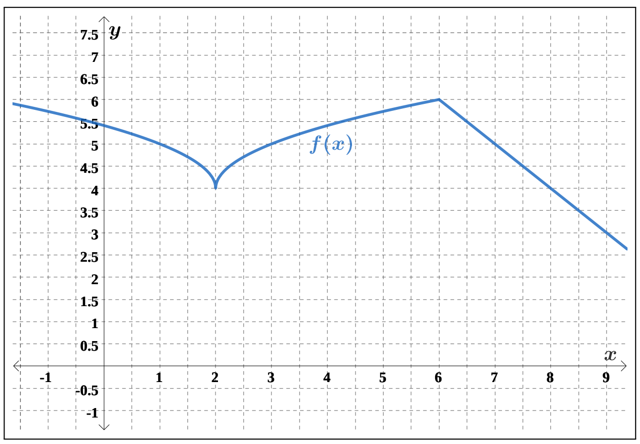
48. A company that makes clothes dryers has a weekly cost function given by $C(x) = 3000 + 12x + 0.2x^2$, where $C(x)$ is the cost, in dollars, when x dryers are made each week. Find the rate of change of cost when 450 dryers are made each week, and interpret your answer.

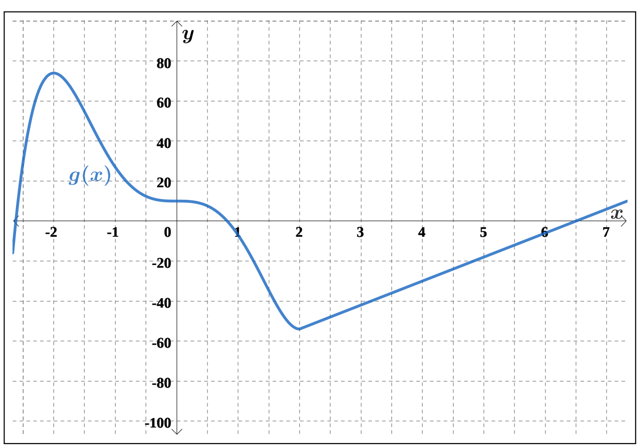
49. An appliance manufacturer that makes and sells microwaves has determined its profit function to be $P(x) = 40x - 0.03x^2 - 5000$ dollars, where x is the number of microwaves made and sold. Find the rate of change of profit when 900 microwaves are made and sold, and interpret your answer.

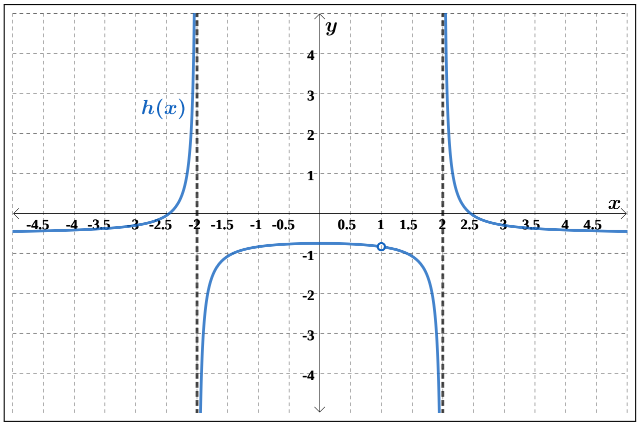
50. A company's total sales, in millions of dollars, after t months is given by $S(t) = 0.5t^2 + 3t + 5$. Find the rate of change of sales after 7 months, and interpret your answer.

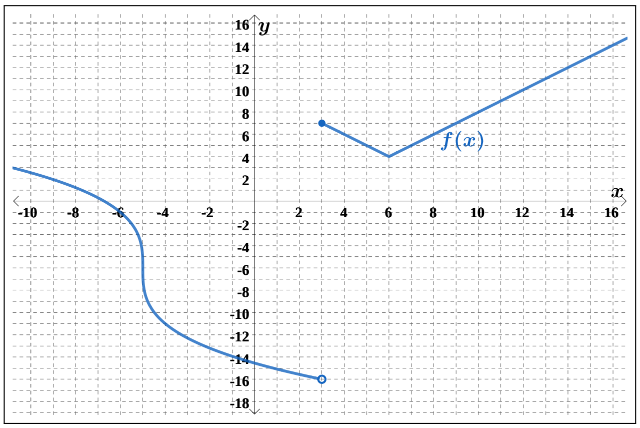
51. The position of an object moving along a horizontal line is given by $s(t) = 3t^2 - 6t + 1$ feet, where t is time in seconds. Find the velocity of the object after 4 seconds.

For Exercises 52 – 55, the graph of a function is shown. Use the graph to determine the x-value(s) where the derivative of the function does not exist and why.

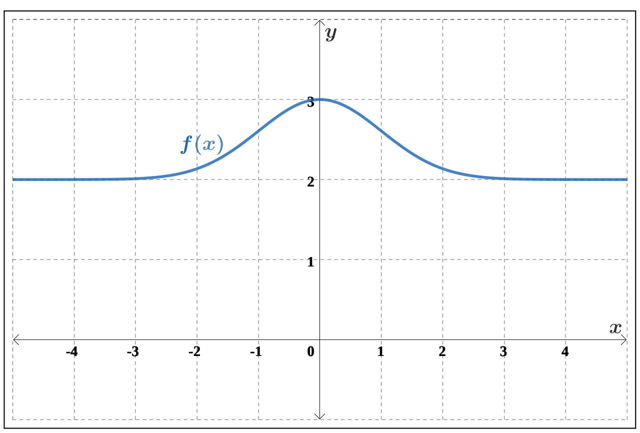
52. 

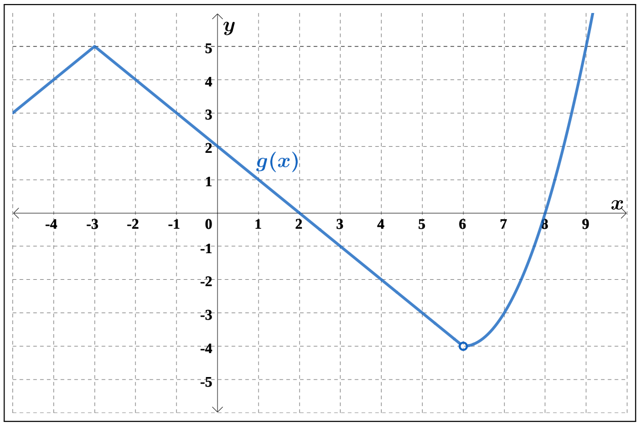
53. 

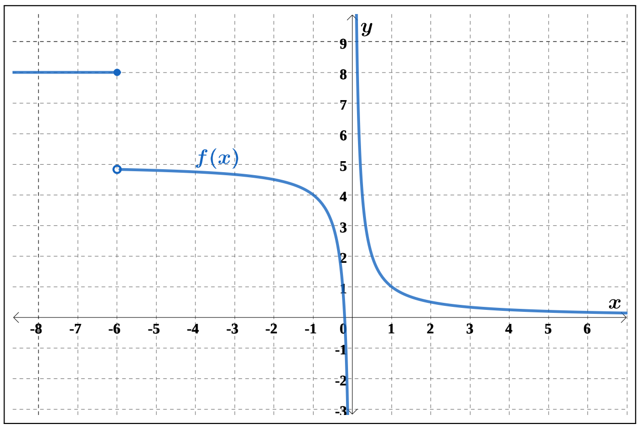
54. 

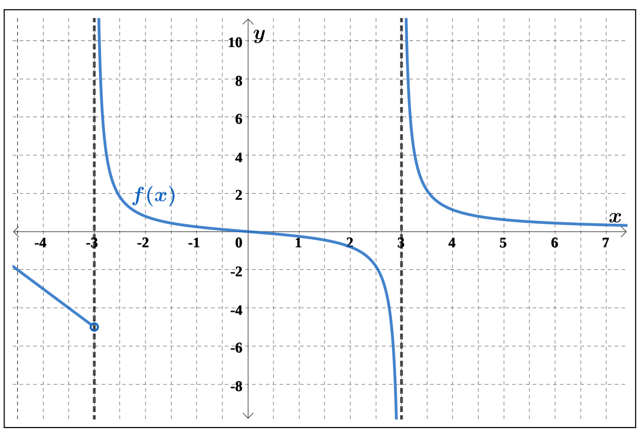
55. 

For Exercises 56 – 59, the graph of a function is shown. Use the graph to sketch the derivative of the function.

56. 

57. 

58. 

59. 

Mastery Practice

60. Given $f(x) = ax + b$, where a and b are positive constants, find $\dfrac{df}{dx}$.

61. Find $\dfrac{d}{dt}\left(-3t^2 - 0.5t + 6\pi\right)$.

62. Given $y = \dfrac{2x}{17-x}$, find $y\,'$.

63. Find $\dfrac{dy}{dx}$ if $y = \dfrac{2-3x}{8x+10}$.

64. Given $f(t) = \dfrac{t^2+4}{t-1}$, find $f\,'(t)$.

65. Given $q(x) = 12 - \sqrt{x}$, find $q\,'(x)$.

66. Find $\dfrac{d}{dx}\left(\sqrt{7-2x}\right)$.

67. Given $h(t) = \sqrt{5t+9} - 4t$, find $\dfrac{dh}{dt}$.

68. Given $y = x^3 + 4x - 2$, find $y\,'$.

69. Given $p(x) = \sqrt{3x+50}$, find the slope of the line tangent to the graph of p at $x = a$, where a is a constant greater than $-50/3$.

For Exercises 70 - 72, find the equation of the line tangent to the graph of the function at the given value.

70. $f(x) = \dfrac{8-6x}{3-2x}$ at $x = 1$

71. $y = \sqrt{4x+9}$ at $x = 4$

72. $h(t) = t^3 - 2t + 1$ at $t = -2$

For Exercises 73 and 74, find the x-value(s) where the graph of the function has a horizontal tangent line.

73. $g(x) = \dfrac{x^2}{3x-1}$

74. $p(x) = ax^2 + bx + c$, where a, b, and c are positive constants.

75. Given $f(x+h) - f(x) = 3h^2 + 4xh + 8h$, find $f\,'(-1)$.

76. Find where the graph of the function $g(t) = \sqrt{8t+1}$ has a slope of $\frac{4}{5}$.

77. Super Rad Bikez specializes in selling glow in the dark bikes for kids. It has determined its price-demand function to be $p(x) = 200 - 0.8x$, where $p(x)$ is the price, in dollars, of each bike when x bikes are sold.

    (a) Find the rate of change of revenue when 150 bikes are sold, and interpret your answer.

    (b) Find the rate of change of revenue when 60 bikes are sold, and interpret your answer.

    (c) Find the rate of change of revenue when 125 bikes are sold, and interpret your answer.

78. A curling iron manufacturer has revenue and cost functions, both in dollars, given by $R(x) = 35x - 0.1x^2$ and $C(x) = 4x + 2000$, respectively, where x is the number of curling irons made and sold.

   (a) Find the rate of change of the manufacturer's profit when 170 curling irons are made and sold, and interpret your answer.

   (b) Based on your answer in part (a), if the current production level is 170 curling irons, should the manufacturer increase production? Why or why not?

79. The position of a particle after t seconds is given by $s(t) = \dfrac{t}{1+t^2}$ feet. Find the velocity function, $v(t)$, of the particle.

80. A company's total sales, in millions of dollars, after t months is given by $S(t) = 2\sqrt{t} + 17$. Find the rate of change of sales after one year, and interpret your answer. Round to eight decimal places, if necessary.

81. An arrow shot upward from the ground will have a height of $s(t) = -16t^2 + 128t$ feet after t seconds.

   (a) Find the velocity of the arrow after 2 seconds.

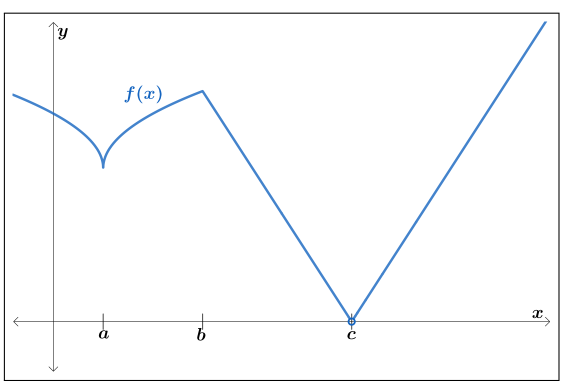
   (b) When will the velocity of the arrow be 0 feet per second?

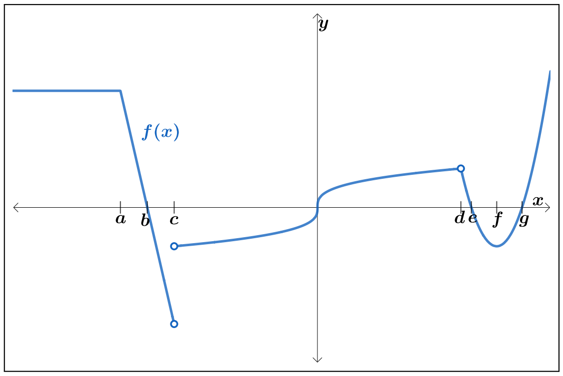
For Exercises 82 and 83, sketch the graph of a function, f, satisfying the given conditions.

82.
- f is continuous on $(-\infty, \infty)$.
- $f'(-3)$ does not exist because there is no tangent line at $x = -3$.
- $f'(1)$ does not exist because there is no tangent line at $x = 1$.

83.
- f is continuous on $(-\infty, 10) \cup (10, \infty)$.
- $f'(2)$ does not exist because there is a vertical tangent line at $x = 2$.
- $f'(10)$ does not exist because there is no tangent line at $x = 10$.

For Exercises 84 and 85, the graph of a function is shown. Use the graph to sketch the derivative of the function.

84. 

85. 

86. Find the equation of the line tangent to the graph of $f(t) = \sqrt{16+t}$ at $t = 0$, and graph both f and the tangent line on the same axes.

Communication Practice

87. Does $\lim_{x\to 0} \frac{f(x+h)-f(x)}{h}$ represent $f'(x)$? Why or why not?

88. Explain how to find the x-value(s) where the graph of a function has a horizontal tangent line.

89. Suppose $P(x)$ is the weekly profit, in dollars, of a company that sells x items each week. Interpret $P'(40) = 80$.

90. Explain why a function fails to be differentiable at a corner.

91. If a function is continuous at $x = c$, is it also differentiable at $x = c$? Explain.

92. If $f'(8)$ exists, is f continuous at $x = 8$? Explain.

93. If $g'(4)$ does not exist because there is a discontinuity at $x = 4$, does that mean the graph of g has a vertical tangent line at $x = 4$? Explain.

2.3 Introductory Derivative Rules and Marginal Analysis

The derivative is a powerful tool, but as we saw in the previous sections, the process of finding the derivative using the limit definition is quite a tedious process. Fortunately (or unfortunately), one thing mathematicians are good at is abstraction.

For instance, instead of working through the limit definition multiple times to find the value of the derivative at several x-values, we abstracted and worked through the process once to find a rule for the derivative function and then used this function to easily find the value of the derivative at several x-values. Now, instead of finding the derivative of a function using the limit definition of the derivative, we will use formulas to quickly find derivatives of a variety of functions!

Although for this and the next two sections we will have to do some algebra that may seem tedious at times, an important thing to remember is that **using the techniques we will learn to find the derivative is easier than using the limit definition of the derivative.**

Learning Objectives:

In this section, you will learn how to use introductory derivative rules to calculate derivatives of functions and solve problems involving real-world applications. Upon completion you will be able to:

- Calculate the derivative of a constant function.
- Calculate the derivative of a power function.
- Calculate the derivative of an exponential function (base e and base b).
- Calculate the derivative of a logarithmic function (base e and base b).
- Calculate the derivative of a function involving sums, differences, and constant multiples of constant, power, exponential, and/or logarithmic functions.
- Calculate the slope of a tangent line using the Introductory Derivative Rules.
- Find the equation of a tangent line using the Introductory Derivative Rules.
- Graph a function and a line tangent to the graph of the function on the same axes.
- Determine the x-value(s) where the graph of a function has a horizontal tangent line using the Introductory Derivative Rules.
- Determine the x-value(s) where the graph of a function has a tangent line of a given slope using the Introductory Derivative Rules.
- Determine the x-value(s) where a function has a specified (instantaneous) rate of change using the Introductory Derivative Rules.
- Calculate the (instantaneous) rate of change of a function involving a real-world scenario, including cost, revenue, profit, and position, using the Introductory Derivative Rules.
- Interpret the meaning of the derivative of a function involving a real-world scenario, including cost, revenue, and profit.
- Calculate marginal cost, revenue, and profit using the Introductory Derivative Rules.
- Interpret the meaning of marginal cost, revenue, and profit.

- Estimate the cost, revenue, or profit of an item using marginal analysis and the Introductory Derivative Rules.
- Compute the exact cost, revenue, or profit of an item.
- Estimate the cost, revenue, or profit of a total number of items using marginal analysis and the Introductory Derivative Rules.

Introductory Derivative Rules

Finding the derivative is, in a way, like analyzing a puzzle that has already been put together. We are given the whole picture (the function), and we need to identify the "pieces" of the function to take the derivative. Let's start by examining the types of "pieces of the puzzle" we will see and how to find the derivative of each piece. Then, we will focus on how to combine the derivatives of the pieces to find the derivatives of more complicated functions.

N *We will not be proving any of these rules rigorously, but we will explain them to some degree. We do want them to make sense after all!*

Let's start by examining the derivative of a linear function: $f(x) = mx+b$, where m is the slope and b is the y-intercept of the graph of the function. Keep in mind that the derivative, $f'(x)$, is the slope of the tangent line. When f is linear, the tangent line is the line itself! Because the slope of the line is m, we can conclude that $f'(x) = m$.

Now, let's look at a constant function: $f(x) = k$, where k is any real number. The graph of a constant function is a horizontal line, and the slope of a horizontal line is zero. So $f'(x) = 0$.

Next, let's explore the relationship between g' and f' if $g(x) = 4f(x)$. This means the graph of g is four times as high as the graph of f. If we calculate the slopes of the secant lines of g and f, we will see that they follow this same relationship: The slopes of the secant lines of g are four times those of f. This will hold true for the tangent lines as well if we take the limit: The slopes of the tangent lines of g are four times those of f! This is true for every tangent line at every point on the graph of g. This means the derivative of g is four times the derivative of f. In other words, $g'(x) = 4f'(x)$. This gives us another rule:

Constants come along for the ride: $\frac{d}{dx}(k\cdot f(x)) = k\left(\frac{d}{dx}(f(x))\right)$, where k is any real number.

A complete list of the Introductory Derivative Rules, which includes the rules we have explored so far as well as the rules for combinations of functions, is given below:

Theorem 2.2 Introductory Derivative Rules

Constant:	$\frac{d}{dx}(k) = 0$	where k is any real number
Power:	$\frac{d}{dx}(x^n) = nx^{n-1}$	where n is any real number
Special Case:	$\frac{d}{dx}(x) = 1$	(because $x = x^1$)
Exponential:	$\frac{d}{dx}(b^x) = b^x \ln(b)$	where b is any positive real number
Special Case:	$\frac{d}{dx}(e^x) = e^x$	(because $\ln(e) = 1$)
Logarithm:	$\frac{d}{dx}(\log_b(x)) = \frac{1}{x\ln(b)}$	where b is any positive real number
Special Case:	$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$	(because $\ln(e) = 1$)
Sum/Difference:	$\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$	where f and g are differentiable functions
Constant Multiple:	$\frac{d}{dx}(k \cdot f(x)) = k\left(\frac{d}{dx}(f(x))\right)$	where k is any real number and f is a differentiable function

N *The Power Rule can be used for* **any** *power! This means it applies to negative exponents and rational exponents as well!*

The Sum/Difference and Constant Multiple Rules combined with the Power Rule allow us to find the derivative of any polynomial and more!

■ **Example 1** Find the derivative of each of the following functions.

a. $f(x) = x^5$

b. $g(x) = \pi^{5/8} + 10x$

c. $f(t) = 3\sqrt[5]{t} - \frac{e^t}{7}$

d. $g(y) = 2.6y^{-4} - \ln(9)$

e. $Q(x) = 8.3\ln(x) - x^{1.1} + x^{15/4}$

f. $P(x) = 14.223\,(\log_{44}(x) + 3^x)$

Solution:

a. This function, $f(x) = x^5$, is difficult to find the derivative of using the limit definition. It can be done, but the algebra gets very messy, very quickly.

Because this is a power function (the variable is in the base), we can use the Power Rule. We can think of the Power Rule as "bring the power down and reduce it by one":

$$\begin{aligned} f(x) &= x^5 \implies \\ f'(x) &= 5 \cdot x^{5-1} \\ &= 5x^4 \end{aligned}$$

b. First, notice $g(x) = \pi^{5/8} + 10x$ consists of a sum of two functions, so we can apply the Sum Rule:

$$g'(x) = \frac{d}{dx}\left(\pi^{\frac{5}{8}}\right) + \frac{d}{dx}(10x)$$

It is important to recognize that $\pi^{5/8}$ is a number! It is approximately 2.05. We may be tempted to use the Power Rule, but it falls under the Constant Rule because there is no x in the term. Thus, $\frac{d}{dx}\left(\pi^{5/8}\right) = 0$. Now, the derivative becomes

$$\begin{aligned} g'(x) &= 0 + \frac{d}{dx}(10x) \\ &= \frac{d}{dx}(10x) \end{aligned}$$

Next, because we have a constant, 10, times a function, x, we can apply the Constant Multiple Rule and bring the constant to the front:

$$= 10\left(\frac{d}{dx}(x)\right)$$

Due to the fact that $x = x^1$, we have a power function and can apply the Power Rule:

$$\begin{aligned} g'(x) &= 10\left(1x^{1-1}\right) \\ &= 10\left(x^0\right) \\ &= 10 \end{aligned}$$

To confirm our answer, notice that the original function is linear. Recall, as discussed previously, that the derivative of a line is just its slope. Because the slope of this line is 10, we can conclude that $g'(x) = 10$.

c. First, notice $f(t) = 3\sqrt[5]{t} - \frac{e^t}{7}$ consists of a difference of two functions, so we can apply the Difference Rule:

$$f'(t) = \frac{d}{dt}\left(3\sqrt[5]{t}\right) - \frac{d}{dt}\left(\frac{e^t}{7}\right)$$

Next, notice that for each term we have a constant times a function. This is fairly easy to see for $3\sqrt[5]{t}$, but for $\frac{e^t}{7}$, it is easier to see once we rewrite it as $\frac{1}{7}e^t$. Now, we will apply the Constant Multiple Rule, which tells us to bring the constant to the front of each term:

$$\begin{aligned} f'(t) &= \frac{d}{dt}\left(3\sqrt[5]{t}\right) - \frac{d}{dt}\left(\frac{1}{7}e^t\right) \\ &= 3\left(\frac{d}{dt}\left(\sqrt[5]{t}\right)\right) - \frac{1}{7}\left(\frac{d}{dt}\left(e^t\right)\right) \end{aligned}$$

The Introductory Derivative Rules state that the derivative of e^t is e^t, but we do not have any derivative rules for a root function. However, because we can rewrite a root function as a power function, $\sqrt[5]{t} = t^{1/5}$. Now, we can apply the Power Rule. Rewriting the root function and applying the rules gives

$$\begin{aligned} f'(t) &= 3\left(\frac{d}{dt}\left(t^{\frac{1}{5}}\right)\right) - \frac{1}{7}\left(\frac{d}{dt}\left(e^t\right)\right) \\ &= 3\left(\frac{1}{5}t^{\frac{1}{5}-1}\right) - \frac{1}{7}e^t \end{aligned}$$

All we have left to do is calculate the exponent of the first term by getting a common denominator:

$$\begin{aligned} \frac{1}{5} - 1 &= \frac{1}{5} - \frac{5}{5} \\ &= \frac{1-5}{5} \\ &= -\frac{4}{5} \end{aligned}$$

Therefore, $f'(t) = \frac{3}{5}t^{-4/5} - \frac{e^t}{7}$.

> *Be careful when subtracting 1 from rational exponents. Take your time and be careful with the arithmetic.*

d. For this function, $g(y) = 2.6y^{-4} - \ln(9)$, note that it does not matter what the independent variable is. We've seen x and t previously, but we may feel a little uncomfortable using y as the independent variable. We treat it the same as we would any other independent variable.

To find $g'(y)$, first notice that $g(y)$ consists of a difference of two functions. So we apply the Difference Rule:

$$g'(y) = \frac{d}{dy}\left(2.6y^{-4}\right) - \frac{d}{dy}(\ln(9))$$

For $\frac{d}{dy}(\ln(9))$, we may be tempted to say this equals $\frac{1}{9}$, but $\ln(9)$ is a constant (like we saw with $\pi^{5/8}$ in part **b**). Thus, $\frac{d}{dy}(\ln(9)) = 0$. The derivative becomes

$$\begin{aligned} g'(y) &= \frac{d}{dy}\left(2.6y^{-4}\right) - 0 \\ &= \frac{d}{dy}\left(2.6y^{-4}\right) \end{aligned}$$

Because we have a constant times a power function, we apply the Constant Multiple Rule and bring the constant 2.6 to the front and then apply the Power Rule:

$$\begin{aligned} g'(y) &= 2.6\left(\frac{d}{dy}\left(y^{-4}\right)\right) \\ &= 2.6\left(-4y^{-4-1}\right) \\ &= -10.4y^{-5} \end{aligned}$$

> *Be careful when finding the derivative of functions with negative exponents! Remember that* $-4-1=-5$, *not* -3.

e. The function $Q(x) = 8.3\ln(x) - x^{1.1} + x^{15/4}$ has several terms subtracted and added. We apply the Difference Rule and Sum Rule to the appropriate terms. Immediately after that, we apply the Constant Multiple Rule to the first term:

$$\begin{aligned} Q'(x) &= \frac{d}{dx}(8.3\ln(x)) - \frac{d}{dx}\left(x^{1.1}\right) + \frac{d}{dx}\left(x^{\frac{15}{4}}\right) \\ &= 8.3\left(\frac{d}{dx}(\ln(x))\right) - \frac{d}{dx}\left(x^{1.1}\right) + \frac{d}{dx}\left(x^{\frac{15}{4}}\right) \end{aligned}$$

Now, we apply the Logarithm Rule to the first term and the Power Rule to the second and third terms:

$$\begin{aligned} Q'(x) &= 8.3\left(\frac{1}{x}\right) - 1.1x^{1.1-1} + \frac{15}{4}x^{\frac{15}{4}-1} \\ &= \frac{8.3}{x} - 1.1x^{0.1} + \frac{15}{4}x^{\frac{11}{4}} \end{aligned}$$

f. The function $P(x) = 14.223\left(\log_{44}(x) + 3^x\right)$ consists of a constant times a function. Applying the Constant Multiple Rule gives

$$P'(x) = 14.223\left(\frac{d}{dx}\left(\log_{44}(x) + 3^x\right)\right)$$

Notice inside the parentheses there is a sum, so we use the Sum Rule:

$$P'(x) = 14.223\left(\frac{d}{dx}(\log_{44}(x)) + \frac{d}{dx}(3^x)\right)$$

Applying the Logarithm Rule to the first term and the Exponential Rule to the second term gives

$$P'(x) = 14.223\left(\frac{1}{x\ln(44)} + 3^x\ln(3)\right)$$

■

Try It # 1:

Find $\dfrac{df}{dx}$, where

a. $f(x) = 4x^3 + 22.$

b. $f(x) = 17x^{10} + e^x - 1.8x + 1003.$

c. $f(x) = 6x^{-9} - 2^x + 4\log_3(x).$

Algebraic Manipulation

Notice that in part **f** of the previous example, we could have distributed the 14.223 first and then applied the Introductory Derivative Rules. We did not need to that, but there are functions in which algebraic manipulation is necessary before we can find the derivative using the Introductory Derivative Rules. In the coming sections, we will learn rules to use when functions are multiplied or divided. But for now, we will algebraically manipulate such functions and apply the Introductory Derivative Rules.

Example 2 Find the derivative of each of the following functions.

a. $y = \dfrac{3}{x}$

b. $f(x) = x^8\left(x^2 - \dfrac{7}{x^2}\right)$

c. $g(x) = \dfrac{x^{10} - 2x^6 - 11x}{x^6}$

d. $h(x) = \left(2x - 5x^2\right)\left(14x^3 - 22\right)$

Solution:

a. We currently do not have a rule for division (that will come in the next section), but that does not mean we cannot find the derivative of $y = \dfrac{3}{x}$ using the Introductory Derivative Rules. But first, we have to do some algebra. Namely, we need to remember that dividing by x is equivalent to multiplying by x^{-1}:

$$\begin{aligned} y &= \frac{3}{x} \\ y &= 3x^{-1} \end{aligned}$$

Now, we have a constant times a power function, so we can apply the Constant Multiple Rule and Power Rule:

$$\begin{aligned} y' &= \frac{d}{dx}(3x^{-1}) \\ &= 3\left(\frac{d}{dx}\left(x^{-1}\right)\right) \\ &= 3\left(-1x^{-1-1}\right) \\ &= -3x^{-2} \end{aligned}$$

b. Similar to part **a**, we do not have a rule for finding the derivative of a product of two functions (yet!), so we start by distributing x^8:

$$\begin{aligned} f(x) &= x^8\left(x^2 - \frac{7}{x^2}\right) \\ &= x^8 \cdot x^2 - \frac{7x^8}{x^2} \\ &= x^{10} - 7x^6 \end{aligned}$$

Using the Difference Rule, Constant Multiple Rule, and Power Rule gives

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(x^{10} - 7x^6\right) \\ &= \frac{d}{dx}\left(x^{10}\right) - \frac{d}{dx}\left(7x^6\right) \\ &= \frac{d}{dx}\left(x^{10}\right) - 7\left(\frac{d}{dx}\left(x^6\right)\right) \\ &= 10x^{10-1} - 7\left(6x^{6-1}\right) \\ &= 10x^9 - 42x^5 \end{aligned}$$

c. Because we do not have a rule for division (yet!), we work to rewrite the function, $g(x) = \dfrac{x^{10} - 2x^6 - 11x}{x^6}$. Due to the fact that there is only one term in the denominator, we begin by "separating" $g(x)$ into three fractions and simplifying each using laws of exponents:

$$\begin{aligned} g(x) &= \frac{x^{10} - 2x^6 - 11x}{x^6} \\ &= \frac{x^{10}}{x^6} - \frac{2x^6}{x^6} - \frac{11x}{x^6} \\ &= x^4 - 2 - 11x^{-5} \end{aligned}$$

Using the Difference Rule, Constant Multiple Rule, Power Rule, and Constant Rule gives

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(x^4 - 2 - 11x^{-5}\right) \\ &= \frac{d}{dx}\left(x^4\right) - \frac{d}{dx}(2) - \frac{d}{dx}\left(11x^{-5}\right) \\ &= \frac{d}{dx}\left(x^4\right) - \frac{d}{dx}(2) - 11\left(\frac{d}{dx}\left(x^{-5}\right)\right) \\ &= 4x^{4-1} - 0 - 11\left(-5x^{-5-1}\right) \\ &= 4x^3 + 55x^{-6} \end{aligned}$$

d. Again, because $h(x) = \left(2x - 5x^2\right)\left(14x^3 - 22\right)$ consists of a product of two functions and we do not have a rule for this type of multiplication (yet!), we must algebraically manipulate the function. We multiply the functions (using FOIL) to get a polynomial in standard form:

$$\begin{aligned} h(x) &= (2x)\left(14x^3\right) + (2x)(-22) + \left(-5x^2\right)\left(14x^3\right) + \left(-5x^2\right)(-22) \\ &= 28x^4 - 44x - 70x^5 + 110x^2 \\ &= -70x^5 + 28x^4 + 110x^2 - 44x \end{aligned}$$

Using the Sum/Difference Rule, Constant Multiple Rule, and Power Rule gives

$$\begin{aligned} h'(x) &= \frac{d}{dx}\left(-70x^5 + 28x^4 + 110x^2 - 44x\right) \\ &= \frac{d}{dx}\left(-70x^5\right) + \frac{d}{dx}\left(28x^4\right) + \frac{d}{dx}\left(110x^2\right) - \frac{d}{dx}(44x) \\ &= -70\left(\frac{d}{dx}\left(x^5\right)\right) + 28\left(\frac{d}{dx}\left(x^4\right)\right) + 110\left(\frac{d}{dx}\left(x^2\right)\right) - 44\left(\frac{d}{dx}\left(x^1\right)\right) \\ &= -70\left(5x^{5-1}\right) + 28\left(4x^{4-1}\right) + 110\left(2x^{2-1}\right) - 44\left(1x^{1-1}\right) \\ &= -350x^4 + 112x^3 + 220x - 44 \end{aligned}$$

▪

Try It # 2:
Find the derivative of each of the following functions.

a. $f(x) = (3 - x)\left(7x^3 - 22\right)$

b. $g(x) = \dfrac{35x^4 - 12x^3 + 2}{9x^2}$

■ **Example 3** Find the equation of the line tangent to the graph of $g(x) = 10 - x^2$ at $x = 2$, and graph the function and the tangent line on the same axes.

Solution:
Recall that to find the equation of the tangent line at $x = 2$, we need the line's slope and the point on the line when $x = 2$.

Remember that the slope of the tangent line at $x = 2$ is the derivative of the function at $x = 2$. With that in mind, we first need to find the derivative of g. Using the Difference Rule, Constant Rule, and Power Rule gives

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(10 - x^2\right) \\ &= \frac{d}{dx}(10) - \frac{d}{dx}\left(x^2\right) \\ &= 0 - 2x^{2-1} \\ &= -2x \end{aligned}$$

To find the slope of the tangent line at $x = 2$, we calculate $g'(2)$:

$$\begin{aligned} g'(2) &= -2(2) \\ &= -4 \end{aligned}$$

Next, we need to find the point on the tangent line when $x = 2$. Remember that the tangent line touches the function at $x = 2$, so we can substitute $x = 2$ into the original function $g(x) = 10 - x^2$ to get the y-value of the point:

$$\begin{aligned} g(2) &= 10 - (2)^2 \\ &= 6 \end{aligned}$$

This tells us that the tangent line passes through the point $(2, 6)$. Thus, the equation of the tangent line is

$$\begin{aligned} y - 6 &= -4(x - 2) \\ y &= -4x + 8 + 6 \\ y &= -4x + 14 \end{aligned}$$

The graphs of $g(x) = 10 - x^2$ and the line tangent to the graph of g at $x = 2$, $y = -4x + 14$, are shown in **Figure 2.4.1**.

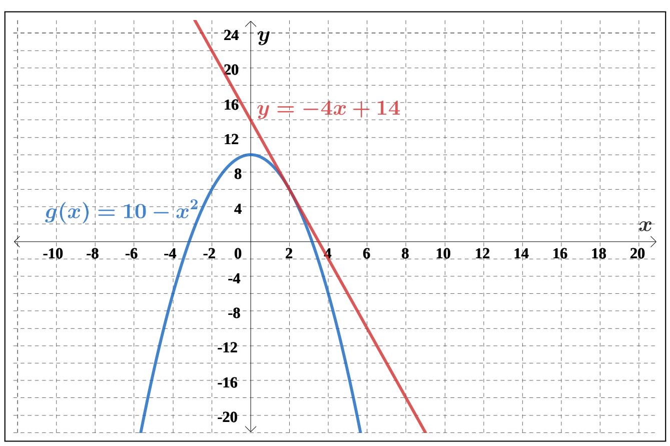


Figure 2.3.1: Graphs of $g(x) = 10 - x^2$ and the tangent line $y = -4x + 14$

■

Try It # 3:
Find the equation of the line tangent to the graph of $f(x) = \sqrt[3]{x}$ at $x = 8$, and graph the function and the tangent line on the same axes.

▪ **Example 4** Better Purchase, a technology store, has a price-demand function given by $p(x) = 300(0.997)^x$, where $p(x)$ is the price, in dollars, of each mouse when x mice are sold. Find the rate of change of price with respect to demand when the demand is 800 mice, and interpret your answer.

Solution:
The rate of change of price with respect to demand is given by the derivative function p'. We will use the Constant Multiple Rule and Exponential Rule to find $p'(x)$:

$$\begin{aligned} p'(x) &= \frac{d}{dx}(300(0.997)^x) \\ &= 300\left(\frac{d}{dx}((0.997)^x)\right) \\ &= 300(0.997)^x \ln(0.997) \end{aligned}$$

Substituting 800 for x to find $p'(800)$ gives

$$\begin{aligned} p'(800) &= 300(0.997)^{800} \ln(0.997) \\ &\approx -\$0.08 \text{ per mouse} \end{aligned}$$

We write a sentence to interpret our answer:

"When 800 mice are demanded, price is decreasing at a rate of $0.08 per mouse."

▪

Try It # 4:
A company has a price-demand function given by $p(x) = 250(0.995)^x$, where $p(x)$ is the price, in dollars, of each item when x items are sold. Find the rate of change of price with respect to demand when 215 items are demanded, and interpret your answer.

▪ **Example 5** The height of a ball thrown upward is given by $s(t) = -16t^2 + 128t$ feet, where t is time in seconds. Find the velocity of the ball after 7 seconds.

Solution:
Recall that velocity is the derivative of the position function. So we find $s'(t)$, or $v(t)$, using the Sum Rule, Constant Multiple Rule, and Power Rule:

$$\begin{aligned} v(t) = s'(t) &= \frac{d}{dt}\left(-16t^2 + 128t\right) \\ &= -16\left(\frac{d}{dt}\left(t^2\right)\right) + 128\left(\frac{d}{dt}\left(t^1\right)\right) \\ &= -16\left(2t^{2-1}\right) + 128\left(1t^{1-1}\right) \\ &= -32t + 128 \end{aligned}$$

Now, we substitute $t = 7$ into the derivative:

$$\begin{aligned} v(7) = s'(7) &= -32 \cdot 7 + 128 \\ &= -96 \text{ feet per second} \end{aligned}$$

After 7 seconds, the ball is moving downward at a rate of 96 feet per second.

▪

The previous example may look familiar because we did it in ***Section 2.2, Example 7****. In that example, we used the limit definition to find the derivative. Notice how much easier it is to use the Introductory Derivative Rules than the limit definition!*

Example 6 Various values of $f(x)$, $g(x)$, $f'(x)$, and $g'(x)$ are given in **Table 2.8**. Use the information in the table to answer each of the following.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-4	-14	-92	19	128
0	0	12	-1	0
4	-2	4	3	16
8	28	172	31	176

Table 2.8: Various values of $f(x)$, $g(x)$, $f'(x)$, and $g'(x)$

a. If $c(x) = g(x) - f(x)$, find $c'(4)$.

b. If $r(x) = 3f(x) + 9^x$, find $r'(0)$.

Solution:

a. We start by using the Difference Rule because $c(x)$ is a difference of two functions. This means $c'(4) = g'(4) - f'(4)$. We find these function values in the table and conclude

$$\begin{aligned} c'(4) &= g'(4) - f'(4) \\ &= 16 - 3 \\ &= 13 \end{aligned}$$

b. Using the Sum Rule, Constant Multiple Rule, and Exponential Rule, we have

$$\begin{aligned} r'(x) &= \frac{d}{dx}(3f(x) + 9^x) \\ &= 3\left(\frac{d}{dx}(f(x))\right) + \frac{d}{dx}(9^x) \\ &= 3f'(x) + 9^x \ln(9) \end{aligned}$$

Substituting $x = 0$ and looking at the table to find $f'(0) = -1$ yields

$$\begin{aligned} r'(0) &= 3f'(0) + 9^0 \ln(9) \\ &= -3 + \ln(9) \end{aligned}$$

Try It # 5:
Using **Table 2.8** from the previous example, answer each of the following.

a. If $p(x) = 14x^3 - 2g(x)$, find $p'(0)$.

b. If $d(x) = 2f(x) + \frac{9}{x}$, find $d'(-4)$.

■ **Example 7** The graphs of the functions f and g are shown in **Figure 2.3.2**. Use the graphs to answer each of the following.

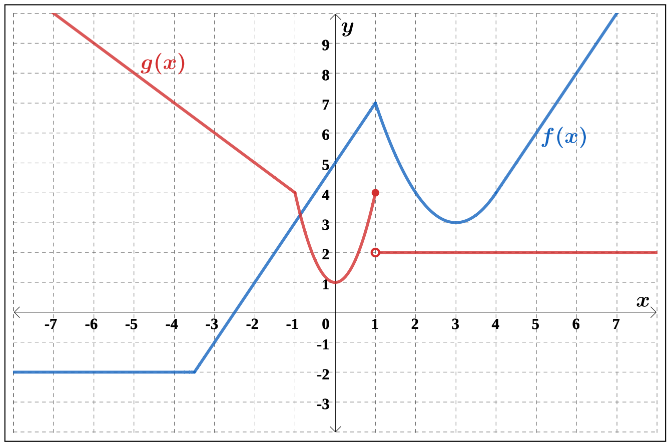


Figure 2.3.2: Graphs of the functions f and g

a. If $h(x) = f(x) + g(x)$, find $h'(0)$.

b. If $j(x) = 4x^2 - 7g(x)$, find $j'(-2)$.

Solution:

a. Because $h(x)$ is the sum of two functions, the Sum Rule gives $h'(x) = f'(x) + g'(x)$. Thus, $h'(0) = f'(0) + g'(0)$.

Recall that $f'(0)$ is the slope of the line tangent to the graph of f at $x = 0$, and $g'(0)$ is the slope of the line tangent to the graph of g at $x = 0$.

Near $x = 0$, the graph of f consists of a linear segment. Thus, the slope of the tanegnt line at $x = 0$ is the slope of this line segment, which is 2. Hence, $f'(0) = 2$.

Next, the graph of g has a horizontal tangent line at $x = 0$. Because the slope of a horizontal line is zero, we conclude that $g'(0) = 0$. Therefore,

$$\begin{aligned} h'(0) &= f'(0) + g'(0) \\ &= 2 + 0 \\ &= 2 \end{aligned}$$

b. Using the Difference Rule, Constant Multiple Rule, and Power Rule gives

$$\begin{aligned} j'(x) &= \frac{d}{dx}\left(4x^2 - 7g(x)\right) \\ &= 4\left(\frac{d}{dx}\left(x^2\right)\right) - 7\left(\frac{d}{dx}(g(x))\right) \\ &= 4\left(2x^{2-1}\right) - 7(g'(x)) \\ &= 8x - 7g'(x) \end{aligned}$$

Substituting -2 for x gives

$$\begin{aligned} j'(-2) &= 8(-2) - 7g'(-2) \\ &= -16 - 7g'(-2) \end{aligned}$$

Near $x = -2$, the graph of g is linear. Thus, $g'(-2)$ is the slope of this line segment, which is -1. So $g'(-2) = -1$, and we have

$$\begin{aligned} j'(-2) &= -16 - 7g'(-2) \\ &= -16 - 7(-1) \\ &= -9 \end{aligned}$$

■

Try It # 6:
Using the graphs of f and g from the previous example (see **Figure 2.3.2**), answer each of the following.

a. If $k(x) = 6f(x) + 3x^2$, find $k'(3)$.

b. If $m(x) = f(x) + 5g(x)$, find $m'(5)$.

■ **Example 8** Determine, algebraically, where $y = 4x^{1/3} + 15$ is differentiable, and write your answer using interval notation.

Solution:
To determine where the function is differentiable, we must find where its derivative, y', is defined.

But, before we do that, we must first consider where the original function y is defined because its derivative will only (possibly) exist where y is defined. Looking at y, we see there are no domain restrictions, so the domain of y is $(-\infty, \infty)$.

Next, we find y'. Using the Sum Rule, Constant Multiple Rule, Power Rule, and Constant Rule gives

$$\begin{aligned} y' &= \frac{d}{dx}\left(4x^{\frac{1}{3}} + 15\right) \\ &= 4\left(\frac{d}{dx}\left(x^{\frac{1}{3}}\right)\right) + \frac{d}{dx}(15) \\ &= 4\left(\frac{1}{3}x^{\frac{1}{3}-1}\right) + 0 \\ &= \frac{4}{3}x^{-\frac{2}{3}} \end{aligned}$$

To find where y' does not exist, we have to consider the domain restrictions to find where y' is undefined. At a glance it may appear that y' has no "issues", but remember that a negative exponent is another way to represent division:

$$\begin{aligned} y' &= \frac{4}{3}x^{-\frac{2}{3}} \\ &= \frac{4}{3x^{\frac{2}{3}}} \end{aligned}$$

Due to the fact that we cannot divide by zero,

$$\begin{aligned} 3x^{\frac{2}{3}} &\neq 0 \\ x^{\frac{2}{3}} &\neq 0 \\ x &\neq 0 \end{aligned}$$

Because the domain of y is all real numbers and y' will exist everywhere except at $x = 0$, we can conclude that y is differentiable on $(-\infty, 0) \cup (0, \infty)$.

■

Try It # 7:
Determine, algebraically, where $y = 20 - 3x^{2/3}$ is differentiable, and write your answer using interval notation.

Marginal Analysis

Remember that, at its heart, the derivative is the rate of change of a function. If that function is cost, revenue, or profit, we can determine how much the cost, revenue, or profit is changing per item when x items are sold. This is referred to as **marginal analysis.**

Definition

- If $C(x)$ is the cost of producing x items, then C' is the **marginal cost function**.
- If $R(x)$ is the revenue from selling x items, then R' is the **marginal revenue function**.
- If $P(x)$ is the profit from selling x items, then P' is the **marginal profit function**.

Let's look at some examples:

Example 9 Suppose that the profit, in dollars, obtained from the sale of x fish-fry feasts each week is given by $P(x) = -0.03x^2 + 8x - 50$. Find the marginal profit when 64 fish-fry feasts are sold, and interpret your answer.

Solution:
To find the marginal profit when 64 fish-fry feasts are sold, we need to calculate $P'(64)$. We first find $P'(x)$:

$$\begin{aligned} P'(x) &= -0.03\left(2x^{2-1}\right) + 8\left(1x^{1-1}\right) - 0 \\ &= -0.06x + 8 \end{aligned}$$

Substituting 64 for x gives

$$\begin{aligned} P'(64) &= -0.06(64) + 8 \\ &= 4.16 \end{aligned}$$

Recall that the units of the derivative function are the units associated with the output of the original function divided by the units associated with the input of the original function. Because $P(x)$ is measured in dollars and x represents the number of fish-fry feasts sold, our units associated with 4.16 are dollars/feast. Thus, the marginal profit is \$4.16 per feast.

We write a sentence to interpret our answer:

"When 64 fish-fry feasts are sold each week, profit is increasing at a rate of \$4.16 per feast."

Try It # 8:
The Best Foot Forward shoe store has a price-demand function given by $p(x) = -0.03x + 45$, where $p(x)$ is the price, in dollars, per pair of Le Boot-Ons shoes when x pairs of shoes are sold. Find the marginal revenue when 200 pairs of shoes are sold, and interpret your answer.

Looking back at the previous example, notice our answer of $4.16 per feast is actually an estimate of the profit from selling the 65^{th} fish-fry feast. This is because the derivative is a rate of change. It tells us the change in profit, in dollars, if one more item is made and sold. So if 64 feasts are currently sold and one more feast is sold (i.e., the 65^{th} feast is sold), then profit increases by approximately $4.16. This is generalized below:

The profit obtained from making and selling the n^{th} item can be **approximated** by $P'(n-1)$.

Now, let's compare our estimate for the profit from selling the 65^{th} feast ($4.16) to the exact profit from selling the 65^{th} feast. To find the exact profit from selling the 65^{th} feast, we only need to use the profit function, $P(x) = -0.03x^2 + 8x - 50$. We find the profit from selling 65 feasts and subtract the profit from selling 64 feasts (this will leave us with the exact profit from selling the 65^{th} feast):

$$\begin{aligned} P(65) - P(64) &= \left(-0.03(65)^2 + 8(65) - 50\right) - \left(-0.03(64)^2 + 8(64) - 50\right) \\ &= (343.25) - (339.12) \\ &= \$4.13 \end{aligned}$$

Note that our approximation of $4.16 is "close" to the exact profit of $4.13. The approximation has the benefit of being easier to compute (once we are comfortable finding derivatives). This is generalized below:

The **exact** profit obtained from making and selling the n^{th} item is $P(n) - P(n-1)$.

Continuing on with our fish-fry fun, we can also use marginal analysis to approximate the profit from selling a total number of items. For example, if we want to estimate the profit from selling 65 feasts, we can find the profit from selling 64 feasts and then add the approximate profit from selling only the 65^{th} feast (i.e., $P'(64)$). This will give us an approximation of the profit from selling all 65 feasts:

$$\begin{aligned} P(65) &\approx P(64) + P'(64) \\ &= \left(-0.03(64)^2 + 8(64) - 50\right) + (-0.06(64) + 8) \\ &= (339.12) + (4.16) \\ &= \$355.16 \end{aligned}$$

This is generalized below:

The profit obtained from making and selling n items, $P(n)$, can be **approximated** by $P(n-1) + P'(n-1)$.

N *Because we are given the rule for $P(x)$, using this approximation is more work than simply finding the exact profit from selling 65 feasts by calculating $P(65)$. You may be asking then, why do we need this approximation? Well, as we've seen in previous sections, we may not always be given a function. We may be given a graph or data points and be expected to find (or approximate) the profit. It is in those situations that this method becomes useful.*

All of these ideas are similar for cost and revenue as well:

Estimating the Cost/Revenue/Profit of a Single Item Using Marginal Analysis

- The approximate cost of making the n^{th} item is $C'(n-1)$.
- The approximate revenue from selling the n^{th} item is $R'(n-1)$.
- The approximate profit from making and selling the n^{th} item is $P'(n-1)$.

The Exact Cost/Revenue/Profit of a Single Item

- The exact cost of making the n^{th} item is $C(n) - C(n-1)$.
- The exact revenue from selling the n^{th} item is $R(n) - R(n-1)$.
- The exact profit from making and selling the n^{th} item is $P(n) - P(n-1)$.

Estimating the Cost/Revenue/Profit of a Total Number of Items, *n*, Using Marginal Analysis

- $C(n) \approx C(n-1) + C\,'(n-1)$
- $R(n) \approx R(n-1) + R\,'(n-1)$
- $P(n) \approx P(n-1) + P\,'(n-1)$

The following examples demonstrate these ideas.

Example 10 A company that makes Barbara Dolls has a weekly cost function, in dollars, of $C(x) = 10{,}000 + 90x - 0.05x^2$, where x is the number of Barbara Dolls produced.

a. Estimate the cost of producing the 723^{rd} Barbara Doll.

b. Find the exact cost of producing the 723^{rd} Barbara Doll.

c. Approximate the cost of producing 723 Barbara Dolls

d. Find the exact cost of producing 723 Barbara Dolls.

Solution:

a. First, note that we are looking for an estimate, not an exact answer. This means we should have a derivative in our calculation. Next, notice that we want to estimate the cost of producing a specific Barbara Doll (the 723^{rd}) and not a group or total number of dolls. Looking at the formulas above, we see that to estimate the cost of the n^{th} item, we find $C\,'(n-1)$. Here, we want to estimate the cost of the 723^{rd} item, so we calculate $C\,'(723-1) = C\,'(722)$.

We must use the Introductory Derivative Rules to find $C\,'(x)$, and then we can calculate $C\,'(722)$:

$$\begin{aligned} C\,'(x) &= 0 + 90\left(1x^{1-1}\right) - 0.05\left(2x^{2-1}\right) \\ &= 90 - 0.1x \end{aligned}$$

Substituting 722 for x gives

$$\begin{aligned} C\,'(722) &= 90 - 0.1(722) \\ &= \$17.80 \text{ per Barbara Doll} \end{aligned}$$

Thus, the approximate cost of producing the 723^{rd} Barbara Doll is \$17.80.

N *Remember, the derivative of the cost function gives us the change in cost if one more item is produced. The value of the derivative when 722 dolls are produced is our estimate for the cost of producing the 723^{rd} doll because if the company currently makes 722 dolls and it makes one more doll, that would be the 723^{rd} doll!*

b. We are asked to find the exact cost of producing a single Barbara Doll (specifically, the 723^{rd} doll). Because this is an exact amount, our formula will not contain any derivatives. Using the formula for finding the exact cost of producing the n^{th} item, where $n = 723$, and the cost function $C(x) = 10,000 + 90x - 0.05x^2$, we have

$$\begin{aligned} C(n) - C(n-1) &= C(723) - C(722) \\ &= \left(10,000 + 90(723) - 0.05(723)^2\right) - \left(10,000 + 90(722) - 0.05(722)^2\right) \\ &= (48,933.55) - (48,915.80) \\ &= \$17.75 \end{aligned}$$

Thus, the exact cost of producing the 723^{rd} Barbara Doll is \$17.75.

N *This formula takes the total cost of making 723 Barbara Dolls and subtracts the total cost of making 722 Barbara Dolls. The result is the dollar amount it costs the company to make the 723^{rd} doll. Note that this exact cost of \$ 17.75 for the 723^{rd} doll is relatively "close" to our estimate in part* ***a****, which we found to be \$ 17.80.*

c. We need to approximate the total cost of producing 723 Barbara Dolls, which is a total number of Barbara Dolls, not just a single doll. Recall the formula for estimating the cost of producing n total items is given by

$$C(n) \approx C(n-1) + C'(n-1)$$

Substituting $n = 723$ and using the functions $C(x) = 10,000 + 90x - 0.05x^2$ and $C'(x) = 90 - 0.1x$ gives

$$\begin{aligned} C(723) &\approx C(722) + C'(722) \\ &= \left(10,000 + 90(722) - 0.05(722)^2\right) + (90 - 0.1(722)) \\ &= (48,915.80) + (17.80) \\ &= \$48,933.60 \end{aligned}$$

Thus, the cost of producing 723 Barbara Dolls is approximately \$48,933.60.

N *This estimate starts with the actual cost of producing 722 Barbara Dolls. Then, we add the value of the derivative when 722 dolls are produced. Why? The derivative gives us the change in cost if the company produces one more doll. So if it is currently making 722 dolls and makes one more doll, the company would be making 723 Barbara Dolls!*

d. To find the exact cost of producing 723 Barbara Dolls, we simply find the value of the cost function $C(x) = 10,000 + 90x - 0.05x^2$ when $x = 723$:

$$\begin{aligned} C(723) &= 10,000 + 90(723) - 0.05(723)^2 \\ &= \$48,933.55 \end{aligned}$$

N *This last part of the example is a reminder about what the cost function tells us. When substituting $x = 723$, the function outputs the cost of making 723 Barbara Dolls (no calculus required!).*

■

Example 11 The company that made the Barbara Dolls in the previous example has a weekly revenue function given by $R(x) = -0.02x^2 + 45x$ dollars, where x is the number of Barbara Dolls sold.

a. Find the exact revenue from selling the 948^{th} Barbara Doll.

b. Approximate the revenue from selling 1057 Barbara Dolls.

c. Estimate the revenue from the sale of the 213^{th} Barbara Doll.

Solution:

a. The exact revenue from selling the 948^{th} Barbara Doll is given by

$$\begin{aligned} R(948) - R(947) &= \left(-0.02(948)^2 + 45(948)\right) - \left(-0.02(947)^2 + 45(947)\right) \\ &= (24,685.92) - (24,678.82) \\ &= \$7.10 \end{aligned}$$

b. To approximate the revenue from selling 1057 Barbara Dolls, we use the formula

$$R(1057) \approx R(1056) + R'(1056)$$

Recalling $R(x) = -0.02x^2 + 45x$ and using the Introductory Derivative Rules to find $R'(x)$ gives

$$\begin{aligned} R'(x) &= -0.02\left(2x^{2-1}\right) + 45\left(1x^{1-1}\right) \\ &= -0.04x + 45 \end{aligned}$$

Therefore,

$$\begin{aligned} R(1057) &\approx R(1056) + R'(1056) \\ &= \left(-0.02(1056)^2 + 45(1056)\right) + (-0.04(1056) + 45) \\ &= (25,217.28) + (2.76) \\ &= \$25,220.04 \end{aligned}$$

Thus, the approximate revenue from selling 1057 Barbara Dolls is $25,220.04.

c. To estimate the revenue from the sale of the 213^{th} Barbara Doll, we calculate

$$\begin{aligned} R'(212) &= -0.04(212) + 45 \\ &= \$36.52 \text{ per Barbara Doll} \end{aligned}$$

Thus, the revenue from the sale of the 213^{th} Barbara Doll is approximately $36.52.

Try It # 9:
Using the revenue function from the previous example, $R(x) = -0.02x^2 + 45x$ dollars when x Barbara Dolls are sold,

a. approximate the revenue from selling the 500^{th} Barbara Doll.

b. find the exact revenue from selling the 500^{th} Barbara Doll.

c. estimate the revenue from selling 500 Barbara Dolls.

d. find the exact revenue from selling 500 Barbara Dolls.

Try It Answers

1. **a.** $\dfrac{df}{dx} = 12x^2$

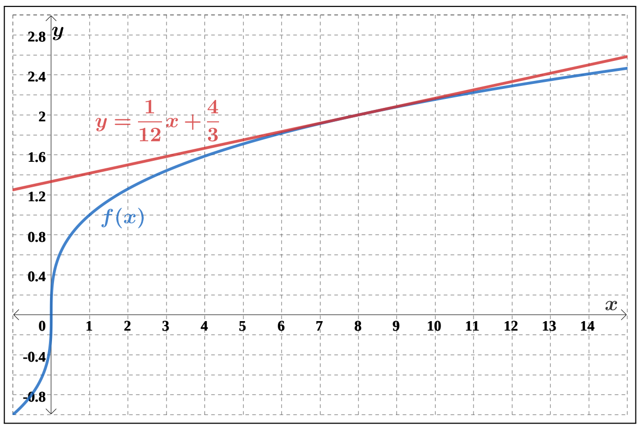
   b. $\dfrac{df}{dx} = 170x^9 + e^x - 1.8$

   c. $\dfrac{df}{dx} = -54x^{-10} - 2^x \ln(2) + 4\left(\dfrac{1}{x\ln(3)}\right)$

2. **a.** $f'(x) = -28x^3 + 63x^2 + 22$

   b. $g'(x) = \dfrac{70}{9}x - \dfrac{4}{3} - \dfrac{4}{9}x^{-3}$

3. $y = \dfrac{1}{12}x + \dfrac{4}{3}$;



4. $-\$0.43$ per item; When 215 items are demanded, price is decreasing at a rate of \$0.43 per item.

5. **a.** 0

   b. 599/16

6. **a.** 18

   b. 2

7. $(-\infty, 0) \cup (0, \infty)$

8. \$33 per pair of shoes; When 200 pairs of shoes are sold, revenue is increasing at a rate of \$33 per pair of shoes.

9. **a.** \$25.04

   b. \$25.02

   c. \$17,500.02

   d. \$17,500

EXERCISES

Basic Skills Practice

For Exercises 1 - 3, find $f'(x)$.

1. $f(x) = 4x^2$

2. $f(x) = -3x^3 + 1$

3. $f(x) = 5x^{-4} - 9x + 12$

For Exercises 4 - 6, find $\dfrac{dy}{dx}$.

4. $y = 6x^2 - 4x + e^x$

5. $y = \ln(x) - x^7 + \sqrt{3}$

6. $y = \dfrac{e^x}{5} - 17x - \ln(x)$

For Exercises 7 - 9, find the slope of the line tangent to the graph of the function at the given x-value.

7. $f(x) = 9x^3 - \frac{1}{2}x + 1$ at $x = -4$

8. $g(x) = -\dfrac{3}{4}x^2 + e^x - 6$ at $x = 0$

9. $h(x) = 2x^5 + \ln(x)$ at $x = 1$

For Exercises 10 - 14, find the equation of the line tangent to the graph of the function at the given x-value.

10. $h(x) = 2x^{-3} + 5x$ at $x = 4$

11. $g = 16x^2 - 9x^3 + 11x - 13$ at $x = -2$

12. $y = -10x^4 + 20x^{-2} - 6$ at at $x = 1$

13. $f(x) = e^x + 4x^5 - 5x^2 + 25$ at $x = 0$

14. $p(x) = 12x^2 - x^{-6} + \ln(x) + 2$ at $x = 1$

15. Given $g(t) = e^t + 5$, find the equation of the line tangent to the graph of g at $t = 0$, and graph both g and the tangent line on the same axes.

For Exercises 16 - 18, find the x-value(s) where the graph of the function has a horizontal tangent line.

16. $f(x) = 4x^3 - 14x^2 + 8x - 13$

17. $f(x) = 5x^4 + 20x^3 - 6$

18. $f(x) = \frac{7}{3}x^3 - 7x + 29$

19. Suppose that the profit, in dollars, obtained from the sale of x barbeque dinners is given by $P(x) = -0.03x^2 + 8x - 50$. Find the rate of change of profit when 40 barbeque dinners are sold.

20. The cost of producing x calculators per week is given by $C(x) = 600 + 100\sqrt{x}$ dollars. Find the marginal cost when 250 calculators are produced each week.

21. A flat iron manufacturer has a revenue function, in dollars, given by $R(x) = 35x - 0.1x^2$, where x is the number of flat irons sold.
    (a) Find the marginal revenue when 160 flat irons are sold.
    (b) Find the rate of change of revenue when 200 flat irons are sold.

22. An arrow shot upward from the ground will have a height of $s(t) = -16t^2 + 128t$ feet after t seconds. Find the velocity of the arrow after 3 seconds.

23. Various values of $f(x)$, $g(x)$, $f'(x)$, and $g'(x)$ are given in the table below. Use the information in the table to answer each of the following.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	7	−4	2
1	−2	8	−2	0
2	−3	7	0	−2
3	−2	4	2	−4

    (a) If $h(x) = f(x) + g(x)$, find $h'(0)$.
    (b) If $j(x) = 3g(x)$, find $j'(1)$.
    (c) If $k(x) = -5g(x) + 7f(x)$, find $k'(3)$.

24. A company that produces cameras has a cost function given by $C(x)$ dollars when x cameras are produced. Use the information in the table below to answer each of the following.

x	$C(x)$	$C'(x)$
20	\$1471.04	\$22.94
21	\$1494.43	\$22.36
22	\$1516.52	\$21.82
23	\$1538.08	\$21.32
24	\$1559.17	\$20.85
25	\$1580.15	\$20.40

(a) Find the rate of change of cost when 22 cameras are produced.

(b) Find the marginal cost when 21 cameras are produced.

(c) Approximate the cost of producing the 23rd camera.

(d) Find the exact cost of producing the 23rd camera.

(e) Estimate the cost if 24 cameras are produced.

(f) Find the exact cost if 24 cameras are produced.

25. A company that sells a popular lawn mower has a revenue function given by $R(x)$ dollars when x lawn mowers are sold. Use the information in the table below to answer each of the following.

x	$R(x)$	$R'(x)$
347	\$116,633.64	\$321.84
348	\$116,955.84	\$321.92
349	\$117,277.96	\$322.00
350	\$117,600.00	\$322.08
351	\$117,921.96	\$322.16
352	\$118,243.84	\$322.24

(a) Find the rate of change of revenue when 351 lawn mowers are sold.

(b) Find the marginal revenue when 349 lawn mowers are sold.

(c) Estimate the revenue from selling the 350th lawn mower.

(d) Find the exact revenue from selling the 350th lawn mower.

(e) Approximate the revenue if 348 lawn mowers are sold.

(f) Find the exact revenue if 348 lawn mowers are sold.

26. An appliance manufacturer that makes and sells high-end blenders has a profit function given by $P(x)$ dollars when x blenders are made and sold. Use the information in the table below to answer each of the following.

x	$P(x)$	$P'(x)$
150	\$54,000.00	\$385.00
151	\$54,384.70	\$384.40
152	\$54,768.80	\$383.80
153	\$55,152.30	\$383.20
154	\$55,535.20	\$382.60
155	\$55,917.50	\$382.00

(a) Find the rate of change of profit when 151 blenders are sold.

(b) Find the marginal profit when 154 blenders are sold.

(c) Approximate the profit from selling the 152^{nd} blender.

(d) Find the exact profit from selling the 152^{nd} blender.

(e) Estimate the profit if 153 blenders are sold.

(f) Find the exact profit if 153 blenders are sold.

Intermediate Skills Practice

For Exercises 27 - 33, find $f'(x)$.

27. $f(x) = \sqrt[4]{x} - 2x^{1/7} + 10e^x - 2\ln(x)$

28. $f(x) = 6\ln(x) - 3x^2 + 25e^x - 5x^{3/8}$

29. $f(x) = -\dfrac{1}{3}x^{5/6} + 4x^2 + 9^x - 1$

30. $f(x) = -7x^3 + x^{2/3} - 5\cdot 4^x - e^3$

31. $f(x) = 2x^3 - 7x + \log_4(x) + \sqrt[8]{19}$

32. $f(x) = \dfrac{7}{11}x^{-2/5} - 3\log_6(x) + \pi^2$

33. $f(x) = 0.03x^{0.2} + \dfrac{8}{13}x^{-26/3} + 6\cdot 100^x$

For Exercises 34 - 37, find $\dfrac{dy}{dx}$.

34. $y = 3x^{-4} - \dfrac{2}{x^2} - \dfrac{3}{7x^4} - 6e^5$

35. $y = \sqrt{x} + 8\log_5(x) + \dfrac{4}{x^{9/10}}$

36. $y = 10\ln(x) - \sqrt[3]{x^2} + \dfrac{3}{x^{-5}}$

37. $y = 8\sqrt[5]{x^3} + 9 \cdot 30^x - \dfrac{27}{x^{-1/4}}$

For Exercises 38 - 43, find the derivative of the function.

38. $f(x) = 3x^2\left(x^6 + 4x^3 - 5x + 2\right)$

39. $g(x) = \left(x^3 + 1\right)(x - 6)$

40. $h(t) = \left(2t^{-2} + 4t - 7\right)(3t + 1)$

41. $y = \dfrac{2x^3 - x^{10} - 21}{x^4}$

42. $f(t) = \dfrac{3t^5 + 2t^{-2} - 4t + 8}{5t^6}$

43. $h(x) = \dfrac{10x^4 - 9x^{-3} + 15}{3x^{-2}}$

For Exercises 44 - 47, find the slope of the line tangent to the graph of the function at the given x-value.

44. $f(x) = 7x^6 - 4\sqrt[4]{x^3} + 2\log(x)$ at $x = 1$

45. $g(x) = 3e^x - \dfrac{2}{x^9}$ at $x = 1$

46. $f(x) = \left(2x^{-3} + 4x^2\right)\left(x^{-1} + 3x\right)$ at $x = -2$

47. $h(x) = \dfrac{-8x^6 + x^{-3} + 11}{-2x^9}$ at $x = 1$

For Exercises 48 - 51 find the equation of the line tangent to the graph of the function at the given x-value.

48. $f(x) = 4\ln(x) - \dfrac{5}{x^2} + 6\sqrt[3]{x^2}$ at $x = 1$

49. $g(x) = 2 \cdot 10^x + 3$ at $x = 0$

50. $f(x) = \left(3x^4 - x^2 - 1\right)\left(6x^{-2} + x\right)$ at $x = -2$

51. $h(x) = \dfrac{2x^{-4} - x^6 - 5}{3x^{-2}}$ at $x = 1$

52. Find the equation of the line tangent to the graph of $f(t) = 3\ln(t) + 2t^2 - 1$ at $t = 1$, and graph both f and the tangent line on the same axes.

For Exercises 53 - 55, find the x-value(s) where the graph of the function has a horizontal tangent line.

53. $f(x) = 2e^x - 5x$

54. $g(x) = \left(2x^2 - 1\right)(4x + 1)$

55. $h(x) = \dfrac{x^8 - 2x^7 + x^6}{x^5}$

For Exercises 56 - 58, find the x-value(s) where the graph of the function has the indicated slope.

56. $f(x) = 3x - 8\ln(x)$; slope is -10

57. $g(x) = \left(x^2 - 2\right)(x + 10)$; slope is 5

58. $h(x) = \dfrac{3x^5 - 4x^3}{x^2}$; slope is 32

59. Various values of $f(x)$, $g(x)$, $f'(x)$, and $g'(x)$ are given in the table below. Use the information in the table to answer each of the following.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
−2	−28	−46	38	64
0	0	6	−2	0
2	−4	2	6	8
4	56	86	62	88

(a) If $h(x) = -\dfrac{5}{6}f(x) + g(x)$, find $h'(-2)$.

(b) If $j(x) = \dfrac{1}{3}x^3 - 4g(x)$, find $j'(4)$.

(c) If $k(x) = \sqrt{2}f(x) + 5x$, find $k'(0)$.

60. The revenue function for a particular karaoke machine for kids is given by $R(x) = 185x - 0.2x^2$, where $R(x)$ is the revenue, in dollars, from selling x karaoke machines.

(a) Find the rate of change of revenue when 225 karaoke machines are sold, and interpret your answer.

(b) Find the marginal revenue when 500 karaoke machines are sold, and interpret your answer.

(c) Approximate the revenue from selling the 145^{th} karaoke machine.

(d) Find the exact revenue from selling the 145^{th} karaoke machine.

(e) Estimate the revenue if 300 karaoke machines are sold.

(f) Find the exact revenue if 300 karaoke machines are sold.

61. The profit function for a company that makes and sells designer shoes is given by $P(x) = -0.3x^2 + 475x - 10,500$, where $P(x)$ is the profit, in dollars, when x pairs of shoes are made and sold.
    (a) Find the rate of change of profit when 975 pairs of shoes are made and sold, and interpret your answer.
    (b) Find the marginal profit when 625 pairs of shoes are made and sold, and interpret your answer.
    (c) Estimate the profit from making and selling the 700^{th} pair of shoes.
    (d) Find the exact profit from making and selling the 700^{th} pair of shoes.
    (e) Approximate the profit if 1532 pairs of shoes are made and sold.
    (f) Find the exact profit if 1532 pairs of shoes are made and sold.

62. A company that produces dishwashers has a weekly cost function given by $C(x) = 3500 + 14x + 0.2x^2$, where $C(x)$ is the cost, in dollars, when x dishwashers are produced each week.
    (a) Find the rate of change of cost when 500 dishwashers are produced each week, and interpret your answer.
    (b) Find the marginal cost when 700 dishwashers are produced each week, and interpret your answer.
    (c) Approximate the cost of producing the 643^{rd} dishwasher.
    (d) Find the exact cost of producing the 643^{rd} dishwasher.
    (e) Estimate the cost of producing 325 dishwashers.
    (f) Find the exact cost of producing 325 dishwashers.

63. A company's total sales after t months is given by $S(t) = 0.02t^4 + 0.03t^3 + 0.15t^2 + 3t + 5$ million dollars. Find the rate of change of sales after 8 months, and interpret your answer.

64. La-Dreamer Dollhouses specializes in selling two-story dollhouses for kids. The company has determined its revenue function is $R(x) = 210x - 0.8x^2$ dollars, where x is the number of dollhouses sold.
    (a) Find the marginal revenue function.
    (b) Find the marginal revenue when 75 dollhouses are sold, and interpret your answer.

65. A company has a cost function given by $C(x) = 2000 + 10x + 0.3x^2$ dollars, where x is the number of items produced. Find the company's exact cost of producing the 50^{th} item.

66. The revenue, in dollars, when x digital clock radios are sold is given by $R(x) = 10x - 0.001x^2$. Estimate the revenue from selling the 452^{nd} clock.

67. The profit function for a skateboard manufacturer is given by $P(x) = 30x - 0.3x^2 - 250$ dollars, where x is the number of skateboards manufactured and sold.
    (a) Estimate the profit from manufacturing and selling the 30^{th} skateboard.
    (b) Find the exact profit from manufacturing and selling the 30^{th} skateboard.
    (c) Estimate the profit from manufacturing and selling 30 skateboards.
    (d) Find the exact profit from manufacturing and selling 30 skateboards.

68. A toaster manufacturer has a revenue function given by $R(x) = 42x - 0.1x^2$ dollars, where x is the number of toasters sold. Estimate the manufacturer's revenue from selling 250 toasters.

69. The position of an object moving along a horizontal line is given by $s(t) = t^3 - 3t^2 - 5t + 6$ feet, where t is time in seconds. Find the velocity of the object after 2 seconds.

Mastery Practice

70. Find $\dfrac{d}{dt}\left(\dfrac{\sqrt[3]{t^2} - 8t^{-1/4} - 5\pi^2\sqrt{t}}{6\sqrt{t}}\right)$.

71. Find $f'(x)$ if $f(x) = \left(3x^{1/4} - 4\sqrt{x}\right)\left(-\dfrac{1}{6}x^{3/4} + \sqrt{x^8}\right)$.

72. Find $\dfrac{dy}{dx}$ if $y = \dfrac{x^{2/3} + 7x^{0.4} - 5\sqrt[4]{x^3}}{x^{-2/3}}$.

73. Given $y = -6(0.43)^x + \dfrac{9}{10}\ln(x)$, find y'.

74. Given $f(t) = \dfrac{-4e^{2t} + 7t^{2/3}e^t - 18e^t}{e^t}$, find $f'(t)$.

75. Given $q(x) = -3x^2\left(12x^3 - 9\sqrt{x}\right)\left(8\sqrt[3]{x^4} - 7x^6\right) + 7\pi^2$, find $q'(x)$.

76. Find $\dfrac{d}{dt}\left(\left(\dfrac{2}{3t^2} - \dfrac{1}{t^3} - 10t^{-7}\right)\left(t + 9t^5\right)\right)$.

77. Given $h(x) = \dfrac{\frac{7}{8}(4^{-x}) - 6x^2}{x^2(4^{-x})}$, find $\dfrac{dh}{dx}$.

78. Given $y = \dfrac{\left(3x^3 - 4x^{-2}\right)\left(\sqrt{x} - 4x^5\right)}{x^3}$, find y'.

For Exercises 79 and 80, find the equation of the line tangent to the graph of the function at the given value.

79. $f(x) = \sqrt{x}\left(7x^{-3.5} + \dfrac{1}{4}\sqrt{x}\right)\left(x^6 - 7\right)$ at $x = 1$

80. $h(t) = \dfrac{4e^{-t}\ln(t) + 5}{e^{-t}}$ at $t = 1$

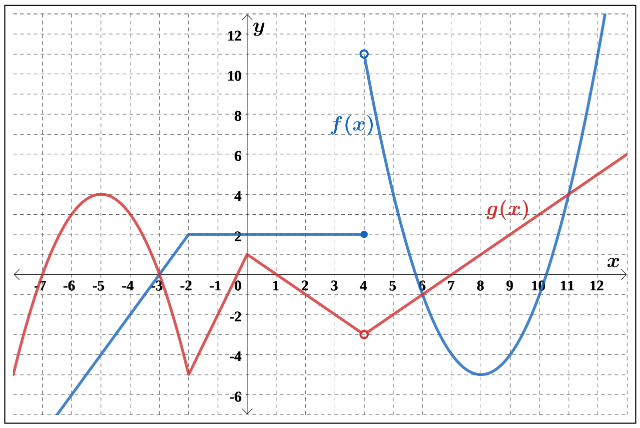
For Exercises 81 and 82, find the x-value(s) where the graph of the function has a horizontal tangent line.

81. $p(x) = \dfrac{2\log_3(x) - 8x}{9}$

82. $g(x) = 2 \cdot e^{x}(e^{-x} + 7) - \dfrac{2}{3x^{-1}}$

83. Find the value(s) of t where the function $g(t) = 2t^2 - 8\ln(t)$ has an instantaneous rate of change of 4.

84. The graphs of the functions f and g are shown below on the same axes. Use the graphs to answer each of the following.



(a) If $h(x) = 0.24f(x) - g(x) + \sqrt[3]{x^2}$, find $h'(-1)$.

(b) If $j(x) = \dfrac{g(x)}{4} - \ln(x) + \sqrt{7}f(x)$, find $j'(8)$.

(c) If $k(x) = 2f(x) - 3\log_5(x)$, find $k'(1)$.

(d) If $m(x) = f(x) + (0.6)^x + 4g(x) - e$, find $m'(2)$.

85. Find the values of the coefficients a, b, and c so that the parabola $f(x) = ax^2 + bx + c$ passes through the point $(1,4)$ and the line tangent to the graph of f at the point $(3,14)$ is $y = 9x - 13$.

86. One Awesome Rainbow, a company that sells custom rainbow headbands, has a price-demand function given by $p(x) = 75(0.999)^x$, where $p(x)$ is the price, in dollars, of each headband when x headbands are sold. Find the rate of change of price when 600 headbands are sold, and interpret your answer.

87. The price-demand and cost functions for the production of cordless drills are given, respectively, by $p(x) = 143 - 0.03x$ and $C(x) = 5{,}000 + 45x$, where x is the number of cordless drills that are sold at a price of $p(x)$ dollars per drill and $C(x)$ is the cost, in dollars, of producing x cordless drills.

(a) Find the marginal cost function.

(b) Estimate the revenue from selling 2000 cordless drills.

(c) Find the exact profit from producing and selling the 1050^{th} cordless drill.

(d) Find $P'(1000)$ and interpret your answer.

88. It costs $C(x) = \sqrt{x}$ dollars to produce x golf balls.

(a) Find the marginal cost when 25 golf balls are produced, and interpret your answer.

(b) Find the marginal cost when 100 golf balls are produced, and interpret your answer.

89. A company that sells umbrellas has determined its price-demand function to be $p(x) = 25 - \sqrt{x}$, where $p(x)$ is the price, in dollars, of each umbrella when x umbrellas are sold.

    (a) Estimate the company's revenue from selling the 50^{th} umbrella.

    (b) Find the company's exact revenue from selling the 50^{th} umbrella.

90. A company has a price-demand function given by $p(x) = -2\sqrt{x} + 50$, where $p(x)$ is the price, in dollars, per item when x items are sold. The company also has a cost function given by $C(x) = 90\sqrt{x} + 12x + 6,000$ dollars when x items are made. If $P(x)$ represents the company's profit, in dollars, when x items are made and sold, find $P'(144)$. Interpret your answer.

91. The position of a hummingbird flying along a straight line after t seconds is given by $s(t) = 3t^3 - 9t$, where $s(t)$ is measured in meters.

    (a) Find the velocity of the bird after 2 seconds.

    (b) When will the velocity of the bird be 0 meters per second?

92. A company's total sales, in millions of dollars, after t months is given by $S(t) = 0.5t^{2/3} + 0.2\sqrt{t} + 6t + 4$. Find the rate of change of sales after two years, and interpret your answer. Round to eight decimal places, if necessary.

93. A hair dryer manufacturer has revenue and cost functions, both in dollars, given by $R(x) = 38x - 0.1x^2$ and $C(x) = 5x + 2000$, respectively, where x is the number of hair dryers made and sold. Estimate the manufacturer's profit from making and selling 175 hair dryers.

94. A company has a revenue function given by $R(x) = x(400 - 0.025x)$ dollars when x items are sold. Find the company's marginal revenue when 200 items are sold, and interpret your answer.

95. A small business that sells customized candles can sell 100 candles each month when the price per candle is \$25. For every dollar the price is raised, 10 fewer candles will be sold. The business has a cost function given by $C(x) = 9x + 1000$ dollars, where x is the number of candles made. Assuming the company's price-demand function is linear,

    (a) estimate the profit from making and selling the 60^{th} candle.

    (b) find the exact profit from making and selling the 60^{th} candle.

    (c) estimate the profit from making and selling 60 candles.

    (d) find the exact profit from making and selling 60 candles.

Communication Practice

96. Explain why $\frac{d}{dt}\left(e^t\right) \neq te^{t-1}$.

97. Explain why $\frac{d}{dx}\left(\pi^5\right) \neq 5\pi^4$.

98. If $P(x)$ is the weekly profit, in dollars, when x items are sold each week, interpret $P(100) + P'(100) = 15,000$.

99. Does $\dfrac{d}{dx}\left(\dfrac{3x^2 - 5^x + \sqrt{x}}{x^4}\right) = \dfrac{6x - 5^x \ln(5) + \frac{1}{2}x^{-1/2}}{4x^3}$? Why or why not?

100. If $C(x)$ is the daily cost, in dollars, when x items are made each day, interpret $C'(25) = 78$.

101. If $R(x)$ is the weekly revenue, in dollars, when x items are sold each week, interpret $R(94) - R(93) = 125$.

102. To approximate the profit from the sell of the n^{th} item, explain why we must calculate the marginal profit when $n - 1$ items are sold.

2.4 The Product and Quotient Rules

In the previous section, we introduced derivative rules which allowed us to find the derivative of some functions without going through the tedious process of using the limit definition of the derivative. One rule we introduced was the Sum Rule for derivatives, which states that the derivative of a sum is the sum of the derivatives. This rule might lead us to think there is a similar rule for finding the derivative of a product of two functions. Let's see if this is indeed the case by looking at the function f below:

$$f(x) = \left(3x^2+1\right)\left(7x^3+4x\right)$$

Notice $f(x)$ is written as a product of two functions. To find $f'(x)$, can we just take the derivative of each factor and multiply the results? Recall from the previous section that

$$\frac{d}{dx}\left(3x^2+1\right) = 6x \quad \text{and} \quad \frac{d}{dx}\left(7x^3+4x\right) = 21x^2+4$$

Is it possible, then, that the derivative of f is simply

$$(6x)\left(21x^2+4\right) = 126x^3+24x \ ?$$

To test this theory, recall from **Section 2.3** that we are able to algebraically manipulate the original function so that we can apply the Introductory Derivative Rules to find the derivative:

$$\begin{aligned} f(x) &= \left(3x^2+1\right)\left(7x^3+4x\right) \\ &= \left(3x^2\right)\left(7x^3\right)+\left(3x^2\right)(4x)+(1)\left(7x^3\right)+(1)(4x) \\ &= 21x^5+12x^3+7x^3+4x \\ &= 21x^5+19x^3+4x \end{aligned}$$

We see that $f(x)$ is now written as a sum of functions, so we can apply the rules from the previous section to find $f'(x)$:

$$\begin{aligned} f'(x) &= 21\left(5x^4\right)+19\left(3x^2\right)+4(1) \\ &= 105x^4+57x^2+4 \end{aligned}$$

Notice this answer is not the same as our hypothesized answer above:

$$105x^4+57x^2+4 \neq 126x^3+24x$$

Thus, we have convinced ourselves that we cannot just take the derivative of each factor and multiply the results!

! *The derivative of a product is **NOT** the product of the derivatives:* $\frac{d}{dx}(f(x)\cdot g(x)) \neq f'(x)\cdot g'(x)$

Even though finding the derivative of a product is not as simple as finding the product of the derivatives, luckily there is a rule for finding the derivative of a product (as well as a rule for finding the derivative of a quotient)!

Learning Objectives:

In this section, you will learn how to use the Product and Quotient Rules to calculate derivatives of functions and solve problems involving real-world applications. Upon completion you will be able to:

- Calculate the derivative of a product of functions using the Product Rule.
- Calculate the derivative of a quotient of functions using the Quotient Rule.
- Calculate the derivative of more complicated functions by combining the Product, Quotient, and Introductory Derivative Rules.
- Calculate the slope of a tangent line using the Product and Quotient Rules.

- Find the equation of a tangent line using the Product and Quotient Rules.
- Graph a function and a line tangent to the graph of the function on the same axes.
- Determine the x-value(s) where the graph of a function has a horizontal tangent line using the Product and Quotient Rules.
- Determine the x-value(s) where the graph of a function has a tangent line of a given slope using the Product and Quotient Rules.
- Determine the x-value(s) where a function has a specified (instantaneous) rate of change using the Product and Quotient Rules.
- Calculate the (instantaneous) rate of change of a function involving a real-world scenario, including cost, revenue, profit, and position, using the Product and Quotient Rules.
- Interpret the meaning of the derivative of a function involving a real-world scenario, including cost, revenue, and profit.
- Calculate marginal cost, revenue, and profit using the Product and Quotient Rules.
- Interpret the meaning of marginal cost, revenue, and profit.
- Estimate the cost, revenue, or profit of an item using marginal analysis and the Product and Quotient Rules.
- Compute the exact cost, revenue, or profit of an item.
- Estimate the cost, revenue, or profit of a total number of items using marginal analysis and the Product and Quotient Rules.
- Calculate average cost, revenue, and profit.
- Interpret the meaning of average cost, revenue, and profit.
- Calculate marginal average cost, revenue, and profit using the Product and Quotient Rules.
- Interpret the meaning of marginal average cost, revenue, and profit.

The Product Rule

As discussed previously, calculating the derivative of a product of two functions is a little more involved than calculating the derivative of a sum or difference of two functions. We will use the **Product Rule** to find the derivative of a product of two functions:

Theorem 2.3 The Product Rule

If f and g are differentiable functions, then

$$\frac{d}{dx}(f(x)\cdot g(x)) = f(x)\left(\frac{d}{dx}(g(x))\right) + g(x)\left(\frac{d}{dx}(f(x))\right)$$

$$= f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

The Product Rule can be stated as "the derivative of a product is the first function times the derivative of the second function, plus the second function times the derivative of the first function." Proving this rule is beyond the scope of this textbook; the important thing is knowing how to use the Product Rule.

▪ **Example 1** Find the derivative of each of the following functions.

a. $f(x) = \left(3x^2 + 1\right)\left(7x^3 + 4x\right)$

b. $g(x) = xe^x$

c. $h(t) = 4t^3 - \sqrt{t}\ln(t)$

Solution:

a. Notice this is the function from the motivating example at the beginning of this section. Applying the Product Rule gives

$$\begin{aligned} f'(x) &= \left(3x^2+1\right)\left(\frac{d}{dx}\left(7x^3+4x\right)\right) + \left(7x^3+4x\right)\left(\frac{d}{dx}\left(3x^2+1\right)\right) \\ &= \left(3x^2+1\right)\left(21x^2+4\right) + \left(7x^3+4x\right)(6x) \end{aligned}$$

In this section, we will find the derivative using the derivative rules and then stop. We tend to not worry about simplifying our answer unless we need to use the derivative to continue solving a problem and we must simplify it to do so. However, we will simplify our answer here to show that it is equivalent to the expression we found in the motivating example using the Introductory Derivative Rules:

$$\begin{aligned} f'(x) &= \left(3x^2\right)\left(21x^2\right) + \left(3x^2\right)(4) + (1)\left(21x^2\right) + (1)(4) + \left(7x^3\right)(6x) + (4x)(6x) \\ &= 63x^4 + 12x^2 + 21x^2 + 4 + 42x^4 + 24x^2 \\ &= 105x^4 + 57x^2 + 4 \end{aligned}$$

This is the same answer we found in the motivating example!

b. Unlike in part **a**, we cannot perform any algebraic steps that would allow us to use the Introductory Derivative Rules to find the derivative of $g(x) = xe^x$. We must use the Product Rule, where x is the first function and e^x is the second function:

$$\begin{aligned} g'(x) &= x\left(\frac{d}{dx}\left(e^x\right)\right) + e^x\left(\frac{d}{dx}(x)\right) \\ &= xe^x + e^x(1) \\ &= xe^x + e^x \end{aligned}$$

c. For this function, $h(t) = 4t^3 - \sqrt{t}\ln(t)$, notice there is a difference of two functions. So we start by applying the Difference Rule from **Section 2.3**:

$$h'(t) = \frac{d}{dt}\left(4t^3\right) - \frac{d}{dt}\left(\sqrt{t}\ln(t)\right)$$

We can use the Power Rule along with the Constant Multiple Rule to find the derivative of $4t^3$. Also, recall that we should always rewrite radicals as power functions in order to apply derivative rules. Thus, we rewrite $\sqrt{t}$ as $t^{1/2}$. This gives us

$$h'(t) = 12t^2 - \frac{d}{dt}\left(t^{\frac{1}{2}} \cdot \ln(t)\right)$$

Because there is a product in the second term, we must use the Product Rule. The first function is $t^{1/2}$ and the second function is $\ln(t)$. This gives

$$\begin{aligned} h'(t) &= 12t^2 - \frac{d}{dt}\left(t^{\frac{1}{2}} \cdot \ln(t)\right) \\ &= 12t^2 - \left(\left(t^{\frac{1}{2}}\right)\left(\frac{d}{dt}(\ln(t))\right) + (\ln(t))\left(\frac{d}{dt}\left(t^{\frac{1}{2}}\right)\right)\right) \\ &= 12t^2 - \left(t^{\frac{1}{2}}\left(\frac{1}{t}\right) + \ln(t)\left(\frac{1}{2}t^{-\frac{1}{2}}\right)\right) \\ &= 12t^2 - \frac{t^{\frac{1}{2}}}{t} - \frac{1}{2}t^{-\frac{1}{2}}\ln(t) \\ &= 12t^2 - t^{-\frac{1}{2}} - \frac{1}{2}t^{-\frac{1}{2}}\ln(t) \end{aligned}$$

Notice that when we applied the Product Rule, we had to write the entire result in parentheses, and then in the final step, we distributed the negative sign. We must always be careful with negative signs because a small slip will cause us to get the wrong derivative!

■

Try It # 1:
Find the derivative of each of the following functions.

a. $f(x) = x\log_8(x)$

b. $R(t) = t^3\left(\sqrt{t} + e^t\right)$

c. $v(t) = \sqrt[4]{t^7} - t^2 5^t$

■ **Example 2** Determine, algebraically, where $y = x^{\frac{1}{5}}\log_7(x)$ is differentiable, and write your answer using interval notation.

Solution:
To determine where the function is differentiable, we must find where its derivative, y', is defined.

But, before we do that, we must first consider where the original function y is defined because its derivative will only (possibly) exist where y is defined. Looking at y, the only domain restriction is $x > 0$ because of the $\log_7(x)$ term (recall the argument of a logarithmic function must be positive). So the domain of y is $(0, \infty)$.

Now, we find the x-values where y' is defined, and assuming these x-values are in the domain of the function y, we will be able to state where the function is differentiable. Thus, we start by finding y'. To do so, we must use the Product Rule:

$$\begin{aligned} y' &= \frac{d}{dx}\left(x^{\frac{1}{5}}\log_7(x)\right) \\ &= \left(x^{\frac{1}{5}}\right)\left(\frac{d}{dx}(\log_7(x))\right) + (\log_7(x))\left(\frac{d}{dx}\left(x^{\frac{1}{5}}\right)\right) \\ &= x^{\frac{1}{5}}\left(\frac{1}{x\ln(7)}\right) + \log_7(x)\left(\frac{1}{5}x^{-\frac{4}{5}}\right) \end{aligned}$$

Because we must continue and find the domain of y', we need to simplify the derivative (some):

$$\begin{aligned} y' &= x^{\frac{1}{5}}\left(\frac{1}{x\ln(7)}\right)+\log_7(x)\left(\frac{1}{5}x^{-\frac{4}{5}}\right) \\ &= \frac{x^{\frac{1}{5}}}{x\ln(7)}+\frac{\log_7(x)}{5x^{\frac{4}{5}}} \\ &= \frac{x^{\frac{1}{5}-1}}{\ln(7)}+\frac{\log_7(x)}{5x^{\frac{4}{5}}} \\ &= \frac{x^{-\frac{4}{5}}}{\ln(7)}+\frac{\log_7(x)}{5x^{\frac{4}{5}}} \\ &= \frac{1}{x^{\frac{4}{5}}\ln(7)}+\frac{\log_7(x)}{5x^{\frac{4}{5}}} \end{aligned}$$

Considering domain restrictions, we see that $x^{4/5} \neq 0$ because we cannot divide by zero. Solving this inequality leads us to conclude that $x \neq 0$. Also, $x > 0$ because of the $\log_7(x)$ term. To satisfy both of these restrictions, y' must have a domain of $(0, \infty)$. Because this domain is a subset of the domain of the orginal function (in fact, they are the same!), we can conclude the function y is differentiable on $(0, \infty)$. Note that this means the function is differentiable at every point in its domain.

▪

Try It # 2:
Determine, algebraically, where $g(x) = x^{\frac{3}{7}}e^x$ is differentiable, and write your answer using interval notation.

The Quotient Rule

We discovered previously that the derivative of a product is **not** the product of the derivatives, but what about quotients? Let's explore this topic by looking at the function f below:

$$f(x) = \frac{5}{x^3}$$

Notice that $f(x)$ is written as a quotient, or ratio, of two functions. To find $f'(x)$, can we just take the derivative of the numerator and the derivative of the denominator and divide the results?

We know that

$$\frac{d}{dx}(5) = 0 \quad \text{and} \quad \frac{d}{dx}\left(x^3\right) = 3x^2$$

Is it possible, then, that the derivative of f is simply

$$\frac{0}{3x^2} = 0 \ ?$$

Recall from the previous section that we are able to rewrite the original function as

$$f(x) = \frac{5}{x^3} = 5x^{-3}$$

Applying the Introductory Derivative Rules to find the derivative gives

$$f'(x) = 5\left(-3x^{-4}\right) = -15x^{-4}$$

Notice this answer is not the same as our hypothesized answer:

$$-15x^{-4} \neq 0$$

Thus, we have convinced ourselves that we cannot just take the derivative of the numerator and the derivative of the denominator and divide the results.

! *The derivative of a quotient is **NOT** the quotient of the derivatives:* $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right) \neq \dfrac{f'(x)}{g'(x)}$

The question now becomes how do we find $\dfrac{d}{dx}\left(\dfrac{f(x)}{g(x)}\right)$? We can develop a formula using the Product Rule. Suppose we let $Q(x) = \dfrac{f(x)}{g(x)}$, and we want to find the derivative $Q'(x)$. If we multiply both sides of $Q(x) = \dfrac{f(x)}{g(x)}$ by $g(x)$, we get $Q(x) \cdot g(x) = f(x)$, or $f(x) = Q(x) \cdot g(x)$. This is a product, so we can use the Product Rule to find $f'(x)$:

$$f'(x) = Q(x) \cdot g'(x) + g(x) \cdot Q'(x)$$

Because our goal is to find a formula for $Q'(x)$, we solve for $Q'(x)$:

$$g(x) \cdot Q'(x) = f'(x) - Q(x) \cdot g'(x)$$

$$Q'(x) = \frac{f'(x) - Q(x) \cdot g'(x)}{g(x)}$$

Next, because we want a formula that only involves $f(x)$ and $g(x)$, we replace $Q(x)$ with how it was originally defined, $\dfrac{f(x)}{g(x)}$:

$$Q'(x) = \frac{f'(x) - \dfrac{f(x)}{g(x)} \cdot g'(x)}{g(x)}$$

Getting a common denominator and algebraically manipulating this expression to avoid having a complex fraction in our result gives

$$\begin{aligned}
Q'(x) &= \frac{f'(x) - \dfrac{f(x) \cdot g'(x)}{g(x)}}{g(x)} \\
&= \frac{\dfrac{g(x)}{g(x)} \cdot f'(x) - \dfrac{f(x) \cdot g'(x)}{g(x)}}{g(x)} \\
&= \frac{\dfrac{g(x) \cdot f'(x)}{g(x)} - \dfrac{f(x) \cdot g'(x)}{g(x)}}{g(x)} \\
&= \frac{\dfrac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{g(x)}}{g(x)} \\
&= \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{(g(x))^2}
\end{aligned}$$

This is called the **Quotient Rule**.

Theorem 2.4 The Quotient Rule

If f and g are differentiable functions, then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)\left(\frac{d}{dx}(f(x))\right) - f(x)\left(\frac{d}{dx}(g(x))\right)}{(g(x))^2}$$

$$= \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{(g(x))^2}$$

The Quotient Rule can be verbalized as "the bottom function times the derivative of the top function, minus the top function times the derivative of the bottom function, all over the bottom function squared." We will practice using the Quotient Rule in the next example.

Example 3 Find the derivative of each of the following functions.

a. $f(x) = \dfrac{5}{x^3}$

b. $y = \dfrac{x+7}{x-7}$

c. $g(x) = \dfrac{\log_9(x)}{\sqrt{x} - 4x^2 - 12}$

Solution:

a. Notice this is the quotient from the motivating example. Applying the Quotient Rule gives

$$f'(x) = \frac{d}{dx}\left(\frac{5}{x^3}\right)$$

$$= \frac{\left(x^3\right)\left(\frac{d}{dx}(5)\right) - (5)\left(\frac{d}{dx}\left(x^3\right)\right)}{(x^3)^2}$$

$$= \frac{\left(x^3\right)(0) - (5)\left(3x^2\right)}{x^6}$$

$$= \frac{-15x^2}{x^6}$$

$$= -\frac{15}{x^4}$$

This answer is equivalent to the answer we found in the motivating example, $f'(x) = -15x^{-4}$.

N *It is always a good idea to rewrite a function like this when possible to avoid using the Quotient Rule because there is less room for error when using the Introductory Derivative Rules instead.*

b. The function $y = \dfrac{x+7}{x-7}$ consists of a quotient of two functions. But in this case, we cannot rewrite the function in order to use the Introductory Derivative Rules because there is more than one term in the denominator. So we use the Quotient Rule:

$$y' = \frac{(x-7)\left(\dfrac{d}{dx}(x+7)\right)-(x+7)\left(\dfrac{d}{dx}(x-7)\right)}{(x-7)^2}$$

$$= \frac{(x-7)(1)-(x+7)(1)}{(x-7)^2}$$

$$= \frac{x-7-x-7}{(x-7)^2}$$

$$= \frac{-14}{(x-7)^2}$$

c. The function $g(x) = \dfrac{\log_9(x)}{\sqrt{x}-4x^2-12}$ also consists of a quotient of two functions, so we apply the Quotient Rule. Also, there is a radical, so we rewrite it as a power function ($\sqrt{x} = x^{1/2}$):

$$g'(x) = \frac{\left(x^{\frac{1}{2}}-4x^2-12\right)\left(\dfrac{d}{dx}(\log_9(x))\right)-(\log_9(x))\left(\dfrac{d}{dx}\left(x^{\frac{1}{2}}-4x^2-12\right)\right)}{\left(x^{\frac{1}{2}}-4x^2-12\right)^2}$$

$$= \frac{\left(x^{\frac{1}{2}}-4x^2-12\right)\left(\dfrac{1}{x\ln(9)}\right)-(\log_9(x))\left(\frac{1}{2}x^{-\frac{1}{2}}-8x\right)}{\left(x^{\frac{1}{2}}-4x^2-12\right)^2}$$

N *As stated previously, we will refrain from simplifying this derivative. This example asks us to find the derivative, and that is what we have done. If we need to use the derivative for some application, it might be necessary to simplify it to continue solving the problem. But for this example, this is the answer we are looking for.*

▪

Try It # 3:
Find the derivative of each of the following functions.

a. $f(x) = \dfrac{x^2-3x+1}{2x^3+4x-12}$

b. $P(x) = \dfrac{14}{\ln(x)}$

c. $C(x) = \dfrac{5x^7}{x^3+3^x}$

In the previous examples, each function involved only one application of the Product Rule or Quotient Rule. Sometimes a function may involve a product and a quotient, or even two products or two quotients. We will practice finding the derivative of these more involved functions in the next example.

Example 4 Find the derivative of each of the following functions:

a. $g(t) = \dfrac{7t^{10}e^t}{3t^4 - 2t^3 + t^2 - 1}$

b. $H(t) = \left(\dfrac{t^2 - 22}{4t^3 + 8t}\right)(\ln(t) + 3t)$

c. $f(t) = \left(2t^8 + 13t^2\right)(\log_3(t))\,14^t$

Solution:

a. Observe that $g(t)$ is a quotient of two functions, so we apply the Quotient Rule:

$$g'(t) = \frac{\left(3t^4 - 2t^3 + t^2 - 1\right)\left(\frac{d}{dt}\left(7t^{10}e^t\right)\right) - \left(7t^{10}e^t\right)\left(\frac{d}{dt}\left(3t^4 - 2t^3 + t^2 - 1\right)\right)}{\left(3t^4 - 2t^3 + t^2 - 1\right)^2}$$

Notice that there are only two parts that involve more work. We must find the derivative of $7t^{10}e^t$ and the derivative of $3t^4 - 2t^3 + t^2 - 1$. The derivative of the latter just involves the Introductory Derivative Rules:

$$\frac{d}{dt}\left(3t^4 - 2t^3 + t^2 - 1\right) = 12t^3 - 6t^2 + 2t$$

Because $7t^{10}e^t$ is a product of two functions, we must use the Product Rule to find its derivative:

$$\begin{aligned}\frac{d}{dt}\left(7t^{10}e^t\right) &= 7t^{10}\left(\frac{d}{dt}\left(e^t\right)\right) + e^t\left(\frac{d}{dt}\left(7t^{10}\right)\right)\\ &= 7t^{10}e^t + e^t\left(70t^9\right)\\ &= 7t^{10}e^t + 70t^9e^t\end{aligned}$$

Substituting both of these into $g'(t)$ gives

$$g'(t) = \frac{\left(3t^4 - 2t^3 + t^2 - 1\right)\left(7t^{10}e^t + 70t^9e^t\right) - \left(7t^{10}e^t\right)\left(12t^3 - 6t^2 + 2t\right)}{\left(3t^4 - 2t^3 + t^2 - 1\right)^2}$$

b. We start with the Product Rule because, at the heart of it, $H(t) = \left(\dfrac{t^2 - 22}{4t^3 + 8t}\right)(\ln(t) + 3t)$ is a product of two functions:

$$H'(t) = \left(\frac{t^2 - 22}{4t^3 + 8t}\right)\left(\frac{d}{dt}(\ln(t) + 3t)\right) + (\ln(t) + 3t)\left(\frac{d}{dt}\left(\frac{t^2 - 22}{4t^3 + 8t}\right)\right)$$

Again, notice there are only two parts that involve more work. We must find the derivative of $\ln(t) + 3t$ and the derivative of $\dfrac{t^2 - 22}{4t^3 + 8t}$. The derivative of $\ln(t) + 3t$ involves the Introductory Derivative Rules:

$$\frac{d}{dt}(\ln(t) + 3t) = \frac{1}{t} + 3$$

Because $\dfrac{t^2-22}{4t^3+8t}$ is a quotient of two functions, we use the Quotient Rule to find its derivative:

$$\frac{d}{dt}\left(\frac{t^2-22}{4t^3+8t}\right)=\frac{\left(4t^3+8t\right)\left(\frac{d}{dt}\left(t^2-22\right)\right)-\left(t^2-22\right)\left(\frac{d}{dt}\left(4t^3+8t\right)\right)}{\left(4t^3+8t\right)^2}$$

$$=\frac{\left(4t^3+8t\right)(2t)-\left(t^2-22\right)\left(12t^2+8\right)}{\left(4t^3+8t\right)^2}$$

Substituting both of these into $H'(t)$ gives

$$H'(t)=\left(\frac{t^2-22}{4t^3+8t}\right)\left(\frac{1}{t}+3\right)+(\ln(t)+3t)\left(\frac{\left(4t^3+8t\right)(2t)-\left(t^2-22\right)\left(12t^2+8\right)}{\left(4t^3+8t\right)^2}\right)$$

c. Notice $f(t)=\left(2t^8+13t^2\right)(\log_3(t))\,14^t$ is the product of three functions. We will start by applying the Product Rule to the two functions $\left(2t^8+13t^2\right)\log_3(t)$ and 14^t:

$$f'(t)=\left(\left(2t^8+13t^2\right)\log_3(t)\right)\left(\frac{d}{dt}\left(14^t\right)\right)+\left(14^t\right)\left(\frac{d}{dt}\left(\left(2t^8+13t^2\right)\log_3(t)\right)\right)$$

Again, there are only two parts that involve more work. We must find the derivative of 14^t and the derivative of $\left(2t^8+13t^2\right)\log_3(t)$. Applying the Introductory Derivative Rules to find the derivative of 14^t gives

$$\frac{d}{dt}\left(14^t\right)=14^t\ln(14)$$

To find the derivative of $\left(2t^8+13t^2\right)\log_3(t)$, we need to use the Product Rule a second time:

$$\begin{aligned}\frac{d}{dt}\left(\left(2t^8+13t^2\right)\log_3(t)\right)&=\left(2t^8+13t^2\right)\left(\frac{d}{dt}(\log_3(t))\right)+(\log_3(t))\left(\frac{d}{dt}\left(2t^8+13t^2\right)\right)\\&=\left(2t^8+13t^2\right)\left(\frac{1}{t\ln(3)}\right)+(\log_3(t))\left(16t^7+26t\right)\end{aligned}$$

Substituting both of these into $f'(t)$ yields

$$f'(t)=\left(\left(2t^8+13t^2\right)\log_3(t)\right)\left(14^t\ln(14)\right)+\left(14^t\right)\left(\left(2t^8+13t^2\right)\left(\frac{1}{t\ln(3)}\right)+(\log_3(t))\left(16t^7+26t\right)\right)$$

N *Note the difference when starting parts **a** and **b**. In part **a**, we started with the Quotient Rule because the most "outside" part of the function (or the last thing we would do when computing it) is division, whereas in part **b**, the most "outside" part of the function is multiplication. Thus, we started with the Product Rule.*

■

Try It # 4:
Find the derivative of each of the following functions.

a. $F(x) = \dfrac{3x^4 - 2x^2 + 2^x - e}{(x-3)(\ln(x))}$

b. $s(t) = \left(6t^3 + 4t\right)\left(\dfrac{10 - 2\log_4(t)}{50 - t^8}\right)$

c. $f(t) = \left(2t^8 + 13t^2\right)(\log_3(t))\,14^t$; Use $2t^8 + 13t^2$ and $(\log_3(t))\,14^t$ as the two starting functions instead of how the problem was worked in part **c** of the previous example to see that you get the same answer!

N *For some, the solutions in this section might seem a little overwhelming. It is highly recommended that when you approach these problems you take one step at a time. If you try to do it all at once, there is a greater chance of making a mistake. Whether you are applying the Product Rule or the Quotient Rule, you eventually are just applying the Introductory Derivative Rules from the previous section, which are much easier to compute.*

■ **Example 5** Find the equation of the line tangent to the graph of $f(x) = \dfrac{30x\ln(x)}{3x^2+12}$ at $x = 1$, and graph the function and the tangent line on the same axes.

Solution:
Remember, to find the equation of a tangent line (or any line), we need its slope and a point on the line. Thus, our first order of business is to find the derivative of f because we know that $f'(1)$ will give us the slope of the tangent line at $x = 1$. $f(x)$ is a quotient of two functions, so we apply the Quotient Rule:

$$f'(x) = \frac{\left(3x^2+12\right)\left(\dfrac{d}{dx}(30x\ln(x))\right) - (30x\ln(x))\left(\dfrac{d}{dx}\left(3x^2+12\right)\right)}{\left(3x^2+12\right)^2}$$

We know that $\dfrac{d}{dx}\left(3x^2+12\right) = 6x$, but because $30x\ln(x)$ is a product of two functions, we must apply the Product Rule to find its derivative:

$$\begin{aligned}
\frac{d}{dx}(30x\ln(x)) &= 30x\left(\frac{d}{dx}(\ln(x))\right) + (\ln(x))\left(\frac{d}{dx}(30x)\right)\\
&= 30x\left(\frac{1}{x}\right) + \ln(x)(30)\\
&= 30\cancel{x}\left(\frac{1}{\cancel{x}}\right) + 30\ln(x)\\
&= 30 + 30\ln(x)
\end{aligned}$$

Substituting both of these into $f'(x)$ gives

$$f'(x) = \frac{\left(3x^2+12\right)(30+30\ln(x)) - (30x\ln(x))(6x)}{\left(3x^2+12\right)^2}$$

Now, we substitute 1 for x to get the slope of the tangent line at $x = 1$:

$$\begin{aligned} f'(1) &= \frac{\left(3(1)^2+12\right)(30+30\ln(1)) - (30(1)\ln(1))(6(1))}{\left(3(1)^2+12\right)^2} \\ &= \frac{(15)(30+0)-0}{(15)^2} \\ &= 2 \end{aligned}$$

Next, we need to find a point on the line. We know $x = 1$, and we can find the corresponding y-value by evaluating the original function $f(x) = \dfrac{30x\ln(x)}{3x^2+12}$ at $x = 1$:

$$\begin{aligned} f(1) &= \frac{30(1)\ln(1)}{3(1)^2+12} \\ &= \frac{0}{15} \\ &= 0 \end{aligned}$$

Thus, the tangent line passes through the point $(1,0)$ and has a slope of 2. Using the point-slope form of the equation of a line gives

$$\begin{aligned} y - 0 &= 2(x-1) \\ y &= 2x - 2 \end{aligned}$$

The graphs of $f(x) = \dfrac{30x\ln(x)}{3x^2+12}$ and $y = 2x - 2$ are shown in **Figure 2.4.1**.

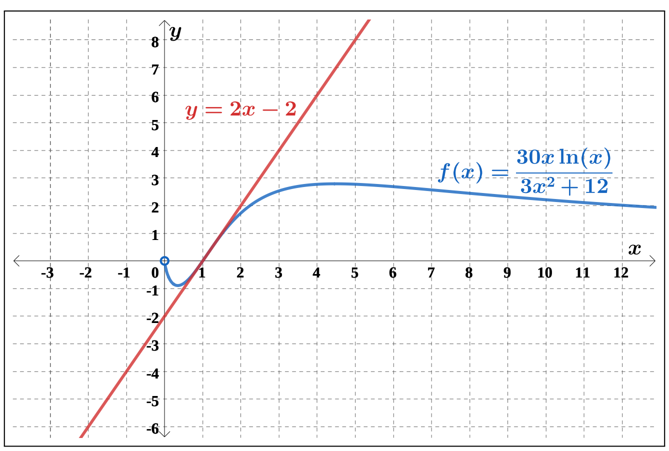


Figure 2.4.1: Graphs of $f(x) = \dfrac{30x\ln(x)}{3x^2+12}$ and the line tangent to the graph of f at $x = 1$

■

Try It # 5:
Find the equation of the line tangent to the graph of $f(x) = \dfrac{2x+3}{x^2+1}$ at $x = -2$.

▪ **Example 6** **Table 2.9** shows values of $f(x)$, $g(x)$, $f'(x)$, and $g'(x)$ for certain values of x. Use the information in the table to answer each of the following.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-4	-14	-92	19	128
0	1	12	-1	0
4	-2	4	3	16
8	28	172	31	176

Table 2.9: Various values of $f(x)$, $g(x)$, $f'(x)$, and $g'(x)$

a. If $T(x) = \dfrac{f(x)}{g(x)}$, find $T'(8)$.

b. If $Z(x) = x^2 \cdot g(x) - \log_3(x)$, find $Z'(4)$.

Solution:

a. $T(x)$ is a quotient of two functions, so we will start by applying the Quotient Rule to find $T'(x)$:

$$T'(x) = \frac{g(x)\cdot f'(x) - f(x)\cdot g'(x)}{(g(x))^2}$$

Substituting $x = 8$ and then finding the appropriate function values in the table gives

$$\begin{aligned} T'(8) &= \frac{g(8)\cdot f'(8) - f(8)\cdot g'(8)}{(g(8))^2} \\ &= \frac{(172)(31)-(28)(176)}{(172)^2} \\ &= \frac{404}{29584} = \frac{101}{7396} \end{aligned}$$

b. To find $Z'(x)$, where $Z(x) = x^2 \cdot g(x) - \log_3(x)$, we start with the Difference Rule:

$$Z'(x) = \frac{d}{dx}\left(x^2\cdot g(x)\right) - \frac{d}{dx}(\log_3(x))$$

To find the derivative of the first term of the difference, we must use the Product Rule. We will also use the Introductory Derivative Rules when finding both derivatives:

$$\begin{aligned} Z'(x) &= \frac{d}{dx}\left(x^2\cdot g(x)\right) - \frac{d}{dx}(\log_3(x)) \\ &= x^2\cdot g'(x) + g(x)\cdot\frac{d}{dx}\left(x^2\right) - \frac{d}{dx}(\log_3(x)) \\ &= x^2\cdot g'(x) + g(x)\cdot(2x) - \frac{1}{x\ln(3)} \end{aligned}$$

Substituting $x = 4$ into $Z'(x) = x^2 \cdot g'(x) + g(x) \cdot (2x) - \dfrac{1}{x\ln(3)}$ and then finding the appropriate function values in the table gives

$$\begin{aligned} Z'(4) &= (4)^2 \cdot g'(4) + g(4) \cdot (2 \cdot 4) - \frac{1}{4\ln(3)} \\ &= 16 \cdot 16 + 4 \cdot 8 - \frac{1}{4\ln(3)} \\ &= 288 - \frac{1}{4\ln(3)} \end{aligned}$$

▪

Try It # 6:
Using **Table 2.9** from the previous example, answer each of the following.

a. If $R(x) = f(x) \cdot f(x) + 2^x \cdot g(x)$, find $R'(4)$.

b. If $C(x) = \dfrac{x^3 + g(x)}{x^3 - f(x)}$, find $C'(0)$.

▪ **Example 7** The graphs of the functions f and g are shown in **Figure 2.4.2**. Use the graphs to answer each of the following.

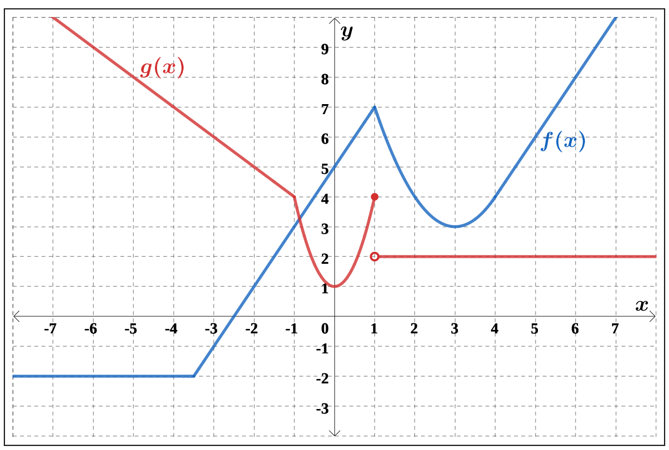


Figure 2.4.2: Graphs of the functions f and g

a. If $h(x) = f(x) \cdot g(x)$, find $h'(0)$.

b. If $j(x) = \dfrac{g(x) - 7^x}{6f(x)}$, find $j'(3)$.

Solution:

a. Using the Product Rule to find $h'(x)$ gives

$$h'(x) = f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

Substituting $x = 0$ we get

$$h'(0) = f(0) \cdot g'(0) + g(0) \cdot f'(0)$$

Looking at **Figure 2.4.2**, we see $f(0) = 5$ and $g(0) = 1$. Recall that these are the y-values of the points on the graphs corresponding to $x = 0$.

To find $g'(0)$, we need to determine the slope of the line tangent to the graph of g at $x = 0$. Because this tangent line is horizontal, we conclude that $g'(0) = 0$. Now, notice the graph of f is linear near $x = 0$. So $f'(0)$ is just the slope of the line. Thus, $f'(0) = 2$. Substituting this information into $h'(0)$ gives

$$\begin{aligned} h'(0) &= f(0) \cdot g'(0) + g(0) \cdot f'(0) \\ &= 5 \cdot 0 + 1 \cdot 2 \\ &= 2 \end{aligned}$$

b. To find the derivative of $j(x) = \dfrac{g(x) - 7^x}{6f(x)}$, we use the Quotient Rule:

$$\begin{aligned} j'(x) &= \frac{(6f(x))\left(\frac{d}{dx}(g(x) - 7^x)\right) - (g(x) - 7^x)\left(\frac{d}{dx}(6f(x))\right)}{(6f(x))^2} \\ &= \frac{(6f(x))(g'(x) - 7^x \ln(7)) - (g(x) - 7^x)(6f'(x))}{(6f(x))^2} \end{aligned}$$

Substituting 3 for x gives

$$j'(3) = \frac{(6f(3))\left(g'(3) - 7^3 \ln(7)\right) - \left(g(3) - 7^3\right)(6f'(3))}{(6f(3))^2}$$

Now, we must find $f(3)$, $f'(3)$, $g(3)$, and $g'(3)$. Looking at **Figure 2.4.2**, we see $f(3) = 3$ and $g(3) = 2$. We also note that $g'(3) = 0$ because the graph of g is a line with a slope of 0 near $x = 3$. Lastly, $f'(3) = 0$ because the graph of f has a horizontal tangent line at $x = 3$. Substituting these values into $j'(3)$ gives

$$\begin{aligned} j'(3) &= \frac{(6(3))(0 - 343\ln(7)) - (2 - 343)(6(0))}{(6(3))^2} \\ &= \frac{(18)(-343\ln(7)) - 0}{324} \\ &= \frac{-6174\ln(7)}{324} \\ &= -\frac{343\ln(7)}{18} \end{aligned}$$

▪

Try It # 7:
Using **Figure 2.4.2** from the previous example, answer each of the following.

a. If $k(x) = \dfrac{f(x)}{g(x)}$, find $k'(-4)$.

b. If $m(x) = 3f(x) \cdot g(x) + e^x$, find $m'(0)$.

Now that we are able to find the derivative of slightly more complicated functions involving products and quotients, let's revisit some of the marginal analysis applications we discussed in **Section 2.3**.

Example 8 Better Purchase, a technology store, has a price-demand function given by $p(x) = 300(0.997)^x$, where $p(x)$ is the price, in dollars, of each mouse when x mice are sold. Find the marginal revenue when 812 mice are sold, and interpret your answer.

Solution:
Recall that the marginal revenue is the derivative of the revenue function. Thus, we need to find $R'(812)$. However, first we need to find the revenue function, R, because we were only given the price-demand function, p, in the problem. Remember that

$$\begin{aligned} R(x) &= x \cdot p(x) \\ &= x\left(300(0.997)^x\right) \\ &= 300x(0.997)^x \end{aligned}$$

Thus, $R(x) = 300x(0.997)^x$ gives the revenue, in dollars, when x mice are sold.

Because $R(x)$ is a product of two functions, we use the Product Rule to find $R'(x)$:

$$\begin{aligned} R'(x) &= (300x)\left(\frac{d}{dx}((0.997)^x)\right) + ((0.997)^x)\left(\frac{d}{dx}(300x)\right) \\ &= 300x\left((0.997)^x \ln(0.997)\right) + ((0.997)^x)(300) \end{aligned}$$

Substituting $x = 812$ gives

$$\begin{aligned} R'(812) &= 300(812)(0.997)^{812} \ln(0.997) + 300(0.997)^{812} \\ &\approx -\$37.66 \text{ per mouse} \end{aligned}$$

This means that when Better Purchase sells 812 mice, their revenue is decreasing at a rate of \$37.66 per mouse.

Recall in the previous section we learned $R'(812)$ can be used to estimate the revenue from selling the 813^{th} mouse. Thus, we could also say that Better Purchase loses approximately \$37.66 in revenue when selling the 813^{th} mouse.

■

Average and Marginal Average Functions

Now that we have more tools in our toolbox to help us find derivatives, we can expand upon the ideas we discussed in **Section 2.3**.

Recall that the average of a group of numbers can be found by adding all the numbers and dividing by how many numbers there are. We can do the same thing to find the average cost of an item: find the total cost of producing all the items and divide by the number of items produced. Likewise, we can define the **average cost function** to be the cost function, $C(x)$, divided by the number of items produced, x. We can define the **average revenue function** and **average profit function** similarly:

Definition

- The **average cost function** has an input of the number of items produced and an output of the average cost of producing each item.

$$\overline{C}(x) = \frac{C(x)}{x}$$

- The **average revenue function** has an input of the number of items sold and an output of the average revenue from selling each item.

$$\overline{R}(x) = \frac{R(x)}{x}$$

- The **average profit function** has an input of the number of items produced and sold and an output of the average profit of producing and selling each item.

$$\overline{P}(x) = \frac{P(x)}{x}$$

■

In the previous section, we found the marginal cost, marginal revenue, and marginal profit functions by finding the derivative of the cost, revenue, and profit functions, respectively. We can define similar functions by taking the derivative of the average cost, average revenue, and average profit functions.

Definition

- The **marginal average cost function** is the derivative of the average cost function, $\overline{C}\,'$.
- The **marginal average revenue function** is the derivative of the average revenue function, $\overline{R}\,'$.
- The **marginal average profit function** is the derivative of the average profit function, $\overline{P}\,'$.

■

Remember that a derivative is a rate of change, so the marginal average cost function, for instance, measures the rate of change of the average cost when x items are produced.

■ **Example 9** A company that makes Barbara Dolls has a weekly cost function, in dollars, given by $C(x) = 6000 + 13x + 0.02x^2$, where x is the number of dolls produced.

a. Find $\overline{C}(1024)$ and interpret your answer.

b. Find the marginal average cost at a production level of 1024 dolls, and interpret your answer.

Solution:

a. The average cost function is given by

$$\overline{C}(x) = \frac{C(x)}{x} = \frac{6000 + 13x + 0.02x^2}{x}$$

Substituting $x = 1024$ into $\overline{C}(x) = \dfrac{6000+13x+0.02x^2}{x}$ gives

$$\begin{aligned}\overline{C}(1024) &= \frac{6000+13(1024)+0.02(1024)^2}{1024}\\ &= \frac{40,283.52}{1024}\\ &\approx \$39.34 \text{ per doll}\end{aligned}$$

Thus, when 1024 Barbara dolls are produced, the average cost per doll is \$39.34.

b. Here, we must find the marginal average cost at a production level of 1024 dolls. The marginal average cost is the derivative of the average cost function, so first we need to find $\overline{C}'(x)$.

There are two ways we can do this. We can use the Quotient Rule to find the derivative, or we can manipulate the function algebraically and then use the Introductory Derivative Rules. For this example, we will algebraically manipulate the function to find the derivative:

$$\begin{aligned}\overline{C}(x) &= \frac{6000}{x} + \frac{13x}{x} + \frac{0.02x^2}{x}\\ &= 6000^{-1} + \frac{13\cancel{x}}{\cancel{x}} + \frac{0.02x^{\cancel{2}}}{\cancel{x}}\\ &= 6000x^{-1} + 13 + 0.02x\end{aligned}$$

Next, we take the derivative:

$$\begin{aligned}\overline{C}'(x) &= -6000x^{-2} + 0 + 0.02\\ &= -6000x^{-2} + 0.02\end{aligned}$$

Substituting $x = 1024$ gives

$$\begin{aligned}\overline{C}'(1024) &= -6000(1024)^{-2} + 0.02\\ &\approx -\$0.01 \text{ per doll per doll}\end{aligned}$$

When 1024 Barbara Dolls are produced, the average cost per doll is decreasing at a rate of \$0.01 per doll.

■

■ **Example 10** The company in the previous example makes an accessory box for their Barbara Dolls. Their weekly revenue function, in dollars, is given by $R(x) = 500 + 610\ln(x)$, where x is the number of boxes sold.

a. Find $R'(300)$ and interpret your answer.

b. Find $\overline{R}(300)$ and interpret your answer.

c. Find $\overline{R}'(300)$ and interpret your answer.

Solution:

a. Notice that we are asked to find the marginal revenue, **not** the marginal average revenue. Hence, we take the derivative of the revenue function:

$$R'(x) = 0 + 610\left(\frac{1}{x}\right) = \frac{610}{x}$$

Substituting $x = 300$ into $R'(x) = \dfrac{610}{x}$ gives

$$\begin{aligned} R'(300) &= \frac{610}{300} \\ &\approx \$2.03 \text{ per box} \end{aligned}$$

Thus, when 300 accessory boxes are sold, revenue is increasing at a rate of \$2.03 per box.

b. We are asked to find $\overline{R}(300)$, which is the average revenue when $x = 300$. To find the average revenue function, we divide the revenue function by x:

$$\overline{R}(x) = \frac{500 + 610\ln(x)}{x}$$

Substituting $x = 300$ gives

$$\begin{aligned} \overline{R}(300) &= \frac{500 + 610\ln(300)}{300} \\ &\approx \$13.26 \text{ per box} \end{aligned}$$

When 300 boxes are sold, the average revenue per box is \$13.26.

c. We are asked to find $\overline{R}'(300)$, so we need to find the derivative of the average revenue function (i.e., we need to find the marginal average revenue function). Recall that we found the average revenue function in part **b**:

$$\overline{R}(x) = \frac{500 + 610\ln(x)}{x}$$

Unlike the previous example, we will have to use the Quotient Rule to find $\overline{R}'(x)$. Even though there is only one term in the denominator, x, we cannot algebraically manipulate the function and use the Introductory Derivative Rules because of the $\ln(x)$ term in the numerator. Using the Quotient Rule gives

$$\begin{aligned} \overline{R}'(x) &= \frac{(x)\left(\frac{d}{dx}(500 + 610\ln(x))\right) - (500 + 610\ln(x))\left(\frac{d}{dx}(x)\right)}{x^2} \\ &= \frac{(x)\left(\frac{610}{x}\right) - (500 + 610\ln(x))(1)}{x^2} \\ &= \frac{\cancel{(x)}\frac{610}{\cancel{x}} - (500 + 610\ln(x))}{x^2} \\ &= \frac{610 - 500 - 610\ln(x)}{x^2} \\ &= \frac{110 - 610\ln(x)}{x^2} \end{aligned}$$

Substituting $x = 300$ into $\overline{R}\,'(x) = \dfrac{110 - 610\ln(x)}{x^2}$ gives

$$\overline{R}\,'(300) = \frac{110 - 610\ln(300)}{(300)^2} \approx -\$0.04 \text{ per box per box}$$

When 300 boxes are sold, the average revenue per box is decreasing at a rate of $0.04 per box.

▪

Try It # 8:
If the weekly cost function, in dollars, for producing the accessory boxes for the Barbara Dolls in the previous example is given by $C(x) = 90 + 6x + 0.04x^2$, where x is the number of boxes produced, find and interpret each of the following.

a. $C\,'(250)$

b. $\overline{C}\,'(250)$

Try It Answers

1. **a.** $f\,'(x) = x\left(\dfrac{1}{x\ln(8)}\right) + \log_8(x)$

   b. $R\,'(t) = t^3\left(\dfrac{1}{2}t^{-1/2} + e^t\right) + \left(\sqrt{t} + e^t\right)\left(3t^2\right)$

   c. $v\,'(t) = \dfrac{7}{4}t^{3/4} - t^2\left(5^t\ln(t)\right) - 5^t(2t)$

2. $(-\infty, 0) \cup (0, \infty)$

3. **a.** $f\,'(x) = \dfrac{\left(2x^3 + 4x - 12\right)(2x - 3) - \left(x^2 - 3x + 1\right)\left(6x^2 + 4\right)}{\left(2x^3 + 4x - 12\right)^2}$

   b. $P\,'(x) = \dfrac{-14\left(\dfrac{1}{x}\right)}{(\ln(x))^2}$

   c. $C\,'(x) = \dfrac{\left(x^3 + 3^x\right)\left(35x^6\right) - \left(5x^7\right)\left(3x^2 + 3^x\ln(3)\right)}{\left(x^3 + 3^x\right)^2}$

4. **a.** $F'(x) = \dfrac{((x-3)(\ln(x)))\left(12x^3 - 4x + 2^x \ln(2)\right) - \left(3x^4 - 2x^2 + 2^x - e\right)\left((x-3)\left(\frac{1}{x}\right) + \ln(x)\right)}{((x-3)(\ln(x)))^2}$

   b. $s'(t) = \left(6t^3 + 4t\right)\left(\dfrac{\left(50 - t^8\right)\left(\frac{-2}{t\ln(4)}\right) - (10 - 2\log_4(t))\left(-8t^7\right)}{\left(50 - t^8\right)^2}\right) + \left(\dfrac{10 - 2\log_4(t)}{50 - t^8}\right)\left(18t^2 + 4\right)$

   c. $f'(t) = \left(2t^8 + 13t^2\right)\left((\log_3(t))\left(14^t \ln(14)\right) + 14^t\left(\frac{1}{t\ln(3)}\right)\right) + \left((\log_3(t))\, 14^t\right)\left(16t^7 + 26t\right)$

5. $y = \dfrac{6}{25}x + \dfrac{7}{25}$

6. **a.** $244 + 64\ln(2)$

   b. -12

7. **a.** $-2/49$

   b. 7

8. **a.** \$26.00 per box; When 250 accessory boxes are produced, cost is increasing at a rate of \$26.00 per box.

   b. \$0.04 per box per box; When 250 accessory boxes are produced, the average cost per box is increasing at a rate of \$0.04 per box.

EXERCISES

Basic Skills Practice

For Exercises 1 - 3, use the Product Rule to find $f'(x)$.

1. $f(x) = xe^x$

2. $f(x) = (5+x)\ln(x)$

3. $f(x) = 8x\log_2(x)$

For Exercises 4 - 6, use the Quotient Rule to find $f'(x)$.

4. $f(x) = \dfrac{7x}{x-2}$

5. $f(x) = \dfrac{12x}{3x+5}$

6. $f(x) = \dfrac{5x^2-2x+3}{e^x-8}$

For Exercises 7 - 10, find the slope of the line tangent to the graph of the function at the given x-value.

7. $f(x) = 8^x(x-2)$ at $x = 2$

8. $g(x) = \dfrac{x+3}{x-3}$ at $x = -3$

9. $h(x) = \dfrac{x}{7x^2+14}$ at $x = 4$

10. $f(x) = (\ln(x))(4x-18)$ at $x = 1$

For Exercises 11 - 13, find the x-value(s) where the graph of f has a horizontal tangent line.

11. $f(x) = \dfrac{4x+8}{e^x}$

12. $f(x) = e^x\left(x^2-2x-7\right)$

13. $f(x) = 4x^2\ln(x)$

For Exercises 14 - 16, x is the number of items made and sold by a company, and the output of the given function is in dollars.

14. If the cost function is given by $C(x) = 4000 + 32x + 0.5x^2$, find
    (a) the average cost function, $\overline{C}(x)$.
    (b) the marginal average cost function, $\overline{C}\,'(x)$.

15. If the revenue function is given by $R(x) = 300x^2(0.997)^x$, find
    (a) the average revenue function, $\overline{R}(x)$.
    (b) the marginal average revenue function, $\overline{R}\,'(x)$.

16. If the profit function is given by $P(x) = 100\sqrt{x} - 600$, find
    (a) the average profit function, $\overline{P}(x)$.
    (b) the marginal average profit function, $\overline{P}\,'(x)$.

Intermediate Skills Practice

For Exercises 17 - 19, find $f\,'(x)$.

17. $f(x) = 5x \cdot 8^x + 2x^2$

18. $f(x) = \left(14x^{\frac{7}{3}} + 90x\right)\left(15x^3 - \log_4(x)\right)$

19. $f(x) = \dfrac{\sqrt{x} + 3x^5}{7x^{-2} + 15x}$

For Exercises 20 - 22, find $\dfrac{dy}{dx}$.

20. $y = \dfrac{14x^{0.9} - 15x^{-0.9}}{16x^4 - e^x}$

21. $y = \dfrac{xe^x}{x - 7}$

22. $y = \left(\dfrac{5^x}{x^2 + 1}\right)\left(x^{50} + 6x^{70}\right)$

For Exercises 23 - 25, find the equation of the line tangent to the graph of the function at the given x-value.

23. $f(x) = \dfrac{3x^2 - 8}{4 - x^2}$ at $x = 0$

24. $g(x) = 15x^3 \ln(x)$ at $x = 1$

25. $h(x) = \dfrac{x^4 - 16}{x \cdot 8^x}$ at $x = -2$

26. Given $f(x) = \dfrac{\log_3(x)}{10 - x}$, find the equation of the line tangent to the graph of f at $x = 9$. Use technology to graph f and the tangent line on the same axes.

For Exercises 27 and 28, find the x-value(s) where the graph of f has a horizontal tangent line.

27. $f(x) = e^x\left(x^2 - 2x - 2\right)$

28. $f(x) = \dfrac{x^2 - 3}{x^2 - 9}$

For Exercises 29 and 30, find the x-value(s) where f has the given instantaneous rate of change.

29. $f(x) = x\ln(x)$; instantaneous rate of change is 1

30. $f(x) = \dfrac{-20}{x + 2}$; instantaneous rate of change is 5

31. Jenn and Barry Ice Cream started an advertising campaign. The company's sales t months after starting the advertising campaign can be modeled by $S(t) = \dfrac{68t^2 - 952t - 142}{t^2 - 14t - 15}$ thousand dollars. Find a function representing the rate of change of sales t months after the campaign started.

32. The value of a yacht t years after it is bought is given by $V(t) = 4.4 + \dfrac{192}{3t^2 + 50}$ ten thousand dollars. Find $V'(12)$ and interpret your answer. Round to six decimal places, if necessary.

33. Sara sells seashells by the seashore. When she sells c shells, her profit is given by $P(c) = \sqrt{c} \cdot (\ln(c) - 2) - 10$ hundred dollars. Find the rate of change of profit when Sara sells 40 seashells, and interpret your answer. Round to four decimal places, if necessary.

34. The company Incensitive makes prank bad smelling incense sticks. The daily revenue function of the company is given by $R(x) = 10x^2 \cdot (0.878)^x$ dollars when x packages of incense sticks are sold.
    (a) Find the company's marginal revenue function.
    (b) Find $R'(25)$ and interpret your answer.
    (c) Find the marginal revenue when 18 packages of incense sticks are sold, and interpret your answer.
    (d) Approximate the revenue from selling the 15^{th} package of incense sticks.
    (e) Find the exact revenue from selling the 12^{th} package of incense sticks.
    (f) Estimate the revenue if 30 packages of incense sticks are sold.
    (g) Find the exact revenue if 18 packages of incense sticks are sold.

35. Fony is a videogame company that makes the Gamestation console. The company's weekly cost function for making x thousand consoles is given by $C(x) = \dfrac{2300x^3 + 90x^2 + 21250}{x^3 + 125}$ thousand dollars. Find the marginal cost function for Fony's consoles.

36. An adult blue dragon is observed flying over a city. Its position, in feet, is given by $s(t) = \dfrac{40t^2 + 40t + 67}{t^2 + 1}$, where t is the number of seconds after it passes overhead. Find a function v representing the velocity of the dragon t seconds after it passes overhead.

37. The table below shows values of $f(x)$, $g(x)$, $f\,'(x)$, and $g\,'(x)$ for certain values of x. Use the information in the table to find each of the following.

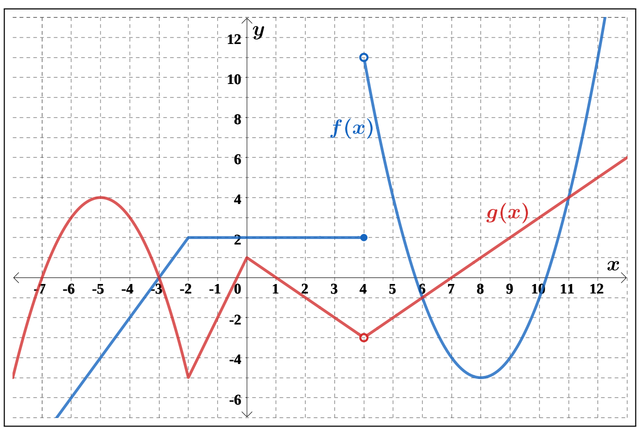
x	$f(x)$	$g(x)$	$f\,'(x)$	$g\,'(x)$
13	16	15	13	13
14	15	15	14	15
15	17	17	14	14
16	16	13	15	13
17	15	13	14	14

(a) $h\,'(15)$, where $h(x) = f(x) \cdot g(x)$

(b) $j\,'(13)$, where $j(x) = \dfrac{f(x) - x^2}{g(x)}$

(c) $k\,'(16)$, where $k(x) = \dfrac{g(x) \cdot f(x)}{e^x - 9}$

38. The graphs of the functions f and g are shown below on the same axes. Use the graphs to find each of the following.



(a) $h\,'(-5)$, where $h(x) = g(x)f(x)$

(b) $j\,'(8)$, where $j(x) = \dfrac{f(x)}{2x} - \dfrac{9}{g(x)}$

39. A company that makes clothes dryers has a weekly cost function, in dollars, given by $C(x) = 3000 + 12x + 0.2x^2$, where x is the number of dryers produced.

    (a) Find $\overline{C}(323)$ and interpret your answer.

    (b) Find the marginal average cost at a production level of 227 dryers, and interpret your answer.

    (c) Find the exact cost of producing 933 dryers.

40. Isaac Ay makes build-it-yourself desks. He has a revenue function, in dollars, given by $R(x) = 0.383x^2(0.997)^x$, where x is the number of desks sold.

    (a) Find $\overline{R}(60)$ and interpret your answer.

    (b) Find the marginal average revenue when 237 desks are sold, and interpret your answer.

    (c) Find the exact revenue from selling 97 desks.

41. A company that makes designer hairbrushes has a weekly profit function, in dollars, of $P(x) = 410 - 11x - 0.4x^2 + 610\ln(x)$, where x is the number of hairbrushes made and sold.

    (a) Find $\overline{P}(20)$ and interpret your answer.

    (b) Find the marginal average profit when 16 hairbrushes are made and sold, and interpret your answer.

    (c) Find the exact profit from making and selling 30 hairbrushes.

Mastery Practice

For Exercises 42 - 46, find $f'(x)$.

42. $f(x) = \left(\sqrt{x} - 18x\right) \cdot 2(10)^x$

43. $f(x) = \left(\dfrac{9x^3 + 1}{\ln(x) + 19x^6}\right)(e^x - 7)$

44. $f(x) = x^2 \cdot (\log_5(x)) \cdot e^x$

45. $f(x) = \left(4x^3 - 8x\right)^{-1}(18(0.5^x) + 37)$

46. $f(x) = \dfrac{4x^{1/4} + \pi^3}{\left(\sqrt[3]{x^2} + 8e^x\right)(2\log(x) - 9.7)}$

47. The table below shows values of $f(x)$, $g(x)$, $f'(x)$, and $g'(x)$ for certain values of x. Use the information in the table to find each of the following.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-2	1	0	-2	-2
-1	0	0	-1	0
0	2	2	-1	-1
1	1	-2	0	-2
2	0	-2	-1	1

(a) $h'(1)$, where $h(x) = \dfrac{f(x)\cdot g(x)}{f(x)+g(x)}$

(b) $j'(-2)$, where $j(x) = f(x)\cdot(8x - g(x))$

(c) $k'(1)$, where $k(x) = 3g(x)\cdot 4^x$

For Exercises 48 and 49, find the equation of the line tangent to the graph of the function at the given x-value. Use technology to graph the function and the tangent line on the same axes.

48. $g(x) = 8x^3e^x$ at $x = -1$

49. $h(x) = \dfrac{x^2 + 13x - 8}{x + 5\sqrt{x}}$ at $x = 4$

For Exercises 50 and 51, find the x-value(s) where the graph of the function has a horizontal tangent line.

50. $f(x) = \dfrac{8x+8}{e^x}$

51. $g(x) = e^x\left(x^2 - 3\right) - 31$

For Exercises 52 and 53, find the x-value(s) where the graph of the function has the indicated slope.

52. $f(x) = e^x\left(x^2 - 6x - 15\right) - 8x$; slope is -8

53. $g(x) = \dfrac{x-4}{3-x}$; slope is $-\dfrac{1}{9}$

54. A snowball is thrown with a velocity given by $v(t) = \dfrac{1.87t^2 + 4t + 4.83}{t^2+1}$ feet per second t seconds after being thrown. At what rate is the snowball decelerating 2 seconds after it is thrown? Round your answer to three decimal places, if necessary.

55. Mr. Wutz owns a hot dog business named Wutz UpDogs. The daily profit function when x hot dogs are sold is given by $P(x) = x^2(0.93)^x - 30$ dollars. Calculate $P'(20)$ and interpret your answer.

56. Mr. Wutz, the owner of Wutz UpDogs, started an advertising campaign. He found that his sales, in dollars, t weeks after the campaign started is given by $S(t) = 10\left(\frac{4t+7}{(1.2)^t}\right)$. Find the rate of change of sales two weeks after the campaign started.

57. A company has determined their yearly cost function is given by $C(x) = 500 + 610\ln(x)$ dollars when x pounds of product are made. Find the marginal average cost when 300 pounds of product are made, and interpret your answer.

58. The general store Nookington's has a department selling shovels. Tom Nook, the proprietor, determines the department's monthly revenue function is $R(x) = 24x - 5x \cdot \log(x)$ dollars, where x is the number of shovels sold.

    (a) Find the marginal revenue when 50 shovels are sold, and interpret your answer.

    (b) Approximate the revenue from selling the 50^{th} shovel.

    (c) Find the exact revenue from selling the 50^{th} shovel.

    (d) Estimate the revenue if 50 shovels are sold.

    (e) Find the exact revenue if 50 shovels are sold.

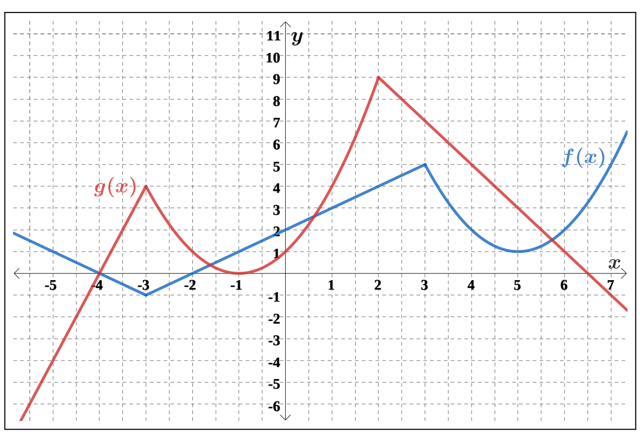
59. A company that makes and sells dishwashers has a monthly profit function, in dollars, given by $P(x) = -0.003x^3 + 0.41x^2 + 102x - 1850$, where x is the number of dishwashers made and sold.

    (a) Find $\overline{P}(234)$ and interpret your answer.

    (b) Find the marginal average profit when 209 dishwashers are made and sold, and interpret your answer.

    (c) Find the exact profit from making and selling 157 dishwashers.

60. The graphs of the functions f and g are shown below on the same axes. Use the graphs to find each of the following.



    (a) $h'(-1)$, where $h(x) = \dfrac{f(x)+g(x)}{f(x)-g(x)}$

    (b) $k'(5)$, where $k(x) = x^2 f(x) + x^3 g(x)$

Communication Practice

61. Does the derivative of a product of functions equal the product of their derivatives? Explain.

62. Does $\frac{d}{dx}\left(\frac{3x^4}{6^x}\right) = \frac{12x^3}{6^x \ln(6)}$? Explain.

63. Kathryn was helping her classmate, Vanessa, with one of her calculus homework problems. The problem said:

$$\text{"Find } f'(x) \text{ if } f(x) = \frac{\ln(x) - \pi x^3 - 4}{6e^x - x^2}\text{."}$$

Kathryn told Vanessa all she has to do is use the Quotient Rule. She wrote the following on Vanessa's paper:

$$\frac{TB' - BT'}{B^2}$$

If Vanessa uses the rule Kathryn wrote on her paper, will she get the correct answer? Explain.

2.5 The Chain Rule

We have seen a wide variety of techniques for calculating the derivative of a function, but there are still many functions for which our rules will not apply. For some functions, we can algebraically manipulate the function first and then apply our previous rules, but the amount of algebraic manipulation involved becomes unrealistic.

For instance, the derivative of $y = \left(3x^2+1\right)^{99}$ can be found by first multiplying $\left(3x^2+1\right)$ by itself ninety-nine times, but this is not reasonable as this algebra would take hours, if not days, to perform by hand. Even though it is possible to find the derivative that way, it is not in any way reasonable.

But for some functions, we currently have no method, reasonable or unreasonable, which will allow us to find the derivative. For instance, we do not currently have a technique that would allow us to find the derivative of $y = \sqrt{3x^2+1}$. Notice, though, we can write this function as a composite function of the form $y = f(g(x))$, where $f(x) = \sqrt{x}$ and $g(x) = 3x^2+1$, and we can use the Introductory Derivative Rules to find the derivatives of both f and g!

Our focus in this section is to find derivatives of functions that are compositions of functions. We will learn a new rule, called the **Chain Rule**, which will allow us to do just that!

Learning Objectives:
In this section, you will learn how to use the Chain Rule to calculate derivatives of a composite functions, as well as functions that have not yet been composed, and solve problems involving real-world applications. Upon completion you will be able to:

- Calculate the derivative of a composite function using the Chain Rule.
- Calculate the derivative of composite function before it has been composed using the Alternate Form of the Chain Rule.
- Calculate the derivative of logarithmic functions by first expanding them using the Properties of Logarithms.
- Calculate the slope of a tangent line using the Chain Rule.
- Find the equation of a tangent line using the Chain Rule.
- Graph a function and a line tangent to the graph of the function on the same axes.
- Determine the x-value(s) where the graph of a function has a horizontal tangent line using the Chain Rule.
- Determine the x-value(s) where the graph of a function has a tangent line of a given slope using the Chain Rule.
- Determine the x-value(s) where a function has a specified (instantaneous) rate of change using the Chain Rule.
- Calculate the (instantaneous) rate of change of a function involving a real-world scenario, including cost, revenue, profit, and position, using the Chain Rule.
- Interpret the meaning of the derivative of a function involving a real-world scenario, including cost, revenue, and profit.
- Calculate marginal cost, revenue, and profit using the Chain Rule.
- Interpret the meaning of marginal cost, revenue, and profit.
- Estimate the cost, revenue, or profit of an item using marginal analysis and the Chain Rule.
- Compute the exact cost, revenue, or profit of an item.

- Estimate the cost, revenue, or profit of a total number of items using marginal analysis and the Chain Rule.
- Calculate marginal average cost, revenue, and profit using the Chain Rule.
- Interpret the meaning of marginal average cost, revenue, and profit.

To help us develop the Chain Rule, let's look at the derivative of two basic, yet similar, functions. First, consider the function $F_1(x) = \left(3x^2+1\right)^2$. We can use the Introductory Derivative Rules to find the derivative of F_1 by first algebraically manipulating the function:

$$\begin{aligned} F_1(x) &= \left(3x^2+1\right)^2 \\ &= \left(3x^2+1\right)\left(3x^2+1\right) \\ &= 9x^4+6x^2+1 \end{aligned}$$

Because this function is now written as a sum of basic functions, we can easily find the derivative using the Introductory Derivative Rules:

$$F_1{}'(x) = 36x^3+12x$$

We will rewrite this derivative slightly to help with the development of our new rule:

$$\begin{aligned} F_1{}'(x) &= 36x^3+12x \\ &= 12x\left(3x^2+1\right) \\ &= 2\left(3x^2+1\right)(6x) \end{aligned}$$

At this point, you may be able to see how to find the derivative of F_1 without first expanding the function, but before we discuss it, let's look at a second, similar function : $F_2(x) = \left(3x^2+1\right)^3$. Notice the only difference between F_2 and F_1 is now the function $\left(3x^2+1\right)$ is being raised to the third power instead of the second power. Again, we first algebraically manipulate the function:

$$\begin{aligned} F_2(x) &= \left(3x^2+1\right)^3 \\ &= \left(3x^2+1\right)\left(3x^2+1\right)\left(3x^2+1\right) \\ &= \left(3x^2+1\right)\left(9x^4+6x^2+1\right) \\ &= 27x^6+18x^4+3x^2+9x^4+6x^2+1 \\ &= 27x^6+27x^4+9x^2+1 \end{aligned}$$

Now, we can easily find the derivative $F_2{}'(x)$. Again, we will algebraically manipulate the result so it is in our desired form:

$$\begin{aligned} F_2{}'(x) &= 162x^5+108x^3+18x \\ &= 18x\left(9x^4+6x^2+1\right) \\ &= 18x\left(3x^2+1\right)\left(3x^2+1\right) \\ &= 18x\left(3x^2+1\right)^2 \\ &= 3\left(3x^2+1\right)^2(6x) \end{aligned}$$

Our results are summarized below:

$$F_1(x) = \left(3x^2+1\right)^2 \quad \text{and} \quad F_1{}'(x) = 2\left(3x^2+1\right)(6x)$$
$$F_2(x) = \left(3x^2+1\right)^3 \quad \text{and} \quad F_2{}'(x) = 3\left(3x^2+1\right)^2(6x)$$

Notice that for both of these functions, to find the derivative without algebraically manipulating the original function first, all we have to do is bring the power down to the front, leave the function on the inside, $3x^2+1$, alone, subtract one from the original power, and then multiply by the derivative of the "inside" function, $3x^2+1$. This is a special case of the Chain Rule that we will discuss in more detail in this section.

For now, notice both F_1 and F_2 are composite functions of the form $f(g(x))$, where $g(x) = 3x^2+1$ and $f(x) = x^2$ for $F_1(x)$ and $f(x) = x^3$ for $F_2(x)$. Thus, to find the derivative of F_1 and F_2, we found the derivative of the "outside" function, f, evaluated at the "inside" function, g, and then we multiplied by the derivative of the "inside" function, g. This is exactly what the Chain Rule tells us to do, and it is summarized below.

Theorem 2.5 The Chain Rule

If f and g are differentiable functions, then

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

Using the Chain Rule

In the following example, we will use the Chain Rule to find the derivative of each of the functions discussed in the introduction.

Example 1 Find the derivative of each of the following functions.

a. $F_1(x) = \left(3x^2+1\right)^2$

b. $F_2(x) = \left(3x^2+1\right)^3$

c. $F_3(x) = \left(3x^2+1\right)^{99}$

d. $F_4(x) = \sqrt{3x^2+1}$

Solution:

a. Because F_1 can be written as the composite function $F_1(x) = f(g(x))$, where $f(x) = x^2$ and $g(x) = 3x^2+1$, we can use the Chain Rule to find $F_1{}'(x)$:

$$F_1{}'(x) = f'(g(x)) \cdot g'(x)$$

Using the Introductory Derivative Rules, we know $f'(x) = 2x$, Thus,

$$\begin{aligned} f'(g(x)) &= f'\left(3x^2+1\right) \\ &= 2\left(3x^2+1\right) \end{aligned}$$

Again, using the Introductory Derivative Rules, we know $g'(x) = 6x$. Substituting this into the Chain Rule gives

$$\begin{aligned} F_1{}'(x) &= f'(g(x)) \cdot g'(x) \\ &= 2\left(3x^2+1\right)(6x) \end{aligned}$$

Notice this is the same answer we reached previously!

b. $F_2(x) = \left(3x^2+1\right)^3$ can be written as the composite function $F_2(x) = f(g(x))$, where $f(x) = x^3$ and $g(x) = 3x^2+1$. Thus, the Chain Rule can be used to find $F_2{}'(x)$:

$$F_2{}'(x) = f'(g(x)) \cdot g'(x)$$

Again, the Introductory Derivative Rules give us $f'(x) = 3x^2$ and $g'(x) = 6x$, Thus,

$$\begin{aligned} f'(g(x)) &= f'\left(3x^2+1\right) \\ &= 3\left(3x^2+1\right)^2 \end{aligned}$$

Substituting this into the Chain Rule gives

$$\begin{aligned} F_2{}'(x) &= f'(g(x)) \cdot g'(x) \\ &= 3\left(3x^2+1\right)^2(6x) \end{aligned}$$

Again, notice this is the same answer we reached previously!

c. $F_3(x) = \left(3x^2+1\right)^{99}$ can be written as the composite function $F_3(x) = f(g(x))$, where $f(x) = x^{99}$ and $g(x) = 3x^2+1$. Thus, the Chain Rule can be used to find $F_3{}'(x)$:

$$F_3{}'(x) = f'(g(x)) \cdot g'(x)$$

The Introductory Derivative Rules give us $f'(x) = 99x^{98}$ and $g'(x) = 6x$. Thus,

$$\begin{aligned} f'(g(x)) &= f'\left(3x^2+1\right) \\ &= 99\left(3x^2+1\right)^{98} \end{aligned}$$

Substituting this into the Chain Rule gives

$$\begin{aligned} F_3{}'(x) &= f'(g(x)) \cdot g'(x) \\ &= 99\left(3x^2+1\right)^{98}(6x) \end{aligned}$$

Using the Chain Rule is much easier than expanding the original function like we discussed at the beginning of this section!

d. Although $F_4(x) = \sqrt{3x^2+1}$ might appear to be very different than the other functions, if we rewrite the radical as a power function, we see that the same ideas hold:

$$F_4(x) = \sqrt{3x^2+1} = \left(3x^2+1\right)^{\frac{1}{2}}$$

Thus, $F_4(x)$ can be written as the composite function $F_4(x) = f(g(x))$, where $f(x) = x^{1/2}$ and $g(x) = 3x^2+1$. Thus, the Chain Rule can be used to find $F_4{}'(x)$:

$$F_4{}'(x) = f'(g(x)) \cdot g'(x)$$

Again the Introductory Derivative Rules give us $f'(x) = \frac{1}{2}x^{-1/2}$ and $g'(x) = 6x$, Thus,

$$\begin{aligned} f'(g(x)) &= f'\left(3x^2+1\right) \\ &= \frac{1}{2}\left(3x^2+1\right)^{-\frac{1}{2}} \end{aligned}$$

Substituting this into the Chain Rule gives

$$\begin{aligned} F_4{}'(x) &= f'(g(x)) \cdot g'(x) \\ &= \frac{1}{2}\left(3x^2+1\right)^{-\frac{1}{2}}(6x) \end{aligned}$$

▪

Generalized Power Rule

The functions in the previous example all represent a specific case of the Chain Rule where the "outside" function is a power function. This specific case is summarized below.

Theorem 2.6 The Generalized Power Rule

If g is a differentiable function and n is any real number, then

$$\frac{d}{dx}\left((g(x))^n\right) = n\,(g(x))^{n-1}\left(\frac{d}{dx}(g(x))\right)$$

$$= n\,(g(x))^{n-1} \cdot g'(x)$$

In the above theorem, if $g(x) = x$, then we would have $\frac{d}{dx}(x^n) = nx^{n-1}\left(\frac{d}{dx}(x)\right) = nx^{n-1}(1) = nx^{n-1}$. Notice this is the Power Rule we know and love from **Section 2.3***!*

Let's get more practice using the Chain Rule!

Example 2 Find the derivative of each of the following functions.

a. $y = \sqrt{1-x}$

b. $y = \left(\frac{2x^5+15}{\ln(x)}\right)^8$

c. $y = e^{8-4x^3}$

Solution:

a. We begin by rewriting the function as $y = (1-x)^{1/2}$. The Chain Rule, or more specifically, the Generalized Power Rule, gives

$$\begin{aligned} y' &= \frac{1}{2}(1-x)^{\frac{1}{2}-1}\left(\frac{d}{dx}(1-x)\right) \\ &= \frac{1}{2}(1-x)^{-\frac{1}{2}}(-1) \\ &= -\frac{1}{2}(1-x)^{-\frac{1}{2}} \end{aligned}$$

b. The most "outside" part of $y = \left(\frac{2x^5+15}{\ln(x)}\right)^8$ is a power, so we use the Generalized Power Rule:

$$y' = 8\left(\frac{2x^5+15}{\ln(x)}\right)^{8-1}\left(\frac{d}{dx}\left(\frac{2x^5+15}{\ln(x)}\right)\right)$$

To find the derivative of the "inside" function, we need to use the Quotient Rule:

$$\frac{d}{dx}\left(\frac{2x^5+15}{\ln(x)}\right) = \frac{(\ln(x))\left(\frac{d}{dx}\left(2x^5+15\right)\right) - \left(2x^5+15\right)\left(\frac{d}{dx}(\ln(x))\right)}{(\ln(x))^2}$$

$$= \frac{(\ln(x))\left(10x^4\right) - \left(2x^5+15\right)\left(\frac{1}{x}\right)}{(\ln(x))^2}$$

Now, we substitute this result into the derivative to get the final answer:

$$\begin{aligned} y' &= 8\left(\frac{2x^5+15}{\ln(x)}\right)^{8-1}\left(\frac{d}{dx}\left(\frac{2x^5+15}{\ln(x)}\right)\right) \\ &= 8\left(\frac{2x^5+15}{\ln(x)}\right)^{7}\left(\frac{(\ln(x))\left(10x^4\right) - \left(2x^5+15\right)\left(\frac{1}{x}\right)}{(\ln(x))^2}\right) \end{aligned}$$

c. Notice $y = e^{8-4x^3}$ has an x in the exponent. Thus, we are no longer dealing with a power function, but an exponential function instead. Therefore, we return to the general form of the Chain Rule to find the derivative:

$$\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$$

Because the Chain Rule allows us to find the derivative of a composition of two functions, we need to be able to identify the two functions. We have seen e^x in previous sections, but now we have a more involved function of x where we typically see just x. Thus, the "outside" function is $f(x) = e^x$ and the "inside" function is $g(x) = 8 - 4x^3$ as shown below:

$$\begin{aligned} f(g(x)) &= f\left(8-4x^3\right) \\ &= e^{8-4x^3} \end{aligned}$$

Because $f'(x) = e^x$, we have

$$\begin{aligned} f'(g(x)) &= f'\left(8-4x^3\right) \\ &= e^{8-4x^3} \end{aligned}$$

We also need the derivative of g:

$$g'(x) = -12x^2$$

Substituting all of these pieces into the Chain Rule gives

$$\begin{aligned} y' &= f'(g(x)) \cdot g'(x) \\ &= e^{8-4x^3}\left(-12x^2\right) \\ &= -12x^2 e^{8-4x^3} \end{aligned}$$

■

Try It # 1:
Find the derivative of each of the following functions.

a. $y = 3(2x + e^x)^5$

b. $y = \left(14x^{-3} - \log_3(x)\right)^9 \cdot \left(x^2 - 1\right)^{\frac{2}{7}}$

Generalized Exponential (base *e*) Rule

The solution to part **c** of the previous example leads us to another specific case of the Chain Rule where the "outside" function is $f(x) = e^x$. We refer to this rule as the **Generalized Exponential (base** *e***) Rule**!

Theorem 2.7 The Generalized Exponential (base *e*) Rule

If g is a differentiable function, then

$$\frac{d}{dx}\left(e^{g(x)}\right) = e^{g(x)}\left(\frac{d}{dx}(g(x))\right)$$

$$= e^{g(x)} \cdot g\,'(x)$$

In the above theorem, if $g(x) = x$, then we would have $\frac{d}{dx}(e^x) = e^x\left(\frac{d}{dx}(x)\right) = e^x(1) = e^x$. Notice this is the Exponential (base e) Rule we know and love from **Section 2.3***!*

Let's get more practice using the Chain Rule!

■ **Example 3** Find the derivative of each of the following functions.

a. $y = e^{7+2x^4-\sqrt{x}}$

b. $f(x) = \left(e^{3-\frac{2}{x}}\right)(x-4)$

c. $C(x) = 4^{13x^2-12x}$

Solution:

a. Here, we have a composition of two functions where the "outside" function is e^x. Thus, we can use the Generalized Exponential (base e) Rule:

$$\begin{aligned} y\,' &= \frac{d}{dx}\left(e^{7+2x^4-\sqrt{x}}\right) \\ &= e^{7+2x^4-\sqrt{x}}\left(\frac{d}{dx}\left(7+2x^4-\sqrt{x}\right)\right) \\ &= e^{7+2x^4-\sqrt{x}}\left(\frac{d}{dx}\left(7+2x^4-x^{\frac{1}{2}}\right)\right) \\ &= e^{7+2x^4-\sqrt{x}}\left(8x^3-\frac{1}{2}x^{-\frac{1}{2}}\right) \end{aligned}$$

b. Because $f(x) = \left(e^{3-\frac{2}{x}}\right)(x-4)$ is a product of two functions, we start with the Product Rule:

$$f'(x) = \left(e^{3-\frac{2}{x}}\right)\left(\frac{d}{dx}(x-4)\right) + (x-4)\left(\frac{d}{dx}\left(e^{3-\frac{2}{x}}\right)\right)$$

We can use the Introductory Derivative Rules to find the the derivative of $x-4$: $\frac{d}{dx}(x-4) = 1$, but we must use the Generalized Exponential (base e) Rule to find the derivative of $e^{3-\frac{2}{x}}$:

$$\begin{aligned}\frac{d}{dx}\left(e^{3-\frac{2}{x}}\right) &= e^{3-\frac{2}{x}}\left(\frac{d}{dx}\left(3-\frac{2}{x}\right)\right)\\ &= e^{3-\frac{2}{x}}\left(\frac{d}{dx}\left(3-2x^{-1}\right)\right)\\ &= e^{3-\frac{2}{x}}\left(2x^{-2}\right)\end{aligned}$$

Substituting both derivatives into $f'(x)$ gives

$$\begin{aligned}f'(x) &= \left(e^{3-\frac{2}{x}}\right)\left(\frac{d}{dx}(x-4)\right) + (x-4)\left(\frac{d}{dx}\left(e^{3-\frac{2}{x}}\right)\right)\\ &= \left(e^{3-\frac{2}{x}}\right)(1) + (x-4)\left(e^{3-\frac{2}{x}}\left(2x^{-2}\right)\right)\\ &= e^{3-\frac{2}{x}} + 2x^{-2}(x-4)e^{3-\frac{2}{x}}\end{aligned}$$

c. $C(x) = 4^{13x^2-12x}$ is not of a form in which we can use the Generalized Exponential (base e) Rule, so we will have to go back to the general form for the Chain Rule. $C(x)$ can be viewed as the composition function, $f(g(x))$, where $f(x) = 4^x$ and $g(x) = 13x^2 - 12x$.

Because $f'(x) = 4^x \ln(4)$, we have

$$\begin{aligned}f'(g(x)) &= f'\left(13x^2-12x\right)\\ &= 4^{13x^2-12x}(\ln(4))\end{aligned}$$

Next, we need to find $g'(x)$:

$$g'(x) = 26x - 12$$

Substituting these pieces into the general form of the Chain Rule gives

$$\begin{aligned}C'(x) &= f'(g(x)) \cdot g'(x)\\ &= 4^{13x^2-12x}(\ln(4))(26x-12)\end{aligned}$$

▪

Try It # 2:
Find the derivative of each of the following functions.

a. $y = 14e^{-2x^2+2x^{-2}}$

b. $y = \dfrac{e^{2x^{-6}+3x^5}}{(\ln(x))^2}$

Generalized Exponential (base b) Rule

The solution to part **c** of the previous example leads us to an even more generalized exponential derivative rule that applies to exponential functions of any base: the **Generalized Exponential (base b) Rule**!

Theorem 2.8 The Generalized Exponential (base b) Rule

If g is a differentiable function and b is any positive real number, then

$$\frac{d}{dx}\left(b^{g(x)}\right) = b^{g(x)}(\ln(b))\left(\frac{d}{dx}(g(x))\right)$$

$$= b^{g(x)}(\ln(b)) \cdot g'(x)$$

In the above theorem, if $g(x) = x$, then we would have $\frac{d}{dx}(b^x) = b^x(\ln(b))\left(\frac{d}{dx}(x)\right) = b^x(\ln(b))(1) = b^x \ln(b)$. Notice this is the other Exponential Rule we know and love from **Section 2.3**! *Notice also that this formula gives the same result as the Generalized Exponential (base e) Rule when $b = e$ because* $\ln(e) = 1$.

Let's get more practice using the Chain Rule!

Example 4 Find $\frac{dy}{dx}$ for each of the following functions.

a. $y = 6^{8^x - 22x^{10}}$

b. $y = 10^{\sqrt[3]{x^2+7}}$

c. $y = 2\ln\left(x^7 - x^{\frac{1}{7}}\right)$

Solution:

a. This function is of a form in which we can use the Generalized Exponential (base b) Rule. Here, $b = 6$ and $g(x) = 8^x - 22x^{10}$, so

$$\begin{aligned}\frac{dy}{dx} &= 6^{8^x - 22x^{10}}(\ln(6))\left(\frac{d}{dx}\left(8^x - 22x^{10}\right)\right)\\ &= 6^{8^x - 22x^{10}}(\ln(6))\left(8^x(\ln(8)) - 220x^9\right)\end{aligned}$$

b. The "outside" part of $y = 10^{\sqrt[3]{x^2+7}}$ is 10 raised to a power, so we start with the Generalized Exponential Rule with base 10:

$$\begin{aligned}\frac{dy}{dx} &= 10^{\sqrt[3]{x^2+7}}(\ln(10))\left(\frac{d}{dx}\left(\sqrt[3]{x^2+7}\right)\right)\\ &= 10^{\sqrt[3]{x^2+7}}(\ln(10))\left(\frac{d}{dx}\left(\left(x^2+7\right)^{\frac{1}{3}}\right)\right)\end{aligned}$$

Next, we apply the Generalized Power Rule to find $\frac{d}{dx}\left(\left(x^2+7\right)^{\frac{1}{3}}\right)$:

$$\frac{d}{dx}\left(\left(x^2+7\right)^{\frac{1}{3}}\right) = \frac{1}{3}\left(x^2+7\right)^{-\frac{2}{3}}(2x)$$

Substituting this into the derivative gives

$$\begin{aligned}\frac{dy}{dx} &= 10^{\sqrt[3]{x^2+7}}(\ln(10))\left(\frac{d}{dx}\left(\left(x^2+7\right)^{\frac{1}{3}}\right)\right)\\ &= 10^{\sqrt[3]{x^2+7}}(\ln(10))\left(\frac{1}{3}\left(x^2+7\right)^{-\frac{2}{3}}(2x)\right)\end{aligned}$$

c. For the function $y = 2\ln\left(x^7 - x^{\frac{1}{7}}\right)$, the previous case of the Chain Rule does not apply. Once again, we must go back and use the general form of the Chain Rule. Perhaps you see the pattern by now and notice that we will get a rule for natural logarithms next!

Using the parlance of the general Chain Rule, we let $f(x) = \ln(x)$ and $g(x) = x^7 - x^{\frac{1}{7}}$. You may be curious about the 2 in the original function. Remember that constants come along for the ride, so we will find the derivative of $h(x) = f(g(x)) = \ln\left(x^7 - x^{\frac{1}{7}}\right)$, and afterwards, we will multiply by 2 to get the final answer.

Because $f'(x) = \frac{1}{x}$, we have

$$\begin{aligned}f'(g(x)) &= f'\left(x^7 - x^{\frac{1}{7}}\right)\\ &= \frac{1}{x^7 - x^{\frac{1}{7}}}\end{aligned}$$

Next, we need to find $g'(x)$:

$$g'(x) = 7x^6 - \frac{1}{7}x^{-\frac{6}{7}}$$

Therefore,

$$\begin{aligned}h'(x) &= f'(g(x)) \cdot g'(x)\\ &= \frac{1}{x^7 - x^{\frac{1}{7}}}\left(7x^6 - \frac{1}{7}x^{-\frac{6}{7}}\right)\\ &= \frac{7x^6 - \frac{1}{7}x^{-\frac{6}{7}}}{x^7 - x^{\frac{1}{7}}}\end{aligned}$$

Multiplying by the constant 2 gives the final answer:

$$\frac{dy}{dx} = 2\left(\frac{7x^6 - \frac{1}{7}x^{-\frac{6}{7}}}{x^7 - x^{\frac{1}{7}}}\right)$$

▪

Try It # 3:

Find $\frac{dy}{dx}$ of each of the following functions.

a. $y = 14^{e^x + \log_4(x) + 4x^8}$

b. $y = \left(2^{5x^3 - 22}\right)\left(3x^{13} - 3^{x^2}\right)$

Generalized Natural Logarithm Rule

The solution to part **c** of the previous example leads us to another specific case of the Chain Rule in which the "outside" function is $f(x) = \ln(x)$. We refer to this rule as the **Generalized Natural Logarithm Rule**!

Theorem 2.9 The Generalized Natural Logarithm Rule

If g is a differentiable function, then

$$\frac{d}{dx}(\ln(g(x))) = \frac{1}{g(x)}\left(\frac{d}{dx}(g(x))\right)$$

$$= \frac{1}{g(x)} \cdot g'(x) = \frac{g'(x)}{g(x)}$$

In the above theorem, if $g(x) = x$, then we would have $\frac{d}{dx}(\ln(x)) = \frac{1}{x}\left(\frac{d}{dx}(x)\right) = \frac{1}{x}(1) = \frac{1}{x}$. Notice this is the Natural Logarithm Rule we know and love from **Section 2.3***!!*

Let's get more practice using the Chain Rule!

Example 5 Differentiate each of the following functions.

a. $y = \ln\left(x^2 + x - x^{-1} - x^{-2}\right)$

b. $f(t) = \ln\left(\frac{\left(t^2+1\right)^5}{e^t}\right)$

c. $y = \log\left(\left(4x^2 - 12x\right)^9\right)$

Solution:

a. If we let $g(x) = x^2 + x - x^{-1} - x^{-2}$, then we can use the Generalized Natural Logarithm Rule. Because $g'(x) = 2x + 1 + x^{-2} + 2x^{-3}$, we have

$$y' = \frac{2x + 1 + x^{-2} + 2x^{-3}}{x^2 + x - x^{-1} - x^{-2}}$$

b. For $f(t) = \ln\left(\frac{\left(t^2+1\right)^5}{e^t}\right)$, there are two ways we can find $f'(t)$. One option is to use the Generalized Natural Logarithm Rule, then the Quotient Rule to find the derivative of the "inside" function, and then the Generalized Power Rule to find the derivative of the numerator of the "inside" function.

However, we have another option: algebraically manipulate the function first using the Properties of Logarithms to make it easier to find its derivative.

We will demonstrate the second option because it is much easier. Rewriting the function using the Properties of Logarithms gives

$$\begin{aligned} f(t) &= \ln\left(\frac{\left(t^2+1\right)^5}{e^t}\right) \\ &= \ln\left(\left(t^2+1\right)^5\right) - \ln\left(e^t\right) \\ &= 5\ln\left(t^2+1\right) - t \end{aligned}$$

Even before we start taking the derivative, we can see how this technique is easier. We still need to use the Generalized Natural Logarithm Rule, but we eliminated the need for the Quotient Rule and Generalized Power Rule:

$$\begin{aligned} f'(t) &= 5\left(\frac{d}{dt}\left(\ln\left(t^2+1\right)\right)\right) - \left(\frac{d}{dt}(t)\right) \\ &= 5\left(\frac{\frac{d}{dt}\left(t^2+1\right)}{t^2+1}\right) - 1 \\ &= 5\left(\frac{2t}{t^2+1}\right) - 1 \\ &= \frac{10t}{t^2+1} - 1 \end{aligned}$$

As shown in this example, using the Properties of Logarithms to algebraically manipulate the function first often reduces the need for more complicated derivative rules. You do have the tools to attempt to find the derivative directly, and doing so should make you appreciate the approach we took here.

c. Recall there are only two instances where we do not have to write the base of a logarithm. The natural logarithm, $\ln(x)$, implies a base of e, and the common logarithm, $\log(x)$, implies a base of 10. Here, for the function $y = \log\left(\left(4x^2 - 12x\right)^9\right)$, we have the second case. Thus, we cannot use the Generalized Natural Logarithm Rule. As you probably expect by now, this solution will lead us to the **Generalized Logarithm (base *b*) Rule**!

Before we find the derivative, let's first apply the Properties of Logarithms to rewrite the function:

$$\begin{aligned} y &= \log\left(\left(4x^2-12x\right)^9\right) \\ &= 9\log\left(4x^2-12x\right) \end{aligned}$$

Notice we are dealing with a composition of functions, and thus, we need to use the general form of the Chain Rule to find the derivative. The "outside" function is $f(x) = 9\log(x)$, and the "inside" function is $g(x) = 4x^2 - 12x$.

Because $f'(x) = 9 \cdot \frac{1}{x} \cdot \frac{1}{\ln(10)}$, we have

$$\begin{aligned} f'(g(x)) &= f'\left(4x^2-12x\right) \\ &= 9 \cdot \frac{1}{4x^2-12x} \cdot \frac{1}{\ln(10)} \\ &= \frac{9}{(4x^2-12x)(\ln(10))} \end{aligned}$$

Next, we need to find $g'(x)$. Recalling $g(x) = 4x^2 - 12x$ gives

$$g'(x) = 8x - 12$$

Therefore, recalling $f'(g(x)) = \dfrac{9}{(4x^2 - 12x)(\ln(10))}$, we have

$$\begin{aligned} y' &= f'(g(x)) \cdot g'(x) \\ &= \left(\frac{9}{(4x^2 - 12x)(\ln(10))}\right)(8x - 12) \\ &= \frac{9(8x-12)}{(4x^2 - 12x)(\ln(10))} \\ &= \frac{72x - 108}{(4x^2 - 12x)(\ln(10))} \end{aligned}$$

▪

Try It # 4:
Find the derivative of each of the following functions:

a. $y = \ln\left(3x^2 - 2x^3 + x + 12\right)$

b. $y = \ln\left(\dfrac{e^{-4x^{12}}}{(x+3)(x-3)^2}\right)$

Generalized Logarithm (base *b*) Rule

The general form of the rule from part **c** of the previous example is

Theorem 2.10 The Generalized Logarithm (base *b*) Rule

If g is a differentiable function and b is any positive real number, then

$$\begin{aligned} \frac{d}{dx}(\log_b(g(x))) &= \left(\frac{1}{g(x)}\right)\left(\frac{1}{\ln(b)}\right)\left(\frac{d}{dx}(g(x))\right) \\ &= \left(\frac{1}{g(x)}\right)\left(\frac{1}{\ln(b)}\right)(g'(x)) \\ &= \frac{g'(x)}{(g(x))(\ln(b))} \end{aligned}$$

In the above theorem, if $g(x) = x$, then we would have $\frac{d}{dx}(\log_b(x)) = \frac{1}{x}\frac{1}{\ln(b)}\left(\frac{d}{dx}(x)\right) = \frac{1}{x}\frac{1}{\ln(b)}(1) = \frac{1}{x\ln(b)}$. Notice this is the Logarithm (base b) Rule we know and love from **Section 2.3***!!*

Let's get more practicing using the Chain Rule!

■ **Example 6** Find the derivative of each of the following functions.

a. $H(x) = \log_{18}\left(x^{\frac{5}{2}} - 22x^5 + 55x^{\frac{1}{2}} + 52x^2\right)$

b. $y = \left(\log\left(x^3 - 187\right)\right)\left(\log_3\left(4x^5 - 1\right)\right)$

c. $F(x) = \log_5\left(\dfrac{25^{x^2}}{\left(5x^3 - 10\right)^2 (x+1)}\right)$

d. $B(t) = \log\left(\left(\dfrac{4t^4 - 3t^3 + 2}{10^{(5t^8 - 27)}}\right)^8\right)$

Solution:

a. The "inside" function here is $g(x) = x^{\frac{5}{2}} - 22x^5 + 55x^{\frac{1}{2}} + 52x^2$, so

$$g'(x) = \frac{5}{2}x^{\frac{3}{2}} - 110x^4 + \frac{55}{2}x^{-\frac{1}{2}} + 104x$$

Using the Generalized Logarithm Rule with base 18, we have

$$H'(x) = \frac{\frac{5}{2}x^{\frac{3}{2}} - 110x^4 + \frac{55}{2}x^{-\frac{1}{2}} + 104x}{\left(x^{\frac{5}{2}} - 22x^5 + 55x^{\frac{1}{2}} + 52x^2\right)(\ln(18))}$$

b. As we have seen in previous examples, we often have to apply more than one derivative rule to a function in order to find its derivative. For instance, the most "outside" part of this function, $y = \left(\log\left(x^3 - 187\right)\right)\left(\log_3\left(4x^5 - 1\right)\right)$, is multiplication, so we need to start with the Product Rule. Then, to find the derivative of each factor, we will use the Generalized Logarithm Rule (first with base 10 and then with base 3):

$$\begin{aligned}
y' &= \left(\log\left(x^3 - 187\right)\right)\left(\frac{d}{dx}\left(\log_3\left(4x^5 - 1\right)\right)\right) + \left(\log_3\left(4x^5 - 1\right)\right)\left(\frac{d}{dx}\left(\log\left(x^3 - 187\right)\right)\right) \\
&= \left(\log\left(x^3 - 187\right)\right)\left(\frac{\frac{d}{dx}\left(4x^5 - 1\right)}{(4x^5 - 1)(\ln(3))}\right) + \left(\log_3\left(4x^5 - 1\right)\right)\left(\frac{\frac{d}{dx}\left(x^3 - 187\right)}{(x^3 - 187)(\ln(10))}\right) \\
&= \left(\log\left(x^3 - 187\right)\right)\left(\frac{20x^4}{(4x^5 - 1)(\ln(3))}\right) + \left(\log_3\left(4x^5 - 1\right)\right)\left(\frac{3x^2}{(x^3 - 187)(\ln(10))}\right) \\
&= \frac{20x^4 \log\left(x^3 - 187\right)}{(4x^5 - 1)(\ln(3))} + \frac{3x^2 \log_3\left(4x^5 - 1\right)}{(x^3 - 187)(\ln(10))}
\end{aligned}$$

c. $F(x) = \log_5\left(\dfrac{25^{x^2}}{\left(5x^3 - 10\right)^2 (x+1)}\right)$ is a function that we could immediately start differentiating using the derivative rules, but we will save time and effort by using the Properties of Logarithms to rewrite the function first:

$$\begin{aligned}
F(x) &= \log_5\left(\frac{25^{x^2}}{\left(5x^3-10\right)^2(x+1)}\right) \\
&= \log_5\left(25^{x^2}\right) - \log_5\left(\left(5x^3-10\right)^2(x+1)\right) \\
&= \log_5\left(25^{x^2}\right) - \left(\log_5\left(5x^3-10\right)^2 + \log_5(x+1)\right) \\
&= \log_5\left(25^{x^2}\right) - \log_5\left(5x^3-10\right)^2 - \log_5(x+1) \\
&= x^2 \cdot \log_5(25) - 2\log_5\left(5x^3-10\right) - \log_5(x+1) \\
&= 2x^2 - 2\log_5\left(5x^3-10\right) - \log_5(x+1)
\end{aligned}$$

Taking the derivative gives

$$\begin{aligned}
F'(x) &= 4x - 2\left(\frac{\frac{d}{dx}\left(5x^3-10\right)}{\left(5x^3-10\right)(\ln(5))}\right) - \frac{\frac{d}{dx}(x+1)}{(x+1)(\ln(5))} \\
&= 4x - 2\left(\frac{15x^2}{\left(5x^3-10\right)(\ln(5))}\right) - \frac{1}{(x+1)(\ln(5))} \\
&= 4x - \frac{30x^2}{\left(5x^3-10\right)(\ln(5))} - \frac{1}{(x+1)(\ln(5))}
\end{aligned}$$

d. To find the derivative of $B(t) = \log\left(\left(\frac{4t^4-3t^3+2}{10^{(5t^8-27)}}\right)^8\right)$, we will use the Properties of Logarithms to rewrite the function first:

$$\begin{aligned}
B(t) &= \log\left(\left(\frac{4t^4-3t^3+2}{10^{(5t^8-27)}}\right)^8\right) \\
&= 8\log\left(\frac{4t^4-3t^3+2}{10^{(5t^8-27)}}\right) \\
&= 8\left(\log\left(4t^4-3t^3+2\right) - \log\left(10^{(5t^8-27)}\right)\right) \\
&= 8\left(\log\left(4t^4-3t^3+2\right) - \log_{10}\left(10^{(5t^8-27)}\right)\right) \\
&= 8\left(\log\left(4t^4-3t^3+2\right) - \left(5t^8-27\right)\right) \\
&= 8\log\left(4t^4-3t^3+2\right) - 8\left(5t^8-27\right) \\
&= 8\log\left(4t^4-3t^3+2\right) - 40t^8 + 216
\end{aligned}$$

Now, to take the derivative, we will use the Generalized Logarithm Rule with base 10:

$$\begin{aligned}
B'(t) &= 8\left(\frac{\frac{d}{dt}\left(4t^4-3t^3+2\right)}{\left(4t^4-3t^3+2\right)(\ln(10))}\right) - 320t^7 + 0 \\
&= 8\left(\frac{16t^3-9t^2}{\left(4t^4-3t^3+2\right)(\ln(10))}\right) - 320t^7 \\
&= \frac{128t^3-72t^2}{\left(4t^4-3t^3+2\right)(\ln(10))} - 320t^7
\end{aligned}$$

■

Try It # 5:
Find the derivative of each of the following functions.

a. $y = \log_4\left(e^{x^3-14x^2+15x} - 8x^2\right)$

b. $y = \log_7\left(\dfrac{x^3+44x-18}{\left(x^5-x^3-x+1\right)343^{(x^4+22)}}\right)$

All of the specific cases of the Chain Rule we have encountered are summarized below.

Specific Formulas for the Chain Rule

If g is a differentiable function, n is any real number, and b is any positive real number, then

- **Generalized Power Rule:** If $y = (g(x))^n$, then $y' = n(g(x))^{n-1} \cdot g'(x)$.
- **Generalized Exponential (base e) Rule:** If $y = e^{g(x)}$, then $y' = e^{g(x)} \cdot g'(x)$.
- **Generalized Exponential (base b) Rule:** If $y = b^{g(x)}$, then $y' = b^{g(x)}(\ln(b)) \cdot g'(x)$.
- **Generalized Natural Logarithm Rule:** If $y = \ln(g(x))$, then $y' = \dfrac{g'(x)}{g(x)}$.
- **Generalized Logarithm (base b) Rule:** If $y = \log_b(g(x))$, then $y' = \dfrac{g'(x)}{(g(x))(\ln(b))}$.

Applications

Example 7 Find the equation of the line tangent to the graph of the following function, f, at $x = 1$:

$$f(x) = \frac{\ln(2-x)}{\sqrt{9x^2-5x^3}}$$

Solution:
Remember that in order to find the equation of any line, we need its slope and a point on the line. To find the point for this line, we calculate $f(1)$:

$$f(1) = \frac{\ln(2-1)}{\sqrt{9(1)^2-5(1)^3}} = \frac{\ln(1)}{\sqrt{4}} = 0$$

Thus, the tangent line passes through the point $(1,0)$.

Next, we need the slope of the tangent line. To find the slope of the tangent line, we calculate $f'(1)$. But first, we must find $f'(x)$. Because the outermost operation of $f(x) = \dfrac{\ln(2-x)}{\sqrt{9x^2-5x^3}}$ is division of two functions, we begin with the Quotient Rule:

$$\begin{aligned} f'(x) &= \frac{\sqrt{9x^2-5x^3}\left(\frac{d}{dx}(\ln(2-x))\right) - (\ln(2-x))\left(\frac{d}{dx}\left(\sqrt{9x^2-5x^3}\right)\right)}{\left(\sqrt{9x^2-5x^3}\right)^2} \\ &= \frac{\sqrt{9x^2-5x^3}\left(\frac{d}{dx}(\ln(2-x))\right) - (\ln(2-x))\left(\frac{d}{dx}\left(\left(9x^2-5x^3\right)^{\frac{1}{2}}\right)\right)}{9x^2-5x^3} \end{aligned}$$

To find $\frac{d}{dx}(\ln(2-x))$, we use the Chain Rule where the "inside" function is $g(x)=2-x$, making $g'(x)=-1$. This yields

$$\begin{aligned}\frac{d}{dx}(\ln(2-x)) &= \frac{g'(x)}{g(x)}\\ &= \frac{-1}{2-x}\end{aligned}$$

To find $\frac{d}{dx}\left(\left(9x^2-5x^3\right)^{1/2}\right)$, the "inside" function is $g(x)=9x^2-5x^3$, making $g'(x)=18x-15x^2$. This yields

$$\begin{aligned}\frac{d}{dx}\left(\left(9x^2-5x^3\right)^{\frac{1}{2}}\right) &= \frac{1}{2}(g(x))^{-\frac{1}{2}}g'(x)\\ &= \frac{1}{2}\left(9x^2-5x^3\right)^{-\frac{1}{2}}\left(18x-15x^2\right)\end{aligned}$$

Substituting these derivatives into $f'(x)$ gives

$$\begin{aligned}f'(x) &= \frac{\sqrt{9x^2-5x^3}\left(\frac{d}{dx}(\ln(2-x))\right)-(\ln(2-x))\left(\frac{d}{dx}\left(\left(9x^2-5x^3\right)^{\frac{1}{2}}\right)\right)}{9x^2-5x^3}\\ &= \frac{\sqrt{9x^2-5x^3}\left(\frac{-1}{2-x}\right)-(\ln(2-x))\left(\frac{1}{2}\left(9x^2-5x^3\right)^{-\frac{1}{2}}\left(18x-15x^2\right)\right)}{9x^2-5x^3}\end{aligned}$$

Now, we can calculate $f'(1)$ to get the slope:

$$\begin{aligned}f'(1) &= \frac{\sqrt{9(1)^2-5(1)^3}\left(\frac{-1}{2-1}\right)-(\ln(2-1))\left(\frac{1}{2}\left(9(1)^2-5(1)^3\right)^{-\frac{1}{2}}\left(18(1)-15(1)^2\right)\right)}{9(1)^2-5(1)^3}\\ &= \frac{\sqrt{4}(-1)-\ln(1)\left(\frac{1}{2}\cdot 4^{-\frac{1}{2}}\cdot 3\right)}{4}\\ &= \frac{-2-0}{4}\\ &= \frac{-2}{4} = -\frac{1}{2}\end{aligned}$$

Using the point-slope form of the equation of a line gives

$$\begin{aligned}y-0 &= -\frac{1}{2}(x-1)\\ y &= -\frac{1}{2}x+\frac{1}{2}\end{aligned}$$

▪

Try It # 6:
Find the equation of the line to the graph of the following function, f, at $x=3$:

$$f(x) = \left((\ln(10))x^2\right)\left(\log\left(x^2+1\right)\right)-\frac{27}{5}x$$

■ **Example 8** **Table 2.10** shows values of $f(x)$, $g(x)$, $f'(x)$, and $g'(x)$ for certain values of x. Use the information in the table to answer each of the following.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	2	1	3	4
2	3	4	1	2
3	1	3	3	3
4	3	2	5	2

Table 2.10: Various values of $f(x)$, $g(x)$, $f'(x)$, and $g'(x)$

a. If $w(x) = g(f(2x))$, find $w'(1)$.

b. If $n(x) = 3x \cdot \log_9(f(x))$, find $n'(3)$.

Solution:

a. The function $w(x) = g(f(2x))$ is a composition of three functions, so we will apply the Chain Rule two times.

First, we have

$$w'(x) = g'(f(2x)) \cdot \frac{d}{dx}(f(2x))$$

Applying the Chain Rule again to find the derivative of $\frac{d}{dx}(f(2x))$ gives

$$\begin{aligned} \frac{d}{dx}(f(2x)) &= f'(2x)\left(\frac{d}{dx}(2x)\right) \\ &= f'(2x) \cdot 2 \end{aligned}$$

Therefore,

$$\begin{aligned} w'(x) &= g'(f(2x)) \cdot \frac{d}{dx}(f(2x)) \\ &= g'(f(2x)) \cdot f'(2x) \cdot 2 \end{aligned}$$

Lastly, we substitute $x = 1$ and use the relevant function values from **Table 2.10**:

$$\begin{aligned} w'(1) &= g'(f(2)) \cdot f'(2) \cdot 2 \\ &= g'(3) \cdot 1 \cdot 2 \\ &= 3 \cdot 1 \cdot 2 \\ &= 6 \end{aligned}$$

b. For $n(x) = 3x \cdot \log_9(f(x))$, the outermost operation is multiplication of two functions. So we start with the Product Rule:

$$n'(x) = 3x \cdot \frac{d}{dx}(\log_9(f(x))) + (\log_9(f(x))) \cdot 3$$

To find $\frac{d}{dx}(\log_9(f(x)))$, we use the Generalized Logarithm Rule with base 9:

$$\frac{d}{dx}(\log_9(f(x))) = \frac{f'(x)}{(f(x))(\ln(9))}$$

Substituting this derivative, we have

$$\begin{aligned} n'(x) &= 3x \cdot \frac{d}{dx}(\log_9(f(x))) + (\log_9(f(x))) \cdot 3 \\ &= 3x \cdot \frac{f'(x)}{(f(x))(\ln(9))} + (\log_9(f(x))) \cdot 3 \end{aligned}$$

Substituting $x = 3$ and using the relevant function values from the table gives

$$\begin{aligned} n'(3) &= 3(3) \cdot \frac{f'(3)}{(f(3))(\ln(9))} + (\log_9(f(3))) \cdot 3 \\ &= 9 \cdot \frac{3}{1(\ln(9))} + (\log_9(1)) \cdot 3 \\ &= \frac{27}{\ln(9)} + 0 \\ &= \frac{27}{\ln(9)} \end{aligned}$$

■

Try It # 7:
Using **Table 2.10** from the previous example, answer each of the following.

a. If $V(x) = e^{(f(x))^2}$, find $V'(4)$.

b. If $C(x) = g(x^3 - 5)$, find $C'(2)$.

■ **Example 9** The graphs of the functions f and g are shown in **Figure 2.5.1**. Use the graphs to answer each of the following.

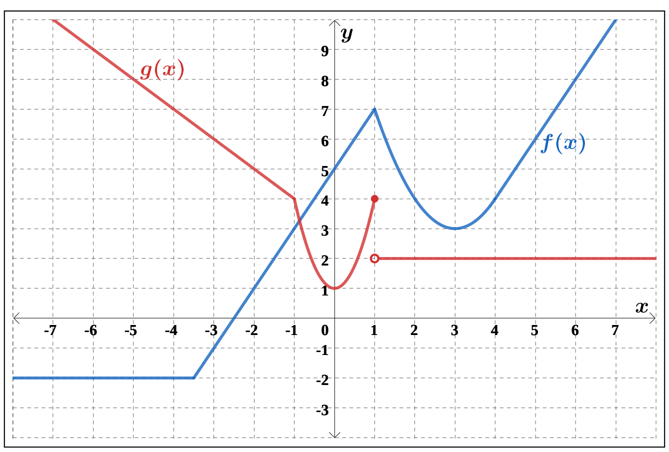


Figure 2.5.1: Graphs of the functions f and g

a. If $h(x) = f(g(x))$, find $h'(-3)$.

b. If $j(x) = e^{f(x) \cdot g(x^2)}$, find $j'(0)$.

Solution:

a. For the function $h(x) = f(g(x))$, the Chain Rule tells us $h'(x) = f'(g(x)) \cdot g'(x)$. Therefore, $h'(-3) = f'(g(-3)) \cdot g'(-3)$.

To calculate $h'(-3)$, we will first find $g(-3)$. Looking at the graph directly, we see that $g(-3) = 6$. This means $f'(g(-3)) = f'(6)$. So now we know

$$\begin{aligned} h'(3) &= f'(g(-3)) \cdot g'(-3) \\ &= f'(6) \cdot g'(-3) \end{aligned}$$

Next, we must find $f'(6)$ and $g'(-3)$. $f'(6)$ is the slope of the graph of f at $x = 6$, and $g'(-3)$ is the slope of the graph of g at $x = -3$. Because both graphs are linear near these x-values, can calculate their slopes (rise over run). We see $f'(6) = 2$ and $g'(-3) = -1$.

Thus,

$$\begin{aligned} h'(-3) &= f'(g(-3)) \cdot g'(-3) \\ &= f'(6) \cdot g'(-3) \\ &= (2)(-1) \\ &= -2 \end{aligned}$$

b. To find $j'(x)$, where $j(x) = e^{f(x) \cdot g(x^2)}$, we start with the Chain Rule. In particular, we use the Generalized Exponential (base e) Rule with the "inside" function being $f(x) \cdot g\left(x^2\right)$. To find the derivative of $f(x) \cdot g\left(x^2\right)$, we will use the Product Rule. Then, we will use the Chain Rule *again* to find the derivative of $g\left(x^2\right)$:

$$\begin{aligned} j'(x) &= e^{f(x) \cdot g(x^2)} \left(\frac{d}{dx} \left(f(x) \cdot g\left(x^2\right) \right) \right) \\ &= e^{f(x) \cdot g(x^2)} \left(f(x) \left(\frac{d}{dx} \left(g\left(x^2\right) \right) \right) + g\left(x^2\right) \left(\frac{d}{dx} (f(x)) \right) \right) \\ &= e^{f(x) \cdot g(x^2)} \left(f(x) \left(g'\left(x^2\right) \right) \left(\frac{d}{dx} \left(x^2\right) \right) + g\left(x^2\right) \cdot f'(x) \right) \\ &= e^{f(x) \cdot g(x^2)} \left(f(x) \left(g'\left(x^2\right) \right) (2x) + g\left(x^2\right) \cdot f'(x) \right) \end{aligned}$$

Now, we substitute $x = 0$:

$$\begin{aligned} j'(0) &= e^{f(0) \cdot g(0^2)} \left(f(0) \left(g'\left(0^2\right) \right) (2(0)) + g\left(0^2\right) \cdot f'(0) \right) \\ &= e^{f(0) \cdot g(0)} \left(f(0) (g'(0)) (2(0)) + g(0) f'(0) \right) \\ &= e^{f(0) \cdot g(0)} \left(0 + g(0) f'(0) \right) \\ &= e^{f(0) \cdot g(0)} \left(g(0) f'(0) \right) \end{aligned}$$

Looking at the graphs to obtain the required function values, we see $f(0) = 5$, $g(0) = 1$, and $f'(0) = 2$ (this is a linear part of the graph so we can calculate the slope). This gives us

$$\begin{aligned} j'(0) &= e^{f(0) \cdot g(0)} \left(g(0) f'(0) \right) \\ &= e^{(5)(1)} ((1)(2)) \\ &= 2e^5 \end{aligned}$$

■

Try It # 8:
Using **Figure 2.5.1** from the previous example, answer each of the following.

a. If $k(x) = x^2 - 3(g(x))^4$, find $k'(-2)$.

b. If $m(x) = \dfrac{7f(x)}{\ln\left(x^2+1\right)}$, find $m'(6)$.

Applications

Example 10 The weekly price-demand function of a company that sells x specialized drill bits for solo objects frozen in carbonite is given by $p(x) = 2600e^{-0.2x}$, where $p(x)$ is the price, in dollars, of each drill bit.

a. Find $p'(10)$ and interpret your answer.

b. Find the marginal revenue when 10 drill bits are sold each week, and interpret your answer.

Solution:

a. Let's start by finding $p'(x)$. Notice this function is of the form $e^{g(x)}$, where $g(x) = -0.2x$. Using the Chain Rule gives

$$\begin{aligned} p'(x) &= 2600e^{g(x)}g'(x) \\ &= 2600e^{-0.2x}(-0.2) \\ &= -520e^{-0.2x} \end{aligned}$$

Substituting $x = 10$ yields

$$\begin{aligned} p'(10) &= -520e^{-0.2(10)} \\ &\approx -\$70.37 \text{ per drill bit} \end{aligned}$$

Thus, when 10 drill bits are sold each week, price is decreasing at a rate of \$70.37 per drill bit.

b. To find the marginal revenue when 10 drill bits are sold each week, recall that the marginal revenue function is the derivative of the revenue function. So first, we must find the revenue function:

$$\begin{aligned} R(x) &= x \cdot p(x) \\ &= x\left(2600e^{-0.2x}\right) \\ &= 2600xe^{-0.2x} \end{aligned}$$

Thus, $R(x) = 2600xe^{-0.2x}$ gives the revenue, in dollars, when x drill bits are sold each week.

Now, to find the derivative $R'(x)$, we need to use the Product Rule as well as the Chain Rule:

$$\begin{aligned} R'(x) &= (2600x)\left(\frac{d}{dx}\left(e^{-0.2x}\right)\right) + \left(e^{-0.2x}\right)\left(\frac{d}{dx}(2600x)\right) \\ &= 2600x\left(-0.2e^{-0.2x}\right) + 2600e^{-0.2x} \\ &= -520xe^{-0.2x} + 2600e^{-0.2x} \end{aligned}$$

Substituting $x = 10$ to find the marginal revenue when 10 drill bits are sold gives

$$\begin{aligned} R'(10) &= -520(10)e^{-0.2(10)} + 2600e^{-0.2(10)} \\ &\approx -\$351.87 \text{ per drill bit} \end{aligned}$$

When 10 drill bits are sold, revenue is decreasing at a rate of \$351.87 per drill bit. Recall this answer also gives us an approximation for the revenue earned from selling *the next* drill bit. In other words, the company will lose approximately \$351.87 in revenue when selling the 11^{th} drill bit.

Example 11 The cost, in dollars, of producing x cameras per week is given by $C(x) = 600 + 100\sqrt{4x+2}$.

a. Estimate the cost when 25 cameras are produced.

b. Find the exact cost of the 16^{th} camera produced.

c. Approximate the cost of the 16^{th} camera produced.

d. Calculate and interpret $\overline{C}\,'(34)$.

Solution:

a. To estimate the cost of producing 25 cameras, we find the total cost of 24 cameras, $C(24)$, and add the approximate cost of producing the 25^{th} camera, which is $C\,'(24)$.

We start by finding $C\,'(x)$ using the Chain Rule, where we have $(g(x))^{\frac{1}{2}}$ with the "inside" function $g(x) = 4x+2$:

$$\begin{aligned} C\,'(x) &= 0 + 100\left(\frac{d}{dx}\left(\sqrt{4x+2}\right)\right) \\ &= 100\left(\frac{1}{2}(g(x))^{-\frac{1}{2}} \cdot g\,'(x)\right) \\ &= 50\,(4x+2)^{-\frac{1}{2}}\,(4) \\ &= 200\,(4x+2)^{-\frac{1}{2}} \end{aligned}$$

Now, we can calculate the approximation:

$$\begin{aligned} C(25) &\approx C(24) + C\,'(24) \\ &= \left(600 + 100\sqrt{4(24)+2}\right) + 200\,(4(24)+2)^{-\frac{1}{2}} \\ &\approx \$1610.15 \end{aligned}$$

Hence, the approximate cost of producing 25 cameras is \$1610.15.

b. Remember that to find the exact cost of producing the 16^{th} camera, we do not need to use calculus. We calculate $C(16) - C(15)$. This difference takes the cost of producing 16 cameras and subtracts the cost of producing the first 15 cameras. This leaves the cost of producing only the 16^{th} camera.

Recalling $C(x) = 600 + 100\sqrt{4x+2}$, we have

$$\begin{aligned} C(16) - C(15) &= \left(600 + 100\sqrt{4(16)+2}\right) - \left(600 + 100\sqrt{4(15)+2}\right) \\ &\approx \$25.00 \end{aligned}$$

Therefore, the exact cost of producing the 16^{th} camera is \$25.00.

c. To approximate the cost of producing the 16^{th} camera, we substitute *one less* item into the derivative. Thus, we calculate $C\,'(15)$ using the derivative $C\,'(x) = 200\,(4x+2)^{-\frac{1}{2}}$ we found previously:

$$\begin{aligned} C\,'(15) &= 200\,(4(15)+2)^{-\frac{1}{2}} \\ &\approx \$25.40 \text{ per camera} \end{aligned}$$

Hence, the approximate cost of producing the 16^{th} camera is \$25.40. This is close to the exact cost of producing the 16^{th} camera, \$25, we found in part **b**.

d. $\overline{C}'(34)$ represents the marginal average cost when 34 cameras are produced. First, we need to find the average cost function, $\overline{C}$, and then we can take the derivative to find the marginal average cost function, $\overline{C}'$.

We find $\overline{C}(x)$ first:

$$\begin{aligned}\overline{C}(x) &= \frac{C(x)}{x}\\ &= \frac{600+100\sqrt{4x+2}}{x}\end{aligned}$$

Now, to take the derivative, we start with the Quotient Rule. When finding the derivative of the numerator, we will substitute the derivative of $600+100\sqrt{4x+2}$ we already found in part **a**:

$$\begin{aligned}\overline{C}'(x) &= \frac{x\left(\frac{d}{dx}\left(600+100\sqrt{4x+2}\right)\right)-\left(600+100\sqrt{4x+2}\right)\left(\frac{d}{dx}(x)\right)}{x^2}\\ &= \frac{x\left(200(4x+2)^{-\frac{1}{2}}\right)-\left(600+100\sqrt{4x+2}\right)}{x^2}\end{aligned}$$

Substituting $x = 34$ to calculate $\overline{C}'(34)$ gives

$$\begin{aligned}\overline{C}'(34) &= \frac{34\left(200(4(34)+2)^{-\frac{1}{2}}\right)-\left(600+100\sqrt{4(34)+2}\right)}{(34)^2}\\ &\approx -\$1.03 \text{ per camera per camera}\end{aligned}$$

Thus, when 34 cameras are produced, the average cost per camera is decreasing at a rate of \$1.03 per camera.

■

■ **Example 12** A bank account is opened with a deposit of \$50,000. The account has an annual interest rate of 8% per year. Find the rate at which the balance of the account is increasing after 6 years if the account is compounded

a. continuously.

b. semiannually.

Solution:

a. Because we want to find a rate of change, we need to take a derivative. However, we are getting ahead of ourselves; first we need a function! The formula for the balance of an account in which interest is compounded continuously is $A = Pe^{rt}$, where P is the principal (or starting) amount, A is the accumulated amount after t years, and r is the interest rate (as a decimal) per year.

! *With formulas for financial math, we always use rates in their* ***decimal*** *form!*

We want a rate of change that is per year, so our independent variable should be t. Letting $P = 50,000$ and $r = 0.08$, we have

$$A(t) = 50,000e^{0.08t}$$

Now, we can take the derivative using the Chain Rule because we have a function of the form $e^{g(t)}$, where $g(t) = 0.08t$:

$$\begin{aligned}A'(t) &= 50,000e^{g(t)}\cdot g'(t)\\ &= 50,000e^{0.08t}(0.08)\\ &= 4000e^{0.08t}\end{aligned}$$

$A'(t) = 4000e^{0.08t}$ gives us the rate at which the amount in the bank account is increasing after t years. We need the rate the balance is increasing after 6 years, so we calculate $A'(6)$:

$$\begin{aligned} A'(6) &= 4000e^{0.08(6)} \\ &\approx \$6464.30 \text{ per year} \end{aligned}$$

After 6 years, the balance of the account is increasing at a rate of \$6464.30 per year.

b. Here, we need to find the rate of change of the balance if the account is compounded semiannually. The formula for an account balance if interest is not compounded continuously is a bit more complicated: $A = P\left(1+\frac{r}{m}\right)^{mt}$, where $A,\ P,\ r,$ and t still represent the same quantities as before, and m is the number of compounding periods per year. Because the account compounds semiannually, or two times per year, $m = 2$ and the function with time as the independent variable is

$$\begin{aligned} A(t) &= 50,000\left(1+\frac{0.08}{2}\right)^{2t} \\ &= 50,000(1+0.04)^{2t} \\ &= 50,000(1.04)^{2t} \end{aligned}$$

Now, we can take the derivative with respect to time using the Chain Rule because the function is of the form $1.04^{g(t)}$, where $g(t) = 2t$:

$$\begin{aligned} A'(t) &= 50,000(1.04)^{g(t)}(\ln(1.04)) \cdot g'(t) \\ &= 50,000(1.04)^{2t}(\ln(1.04))(2) \\ &= 100,000(\ln(1.04))(1.04)^{2t} \end{aligned}$$

Thus,

$$\begin{aligned} A'(6) &= 100,000(\ln(1.04))(1.04)^{2(6)} \\ &\approx \$6279.36 \text{ per year} \end{aligned}$$

After 6 years, the balance of the account is increasing at a rate of \$6279.36 per year.

These answers do not give us the amount of money in the accounts after 6 years! These values give us the **rate of change** *of the account balances with respect to time.*

■

Try It # 9:
Find each of the following.

a. Katya deposits \$500 into an account that compounds monthly and has an annual interest rate of 4.2% per year. How fast is the balance of the account increasing after 3 years?

b. Amy deposits \$1300 into an account that compounds continuously and has an annual interest rate of 3.3% per year. How fast is the balance of the account increasing after 12 years?

Alternate Form of the Chain Rule

All of the Generalized Chain Rule formulas discussed in this section are useful when we need to find the derivative of a function that consists of two functions that have already been composed and the result is an "outside" function and an "inside" function. There is another form of the Chain Rule that can be used to find the derivative of a function that consists of functions that have not been composed yet:

Theorem 2.11 Alternate Form of the Chain Rule

If f and g are differentiable functions with $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Even though most prefer LaGrange notation for derivatives, here we use the Leibnitz notation because we need to pay attention to the variable that we are taking the derivative with respect to. For instance, $\frac{dy}{du}$ tells us the independent variable is u. Let's look at an example using the **Alternate Form of the Chain Rule**.

Example 13 If $y = 5u^4 - 9e^u$ and $u = 7 + 12\ln(x)$, find $\frac{dy}{dx}$ using the Alternate Form of the Chain Rule.

Solution:

Because we are given two separate functions that have not been composed, using the Alternate Form of the Chain Rule to find the derivative $\frac{dy}{dx}$ makes sense! To find $\frac{dy}{dx}$, we find the derivative of each function first:

$$\frac{dy}{du} = 20u^3 - 9e^u \quad \text{and} \quad \frac{du}{dx} = 0 + 12\left(\frac{1}{x}\right) = \frac{12}{x}$$

The Alternate Form of the Chain Rule gives

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \cdot \frac{du}{dx} \\ &= \left(20u^3 - 9e^u\right)\left(\frac{12}{x}\right) \end{aligned}$$

We want to find $\frac{dy}{dx}$, and paying attention to the notation tells us that we need to find the derivative of the function y with respect to x (not u). Thus, we must substitute for u so the answer is entirely in terms of the variable x. Noting that $u = 7 + 12\ln(x)$, we have

$$\begin{aligned} \frac{dy}{dx} &= \left(20u^3 - 9e^u\right)\left(\frac{12}{x}\right) \\ &= \left(20(7 + 12\ln(x))^3 - 9e^{7+12\ln(x)}\right)\left(\frac{12}{x}\right) \end{aligned}$$

■

Alternatively, we could have used the original form of the Chain Rule in the previous example if we would have composed the functions first. We will demonstrate this technique to show that the two forms of the Chain Rule are equivalent.

Recall that y is a function of u, $y = f(u) = 5u^4 - 9e^u$, and u is a function of x, $u = g(x) = 7 + 12\ln(x)$. To work the problem this way, we first need to compose the functions to find $y = f(g(x))$:

$$\begin{aligned} y &= f(g(x)) \\ &= f(7 + 12\ln(x)) \\ &= 5(7 + 12\ln(x))^4 - 9e^{7+12\ln(x)} \end{aligned}$$

Now, we can use the original form of the Chain Rule to find the derivative. For the first term, we will use the Generalized Power Rule with $g(x) = 7 + 12\ln(x)$. For the second term, we will use the Generalized Exponential Rule, where $g(x)$ also equals $7 + 12\ln(x)$.

Writing the function this way, we have

$$\begin{aligned} y &= 5(7+12\ln(x))^4 - 9e^{7+12\ln(x)} \\ &= 5(g(x))^4 - 9e^{g(x)} \end{aligned}$$

Finding the derivative using the corresponding rules gives

$$y' = 20(g(x))^3 \cdot g'(x) - 9e^{g(x)} \cdot g'(x)$$

Substituting $g(x) = 7 + 12\ln(x)$ as well as calculating and substituting $g'(x) = \dfrac{12}{x}$, we have

$$y' = 20(7+12\ln(x))^3 \cdot \left(\frac{12}{x}\right) - 9e^{7+12\ln(x)} \cdot \left(\frac{12}{x}\right)$$

If we factor $\dfrac{12}{x}$ from both terms, we can write our answer as

$$y' = \left(20(7+12\ln(x))^3 - 9e^{7+12\ln(x)}\right) \cdot \left(\frac{12}{x}\right)$$

which is identical to the answer we got using the Alternate Form of the Chain Rule in the previous example!

N *These two forms of the Chain Rule are always equivalent.*

We can also apply the Alternate Form of the Chain Rule regardless of what variables are used as we will see in the next example.

▪ **Example 14** If $r(k) = \sqrt[3]{k} - \pi k$ and $k(a) = \log_6(a) - 2 \cdot 4^a$, find $\dfrac{dr}{da}$ using the Alternate Form of the Chain Rule.

Solution:
To find $\dfrac{dr}{da}$, we find the derivative of each function first:

$$\frac{dr}{dk} = \frac{1}{3}k^{-2/3} - \pi \quad \text{and} \quad \frac{dk}{da} = \frac{1}{a(\ln 6)} - 2\left(4^a \ln 4\right)$$

The Alternate Form of the Chain Rule gives

$$\begin{aligned} \frac{dr}{da} &= \frac{dr}{dk} \cdot \frac{dk}{da} \\ &= \left(\frac{1}{3}k^{-2/3} - \pi\right)\left(\frac{1}{a\ln(6)} - 2\left(4^a \ln(4)\right)\right) \end{aligned}$$

Again, because we want to find $\dfrac{dr}{da}$, we need the derivative of the function r with respect to a (not k). Thus, we must substitute for k in the answer. Noting that $k(a) = \log_6(a) - 2 \cdot 4^a$, we have

$$\begin{aligned} \frac{dr}{da} &= \left(\frac{1}{3}k^{-2/3} - \pi\right)\left(\frac{1}{a\ln(6)} - 2\left(4^a \ln(4)\right)\right) \\ &= \left(\frac{1}{3}\left(\log_6(a) - 2 \cdot 4^a\right)^{-2/3} - \pi\right)\left(\frac{1}{a\ln(6)} - 2\left(4^a \ln(4)\right)\right) \end{aligned}$$

▪

Try It # 10:
Find each of the following using the Alternate Form of the Chain Rule.

a. Given $z = 17x + 4x^{-3}$ and $x = 7\ln(y) + e^y$, find $\dfrac{dz}{dy}$.

b. Given $u = \dfrac{2x^6}{\log(x)}$ and $y = \left(5 - u^{16}\right)^{14}$, find $\dfrac{dy}{dx}$.

To summarize, we write both forms of the Chain Rule:

The Chain Rule

If f and g are differentiable functions, then we have the following:

Original Form: If $y = f(g(x))$, then $y' = f'(g(x)) \cdot g'(x)$.

Alternate Form: If $y = f(u)$ and $u = g(x)$, then $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$.

Remember that as complicated as finding derivatives using the Chain Rule, Product Rule, Quotient Rule, and other rules may be, using these rules is still significantly easier than using the limit definition of the derivative.

Enrichment: Logarithmic Differentiation

The Generalized Natural Logarithm Rule is useful for finding derivatives of functions with natural logarithms, but it can also be used to find other derivatives. Consider the function $y = x^x$ whose graph is shown in **Figure 2.5.2**.

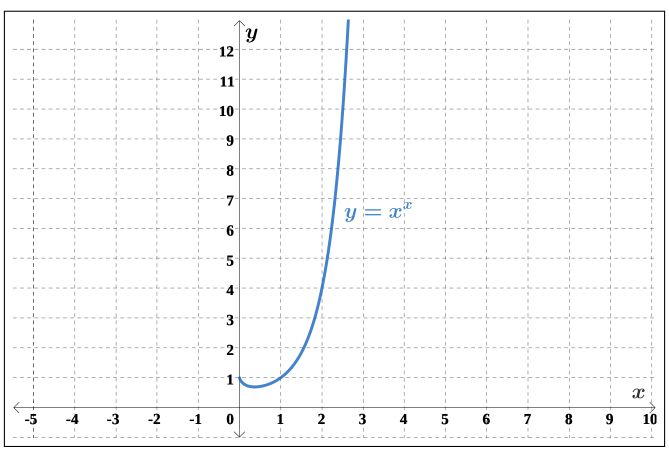


Figure 2.5.2: Graph of $y = x^x$

This function has a domain of $(0, \infty)$, and its graph has a shape similar to that of a parabola (except with a much more rapid increase). The function has no sharp corners or cusps, no discontinuities, and no vertical tangent lines. Therefore, its derivative should exist everywhere on its domain. However, the function $y = x^x$ does not fit into any of our previous rules. It turns out that we actually can find a formula for the derivative, but we have to start with something counterintuitive: making the function *more complicated*!

If we let $g(x) = \ln(x^x)$, then $g(x) = x\ln(x)$, which we can find the derivative of using the Product Rule:

$$\begin{aligned} g'(x) &= x\left(\frac{1}{x}\right) + (\ln(x))(1) \\ &= 1 + \ln(x) \end{aligned}$$

Alternately, we could have used the Generalized Natural Logarithm Rule to find the derivative of g in its original form, $g(x) = \ln(x^x)$. Doing so yields

$$g'(x) = \frac{\frac{d}{dx}(x^x)}{x^x}$$

Now, we have created two ways to represent $g'(x)$:

$$g'(x) = 1 + \ln(x) \quad \text{and} \quad g'(x) = \frac{\frac{d}{dx}(x^x)}{x^x}$$

Setting these two representations equal to each other gives

$$1 + \ln(x) = \frac{\frac{d}{dx}(x^x)}{x^x}$$

Because our original goal was to find the derivative of $y = x^x$ (i.e., find $\frac{d}{dx}(x^x)$), we will solve the equation above for $\frac{d}{dx}(x^x)$. Multiplying both sides of the equation by x^x gives

$$x^x(1 + \ln(x)) = \frac{d}{dx}(x^x)$$

In other words,

$$\frac{d}{dx}(x^x) = x^x(1 + \ln(x))$$

Thus, we have found the derivative of $y = x^x$!

This technique of making something more complicated to access mathematical rules is a common one used in mathematics. In this particular case, it gives rise to a technique called **logarithmic differentiation**.

If we have a function where the independent variable is in both the base and the exponent, such as $y = x^x$, we should write the function as the argument of a natural logarithm. Then, we can use the Properties of Logarithms to rewrite the function and take the derivative using previous techniques (i.e., the Product Rule, Quotient Rule, Chain Rule, etc.). Finding the derivative *again* using the Generalized Natural Logarithm Rule allows us to set the two derivatives equal in order to solve for the desired derivative. We now summarize this process:

Logarithmic Differentiation

To find $f'(x)$ given a function $f(x)$ with the independent variable in both the base and the exponent,

1. Set $g(x) = \ln(f(x))$.
2. Expand $g(x)$ as much as possible using the Properties of Logarithms.
3. Find $g'(x)$ using previous techniques (Product Rule, Quotient Rule, Chain Rule, etc).
4. Because $g'(x) = \dfrac{f'(x)}{f(x)}$ by the Generalized Natural Logarithm Rule as well, solving this equation for $f'(x)$ gives $f'(x) = g'(x) \cdot f(x)$.

Let's see this in action with some examples:

▪ **Example 15** Find the derivative of each of the following functions.

a. $f(x) = 3x^{5x^2-12}$

b. $f(x) = \left(14x^5 + 8x^3 + 1\right)^{4(x+1)^2-15}$

Solution:

a. Because the independent variable, x, is in both the base and the exponent of $f(x)$, we will use logarithmic differentiation:

1. Let $g(x) = \ln\left(3x^{5x^2-12}\right)$.

2. Use the Properties of Logarithms to fully expand $g(x)$:

$$\begin{aligned} g(x) &= \ln\left(3x^{5x^2-12}\right) \\ &= \ln(3) + \ln\left(x^{5x^2-12}\right) \\ &= \ln(3) + \left(5x^2 - 12\right)\ln(x) \end{aligned}$$

3. Find $g'(x)$ using previous derivative rules:

$$\begin{aligned} g'(x) &= \frac{d}{dx}\left(\ln(3) + \left(5x^2-12\right)\ln(x)\right) \\ &= 0 + \left(5x^2-12\right)\left(\frac{d}{dx}(\ln(x))\right) + (\ln(x))\left(\frac{d}{dx}\left(5x^2-12\right)\right) \\ &= \frac{5x^2-12}{x} + (\ln(x))(10x) \end{aligned}$$

4. Because $g'(x) = \dfrac{f'(x)}{f(x)}$ by the Generalized Natural Logarithm Rule as well, solving this equation for $f'(x)$ gives $f'(x) = g'(x) \cdot f(x)$. Thus, we substitute $g'(x) = \dfrac{5x^2-12}{x} + (\ln(x))(10x)$ and $f(x) = 3x^{4x^2-12}$ to find $f'(x)$:

$$\begin{aligned} f'(x) &= g'(x) \cdot f(x) \\ &= \left(\frac{5x^2-12}{x} + (\ln(x))(10x)\right)\left(3x^{5x^2-12}\right) \end{aligned}$$

b. Because the independent variable, x, is in both the base and the exponent of $f(x) = \left(14x^5 + 8x^3 + 1\right)^{4(x+1)^2-15}$, we will use logarithmic differentiation:

1. Let $g(x) = \ln\left(\left(14x^5 + 8x^3 + 1\right)^{4(x+1)^2-15}\right)$.

2. Use the Properties of Logarithms to fully expand $g(x)$:

$$\begin{aligned} g(x) &= \ln\left(\left(14x^5 + 8x^3 + 1\right)^{4(x+1)^2-15}\right) \\ &= \left(4(x+1)^2 - 15\right)\left(\ln\left(14x^5 + 8x^3 + 1\right)\right) \end{aligned}$$

3. Find $g'(x)$ using previous derivative rules:

$$\begin{aligned} g'(x) &= \left(4(x+1)^2 - 15\right)\left(\frac{d}{dx}\left(\ln\left(14x^5 + 8x^3 + 1\right)\right)\right) + \left(\ln\left(14x^5 + 8x^3 + 1\right)\right)\left(\frac{d}{dx}\left(4(x+1)^2 - 15\right)\right) \\ &= \left(4(x+1)^2 - 15\right)\frac{\frac{d}{dx}\left(14x^5 + 8x^3 + 1\right)}{14x^5 + 8x^3 + 1} + \left(\ln\left(14x^5 + 8x^3 + 1\right)\right)\left(2 \cdot 4(x+1)\left(\frac{d}{dx}(x+1)\right) - 0\right) \\ &= \left(4(x+1)^2 - 15\right)\frac{70x^4 + 24x^2}{14x^5 + 8x^3 + 1} + \left(\ln\left(14x^5 + 8x^3 + 1\right)\right)(8(x+1)(1)) \\ &= \left(4(x+1)^2 - 15\right)\frac{70x^4 + 24x^2}{14x^5 + 8x^3 + 1} + \left(\ln\left(14x^5 + 8x^3 + 1\right)\right)(8x+8) \end{aligned}$$

4. Because $g'(x) = \dfrac{f'(x)}{f(x)}$ by the Generalized Natural Logarithm Rule as well, solving this equation for $f'(x)$ gives $f'(x) = g'(x) \cdot f(x)$.

Thus, we substitute $g'(x) = \left(4(x+1)^2 - 15\right)\dfrac{70x^4 + 24x^2}{14x^5 + 8x^3 + 1} + \left(\ln\left(14x^5 + 8x^3 + 1\right)\right)(8x+8)$ and $f(x) = \left(14x^5 + 8x^3 + 1\right)^{4(x+1)^2-15}$ to find $f'(x)$:

$$f'(x) = \left(\left(4(x+1)^2 - 15\right)\frac{70x^4 + 24x^2}{14x^5 + 8x^3 + 1} + \left(\ln\left(14x^5 + 8x^3 + 1\right)\right)(8x+8)\right)\left(\left(14x^5 + 8x^3 + 1\right)^{4(x+1)^2-15}\right)$$

▪

Try It Answers

1. **a.** $y' = 15(2x + e^x)^4(2 + e^x)$

   b. $y' = \left(14x^{-3} - \log_3(x)\right)^9\left(\frac{2}{7}\left(x^2 - 1\right)^{-5/7}(2x)\right) + \left(x^2 - 1\right)^{2/7}\left(9\left(14x^{-3} - \log_3(x)\right)^8\left(-42x^{-4} - \frac{1}{x\ln(3)}\right)\right)$

2. **a.** $y' = 14e^{-2x^2+2x^{-2}}\left(-4x - 4x^{-3}\right)$

   b. $y' = \dfrac{\left((\ln(x))^2\right)\left(e^{2x^{-6}+3x^5}\left(-12x^{-7} + 15x^4\right)\right) - e^{2x^{-6}+3x^5}(2(\ln(x)))\left(\frac{1}{x}\right)}{(\ln(x))^4}$

3. **a.** $y' = 14^{e^x+\log_4(x)+4x^8}\left(e^x + \frac{1}{x\ln(4)} + 32x^7\right)$

b. $y' = \left(2^{5x^3-22}\right)\left(39x^{12} - 3^{x^2}(\ln(3))(2x)\right) + \left(3x^{13} - 3^{x^2}\right)\left(2^{5x^3-22}(\ln(2))\left(15x^2\right)\right)$

4. **a.** $y' = \frac{6x - 6x^2 + 1}{3x^2 - 2x^3 + x + 12}$

b. $y' = -48x^{11} - \frac{1}{x+3} - \frac{2}{x-3}$

5. **a.** $y' = \frac{e^{x^3-14x^2+15x}\left(3x^2 - 28x + 15\right) - 16x}{\left(e^{x^3-14x^2+15x} - 8x^2\right)(\ln(4))}$

b. $y' = \frac{3x^2 + 44}{\left(x^3 + 44x - 18\right)(\ln(7))} - \frac{5x^4 - 3x^2 - 1}{\left(x^5 - x^3 - x + 1\right)(\ln(7))} - 12x^3$

6. $y = (6\ln(10))\,x - 9\ln(10) - \frac{81}{5}$

7. **a.** $30e^9$

b. 36

8. **a.** 1496

b. $\frac{14\ln(37) - \frac{672}{37}}{(\ln(37))^2}$

9. **a.** \$23.77 per year

b. \$63.74 per year

10. **a.** $\frac{dz}{dy} = \left(17 - 12\left(7\ln(y) + e^y\right)^{-4}\right)\left(\frac{7}{y} + e^y\right)$

b. $\frac{dy}{dx} = 14\left(5 - \left(\frac{2x^6}{\log(x)}\right)^{16}\right)^{13}\left(-16\left(\frac{2x^6}{\log(x)}\right)^{15}\right)\left(\frac{(\log(x))\left(12x^5\right) - \left(2x^6\right)\left(\frac{1}{x\ln(10)}\right)}{(\log(x))^2}\right)$

EXERCISES

Basic Skills Practice

For Exercises 1 - 3, use the Generalized Power Rule to find the derivative of the function.

1. $f(x) = (4x+12)^6$

2. $g(x) = \left(12x^3+33x\right)^{4/3}$

3. $f(t) = (8t+e^t)^5$

For Exercises 4 - 7, use the Generalized Exponential Rule to find $\dfrac{dy}{dx}$ for the function.

4. $y = e^{x^2}$

5. $y = e^{\sqrt{x}-\frac{13}{x}}$

6. $y = 8^{9x^5-12x^{-5}}$

7. $y = 10^{2^x+14}$

For Exercises 8 - 11, use the Generalized Logarithm Rule to find $f'(x)$.

8. $f(x) = \ln(4+x)$

9. $f(x) = \ln\left(9x^3-13x^{-3/2}+17\right)$

10. $f(x) = \log_3\left(\dfrac{11}{x^2}+55x^2\right)$

11. $f(x) = \log_5\left(36x^{11}-12x^7+x^3-1\right)$

For Exercises 12 - 14, find the slope of the line tangent to the graph of f at the given x-value.

12. $f(x) = \left(7x^3+55\right)^9$ at $x=-2$

13. $f(x) = 6^{x^{10}-2x^5+12x}$ at $x=0$

14. $f(x) = \log_{11}\left(4x^2+16x\right)$ at $x=1$

For Exercises 15 - 17, find the equation of the line tangent to the graph of the function at the given x-value.

15. $f(x) = \left(x^5 - 2\right)^7$ at $x = 1$

16. $g(x) = e^{x^3+6x^2+12x}$ at $x = 0$

17. $h(x) = \ln\left(x^2 + 3x + 3\right)$ at $x = -1$

18. Given $f(x) = \frac{1}{8}\left(x^3 - 3x^2 + 3x\right)^4$, find the equation of the line tangent to the graph of f at $x = 2$. Use technology to graph f and the tangent line on the same axes.

For Exercises 19 - 21, find the x-value(s) where the graph of f has a horizontal tangent line.

19. $f(x) = \left(x^2 - 4\right)^{10}$

20. $f(x) = e^{x^4-108x}$

21. $f(x) = \ln\left(x^2 + 11x + 31\right)$

For Exercises 22 - 25, use the Alternate Form of the Chain Rule to find $\frac{dy}{dx}$.

22. $y = \sqrt{u}$ and $u = 8x^3 - 8$

23. $y = 8u^3 - 8$ and $u = \sqrt{x}$

24. $y = \ln(z) - 13z$ and $z = x^9 - 3x^3 + 1$

25. $y = \frac{t}{t+1}$ and $t = e^x + 7x$

26. The price-demand function for a brand of diploma frame is given by $p(x) = -(-0.01x - 2)^4 + 303$, where $p(x)$ is the price, in dollars, per frame when x frames are demanded. At what rate is the price changing when 400 frames are demanded?

27. The cost of making x high end toaster ovens is given by $C(x) = x^2 + \sqrt{13x + 160}$ dollars. At what rate is the cost changing when 300 toaster ovens are made. In other words, what is the marginal cost when $x = 300$?

28. The function $s(t) = \frac{-1}{(t^2+1)^3}$ represents the position of a particle, in inches, after t seconds. What is the instantaneous velocity of the particle after 1 second?

Intermediate Skills Practice

For Exercises 29 - 44, find $f'(x)$.

29. $f(x) = (xe^x + 8x)^9$

30. $f(x) = \left(x^3 - 17\log_2(x)\right)^8 (2x + 14)$

31. $f(x) = \sqrt[5]{\left(12x^7 - e^x + \pi\right)^4} \cdot 9^x$

32. $f(x) = \dfrac{10^x + x^{10}}{(3x + 15)^5}$

33. $f(x) = \left(\dfrac{\ln(x) + 17x^4}{x^2 - 3}\right)^6$

34. $f(x) = \dfrac{31x^3 + 7x - 3}{\left(4x^5 + x\right)^6}$

35. $f(x) = x^2 e^{x^3 + 21}$

36. $f(x) = e^{x^3 4^x}$

37. $f(x) = 7^{x/(x-7)}$

38. $f(x) = \dfrac{1 - e^{x^2 - 1}}{(17x + 17)}$

39. $f(x) = \left(x^4 e^{12x^{-3}} - 9\right)^{-1}$

40. $f(x) = \left(8x^2\right)\left(\ln\left(x^2 + 49\right)\right)$

41. $f(x) = (e^x - 1)\left(\log\left(9 - \sqrt{x}\right)\right)$

42. $f(x) = \dfrac{5x^3 - 125x}{\ln(7x + 14)}$

43. $f(x) = \dfrac{\log_4\left(x^2 + 36\right)}{8x^{0.2} - 2x^{0.8}}$

44. $f(x) = \dfrac{\ln\left(6x^2 + 13x - 2\right)}{\left(x^3 - 1\right)e^x}$

For Exercises 45 - 47, (a) use the Properties of Logarithms to fully expand $f(x)$ and (b) find $f'(x)$.

45. $f(x) = \ln\left((x+4)^7 (x-10)^{14}\right)$

46. $f(x) = \log_5\left(\dfrac{(5x-77)^6}{(18x+9)^3}\right)$

47. $f(x) = \ln\left(\dfrac{x^8\left(15-22x^3\right)^5}{\left(3x^2-11\right)^{33}\left(16-x^2\right)^{12}}\right)$

For Exercises 48 - 50, find the slope of the line tangent to the graph of f at the given x-value.

48. $f(x) = (x+10)(3x+22)^8$ at $x = -7$

49. $f(x) = e^{(x-9)^{17}}$ at $x = 10$

50. $f(x) = \dfrac{\ln(e^x+6x)}{\left(x^2-4\right)}$ at $x = 0$

51. Given $f(x) = \ln\left(5x^2+12x+3\right)$, find the equation of the line tangent to the graph of f at $x = 0$. Use technology to graph f and the tangent line on the same axes.

For Exercises 52 and 53, find the x-value(s) where the graph of f has a horizontal tangent line.

52. $f(x) = x^3 e^{x^3}$

53. $f(x) = \ln\left(8^{x^2-4}+1\right)$

For Exercises 54 and 55, find the x-value(s) where the line tangent to the graph of f has the indicated slope.

54. $f(x) = \ln\left(x^2-5x-5\right)$; slope is 1

55. $f(x) = 27x + e^{2x^3+33x^2+168x}$; slope is 27

For Exercises 56 - 59, find $\dfrac{dy}{dx}$.

56. $y = \sqrt{2u+1}$ and $u = 4x^9$

57. $y = 5^u - 13u^3$ and $u = \ln\left(x^2+36\right)$

58. $y = \left(w^2+1\right)e^w$ and $w = 2x^3+5$

59. $y = 3^{t^2-2t+1}$ and $t = 2x^2+32$

60. The daily profit function for the food truck Crepes by Monica is given by the function $P(x)=10\left(x^2-1000\right)^{1/3}-287$ dollars when x crepes are sold. Find $P'(33)$ and interpret your answer.

61. The weekly cost function for Susan's Sticky Notes is given by $C(x)=20\ln\left(3x^2+10x+25\right)$ dollars, where x is the number of boxes of sticky notes produced.

    (a) Find the marginal cost when 160 boxes of sticky notes are produced, and interpret your answer.

    (b) Approximate the cost of producing the 200th box of sticky notes.

    (c) Find the exact cost of producing the 175th box of sticky notes.

    (d) Estimate the cost if 130 boxes of sticky notes are produced.

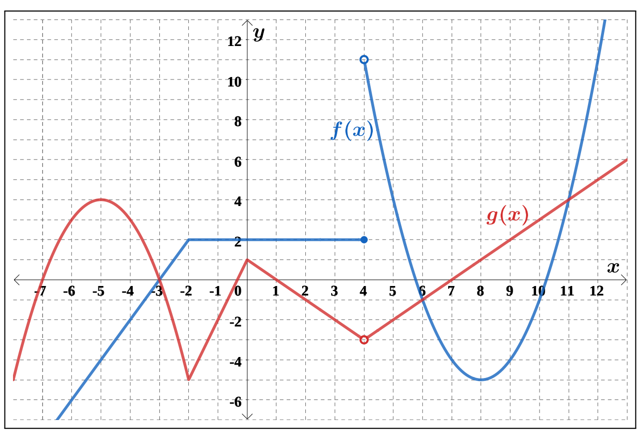
    (e) Find the exact cost if 250 boxes of sticky notes are produced.

62. A corporation deposits \$80,000 in a new bank account that earns interest at an annual rate of 2.7% per year and compounds continuously. The formula for the amount in the account after t years is given by $A(t)=80{,}000e^{0.027t}$ dollars. At what rate is the balance of the account growing after 7 years?

63. Mahesh is saving to take a vacation. He deposited \$1200 in a bank account that earns interest at an annual rate of 3% per year and compounds monthly. The formula for the amount in the account after t years is given by $A(t)=1200(1.0025)^{12t}$ dollars. How fast is Mahesh's account balance growing after 2 years?

64. A pteranodon is flying by. The scientist observing it notices that its position, in feet, from her after t seconds is given by the function $s(t)=30(8t+3)^{2/3}+10$. At what speed is the pteranodon flying after 30 seconds? *Note: The speed is given by the absolute value of the rate of change of* $s(t)$. Round your answer to two decimal places, if necessary.

65. The graphs of the functions f and g are shown below on the same axes. Use the graphs to find each of the following.



    (a) $h'(-1)$, where $h(x)=g(f(x))$

    (b) $j'(-5)$, where $j(x)=3x-f(f(x))$

    (c) $k'(2)$, where $k(x)=g\left(x^3\right)$

66. The table below shows values of $f(x)$, $g(x)$, $f'(x)$, and $g'(x)$ for certain values of x. Use the information in the table to find each of the following.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-2	1	1	-2	-2
-1	2	1	-1	1
0	2	2	-1	-1
1	1	-2	0	-2
2	0	-2	-1	1

(a) $h'(0)$, where $h(x) = g(f(x))$

(b) $j'(-2)$, where $j(x) = \dfrac{f(2x+3)}{g(x)}$

(c) $k'(-1)$, where $k(x) = 3g\left(x^3\right) \cdot \ln(f(x))$

Mastery Practice

For Exercises 67 - 77, find $\dfrac{dy}{dx}$.

67. $y = \log_2\left(\dfrac{\left(14x - 7x^2\right)^8 \sqrt{2x^3 - 1}}{(x-33)^7 \cdot 2^{4x+5}}\right)$

68. $y = \dfrac{\left(11x^5 + 6^{4x^2-16}\right)^7}{x\ln(x)}$

69. $y = \dfrac{8u^3}{\ln(u+1)}$ and $u = (4x-12)^{3/2}$

70. $y = \sqrt[5]{4x^3 - 3x^{-4}}$

71. $y = \dfrac{9e^7 - \sqrt[3]{x^4}}{25^{\sqrt{x^2-3x+4}}}$

72. $y = \left(8 - x^3\right)4^{3x^2-64}$

73. $y = 74t^{17} + 35t^7 - 41t + 2$ and $t = \left(14\ln(x) + \dfrac{9}{x}\right)^3$

74. $y = \ln\left(\dfrac{1}{\sqrt[3]{x^2+16} \cdot (3^x + 19)^{10}}\right)$

75. $y = e^{2x^{11}+44}$

76. $y = \sqrt{\left(\pi x^{-2}\right)\left(\log_4\left(9x^2 - 5e^x\right)\right)}$

77. $y = \sqrt{t+5}$, $t = e^{u^2}$, and $u = \ln\left(x^2+1\right)$

78. Find the equation of the line tangent to the graph of $f(x) = 5^{2x^2-1}$ at $x = 1$. Use technology to graph f and the tangent line on the same axes.

For Exercises 79 - 81, find the x-value(s) where the graph of f has a horizontal tangent line.

79. $f(x) = \dfrac{(x+2)^{13}}{(x-2)^{19}}$

80. $f(x) = \log_8\left(3x^4 - 8x^3 - 18x^2 + 140\right) + 14$

81. $f(x) = x^2 e^{8x^2+100}$

For Exercises 82 and 83, find the x-value(s) where the graph of f has the indicated slope.

82. $f(x) = \log\left(x^2 + 15x + 15\right)$; slope is $\dfrac{1}{\ln(10)}$

83. $f(x) = 23 + 13x - 12^{x^3-48x}$; slope is 13

84. Find the value(s) of x where the function $f(x) = 5\left(x^3-8\right)^{38} - 18x$ has an instantaneous rate of change of -18.

85. A dust particle is $D(t) = 13 - \left(0.015t^3 - 0.75t\right)^{4/3}$ inches above the ground after t seconds. How fast is the particle moving after 6 seconds? Round your answer to three decimal places, if necessary.

86. The monthly cost function for vibranium mining operations in the small nation of Wakanda is given by $C(t) = \log_9\left(0.06t^4 + 23t^2 + 30\right)$ thousands of Wakandan dollars when the mines have been open for t hours this month. Find $C'(50)$ and interpret your answer. Round to three decimal places, if necessary.

87. Pack Men is a moving company specializing in safely transporting arcade game cabinets. The yearly profit is $P(x) = 2.2\sqrt{x^{7/3} + 90} - 3.4x - 200$ hundreds of dollars when x moves are completed. Find the marginal profit when 3299 moves are completed, and interpret your answer. Round to four decimal places, if necessary.

88. The price-demand function for a brand of diploma frame is given by $p(x) = -(-0.01x - 2)^4 + 303$, where $p(x)$ is the price, in dollars, per frame when x frames are demanded.

    (a) Find $R'(150)$, where $R(x)$ is the revenue, in dollars, when x frames are demanded, and interpret your answer.

    (b) Find the marginal average revenue when 150 frames are sold, and interpret your answer.

89. Generic Corp, a manufacturer of doodads, has a cost function given by $C(x) = \left(0.02x^2 + 0.18x + 30.28\right)^{7/9}$ dollars when x doodads are made.

    (a) Find the rate of change of cost when 116 doodads are made, and interpret your answer.

    (b) Approximate the cost of making the 116$^{\text{th}}$ doodad.

    (c) Find the exact cost of making the 116$^{\text{th}}$ doodad.

    (d) Estimate the cost if 116 doodads are made.

    (e) Find the exact cost if 116 doodads are made.

90. A bank account has an initial balance of \$400. The account earns interest at an annual rate of 3.24% per year compounded continuously. How fast is the balance of the account growing after 7 years?

91. Walton Martinez is planning on opening a store and is saving money by placing \$20,000 in a bank account that compounds quarterly and has an annual interest rate of 4.2% per year. At what rate is his account balance growing after 3 years?

92. The table below shows values of $f(x)$, $g(x)$, $f'(x)$, and $g'(x)$ for certain values of x. Use the information in the table to find each of the following.

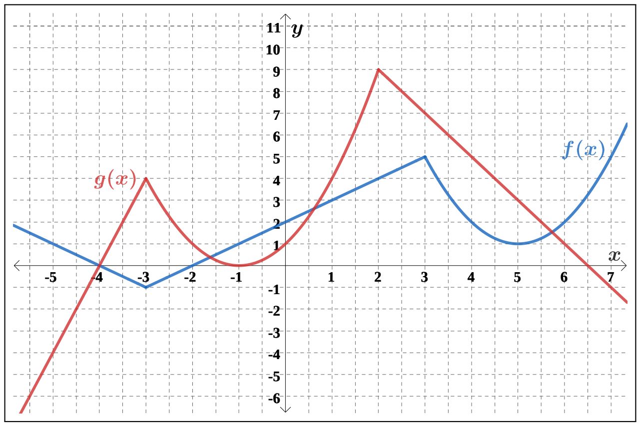
x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
-10	5	5	-8	-5
-5	10	5	-4	DNE
0	10	10	-3	2
5	5	-10	0	-2
10	0	1	-8	3

    (a) $h'(10)$, where $h(x) = (2x + g(f(x)))\left(e^{g(x)}\right)$

    (b) $j'(-5)$, where $j(x) = 4x^3 - \log_2(f(x) + g(x))$

    (c) $k'(5)$, where $k(x) = \dfrac{\ln\left(8x + e^{f(x)}\right)}{f(g(x))}$

93. The graphs of the functions f and g are shown below on the same axes. Use the graphs to find each of the following.



(a) $h'(5)$, where $h(x) = g(f(x))$

(b) $j'(4)$, where $j(x) = 8x - g(g(x))$

(c) $k'(-1)$, where $k(x) = f\left(x^2\right) + g\left(x^3\right)$

Communication Practice

94. When calculating $\dfrac{d}{dx}\left(\sqrt[3]{\dfrac{6x^4 - x^2 + 7}{5x^3 - x + 1}}\right)$, what derivative rule should be used first?

95. Explain why a calculus student may want to use the Properties of Logarithms before using the Chain Rule to find the derivative of a logarithmic function.

96. Does $\dfrac{d}{dx}\left(\left(x^2 + 4x\right)^6\right) = 6(2x+4)^5$? Explain.

97. Describe the circumstance in which one might use the Alternate Form of the Chain Rule.

98. Explain why it is necessary to perform a substitution after using the Alternate Form of the Chain Rule.

99. Sam does not like using the Quotient Rule. Jenn told her she can avoid the Quotient Rule by using the Product Rule and Chain Rule instead. Explain the procedure Jenn was referring to.

100. If Bruce uses the Chain Rule, but his friend uses the Alternate Form of the Chain Rule to find the derivative of the same function, will their answers be mathematically equivalent? Explain.

2.6 Implicit Differentiation and Related Rates

Until now, the functions we needed to differentiate were either given *explicitly*, such as $y = x^2 + e^x$, or it was possible to get an explicit formula for them, such as solving $y^3 - 3x^2 = 5$ for y to get $y = \sqrt[3]{5 + 3x^2}$. Sometimes, however, we will have an equation relating x and y which is either difficult or impossible to solve explicitly for y, such as $y + e^y = x^2$. Even if this is the case, we can still find the derivative, $\frac{dy}{dx}$, of such equations using **implicit differentiation**.

Implicit differentiation is a very useful differentiation technique that allows us to solve **related-rates** problems. If several variables, or quantities, are *related* to each other and all but one of the variables are changing at a known rate, then we can determine how rapidly the other variable must be changing. These types of problems are called related-rates problems.

For example, the area of a circle, A, is related to its radius, r, by the equation $A = \pi r^2$. Because the area is related to the radius, if the area of the circle is increasing over time, then the rate of change of the area with respect to time must be related to the rate of change of the radius with respect to time. In other words, because A is related to r, $\frac{dA}{dt}$ is related to $\frac{dr}{dt}$. Thus, just as we could find the area A given some radius r, we can find $\frac{dA}{dt}$ if we know $\frac{dr}{dt}$.

Learning Objectives:

In this section, you will learn how to differentiate equations implicitly and use implicit differentiation to solve related-rates problems. Upon completion you will be able to:

- Demonstrate the ability to implicitly differentiate equations.
- Calculate the slope of a tangent line using implicit differentiation.
- Find the equation of a tangent line using implicit differentiation.
- Solve related-rates word problems involving cost, revenue, and profit functions.
- Solve related-rates word problems involving geometric applications.

Implicit Differentiation

Before we can learn how to solve related-rates problems, we must first learn how to use implicit differentiation to find the derivative of an equation when the equation cannot be explicitly solved for the dependent variable (typically y).

The main idea behind implicit differentiation is to **treat y as a function of x** even if we cannot explicitly solve for y, which is the case with the equation mentioned in the introduction, $y + e^y = x^2$. Treating y as a function of x does not require any extra work per say, but we need to be very careful when we differentiate and remember to use the Chain Rule.

For instance, let's find $\frac{d}{dx}\left(y^3\right)$ and treat y as a function of x. Thus, we need to find this derivative implicitly. To see this more clearly, let's write y in the form $y(x)$ because we are treating y as a function of x:

$$\frac{d}{dx}\left(y^3\right) = \frac{d}{dx}\left((y(x))^3\right)$$

Using the Chain Rule, we have

$$\begin{aligned}\frac{d}{dx}\left((y(x))^3\right) &= 3(y(x))^2\left(\frac{d}{dx}(y(x))\right)\\ &= 3(y(x))^2 \cdot y'(x)\\ &= 3(y(x))^2 \cdot \frac{dy}{dx} \quad \text{(using Leibniz notation)}\end{aligned}$$

Substituting y back for $y(x)$ gives

$$\begin{aligned}&= 3(y)^2 \cdot \frac{dy}{dx}\\ &= 3y^2 \cdot \frac{dy}{dx}\end{aligned}$$

Thus, essentially all we need to remember when using implicit differentiation is that if we take the derivative of a term involving y, after using the derivative rules we must multiply by $\frac{dy}{dx}$ to incorporate the Chain Rule because we are treating y as a function of x. Note that if we are taking the derivative of a term involving x, we just take the derivative as usual (we do not need to multiply by $\frac{dy}{dx}$).

N *While some textbooks use the notation y' instead of the Leibniz notation $\frac{dy}{dx}$ when using implicit differentiation, we will continue to use $\frac{dy}{dx}$ because it will help us understand the relationships between the variables when working related-rates problems (coming soon!).*

We will now use the ideas outlined previously to establish a method for finding $\frac{dy}{dx}$ using implicit differentiation when we are given an *equation* relating x and y:

Implicit Differentiation Method $\left(\text{to find } \frac{dy}{dx}\right)$

1. Take the derivative of both sides of the equation. Treat the x terms as usual, but when taking the derivative of y terms, apply the usual derivative rules and remember to multiply by $\frac{dy}{dx}$ to incorporate the Chain Rule.
2. Move all the $\frac{dy}{dx}$ terms to one side of the equation and all the remaining terms to the other side.
3. Factor the $\frac{dy}{dx}$ term, and solve for $\frac{dy}{dx}$ by dividing.

Now, let's practice using implicit differentiation!

Example 1 Given $x^3 + y^4 = 36$, find $\frac{dy}{dx}$ using implicit differentiation.

Solution:
We will apply the Implicit Differentiation Method:

1. Take the derivative of both sides of the equation, and remember to use the Chain Rule when taking the derivative of y terms:

$$\frac{d}{dx}\left(x^3 + y^4\right) = \frac{d}{dx}(36) \implies$$
$$3x^2 + 4y^3 \cdot \frac{dy}{dx} = 0$$

2. Move the terms containing $\frac{dy}{dx}$ to one side and all remaining terms to the other side:

$$4y^3 \cdot \frac{dy}{dx} = -3x^2$$

3. Factor and solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = \frac{-3x^2}{4y^3}$$

N *In this example, we did not need to factor $\frac{dy}{dx}$ in step 3 because it only appeared in one term. Therefore, we only needed to divide to solve for $\frac{dy}{dx}$.*

■

N *As seen in the previous example, when using implicit differentiation, the resulting expression for $\frac{dy}{dx}$ is in terms of both the independent variable x and the dependent variable y. Although in some cases it may be possible to express $\frac{dy}{dx}$ in terms of x only, it is generally not possible to do so.*

Try It # 1:
Given $3x^2 + \sqrt[4]{y} = 19$, find $\frac{dy}{dx}$ using implicit differentiation.

■ **Example 2** Given $y^5 + x^2y^3 = 1 + 2x + \ln(y)$, find $\frac{dy}{dx}$ using implicit differentiation.

Solution:
We will apply the Implicit Differentiation Method:

1. Take the derivative of both sides of the equation, and remember to use the Chain Rule when taking the derivative of y terms:

Notice before we take the derivative that there is a product of terms, x^2y^3, in the equation. When taking the derivative of this term, we need to use the Product Rule as well as the Chain Rule when taking the derivative of the y^3 term:

$$\frac{d}{dx}\left(y^5 + x^2y^3\right) = \frac{d}{dx}\left(1 + 2x + \ln(y)\right) \implies$$

$$5y^4 \cdot \frac{dy}{dx} + x^2\left(3y^2 \cdot \frac{dy}{dx}\right) + y^3(2x) = 2 + \frac{1}{y} \cdot \frac{dy}{dx}$$

2. Move the terms containing $\frac{dy}{dx}$ to one side (in this case, the left-hand side) and all remaining terms to the other side (in this case, the right-hand side):

$$5y^4 \cdot \frac{dy}{dx} + x^2\left(3y^2 \cdot \frac{dy}{dx}\right) - \frac{1}{y} \cdot \frac{dy}{dx} = 2 - y^3(2x)$$

$$5y^4 \cdot \frac{dy}{dx} + 3x^2y^2 \cdot \frac{dy}{dx} - \frac{1}{y} \cdot \frac{dy}{dx} = 2 - 2xy^3$$

3. Factor and solve for $\frac{dy}{dx}$:

Factoring $\frac{dy}{dx}$ from the left-hand side of the equation and dividing to solve for $\frac{dy}{dx}$ gives

$$\frac{dy}{dx}\left(5y^4 + 3x^2y^2 - \frac{1}{y}\right) = 2 - 2xy^3$$

$$\frac{dy}{dx} = \frac{2 - 2xy^3}{5y^4 + 3x^2y^2 - \frac{1}{y}}$$

■

In the previous example, we moved all the $\frac{dy}{dx}$ terms to the left-hand side of the equation and all the remaining terms to the right-hand side of the equation in step 2 because there were more $\frac{dy}{dx}$ terms on the left-hand side after completing step 1 than on the right-hand side. We could have moved all the $\frac{dy}{dx}$ terms to the right-hand side of the equation and all the remaining terms to the left-hand side in step 2 instead. Although the answer would have looked slightly different, it would have been mathematically equivalent.

Try It # 2:
Given $x^2y^9 - e^y = 6 - 4x^3$, find $\frac{dy}{dx}$ using implicit differentiation.

■ **Example 3** Given $\sqrt{xy} + 5y = 6 - e^{4y^2}$, find $\frac{dy}{dx}$ using implicit differentiation.

Solution:
We will apply the Implicit Differentiation Method:

1. Take the derivative of both sides of the equation, and remember to use the Chain Rule when taking the derivative of y terms:

Notice before we take the derivative that we need to use the Generalized Power Rule for the term $\sqrt{xy}$, which we can write as $(xy)^{1/2}$, and we need to use the Generalized Exponential (base e) Rule for the term e^{4y^2}. Also, there is a product of terms, xy, in the equation.

Taking the derivative implicitly gives

$$\frac{d}{dx}\left((xy)^{\frac{1}{2}}+5y\right)=\frac{d}{dx}\left(6-e^{4y^2}\right) \implies$$

$$\frac{1}{2}(xy)^{-\frac{1}{2}}\left(x\left(1\cdot\frac{dy}{dx}\right)+y(1)\right)+5\cdot\frac{dy}{dx}=-e^{4y^2}\left(8y\cdot\frac{dy}{dx}\right)$$

Because we will need to move all the $\frac{dy}{dx}$ terms to one side of the equation in the next step, we need to distribute the term $\frac{1}{2}(xy)^{-\frac{1}{2}}$ in order to isolate all the terms containing $\frac{dy}{dx}$. We begin by rewriting $\frac{1}{2}(xy)^{-\frac{1}{2}}$ using laws of exponents and rearranging the terms on the right-hand side of the equation:

$$\frac{1}{2}(xy)^{-\frac{1}{2}}\left(x\left(1\cdot\frac{dy}{dx}\right)+y(1)\right)+5\cdot\frac{dy}{dx}=-e^{4y^2}\left(8y\cdot\frac{dy}{dx}\right)$$

$$\frac{1}{2}\left(x^{-\frac{1}{2}}y^{-\frac{1}{2}}\right)\left(x\cdot\frac{dy}{dx}+y\right)+5\cdot\frac{dy}{dx}=-8ye^{4y^2}\cdot\frac{dy}{dx}$$

$$\frac{1}{2}x^{-\frac{1}{2}}y^{-\frac{1}{2}}\left(x\cdot\frac{dy}{dx}+y\right)+5\cdot\frac{dy}{dx}=-8ye^{4y^2}\cdot\frac{dy}{dx}$$

Now, we will distribute the term $\frac{1}{2}x^{-\frac{1}{2}}y^{-\frac{1}{2}}$ using laws of exponents. Remember when multiplying terms with the same base to add their exponents. When distributing $\frac{1}{2}x^{-\frac{1}{2}}y^{-\frac{1}{2}}$ onto the first term, $x\cdot\frac{dy}{dx}$, we need to add the exponents of the x terms. In other words, $x^{-\frac{1}{2}}\cdot x=x^{-\frac{1}{2}}\cdot x^1=x^{\frac{1}{2}}$. The result is the same when distributing $\frac{1}{2}x^{-\frac{1}{2}}y^{-\frac{1}{2}}$ onto the y term:

$$\frac{1}{2}x^{-\frac{1}{2}}y^{-\frac{1}{2}}\left(x\cdot\frac{dy}{dx}+y\right)+5\cdot\frac{dy}{dx}=-8ye^{4y^2}\cdot\frac{dy}{dx}$$

$$\frac{1}{2}x^{\frac{1}{2}}y^{-\frac{1}{2}}\cdot\frac{dy}{dx}+\frac{1}{2}x^{-\frac{1}{2}}y^{\frac{1}{2}}+5\cdot\frac{dy}{dx}=-8ye^{4y^2}\cdot\frac{dy}{dx}$$

2. Move the terms containing $\frac{dy}{dx}$ to one side (in this case, the left-hand side) and all remaining terms to the other side (in this case, the right-hand side):

$$\frac{1}{2}x^{\frac{1}{2}}y^{-\frac{1}{2}}\cdot\frac{dy}{dx}+5\cdot\frac{dy}{dx}+8ye^{4y^2}\cdot\frac{dy}{dx}=-\frac{1}{2}x^{-\frac{1}{2}}y^{\frac{1}{2}}$$

3. Factor and solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx}\left(\frac{1}{2}x^{\frac{1}{2}}y^{-\frac{1}{2}}+5+8ye^{4y^2}\right)=-\frac{1}{2}x^{-\frac{1}{2}}y^{\frac{1}{2}}$$

$$\frac{dy}{dx}=\frac{-\frac{1}{2}x^{-\frac{1}{2}}y^{\frac{1}{2}}}{\frac{1}{2}x^{\frac{1}{2}}y^{-\frac{1}{2}}+5+8ye^{4y^2}}$$

■

Try It # 3:
Given $7\sqrt{x^2-y^3}+y=8-\ln\left(y^6\right)$, find $\frac{dy}{dx}$ using implicit differentiation.

Example 4 Given $\frac{1+4y^2}{3e^y+\sqrt{x}}=x+2$, find $\frac{dy}{dx}$ using implicit differentiation.

Solution:
We will apply the Implicit Differentiation Method:

1. Take the derivative of both sides of the equation, and remember to use the Chain Rule when taking the derivative of y terms:

Notice that, as written, we would need to use the Quotient Rule when taking the derivative of the left-hand side of the equation. While this is a valid way to proceed, we actually have the option to avoid using it, which will make our calculation easier. Instead, we can multiply both sides of the equation by the denominator $3e^y+\sqrt{x}$. We will demonstrate this technique instead of using the Quotient Rule. Multiplying both sides of the equation by the denominator of the left-hand side gives

$$\begin{aligned}\frac{1+4y^2}{3e^y+\sqrt{x}}&=x+2\\ \left(3e^y+\sqrt{x}\right)\left(\frac{1+4y^2}{3e^y+\sqrt{x}}\right)&=(x+2)\left(3e^y+\sqrt{x}\right)\\ \cancel{\left(3e^y+\sqrt{x}\right)}\left(\frac{1+4y^2}{\cancel{3e^y+\sqrt{x}}}\right)&=(x+2)\left(3e^y+\sqrt{x}\right)\\ 1+4y^2&=(x+2)\left(3e^y+\sqrt{x}\right)\end{aligned}$$

At this point, there are two options: take the derivative of the equation and use the Product Rule to take the derivative of the right-hand side, or continue algebraically manipulating the equation by multiplying the factors on the right-hand side using FOIL. We will demonstrate the latter in order to avoid using the Product Rule. Multiplying the factors gives

$$\begin{aligned}1+4y^2&=(x+2)\left(3e^y+\sqrt{x}\right)\\ 1+4y^2&=x(3e^y)+x\sqrt{x}+2(3e^y)+2\sqrt{x}\\ 1+4y^2&=3xe^y+x^{\frac{3}{2}}+6e^y+2x^{\frac{1}{2}}\end{aligned}$$

Now, taking the derivative of this equation implicitly, and remembering to use the Product Rule for the $3xe^y$ term, gives

$$\frac{d}{dx}\left(1+4y^2\right)=\frac{d}{dx}\left(3xe^y+x^{\frac{3}{2}}+6e^y+2x^{\frac{1}{2}}\right)\implies$$

$$8y\cdot\frac{dy}{dx}=3x\left(e^y\cdot\frac{dy}{dx}\right)+e^y(3)+\frac{3}{2}x^{\frac{1}{2}}+6e^y\cdot\frac{dy}{dx}+x^{-\frac{1}{2}}$$

2. Move the terms containing $\frac{dy}{dx}$ to one side (in this case, the right-hand side) and all remaining terms to the other side (in this case, the left-hand side):

$$-3e^y-\frac{3}{2}x^{\frac{1}{2}}-x^{-\frac{1}{2}}=3xe^y\cdot\frac{dy}{dx}+6e^y\cdot\frac{dy}{dx}-8y\cdot\frac{dy}{dx}$$

3. Factor and solve for $\frac{dy}{dx}$:

$$-3e^y - \frac{3}{2}x^{\frac{1}{2}} - x^{-\frac{1}{2}} = \frac{dy}{dx}(3xe^y + 6e^y - 8y)$$

$$\frac{-3e^y - \frac{3}{2}x^{\frac{1}{2}} - x^{-\frac{1}{2}}}{3xe^y + 6e^y - 8y} = \frac{dy}{dx}, \quad \text{or}$$

$$\frac{dy}{dx} = \frac{-3e^y - \frac{3}{2}x^{\frac{1}{2}} - x^{-\frac{1}{2}}}{3xe^y + 6e^y - 8y}$$

■

Using the Quotient Rule to find the derivative of the curve in the previous example will yield an expression for $\frac{dy}{dx}$ that looks very different from the answer we found. The two different looking expressions will be, in general, not equal! If we try to substitute a randomly selected point (x, y) into each expression, we will usually get different values. However, if we restrict the (x, y) points to be only those described by the original equation, both methods will yield the same value for $\frac{dy}{dx}$.

You should always check to see if the Quotient Rule can be avoided when performing implicit differentiation. Multiplying each side of the equation by the denominator to eliminate the need for the Quotient Rule should make the calculation easier.

Try It # 4:
Given $\frac{5x^3 + 7^y}{8 + \log_2(y)} = 10 - 3x$, find $\frac{dy}{dx}$ using implicit differentiation.

Slopes and Equations of Tangent Lines

We can also use implicit differentiation to find the slope or equation of the line tangent to a curve at a given point.

■ **Example 5** Find the slope of the line tangent to the circle $x^2 + y^2 = 25$ at the point $(3, 4)$ using implicit differentiation.

Solution:
Recall that the slope of the tangent line at a point is given by the value of the derivative at that point. Therefore, we need to find the derivative, $\frac{dy}{dx}$, and evaluate it at the point $(3, 4)$. First, we will use the Implicit Differentiation Method to find $\frac{dy}{dx}$:

1. Take the derivative of both sides of the equation, and remember to use the Chain Rule when taking the derivative of y terms:

$$\frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}(25) \implies$$
$$2x + 2y \cdot \frac{dy}{dx} = 0$$

2. Move the terms containing $\frac{dy}{dx}$ to one side (in this case, the left-hand side) and all remaining terms to the other side (in this case, the right-hand side) :

$$2y \cdot \frac{dy}{dx} = -2x$$

3. Factor and solve for $\frac{dy}{dx}$:

$$\begin{aligned} \frac{dy}{dx} &= \frac{-2x}{2y} \\ &= \frac{-x}{y} \end{aligned}$$

Now, we can find the slope at the point $(3,4)$ by evaluating $\frac{dy}{dx}$ at this point:

$$\begin{aligned} \left.\frac{dy}{dx}\right|_{(3,4)} &= \left.\frac{-x}{y}\right|_{(3,4)} \\ &= \frac{-3}{4} \end{aligned}$$

Thus, the slope of the line tangent to the circle at the point $(3,4)$ is $-\frac{3}{4}$. The graph of the circle $x^2 + y^2 = 25$ is shown in **Figure 2.6.1** along with the graph of the line tangent to the circle at the point $(3,4)$.

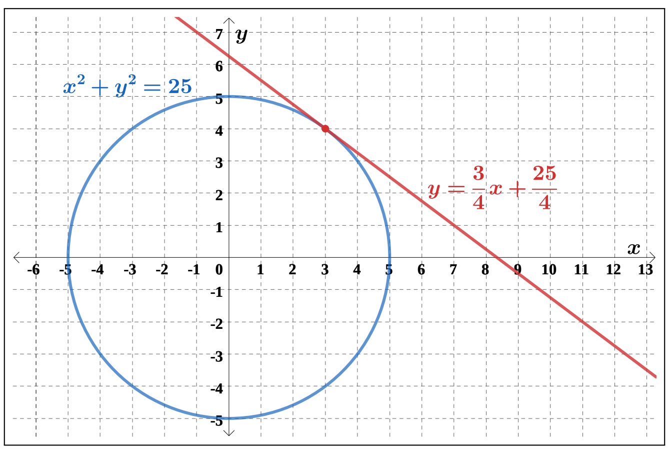


Figure 2.6.1: Graphs of the circle $x^2 + y^2 = 25$ and the line tangent to the curve at the point $(3,4)$

N *Although we were told to use implicit differentiation to find the slope of the tangent line in this example, we could have solved the equation $x^2 + y^2 = 25$ explicitly for y instead. Solving the equation $x^2 + y^2 = 25$ for y gives $y = \pm\sqrt{25 - x^2}$, and we would use the positive root, $y = \sqrt{25 - x^2}$, to find $\frac{dy}{dx}$ because we are looking for the slope at the point $(3,4)$ and $y = \sqrt{25 - x^2}$ represents the curve at this point. This would allow us to find $\frac{dy}{dx}$ without using implicit differentiation. Thus, in some cases, we may have a choice of methods to use to find $\frac{dy}{dx}$. However, many times we cannot explicitly solve for y, and the only way of determining $\frac{dy}{dx}$ is by using implicit differentiation.*

■

Implicit differentiation allows us to find slopes of lines tangent to curves that are clearly not functions (they fail the vertical line test as we can see in the previous example). We are using the idea that portions of y are functions that satisfy the given equation, not that y is actually a function of x.

Example 6 Find the equation of the line tangent to the curve $x^2+2xy+y^2+3x-7y=-2$ at the point $(1,2)$.

Solution:
To find the equation of the line tangent to any curve at a point, we need two pieces of information: the slope of the tangent line and the point. We are given the point, $(1,2)$, so we need to find the slope which is given by $\frac{dy}{dx}$.
We need to use implicit differentiation to find $\frac{dy}{dx}$, so let's apply the Implicit Differentiation Method:

1. Take the derivative of both sides of the equation, and remember to use the Chain Rule when taking the derivative of y terms:

Notice before we take the derivative that there is a product of terms, $2xy=(2x)(y)$, in the equation. Using the Product Rule, as well as the Chain Rule when taking the derivative of y terms, gives

$$\frac{d}{dx}\left(x^2+2xy+y^2+3x-7y\right)=\frac{d}{dx}(-2) \implies$$

$$2x+2x\left(1\cdot\frac{dy}{dx}\right)+y(2)+2y\cdot\frac{dy}{dx}+3-7\cdot\frac{dy}{dx}=0$$

2. Move the terms containing $\frac{dy}{dx}$ to one side (in this case, the left-hand side) and all remaining terms to the other side (in this case, the right-hand side):

$$2x\cdot\frac{dy}{dx}+2y\cdot\frac{dy}{dx}-7\cdot\frac{dy}{dx}=-2x-2y-3$$

3. Factor and solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx}(2x+2y-7)=-2x-2y-3$$
$$\frac{dy}{dx}=\frac{-2x-2y-3}{2x+2y-7}$$

Finding the slope of the tangent line at the point $(1,2)$ by evaluating $\frac{dy}{dx}$ at the point $(1,2)$ gives

$$\left.\frac{dy}{dx}\right|_{(1,2)}=\left.\frac{-2x-2y-3}{2x+2y-7}\right|_{(1,2)}$$

$$=\frac{-2(1)-2(2)-3}{2(1)+2(2)-7}$$

$$=\frac{-9}{-1}=9$$

Thus, the slope of the line tangent to the curve at the point $(1,2)$ is 9.

We can use the point-slope form of the equation of a line, $y - y_1 = m(x - x_1)$, to find the equation of the tangent line:

$$\begin{aligned} y - y_1 &= m(x - x_1) \implies \\ y - 2 &= 9(x - 1) \\ y &= 9x - 9 + 2 \\ &= 9x - 7 \end{aligned}$$

Therefore, the equation of the line tangent to the curve at the point $(1, 2)$ is $y = 9x - 7$. The graph of the curve is shown in **Figure 2.6.2** along with the graph of the line tangent to the curve at the point $(1, 2)$.

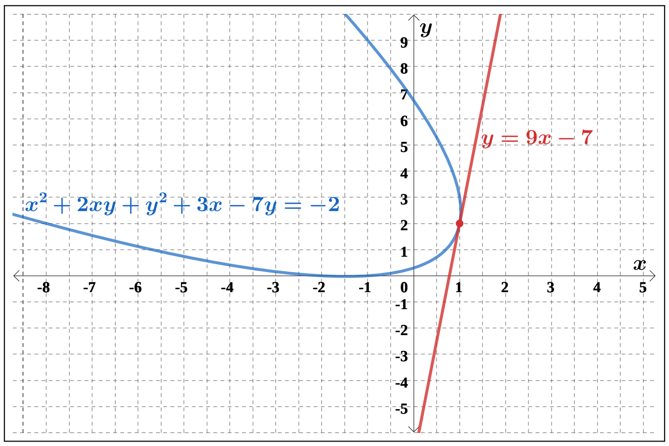


Figure 2.6.2: Graphs of $x^2 + 2xy + y^2 + 3x - 7y = -2$ and the line tangent to the curve at the point $(1, 2)$

▪

Try It # 5:
Find the equation of the line tangent to the curve $y^3 + x^3 - 3xy = 0$ at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$.

Related Rates

In many real-world applications, related quantities are changing with respect to time. For example, consider the circle mentioned in the introduction again. We can say that the rate of change of the area with respect to time, $\dfrac{dA}{dt}$, is related to the rate of change of the radius with respect to time, $\dfrac{dr}{dt}$, because the area, A, is related to the radius, r (because $A = \pi r^2$). In this case, we say that $\dfrac{dA}{dt}$ and $\dfrac{dr}{dt}$ are **related rates**.

We will study several examples of related quantities that are changing with respect to time and learn how to calculate the rate of change of one quantity when we are given the rates of change of the other quantities.

We summarize the strategy for solving related-rates problems below:

Related-Rates Strategy (for solving related-rates problems)

1. Assign variables to all quantities involved in the problem, and draw a picture, if applicable.
2. State, in terms of the variables, the information that is given and the rate to be determined.
3. Find an equation relating the variables introduced in step 1, if necessary.
4. Differentiate both sides of the equation found in step 3 with respect to time using implicit differentiation. Remember to use the Chain Rule when taking the derivative of a term involving a variable other than time, t. This new equation will relate the derivatives.
5. Substitute all known values into the equation found in step 4, and then solve for the unknown rate of change.

Let's apply the above strategy to work an actual example involving, you guessed it, a circle!

▪ **Example 7** Suppose the border of Emtown, a small town in East Texas, is roughly circular, and its radius has been increasing at a rate of 0.1 miles each year. Find how fast the area of the town is increasing when its radius is 5 miles. Round your answer to three decimal places, if necessary.

Solution:
We will apply the Related-Rates Strategy:

1. Assign variables and draw a picture:

Let A represent the area of Emtown (i.e., a circle), in square miles, and r represent its radius, in miles. Also, let t represent time, in years.

We can draw a picture of a circle to help visualize the problem. See **Figure 2.6.3**.

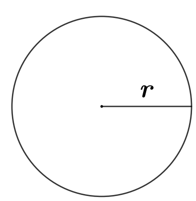


Figure 2.6.3: Border of Emtown is given by a circle with radius r

2. State the information given and the rate to be determined in terms of variables:

We are given that the rate at which the radius is changing with respect to time is 0.1 miles each year. Thus, we know $\frac{dr}{dt} = 0.1$ miles per year. We also know the radius: $r = 5$ miles.

We need to find the rate of change of the area with respect to time: $\frac{dA}{dt}$.

3. Find an equation relating the variables in step 1, if necessary:

To relate the variables A and r, we can use the equation representing the area of a circle:

$$A = \pi r^2$$

4. Differentiate both sides of the equation found in step 3 with respect to time using implicit differentiation:

When we take the derivative of both sides of the equation with respect to time, we must remember that t is the independent variable and use the Chain Rule when taking the derivative of a term involving a variable other than t:

$$\frac{d}{dt}(A) = \frac{d}{dt}\left(\pi r^2\right) \implies$$
$$\frac{dA}{dt} = 2\pi r \cdot \frac{dr}{dt}$$

We now have a new equation relating the derivatives $\frac{dA}{dt}$ and $\frac{dr}{dt}$.

5. Substitute the known values, and solve for the desired rate of change:

Substituting $r = 5$ and $\frac{dr}{dt} = 0.1$ into our new equation in step 4, we get

$$\begin{aligned}\frac{dA}{dt} &= 2\pi r \cdot \frac{dr}{dt} \\ &= 2\pi(5)(0.1) \\ &= \pi \\ &\approx 3.142 \text{ square miles per year}\end{aligned}$$

Thus, when Emtown's radius is 5 miles, the area of the town is increasing at a rate of approximately 3.142 square miles per year.

■

Be careful when solving related-rates problems to not substitute known values too soon. For example, if the value for a changing quantity is substituted into an equation before both sides of the equation are differentiated, then that quantity will behave as a constant and its derivative will not appear in the new equation we found in step 4 (because the derivative of a constant is zero!). That is why it is important to first define variables and use calculus with the variables, and then substitute the known values (i.e., numbers).

Try It # 6:
An expandable sphere is being filled with liquid at a constant rate from a tap (imagine a water balloon connected to a faucet). The radius of the sphere is increasing at a rate of 2 inches per minute when the radius is 3 inches. How fast is the liquid flowing from the tap (i.e., how fast is the volume of the sphere changing) when the radius is 3 inches? Round your answer to three decimal places, if necessary. *Hint: The volume, V, of a sphere is given by $V = \frac{4}{3}\pi r^3$, where r is its radius.*

Similarly, we can also solve related-rates problems involving business applications. For example, although revenue, R, is based on the number of items demanded, x, the number of items demanded is usually a function of time, t. Therefore, if we know the rate at which demand is changing with respect to time, we can determine the rate of change of revenue with respect to time.

We will see this idea in the following examples involving business applications!

■ **Example 8** A company has determined its weekly price-demand equation for selling x million items each week is given by $p^2 = 5200 - x^2$, where p is the price, in dollars, of each item. If weather conditions are driving the price up by \$2 each week, find the rate of change of demand with respect to time when price is \$40 per item, and interpret your answer. Round to the nearest integer, if necessary.

Solution:
Again, we will use the Related-Rates Strategy:

1. Assign variables and draw a picture:

We are told x is the number of items sold, in millions, and p is the price, in dollars, per item. We will let t represent time, in weeks. Drawing a picture is not applicable to business applications in general, so we will continue on to step 2.

2. State the information given and the rate to be determined in terms of variables:

We are given that the rate at which price is changing with respect to time is \$2 each week. Thus, we know $\frac{dp}{dt} = 2$ dollars per week. We also know the price per item: $p = \$40$.

We need to find the rate of change of demand with respect to time: $\frac{dx}{dt}$.

3. Find an equation relating the variables in step 1, if necessary:

We do not need to find an equation relating the variables x and p because we are given the price-demand equation in the problem:

$$p^2 = 5200 - x^2$$

4. Differentiate both sides of the equation found in step 3 with respect to time using implicit differentiation, and remember to use the Chain Rule when taking the derivative of a term involving a variable other than time, t:

$$\frac{d}{dt}\left(p^2\right) = \frac{d}{dt}\left(5200 - x^2\right) \implies$$
$$2p \cdot \frac{dp}{dt} = -2x \cdot \frac{dx}{dt}$$

We now have a new equation relating the derivatives $\frac{dp}{dt}$ and $\frac{dx}{dt}$.

5. Substitute the known values, and solve for the desired rate of change:

Substituting $p = 40$ and $\frac{dp}{dt} = 2$ into our new equation in step 4, we get

$$2p \cdot \frac{dp}{dt} = -2x \cdot \frac{dx}{dt} \implies$$
$$2(40)(2) = -2x \cdot \frac{dx}{dt}$$

Before we can solve for $\frac{dx}{dt}$, we need to find the value of x. Using the given price-demand equation, $p^2 = 5200 - x^2$, and substituting $p = 40$ gives

$$(40)^2 = 5200 - x^2$$
$$1600 = 5200 - x^2$$
$$x^2 = 3600$$
$$\implies x = -60 \text{ and } x = 60$$

Because x represents quantity, in millions, we know $x = 60$ (i.e., we can discard $x = -60$ because it does not make sense to talk about a negative quantity). Now, we substitute this value for x into the equation and solve for $\frac{dx}{dt}$:

$$
\begin{aligned}
2(40)(2) &= -2x \cdot \frac{dx}{dt} \implies \\
2(40)(2) &= -2(60) \cdot \frac{dx}{dt} \\
160 &= -120 \cdot \frac{dx}{dt} \\
\frac{160}{-120} &= \frac{dx}{dt}, \quad \text{or} \\
\frac{dx}{dt} &= -\frac{160}{120} \\
&\approx -1.333 \approx -1 \text{ million items per week}
\end{aligned}
$$

Thus, when price is \$40 per item, demand is decreasing at a rate of approximately 1 million items each week.

▪

▪ **Example 9** A company that manufactures high-end calculators has a weekly revenue equation given by $R = -0.01x^2 + 400x$ dollars, where x is the number of calculators sold each week. If the number of calculators sold is increasing at a rate of 200 calculators per week, find the rate of change of revenue with respect to time when 1000 calculators are sold each week. Interpret your answer.

Solution:
Again, we will use the Related-Rates Strategy:

1. Assign variables and draw a picture:

We are told x is the number of calculators sold each week and R is the company's revenue, in dollars. We will let t represent time, in weeks. Drawing a picture is not applicable to business applications in general, so we will continue on to step 2.

2. State the information given and the rate to be determined in terms of variables:

We are given that the rate of change of the number of calculators sold with respect to time is 200 calculators per week. Thus, we know $\frac{dx}{dt} = 200$ calculators per week. We also know the number of calculators sold each week: $x = 1000$ calculators.

We need to find the rate of change of revenue with respect to time: $\frac{dR}{dt}$.

3. Find an equation relating the variables in step 1, if necessary:

We do not need to find an equation relating the variables x and R because we are given the revenue equation in the problem:

$$R = -0.01x^2 + 400x$$

4. Differentiate both sides of the equation found in step 3 with respect to time using implicit differentiation, and remember to use the Chain Rule when taking the derivative of a term involving a variable other than time, t:

$$
\begin{aligned}
\frac{d}{dt}(R) &= \frac{d}{dt}\left(-0.01x^2 + 400x\right) \implies \\
\frac{dR}{dt} &= -0.02x \cdot \frac{dx}{dt} + 400 \cdot \frac{dx}{dt}
\end{aligned}
$$

We now have a new equation relating the derivatives $\frac{dx}{dt}$ and $\frac{dR}{dt}$.

5. Substitute the known values, and solve for the desired rate of change:

Substituting $x = 1000$ and $\frac{dx}{dt} = 200$ into our new equation in step 4, we get

$$\begin{aligned}\frac{dR}{dt} &= -0.02x \cdot \frac{dx}{dt} + 400 \cdot \frac{dx}{dt} \\ &= -0.02(1000)(200) + 400(200) \\ &= \$76,000 \text{ per week}\end{aligned}$$

Thus, when 1000 calculators are sold each week, revenue is increasing at a rate of \$76,000 per week.

▪

Try It # 7:
The company that sells calculators mentioned in the previous example has a weekly cost equation given by $C = 100x + 10,000$ dollars, where x is the number of calculators produced each week. If the number of calculators produced and sold is continuing to increase at a rate of 200 calculators per week, find the rate of change of profit with respect to time when 6000 calculators are produced and sold each week. Interpret your answer.

We will now return to a related-rates problem involving a geometric application.

▪ **Example 10** A 10-foot ladder is leaning against a wall. If the top of the ladder slides down the wall at a rate of 2 feet per second, how fast is the bottom of the ladder moving along the ground away from the wall when the bottom of the ladder is 5 feet from the wall? Round your answer to three decimal places, if necessary.

Solution:
We will apply the Related-Rates Strategy:

1. Assign variables and draw a picture:

Let x represent the distance from the bottom of the ladder to the wall and y represent the distance from the top of the ladder to the ground, both in feet. Also, let t represent time, in seconds.

We can draw a picture to help visualize the problem. Note that we can also label the length of the ladder, 10 feet (ft). See **Figure 2.6.4**.

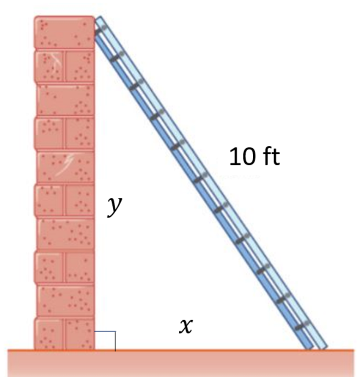


Figure 2.6.4: Ladder leaning against a wall forms a right triangle

2. State the information given and the rate to be determined in terms of variables:

We are given that the rate at which the top of the ladder is sliding down the wall is 2 feet per second. Thus, we know $\frac{dy}{dt} = -2$ feet per second. Note that this rate of change is *negative* because the distance from the top of the ladder to the ground is *decreasing*. We also know the bottom of the ladder is 5 feet from the wall: $x = 5$ feet.

We need to find how fast the bottom of the ladder is moving along the ground away from the wall. In other words, we need to find the rate of change of the distance x with respect to time t: $\frac{dx}{dt}$.

3. Find an equation relating the variables in step 1, if necessary:

To relate the variables x and y, we can use the Pythagorean Theorem because the ladder leaning against the wall forms a right triangle:

$$x^2 + y^2 = 10^2$$

4. Differentiate both sides of the equation found in step 3 with respect to time using implicit differentiation, and remember to use the Chain Rule when taking the derivative of a term involving a variable other than time, t:

$$\begin{aligned} \frac{d}{dt}\left(x^2 + y^2\right) &= \frac{d}{dt}\left(10^2\right) \implies \\ 2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} &= 0 \end{aligned}$$

Dividing by 2 on both sides gives

$$x \cdot \frac{dx}{dt} + y \cdot \frac{dy}{dt} = 0$$

We now have a new equation relating the derivatives $\frac{dx}{dt}$ and $\frac{dy}{dt}$.

5. Substitute the known values, and solve for the desired rate of change:

Substituting $x = 5$ and $\frac{dy}{dt} = -2$ into our new equation in step 4, we get

$$\begin{aligned} x \cdot \frac{dx}{dt} + y \cdot \frac{dy}{dt} &= 0 \implies \\ 5 \cdot \frac{dx}{dt} + y(-2) &= 0 \end{aligned}$$

Before we can solve for $\frac{dx}{dt}$, we need to find the value of y. Using the equation we found in step 3, $x^2 + y^2 = 10^2$, and substituting $x = 5$ gives

$$\begin{aligned} 5^2 + y^2 &= 10^2 \\ 25 + y^2 &= 100 \\ y^2 &= 75 \\ \implies y = -\sqrt{75} &\text{ and } y = \sqrt{75} \end{aligned}$$

Because y represents distance, in feet, $y = \sqrt{75}$. Now, we substitute this value into the equation and solve for $\frac{dx}{dt}$:

$$\begin{aligned} 5 \cdot \frac{dx}{dt} + y(-2) = 0 &\implies \\ 5 \cdot \frac{dx}{dt} + \sqrt{75}(-2) &= 0 \\ 5 \cdot \frac{dx}{dt} &= 2\sqrt{75} \\ \frac{dx}{dt} &= \frac{2\sqrt{75}}{5} \\ &\approx 3.464 \text{ feet per second} \end{aligned}$$

Thus, when the bottom of the ladder is 5 feet from the wall, it is moving away from the wall at a rate of approximately 3.464 feet per second. Note that this rate should be *positive* because the distance from the wall to the bottom of the ladder is *increasing* as the bottom of the ladder slides away from the wall.

▪

Try It # 8:
A 25-foot ladder is leaning against a wall. If the bottom of the ladder is pushed toward the wall at a rate of 1 foot per second and the bottom of the ladder is initially 20 feet away from the wall, how fast does the top of the ladder move up the wall after 5 seconds of pushing? Round your answer to two decimal places, if necessary.

Try It Answers

1. $\frac{dy}{dx} = \frac{-6x}{\frac{1}{4}y^{-3/4}}$

2. $\frac{dy}{dx} = \frac{-12x^2 - 2xy^9}{9x^2y^8 - e^y}$

3. $\frac{dy}{dx} = \frac{-7x\left(x^2 - y^3\right)^{-1/2}}{-\frac{21}{2}y^2\left(x^2 - y^3\right)^{-1/2} + 1 + \frac{6}{y}}$

4. $\frac{dy}{dx} = \frac{15x^2 + 24 + 3\log_2(y)}{\frac{10}{y\ln(2)} - \frac{3x}{y\ln(2)} - 7^y\ln(7)}$

5. $y = -x + 3$

6. 226.195 cubic inches per minute

7. \$36,000 per week; When 6000 calculators are produced and sold each week, profit is increasing at a rate of \$36,000 per week.

8. 0.75 feet per second

EXERCISES

Basic Skills Practice

For Exercises 1 - 5, use implicit differentiation to find $\frac{dy}{dx}$.

1. $x^4 + 3y^2 = 37$

2. $3\sqrt{x} - 4 = e^y + \frac{5}{x}$

3. $\sqrt{7} + 2\ln(y) = \frac{1}{4}x^{-8} - y$

4. $3x^6 + 2(4^y) - 10x = \sqrt[3]{y^2}$

5. $0.1y^{0.4} - \ln(x) = 7\log_2(y) + \pi$

For Exercises 6 - 8, use implicit differentiation to find the slope of the line tangent to the graph of the equation at the given point.

6. $\ln(y) + 9 = \frac{1}{3}x^3$ at $(3, 1)$

7. $-6e^y + \frac{1}{4}y^3 - 2\ln(x) = -8$ at $(e, 0)$

8. $19 - 3x^2 + 9^x = 4\sqrt{y}$ at $(0, 25)$

For Exercises 9 - 11, use implicit differentiation to find the equation of the line tangent to the graph of the equation at the given point.

9. $7y^2 - e^x - 2y = 103$ at $(0, 4)$

10. $0.4\sqrt[3]{y} + 3.6 = \ln(y) + 2x^2$ at $(2, 1)$

11. $-3x + 5^y = -4y^2 - 17$ at $(6, 0)$

For Exercises 12 - 14, suppose x and y are functions of t and are related by the given equation. Use the information to find $\frac{dy}{dt}$.

12. $y = 4x^2 - 5x + 3$, $\frac{dx}{dt} = 2$, and $x = -6$

13. $y = \dfrac{x^3 - 5x^2 + 6^x}{8x - 10}$, $\dfrac{dx}{dt} = -4$, and $x = 0$

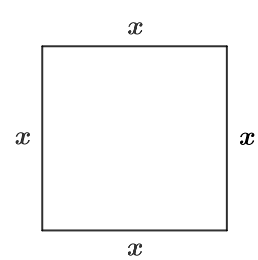
14. $y = (9\ln(x))\sqrt{x^2 + 2x + 1}$, $\dfrac{dx}{dt} = 3$, and $x = 1$

15. No Spots Left sells dishwashers and has a weekly cost equation given by $C = 3500 + 14x + 0.2x^2$, where C is the cost, in dollars, when x dishwashers are sold each week. If the number of dishwashers the company sells is increasing at a rate of 65 dishwashers per week, find the rate at which the company's cost is changing with respect to time when 250 dishwashers are sold each week. In other words, find $\dfrac{dC}{dt}$ if we know $\dfrac{dx}{dt} = 65$ dishwashers per week and $x = 250$ dishwashers.

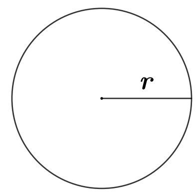
16. The monthly profit equation for a company that sells designer handbags is given by $P = -0.2x^2 + 450x - 10{,}000$, where P is the profit, in dollars, when x handbags are sold each month. If the number of handbags the company sells is increasing at a rate of 135 handbags per month, find the rate at which the company's profit is changing with respect to time when 800 handbags are sold each month. In other words, find $\dfrac{dP}{dt}$ if we know $\dfrac{dx}{dt} = 135$ handbags per month and $x = 800$ handbags.

17. Glow Kicks, a company that sells glow in the dark sneakers, has a weekly revenue equation given by $R = 207x - 0.2x^2$, where R is the revenue, in dollars, from selling x pairs of sneakers each week. If the number of pairs of sneakers the company sells is increasing at a rate of 25 pairs of sneakers per week, find the rate at which the company's revenue is changing with respect to time when 160 pairs of sneakers are sold each week. In other words, find $\dfrac{dR}{dt}$ if we know $\dfrac{dx}{dt} = 25$ pairs of sneakers per week and $x = 160$ pairs of sneakers.

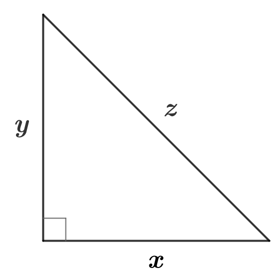
18. The area, in square feet, of the square shown below is given by $A = x^2$, where x is the length, in feet, of each side. If the length of each side of the square is increasing at a rate of 2 feet per second, find the rate at which the area of the square is increasing with respect to time when the sides are 7 feet long. In other words, find $\dfrac{dA}{dt}$ if we know $\dfrac{dx}{dt} = 2$ feet per second and $x = 7$ feet.



19. The area, in square miles, of the circle shown below is given by $A = \pi r^2$, where r is the length, in miles, of the radius. If the radius is increasing at a rate of 0.15 mile per day, find the rate at which the area of the circle is increasing with respect to time when the radius is 3 miles. In other words, find $\frac{dA}{dt}$ if we know $\frac{dr}{dt} = 0.15$ mile per day and $r = 3$ miles.



20. The right triangle shown below has sides of length x and y feet, and it has a hypotenuse of length z feet. If both the sides of length x and y feet are increasing at a rate of 1.5 feet per second, find the rate at which the hypotenuse is increasing with respect to time when the side of length x is 3 feet and the side of length y is 4 feet. In other words, find $\frac{dz}{dt}$ if we know $\frac{dx}{dt} = 1.5$ feet per second, $\frac{dy}{dt} = 1.5$ feet per second, $x = 3$ feet, and $y = 4$ feet. *Hint: By the Pythagorean Theorem, we know* $x^2 + y^2 = z^2$.



Intermediate Skills Practice

For Exercises 21 - 30, use implicit differentiation to find $\frac{dy}{dx}$.

21. $x^2 + xy^3 - 3y^2 = 45$

22. $\frac{1}{4}\ln(y) - xe^y + \pi = \frac{8}{y}$

23. $3yx^{-7} + \log(y) - 5^x\sqrt{y} = 6\ln(x)$

24. $-\frac{1}{3}x^6 + \frac{2x}{y^3} + \sqrt[4]{7}x = 72$

25. $\log_2(x) + \frac{e^y}{\sqrt{x}} = 4e^x$

26. $\dfrac{4x^2}{\ln(y)} - x^3 - 4 = 2y$

27. $e^{2y^3} - 5x^2 + \sqrt{12} = -9\sqrt{y}$

28. $\left(7y^2 - 31\right)^3 - 2(6^x) - 15y^{-3} = 105$

29. $3x^4 - \ln\left(y^2 + 36\right) = \dfrac{4}{y^2} - \ln(x)$

30. $y - \dfrac{1}{10}x^{-2/5} = 2^{\sqrt{y}} - 83$

For Exercises 31 - 39, use implicit differentiation to find $\dfrac{dy}{dx}$.

31. $\left(3y^5 - y\right)\left(0.5x^2 - 6x\right) = 0$

32. $\left(9\sqrt{x} + 2x\right)\left(3y^2 + 14y\right) = 120$

33. $\left(4y^3 - 2y^2 + y\right)\left(x^5 - 3x\right) = 7x^5$

34. $\dfrac{-5y^4}{2x^2 - \frac{1}{3}x^3} = 30$

35. $\dfrac{8x^2}{3\sqrt{y} - 7y^{0.2}} = 6x$

36. $\dfrac{2y}{10x^{-1} + x^{-0.8}} = 5x^2 + 1$

37. $e^{4x^2 + \sqrt{y}} = 13$

38. $\left(2y^3 - x^4\right)^8 = 12x$

39. $\log_2\left(3x^4 - 5y\right) = -48$

For Exercises 40 - 42, find the slope of the line tangent to the graph of the equation at the given point.

40. $\left(x - 3x^2\right)\left(y^4 - 5y\right) - 56 = 0$ at $(-2, 1)$

41. $-\dfrac{5}{21} - \dfrac{2}{3}x^3 = -\dfrac{3x^2}{7^y}$ at $(1, 1)$

42. $\sqrt{x^2+y^2} = -\frac{5}{3}x$ at $(-3,4)$

For Exercises 43 - 45, find the equation of the line tangent to the graph of the equation at the given point.

43. $\sqrt[3]{x}y^4 - e^y + 5x^{2/3} = 19$ at $(8,0)$

44. $\frac{4x}{2y^3-5y^2} = -4$ at $(7,-1)$

45. $\ln(10-y^2)+16 = -2x^3$ at $(-2,3)$

For Exercises 46 - 48, suppose x and y are functions of t and are related by the given equation. Use the information to find $\frac{dy}{dt}$.

46. $x^3y^4 - 2xy = 72$, $\frac{dx}{dt} = 2$, $x = 4$, and $y = -1$

47. $\ln\left(3x^2-5y\right)+16 = x^4$, $\frac{dx}{dt} = 3$, $x = 2$, and $y = \frac{11}{5}$

48. $\frac{\log(y)}{2x^3} = \frac{1}{2}$, $\frac{dx}{dt} = -2$, $x = 1$, and $y = 10$

For Exercises 49 - 51, suppose x and y are functions of t and are related by the given equation. Use the information to find $\frac{dx}{dt}$.

49. $\frac{4\sqrt{y}}{x^2+3x} = \frac{y^2}{8}$, $\frac{dy}{dt} = -1$, $x = 1$, and $y = 4$

50. $e^{3x^2-12y} + 4x^3 = -31$, $\frac{dy}{dt} = 2$, $x = -2$, and $y = 1$

51. $\left(3y^2-2y\right)\left(x^4+6\right) = 7-4y$, $\frac{dy}{dt} = -3$, $x = -1$, and $y = 1$

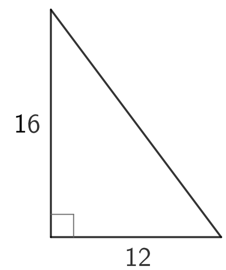
52. Bright Lights, a company that sells night lights, has a monthly revenue equation given by $R = 12x - 0.001x^2$, where R is the revenue, in dollars, from selling x night lights each month. If the number of night lights the company sells is increasing at a rate of 475 night lights per month, find the rate at which the company's revenue is changing with respect to time when 5000 night lights are sold each month. Interpret your answer.

53. The weekly profit equation for a life vest manufacturer is given by $P = 30x - 0.3x^2 - 225$, where P is the profit, in dollars, when x life vests are sold each week. If the number of life vests the company sells is increasing at a rate of 25 vests per week, find the rate at which the company's profit is changing with respect to time when 180 life vests are sold each week. Interpret your answer.

54. The weekly cost equation for a company that produces freezers is given by $C = 3400 + 12x + 0.2x^2$, where C is the cost, in dollars, when x freezers are produced each week. If the number of freezers the company produces is increasing at a rate of 24 freezers per week, find the rate at which the company's cost is changing with respect to time when 315 freezers are produced each week. Interpret your answer.

55. Sleepy Dogs specializes in selling orthopedic beds for large dogs. It has determined its weekly price-demand equation to be $p = 215 - 0.8x$, where x is the number of dog beds sold each week at a price of p dollars each. If the number of beds the company sells is increasing at a rate of 15 beds per week, find the rate of change of price with respect to time when 100 dog beds are sold each week. Interpret your answer.

56. The 12-inch base of a right triangle is growing at a rate of 3 inches per hour, and its 16 inch height is shrinking at a rate of 3 inches per hour.



(a) How fast is the area of the triangle changing?

(b) How fast is the perimeter of the triangle changing?

(c) How fast is the length of the triangle's hypotenuse changing?

57. The base of a triangle is shrinking at a rate of 1 cm/min, and the height of the triangle is increasing at a rate of 5 cm/min. Find the rate at which the area of the triangle changes when its height is 22 cm and its base is 10 cm.

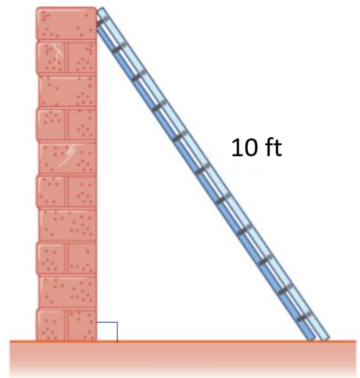
58. The length of a 12-foot by 8-foot rectangle is increasing at a rate of 3 feet per second, and its width is decreasing at a rate of 2 feet per second.

(a) How fast is the perimeter of the rectangle changing?

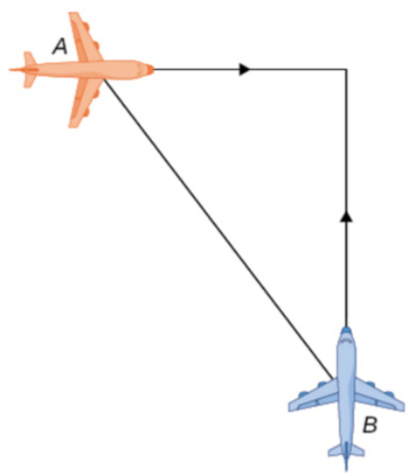
(b) How fast is the area of the rectangle changing?

59. A rock is thrown into a pond causing a circular ripple. If the radius of the ripple is increasing at a rate of 2.5 feet per second, at what rate is its area changing when the radius is 8 feet?

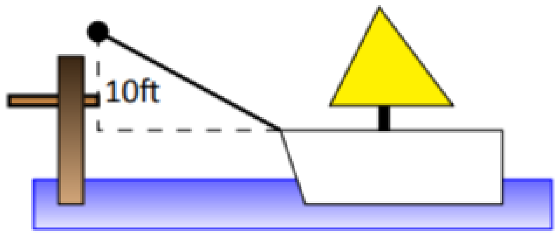
60. A 10-ft ladder is leaning against a wall. If the top of the ladder slides down the wall at a rate of 1.5 ft/sec, how fast is the bottom of the ladder moving along the ground away from the wall when the bottom of the ladder is 6 ft from the wall?



61. Two airplanes are flying in the air at the same height. Airplane A is flying east at 250 mi/h, and airplane B is flying north at 300 mi/h. They are both headed to the same airport which is located 30 miles east of airplane A and 40 miles north of airplane B. At what rate is the distance between the airplanes changing?



62. A boat is being pulled into a dock at a constant rate of 30 ft/min by a winch located 10 ft above the deck of the boat. At what rate is the boat approaching the dock when the boat is



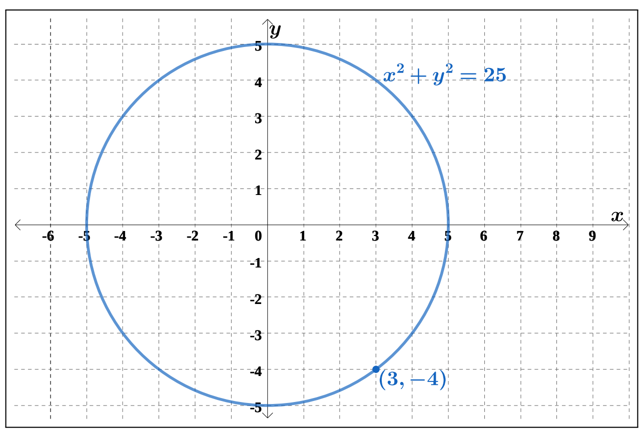
(a) 50 ft away from the dock? Round your answer to three decimal places, if necessary.

(b) 15 ft away from the dock? Round your answer to three decimal places, if necessary.

(c) 1 ft from the dock? Round your answer to three decimal places, if necessary.

63. Two ships leave the same port at the same time. After one hour, ship A is 3 miles north of port traveling north at 6 miles per hour. Ship B is 4 miles east of port traveling back to port at 7 miles per hour. At what rate is the distance between the two ships decreasing?

64. A point is moving along the graph of $x^2 + y^2 = 25$. When the point is at $(3, -4)$, its x-coordinate is decreasing at a rate of 1.2 units per second. How fast is its y-coordinate changing at that time?



Mastery Practice

For Exercises 65 - 73, find $\frac{dy}{dx}$.

65. $\dfrac{y^2 - \frac{6}{11}y^3}{-0.1x^{0.2} + \log_5(x)} = 47$

66. $\left(\sqrt{x} + 3x\right)\left(4\ln(y) - 3y^2 + y\right) = 2y^3 + e^2$

67. $\dfrac{2x^2+4x^{5/4}}{10^y-e^y} = \sqrt{x}+2$

68. $\log_2\left(x^2+y^2\right)+x^3y^4+\sqrt[3]{y^5}=107$

69. $e^{4y^2-\pi}\left(3\ln(x)-y^5\right)=-7y^2$

70. $\dfrac{5\ln(y)}{2y^{-1/9}+7^x}=8\sqrt[4]{x^3}$

71. $\log_6\left(3y^5\right)-e^{2xy}=7$

72. $(4^y-2x)\left(\ln(y)-x^5\right)=3x-\dfrac{8}{y^2}$

73. $\dfrac{\ln(x)}{x^2-\sqrt{y}}=y$

For Exercises 74 - 76, find the slope of the line tangent to the graph of the equation at the given point.

74. $\dfrac{12e^y-(y+1)^{-2/5}}{7+4(e^y)}-1=\sqrt{x}$ at $(1,0)$

75. $\left(2x^3+3y^4\right)\left(\dfrac{1}{5}y^3+0.5x^2+1\right)=-\dfrac{182}{5}$ at $(-2,-1)$

76. $\log_5\left(x^2-\dfrac{y}{4}\right)+2y^2-287=\dfrac{x^3}{8}$ at $(2,12)$

For Exercises 77 - 79, find the equation of the line tangent to the graph of the equation at the given point.

77. $e^{y-1}\sqrt{2y^3-x}=1$ at $(1,1)$

78. $\dfrac{10y}{2y^2-3x}=0.2y$ at $(0,5)$

79. $\ln(xy)=3y^2-\dfrac{3}{4}$ at $\left(2,\frac{1}{2}\right)$

80. Suppose $\dfrac{3\sqrt{y}}{x^2-e^{3x}}=-6$ and x and y are functions of t. Find $\dfrac{dy}{dt}$ if $\dfrac{dx}{dt}=7$, $x=0$, and $y=4$.

81. Suppose $e^{x/y}=e^2$ and x and y are functions of t. Find $\dfrac{dx}{dt}$ if $\dfrac{dy}{dt}=-2$, $x=2$, and $y=1$.

82. A company that sells toy microphones has determined its monthly price-demand equation to be $p = 24 - \sqrt{x}$, where x is the number of microphones sold each month at a price of p dollars each. If the number of microphones the company sells is increasing at a rate of 128 microphones per month, find the rate of change of the price of each microphone with respect to time when 2500 microphones are sold each month. Interpret your answer.

83. An office lamp manufacturer has weekly revenue and cost equations, both in dollars, given by $R = 38x - 0.1x^2$ and $C = 5x + 2000$, respectively, where x is the number of lamps made and sold each week. If the number of lamps made and sold is increasing at a rate of 30 lamps per week, find the rate of change of the company's profit with respect to time when 200 lamps are made and sold each week. Interpret your answer.

84. A jeweler that sells custom necklaces has a monthly price-demand equation given by $p = 350(0.999)^x$, where x is the number of necklaces sold each month at a price of p dollars each. If the number of necklaces the jeweler sells is increasing at a rate of 185 necklaces per month, find the rate at which the company's revenue is changing with respect to time when 2500 necklaces are sold each month. Interpret your answer.

85. The weekly cost equation, in dollars, of a company that manufactures toasters is given by $C = \dfrac{1200x + 42 + x^3}{6x}$, where x is the number of toasters manufactured each week and $x > 0$. If the number of toasters the company manufactures is increasing at a rate of 45 toasters per week, find the rate at which the company's cost is changing with respect to time when 375 toasters are manufactured each week. Interpret your answer.

86. A company has a weekly price-demand equation given by $p = \dfrac{900}{\sqrt{x}}$, where x is the number of items sold each week at a price of p dollars per item and $x > 0$. The company also has a weekly cost equation given by $C = 80\sqrt{x} + 10x + 7{,}000$ dollars, where x is the number of items produced each week. If the number of items produced and sold is increasing at a rate of 95 items per week, find the rate at which the company's profit is increasing with respect to time when 450 items are produced and sold each week. Interpret your answer.

87. A manufacturer determines the relationship between the demand and price for an item is given by $2x^2 + 6xp + 30p^2 = 125{,}000$, where x is the number of items demanded each month at a price of p dollars each. If the price per item is increasing at a rate of \$3.50 per month, find the rate of change of the demand with respect to time when the price is \$50 per item. Interpret your answer. Round to the nearest integer, if necessary.

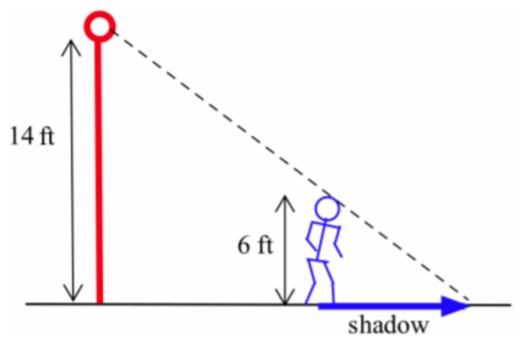
88. A manufacturer determines the relationship between the demand and price for an item is given by $3x^2 + 3xp + 40p^2 = 160{,}000$, where x is the number of items demanded each month at a price of p dollars each. If the number of items demanded is decreasing at a rate of 10 items per month, find the rate of change of the price with respect to time when demand is 200 items each month. Interpret your answer.

89. The cost, in dollars, to produce x boxes of mint cookies each week is given by $C = 0.0001x^3 - 0.02x^2 + 3x + 300$. In t weeks, production is estimated to be $x = 1600 + 100t$ boxes of cookies. How fast is cost increasing with respect to time in two weeks? Interpret your answer.

90. A rainbow trout has taken the fly at the end of a 60-foot fishing line, and the line is being reeled in at a rate of 30 feet per minute. If the tip of the rod is 10 feet above the water and the trout is at the surface of the water, how fast is the trout being pulled toward the angler? Round your answer to three decimal places, if necessary.

91. A 20-foot ladder is leaning against a wall. The bottom of the ladder slides away from the wall at a rate of 3 feet per second. How fast is the top of the ladder sliding down the wall when the bottom of the ladder is 15 feet from the wall? Round your answer to three decimal places, if necessary.

92. A 6-foot-tall person is walking away from a 14-foot lamp post at a rate of 3 feet per second. When the person is 10 feet away from the lamp post, how fast is the length of the person's shadow changing?



93. A helicopter sitting 10 feet off the ground rises directly into the air at a constant rate of 25 feet per second. You are running on the ground starting directly under the helicopter at a constant rate of 10 feet per second. Find the rate of change of the distance between the helicopter and yourself after 5 seconds. Round your answer to three decimal places, if necessary.

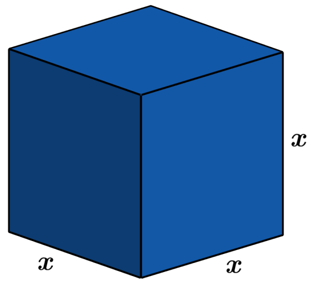
94. A triangle has a height that is increasing at a rate of 2 cm/s, and its area is increasing at a rate of 4 cm^2/s. Find the rate at which the length of the base of the triangle is changing when the height of the triangle is 4 cm and its area is 20 cm^2.

95. Two cars start moving from the same point. One car travels south at a constant rate of 40 miles per hour, and the other car travels west at a constant rate of 25 miles per hour. After three hours, how fast is the distance between the cars increasing? Round your answer to three decimal places, if necessary.

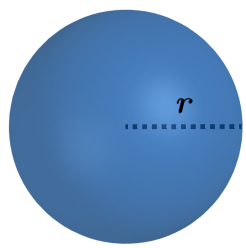
96. A 5-ft-tall person walks toward a wall at a rate of 2 ft/s. A spotlight is located on the ground behind the person 40 ft from the wall. How fast does the height of the person's shadow on the wall change when the person is 10 ft from the wall?

97. Using the previous exercise, what is the rate at which the height of the shadow changes when the person is 10 ft from the wall if the person is walking away from the wall at a rate of 2 ft/s?

98. The volume of a cube is given by $V = x^3$, where x is the length of each side. If the length of each side of a cube increases at a rate of $\frac{1}{2}$ m/s, find the rate at which the volume of the cube increases when each side of the cube is 4 m long.



99. The volume of a sphere is given by $V = \dfrac{4}{3}\pi r^3$, where r is its radius. Determine how fast the volume of a spherical balloon is increasing if its radius is increasing at a rate of 2.5 centimeters per minute when the radius is 8 centimeters.



100. The volume of a cube decreases at a rate of 10 m^3/s. Find the rate at which the length of each side of the cube changes when each side of the cube is 2 m long.

101. The radius of a sphere increases at a rate of 1.5 meters per second. How fast is the volume of the sphere increasing when its radius is 20 meters?

102. A spherical balloon is inflated with air flowing at a rate of 10 cm^3/s. How fast is the radius of the balloon increasing when the radius is

   (a) 1 cm? Round your answer to five decimal places, if necessary.

   (b) 10 cm? Round your answer to five decimal places, if necessary.

   (c) 100 cm? Round your answer to five decimal places, if necessary.

103. The radius of a sphere is increasing at a rate of 9 cm/s. Find the radius of the sphere when the volume and the radius of the sphere are increasing at the same numerical rate. Round your answer to three decimal places, if necessary.

Communication Practice

104. Explain, in general terms, when it is necessary to use implicit differentiation to find $\frac{dy}{dx}$.

105. If $y+3x^4-e^x=7-\ln(x)$, do we need to use implicit differentiation to find $\frac{dy}{dx}$? Explain.

106. When using implicit differentiation to find $\frac{dy}{dx}$, what major assumption are we making about y (with regard to its relationship with x)?

107. Briefly describe the method for finding $\frac{dy}{dx}$ implicitly.

108. Explain, in general terms, what a related-rates problem is.

109. Briefly describe the steps for solving a related-rates problem.

Chapter 3

3. Curve Sketching & Optimization

In theory and applications, we often want to maximize or minimize some quantity. A manufacturer may want to maximize profit or minimize cost. An engineer may want to maximize the speed of a new computer or minimize the amount of heat produced by an appliance. A student may want to maximize their grade in calculus or minimize the number of study hours needed to earn a particular grade.

Without calculus, we only know how to find optimal points in a few specific cases. For example, we know how to find the vertex of a parabola, but what if we need to maximize or minimize an unfamiliar function?

The best way we have to find an optimal solution without using calculus is to examine the graph of the function, perhaps using technology. However, our view of the graph will depend on the viewing window we choose, and we might miss something important! In addition, we will probably only get an approximation by examining the graph of the function and not an exact answer.

Calculus provides ways of drastically narrowing the number of points we need to examine to find the exact locations of maxima and minima, while at the same time ensuring that we have not missed anything important.

In this chapter, we will explore how we can use calculus (specifically, use first and second derivatives) to investigate the behavior of functions, and we will apply this knowledge to locate the extreme values and inflection points of functions. In addition, we will use the information we obtain from analyzing a function's first and second derivatives to create a rough sketch of the graph of the function (without using technology!). Finally, we will apply our knowledge about first and second derivatives to find optimal solutions to real-world problems using a process called **optimization**.

3.1 Analyzing Graphs with the First Derivative

Recall from **Section 2.2** that a function is **increasing** where its derivative is positive, and it is **decreasing** where its derivative is negative. The derivative can also provide other useful information about a function. For instance, looking at the graph of the function f shown in **Figure 3.1.1** below, we see that the largest y-value of $f(x)$ occurs at $x = 3$. There is also a horizontal tangent line at $x = 3$, which means $f'(3) = 0$. This is not a coincidence!

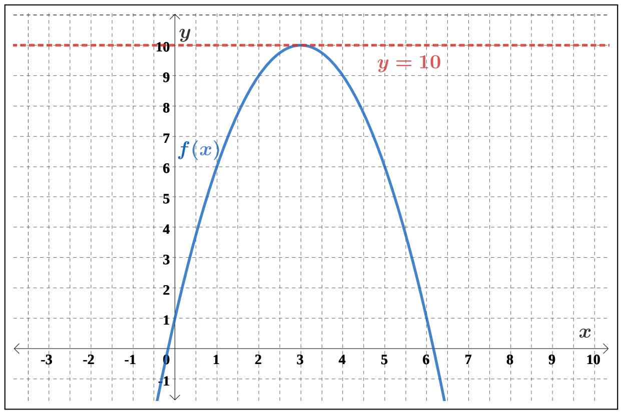


Figure 3.1.1: Graph of a function f in which $f'(3) = 0$ and the largest y-value of $f(x)$ occurs at $x = 3$

There is a close relation between the x-values where the derivative equals zero and minimums and maximums of a function. Be warned, however, that it's not as simple as you may think. Consider the graph of $g(x) = x^3 - 9x^2 + 27x - 22$ shown in **Figure 3.1.2**:

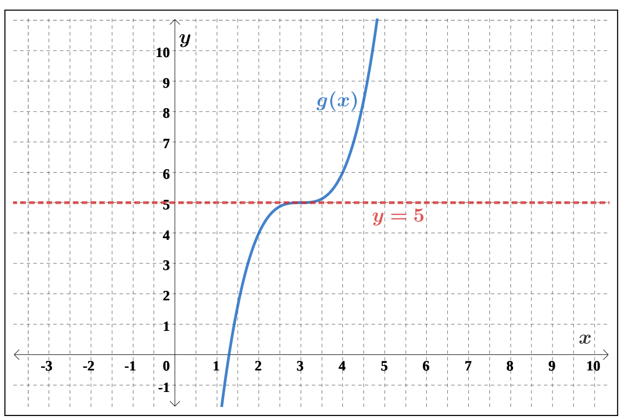


Figure 3.1.2: Graph of a function g in which $g'(3) = 0$, but there is no largest (or smallest) y-value of $g(x)$ occurring at $x = 3$

Notice the function g shown in **Figure 3.1.2** has a horizontal tangent line at $x = 3$, but the point $(3, 5)$ *is not a* ***local extremum***! In other words, the point $(3, 5)$ is not at the bottom of a valley or at the top of a hill. In this section, we will explore the relationship between the derivative and **local extrema** of a function as well as learn a test for finding local extrema.

Learning Objectives:

In this section, you will learn how to apply the Increasing/Decreasing Test to find the intervals where a function is increasing/decreasing, apply the First Derivative Test to find local extrema, and solve problems involving real-world applications. Upon completion, you will be able to:

- Define partition number of f'.
- Determine the intervals where a function is increasing/decreasing using the Increasing/Decreasing Test.

- Define critical value of f.
- Find the critical values of a function f given its rule.
- Specify the conditions necessary for a function to have a local extremum.
- Prove that a critical value of a function may not coincide with a local extremum by constructing a graph of a function.
- Distinguish between critical values of f and partition numbers of f'.
- Apply the First Derivative Test to find the local extrema of a function f given its rule.
- Apply the First Derivative Test to find the local extrema of a function f given the rule for its derivative f' and the domain of f.
- Analyze the graph of f to determine the partition numbers of f', the critical values of f, the intervals where f is increasing/decreasing, and the local extrema of f.
- Analyze the graph of f' to determine the partition numbers of f', the critical values of f, the intervals where f is increasing/decreasing, and the x-values where f has local extrema.
- Demonstrate knowledge of the First Derivative Test by applying it to real-world applications including cost, revenue, and profit.

Increasing/Decreasing Intervals

We will begin our journey by recalling that if the graph of a function has a positive slope on an interval, then the function is increasing on the interval. Likewise, if the graph of a function has a negative slope on an interval, then the function is decreasing on the interval:

Theorem 3.1 Increasing/Decreasing Test

Suppose the function f is differentiable on an interval.

- If $f'(x) > 0$ on the interval, then ***f* is increasing** on the interval.
- If $f'(x) < 0$ on the interval, then ***f* is decreasing** on the interval.

To find on what intervals a function is increasing and decreasing, we will create a sign chart of $f'(x)$. The sign chart will be *partitioned* into intervals by "important" x-values. We will then test x-values within each interval to determine whether the derivative of the function is positive or negative on the interval. This, in turn, will tell us if the function is increasing or decreasing on that interval.

The big question is, what are the "important" x-values we need to put on our sign chart, and how do we calculate them?

The "important" x-values are those where the graph of f might possibly switch from increasing to decreasing or decreasing to increasing, or, restating the problem, the x-values where $f'(x)$ changes sign. It turns out that if $f'(x)$ changes sign at a particular x-value, then either $f'(x) = 0$ or if $f'(x)$ does not exist at that x-value. We have seen an example where $f'(x) = 0$ and the function f switches from increasing to decreasing previously in **Figure 3.1.1** (at $x = 3$). The graph of f shown in **Figure 3.1.3** demonstrates the other possibility:

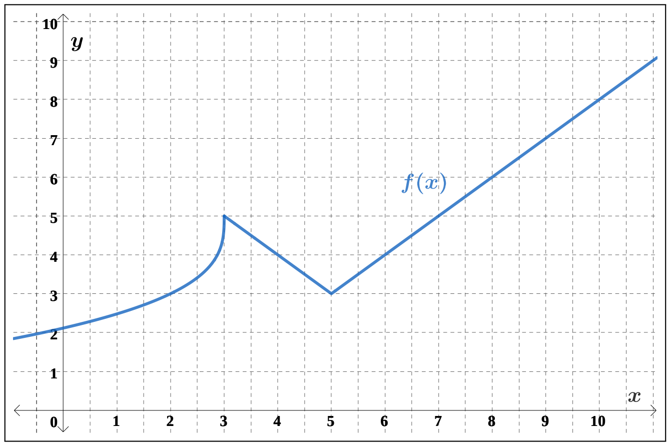


Figure 3.1.3: Graph of a function f in which $f'(3)$ and $f'(5)$ do not exist and f switches between increasing/decreasing at both $x = 3$ and $x = 5$

At $x = 3$, f has a cusp and switches from increasing to decreasing, and at $x = 5$, f has a corner with the opposite switch.

It is also possible $f'(x)$ does not exist at an x-value because f is undefined at that x-value. However, it is still possible for f to switch from increasing to decreasing, or the other way around. For example, consider the graph of f shown in **Figure 3.1.4**. The function is increasing on the interval $(-\infty, 4)$ and decreasing on the interval $(4, \infty)$:

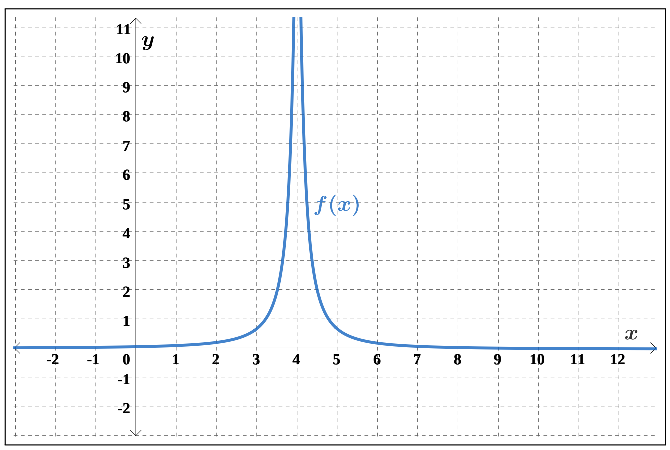


Figure 3.1.4: Graph of a function f in which $f'(4)$ does not exist, but f switches from increasing to decreasing at $x = 4$

We claim that the only way a function f can switch between increasing and decreasing at a particular x-value is if $f'(x) = 0$ or $f'(x)$ does not exist at that x-value. How do we know these are the only two possibilities? Well, the only ways the derivative f' can switch from negative to positive or positive to negative is to pass through a point where $f'(x) = 0$ or change sign by the graph of f' being discontinuous and "jumping across" the x-axis.

For a non-mathematical example, consider that the only ways to cross a river are to get wet or teleport. This is a special case of a fact that mathematicians call the Intermediate Value Theorem. If the river is the line $y = 0$, then the only way for $f'(x)$ to "cross the river" (i.e., switch from positive to negative or negative to positive) is for it to get wet (in which case $f'(x) = 0$ at that point) or for it to teleport (in which case f' is discontinuous, which also means it does not exist).

N *For the purposes of this textbook, we assume that the individual trying to cross the river does not have a horse, a boat, or the ability to build a bridge. The individual also does not have a strong friend to throw them across the river, the ability to grow wings and fly, access to a hot air balloon, or any other number of ways to go over, under, or around the river.*

! *Just because we get wet does not mean we cross the river! See **Figure 3.1.2** for an example where $f'(3) = 0$, but the derivative does not change sign at $x = 3$.*

Thus, the important x-values where f *might* switch between increasing and decreasing are the x-values where $f'(x) = 0$ or $f'(x)$ does not exist. We call these x-values **partition numbers of f'**, and we will use them to *partition* the sign chart of $f'(x)$:

Definition

Partition numbers of f' are the x-values where $f'(x) = 0$ or $f'(x)$ does not exist. ▪

N *For the function $f(x) = \ln x$, the definition above includes all x-values less than or equal to zero as partition numbers of f' (because the domain of f is $x > 0$, the derivative f' certainly does not exist for $x \leq 0$). However, having all of these x-values as partition numbers will not meaningfully partition a number line into discrete intervals that we can use to create a sign chart of $f'(x)$ (which is the goal). Although, because $f'(0)$ does not exist and there is a meaningful transition happening at $x = 0$ (the derivative changes from not existing to existing at $x = 0$), we consider $x = 0$ to be a partition number of f'. Also, $x = 0$ would be needed to partition a number line into the proper intervals when creating a sign chart of $f'(x)$. The definition of partition numbers of f' can be made more mathematically precise for cases like this, but it is beyond the scope of this textbook. We will use the definition above and make judgement calls about what is or is not an "important" x-value when determining the partition numbers of f'.*

Combining all of this information, we now have a process for finding where a function is increasing/decreasing:

Finding Intervals Where f is Increasing/Decreasing Using the Increasing/Decreasing Test

1. Determine the domain of f.
2. Find the partition numbers of f'. Recall that these are the x-values where $f'(x) = 0$ or $f'(x)$ does not exist.
3. Create a sign chart of $f'(x)$ using the partition numbers of f' to divide the sign chart (number line) into intervals. Then,
   - Select any x-value in each interval, and evaluate the derivative f' at each x-value to determine whether $f'(x)$ is positive or negative on each interval. Indicate whether $f'(x)$ is positive or negative on each interval by writing "+" or "-".
   - Apply the Increasing/Decreasing Test to find the intervals where f is increasing and the intervals where it is decreasing:
     - If $f'(x)$ is positive, then f is increasing.
     - If $f'(x)$ is negative, then f is decreasing.

We demonstrate using this process in the following example.

■ **Example 1** Find the intervals where $f(x) = x^3 + x^2 - x + 1$ is increasing/decreasing.

Solution:
We will use the three-step process outlined previously to apply the Increasing/Decreasing Test:

1. Determine the domain of f:

The function f is a polynomial, so its domain is $(-\infty, \infty)$.

2. Find the partition numbers of f':

We start by finding $f'(x)$:

$$f'(x) = 3x^2 + 2x - 1$$

To find the partition numbers of f', we find the x-values where $f'(x) = 0$ or $f'(x)$ does not exist. Because f' is a polynomial and has a domain of all real numbers, it will exist everywhere. Thus, we only need to find the x-values where $f'(x) = 0$:

$$\begin{aligned} f'(x) &= 0 \implies \\ 3x^2 + 2x - 1 &= 0 \\ (3x-1)(x+1) &= 0 \end{aligned}$$

This means we need to solve $3x - 1 = 0$ and $x + 1 = 0$. Starting with $3x - 1 = 0$, we have

$$\begin{aligned} 3x - 1 &= 0 \\ 3x &= 1 \\ x &= \frac{1}{3} \end{aligned}$$

Next,

$$\begin{aligned} x + 1 &= 0 \\ x &= -1 \end{aligned}$$

Thus, there are two partition numbers of f': $x = -1$ and $x = \frac{1}{3}$.

3. Create a sign chart of $f'(x)$:

We will place the partition numbers of f' on a number line, with both $x = -1$ and $x = \frac{1}{3}$ having solid dots to indicate that they are in the domain of f.

Now, we need to determine the sign of $f'(x)$ on the intervals created by the partition numbers of f': $(-\infty, -1)$, $\left(-1, \frac{1}{3}\right)$, and $\left(\frac{1}{3}, \infty\right)$. To do determine the sign of $f'(x)$ on each interval, we will select an x-value in each interval to test. We will choose $x = -2,\ 0,$ and 2, but any x-value in each interval will give the same information:

$$\begin{aligned} f'(x) &= 3x^2 + 2x - 1 \implies \\ f'(-2) &= 3(-2)^2 + 2(-2) - 1 = 7 > 0 \\ f'(0) &= 3(0)^2 + 2(0) - 1 = -1 < 0 \\ f'(2) &= 3(2)^2 + 2(2) - 1 = 15 > 0 \end{aligned}$$

Using this information, we can fill in the sign chart of $f'(x)$. Because we are also interested in the information this yields for f, we will include that information below the number line. See **Figure 3.1.5**.

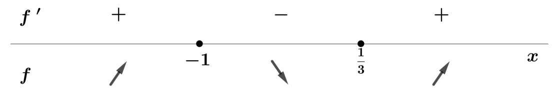


Figure 3.1.5: Sign chart of $f'(x)$ with the corresponding information for $f(x) = x^3 + x^2 - x + 1$

From here, we can determine the answer: The function $f(x) = x^3 + x^2 - x + 1$ is increasing on $(-\infty, -1)$ and $\left(\frac{1}{3}, \infty\right)$, and it is decreasing on $\left(-1, \frac{1}{3}\right)$.

We can check our answer by looking at the graph of the function $f(x) = x^3 + x^2 - x + 1$ shown in **Figure 3.1.6**:

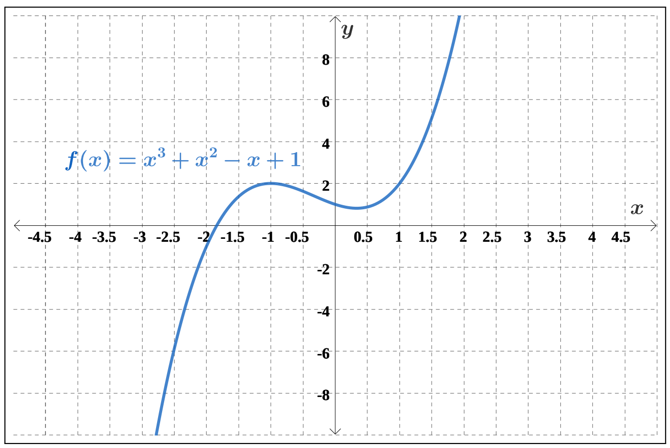


Figure 3.1.6: Graph of $f(x) = x^3 + x^2 - x + 1$

■

When applying the Increasing/Decreasing Test, it is not necessary to include solid dots or open circles on the number line to indicate whether or not the partition numbers of f' are in the domain of f when creating the sign chart of $f'(x)$ (like we did in step 3 of the previous example). However, it is good practice to do so because we will need to know whether or not the partition numbers of f' are in the domain of f when applying future tests.

■ **Example 2** Find the intervals where $f(x) = (5 - x)^{\frac{1}{3}}$ is increasing/decreasing.

Solution:
We will use the three-step process outlined previously to apply the Increasing/Decreasing Test:

1. Determine the domain of f:

This function is not a polynomial like the function in the previous example, so we need to be careful. Let's recall the domain restrictions discussed previously in **Section 1.4**:

Domain Restrictions

1. The denominator must be nonzero.
2. The argument of an even root must be nonnegative.
3. The argument of a logarithm (of any base) must be positive.

This function does not contain division, nor does it have a logarithm. There is a root (fractional power), but it is an *odd* root, not an even root. Thus, the domain is $(-\infty, \infty)$.

2. Find the partition numbers of $f\,'$:

We begin by finding the derivative using the Chain Rule:

$$\begin{aligned} f\,'(x) &= \frac{1}{3}(5-x)^{-\frac{2}{3}}\left(\frac{d}{dx}(5-x)\right) \\ &= \frac{1}{3}(5-x)^{-\frac{2}{3}}(-1) \\ &= -\frac{1}{3}(5-x)^{-\frac{2}{3}} \end{aligned}$$

If we were just asked to find the derivative, we could stop here and leave the function $f\,'$ in its current form. However, because we have to *continue using the derivative* to find the x-values where $f\,'(x) = 0$ or $f\,'(x)$ does not exist, we must use algebraic manipulation to simplify the derivative. In this particular case, we will use the laws of exponents to ensure $f\,'$ does not have any negative exponents:

$$\begin{aligned} f\,'(x) &= -\frac{1}{3}(5-x)^{-\frac{2}{3}} \\ &= \frac{-1}{3(5-x)^{\frac{2}{3}}} \end{aligned}$$

To find the partition numbers of $f\,'$, we find the x-values where $f\,'(x) = 0$ or $f\,'(x)$ does not exist. Because $f\,'(x)$ is a quotient, or ratio, of two functions, it will equal zero where the function in its numerator equals zero:

$$\begin{aligned} f\,'(x) = 0 &\implies \\ -1 &= 0 \end{aligned}$$

Because $-1 \neq 0$, there are no partition numbers where the derivative equals zero.

Similarly, $f\,'(x)$ will not exist when the function in its denominator equals zero:

$$\begin{aligned} f\,'(x) \text{ DNE } &\implies \\ 3(5-x)^{\frac{2}{3}} &= 0 \\ (5-x)^{\frac{2}{3}} &= 0 \\ \left((5-x)^{\frac{2}{3}}\right)^{\frac{3}{2}} &= 0^{\frac{3}{2}} \\ 5-x &= 0 \\ x &= 5 \end{aligned}$$

The derivative does not exist at $x = 5$, and it is the only partition number of $f\,'$.

3. Create a sign chart of $f\,'(x)$:

We will place the partition number of $f\,'$ on a number line with a solid dot to indicate that $x = 5$ is in the domain of f.

Now, we need to determine the sign of $f\,'(x)$ on the intervals $(-\infty, 5)$ and $(5, \infty)$ by selecting an x-value in each interval to test. We will choose $x = 0$ and $x = 13$, but any x-value in each interval will give the same information:

$$f'(x) = \frac{-1}{3(5-x)^{\frac{2}{3}}} \implies$$

$$f'(0) = \frac{-1}{3(5-0)^{\frac{2}{3}}} \approx -0.114 < 0$$

$$f'(13) = \frac{-1}{3(5-13)^{\frac{2}{3}}} = -\frac{1}{12} < 0$$

Using this information, we can create the sign chart of $f'(x)$. See **Figure 3.1.7**.

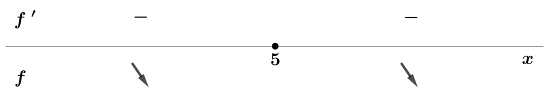


Figure 3.1.7: Sign chart of $f'(x)$ with the corresponding information for $f(x) = (5-x)^{1/3}$

In conclusion, the function $f(x) = (5-x)^{\frac{1}{3}}$ is decreasing on $(-\infty, 5)$ and $(5, \infty)$.

■

The sign chart of $f'(x)$ will not always have alternating signs! Notice in the previous example that $f'(x)$ did not change signs at $x = 5$.

Try It # 1:

Find the intervals where $f(x) = \dfrac{1}{(x-4)^2}$ is increasing/decreasing.

Local Extrema

We can continue using the ideas developed previously to find where a function has hills or valleys, or more formally, local extrema.

Definition

On an open interval containing $x = c$,

- $f(c)$ is a **local maximum** if $f(c) \geq f(x)$ for all x near c.
- $f(c)$ is a **local minimum** if $f(c) \leq f(x)$ for all x near c.
- $f(c)$ is a **local extremum** if $f(c)$ is a local maximum or minimum.

■

The plurals of maximum, minimum, and extremum are maxima, minima, and extrema, respectively. Also, note that some texts may refer to local extrema as relative extrema.

Local extrema need to only be the largest (or smallest) function values in some neighborhood of a point. For instance, the David G. Eller Oceanography & Meteorology Building (at 151 feet tall) is the tallest building on Texas A&M University's College Station campus, so it is *locally* the tallest building. However, it is not the tallest building in the world, nor is it even the tallest building in the state of Texas. The J.P. Morgan Chase Tower in Houston is the tallest building in Texas at 1002 feet tall!

N *As of the writing of this textbook, the Burj Khalifa in Dubai (at 2717 feet tall) is the tallest building in the world. This serves as an example of an* ***absolute maximum****, which is something we will discuss more fully in* ***Section 3.4****.*

Next, we will focus on how to find the local extrema of a function. Let's look at the graph of f we discussed in the introduction, which is shown again in **Figure 3.1.8**:

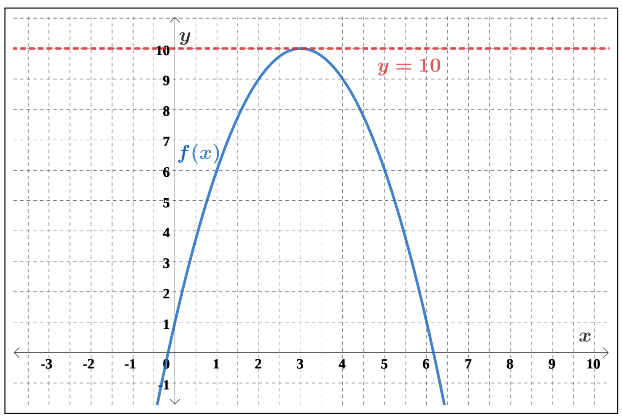


Figure 3.1.8: Graph of a function f in which $f'(3) = 0$ and the largest y-value of $f(x)$ occurs at $x = 3$

The graph of f has a local maximum at $x = 3$, and $x = 3$ is a partition number of $f'(x)$ (because $f'(3) = 0$) that is in the domain of f. Notice also that $f'(x)$ switches from positive to negative at $x = 3$. Based on this information, to find local extrema, we should find the partition numbers of f' that are *in the domain of the function* f and investigate whether or not $f'(x)$ changes sign at these x-values.

The x-values of this special subset of partition numbers of f' that are in the domain of f we will investigate are called **critical values** of f:

Definition

A function f has a **critical value** at $x = c$ if either $f'(c) = 0$ or $f'(c)$ does not exist and $x = c$ is in the domain of f. ▪

We must be careful, though, because a critical value will only result in a local extremum if $f'(x)$ changes sign at the critical value. For example, let's look at the function g we discussed in the introduction, which is shown again in **Figure 3.1.9**. $x = 3$ is critical value of g because $g'(3) = 0$ and $x = 3$ is in the domain of g, but there is no local extremum at $x = 3$:

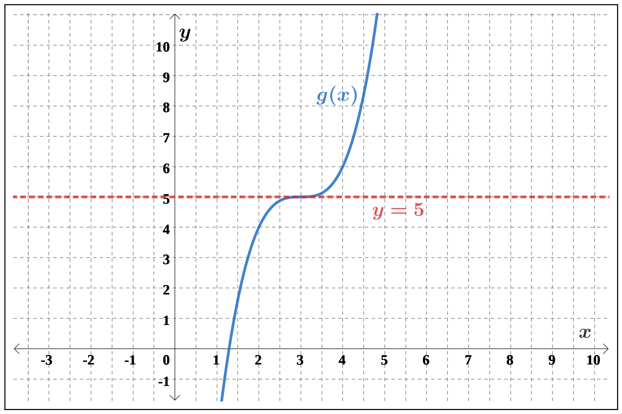


Figure 3.1.9: Graph of a function g in which $g'(3) = 0$, but there is no largest (or smallest) y-value of g occurring at $x = 3$

Before we learn how to determine whether or not a critical value results in a local extremum, let's practice finding critical values.

■ **Example 3** Find the critical values of the following functions, if they exist.

a. $f(x) = x^3 - 6x^2 + 9x + 2$

b. $f(x) = e^{-x}(x-8)$

c. $f(x) = \dfrac{x}{(x-4)^2}$

Solution:

a. The critical values are the partition numbers of f' that are in the domain of f, so we will start by finding the domain of the function. This function is a polynomial, so its domain is all real numbers. Thus, any partition number of f' we find will be a critical value of f as well!

Our next step is to find $f'(x)$:

$$f'(x) = 3x^2 - 12x + 9$$

Recall that to find the partition numbers of f', we need to find the x-values where $f'(x) = 0$ or $f'(x)$ does not exist. Because the derivative f' is a polynomial, it will exist everywhere. So we only need to find the x-values where $f'(x) = 0$:

$$\begin{aligned} f'(x) &= 0 \implies \\ 3x^2 - 12x + 9 &= 0 \\ 3(x^2 - 4x + 3) &= 0 \\ 3(x-1)(x-3) &= 0 \\ &\implies x = 1 \text{ and } x = 3 \end{aligned}$$

Thus, $x = 1$ and $x = 3$ are the partition numbers of f'. Because the domain of f is $(-\infty, \infty)$, $x = 1$ and $x = 3$ are also critical values of f.

b. Again, we will start by finding the domain of $f(x) = e^{-x}(x-8)$, and then we will find the partition numbers of f' and compare the two to determine the critical values of f.

Considering domain restrictions, we see that the function has no logarithms or even roots. There is division hidden in the function by a negative exponent: $e^{-x} = \dfrac{1}{e^x}$. However, because $e^x > 0$, there are no x-values that will give division by zero. Therefore, the domain of f is $(-\infty, \infty)$.

Next, we find the derivative of the function in order to find the partition numbers of f'. We start with the Product Rule:

$$\begin{aligned} f'(x) &= (x-8)\left(\frac{d}{dx}(e^{-x})\right) + e^{-x}\left(\frac{d}{dx}(x-8)\right) \\ &= (x-8)e^{-x}\left(\frac{d}{dx}(-x)\right) + e^{-x}(1) \\ &= (x-8)e^{-x}(-1) + e^{-x} \end{aligned}$$

Again, in the previous chapter, we could stop here and leave the derivative in this form. However, because we have to find the partition numbers of f' by setting the derivative equal to zero and finding where it does not exist, we need to continue and algebraically manipulate the function by factoring e^{-x} from both terms:

$$\begin{aligned} f'(x) &= (x-8)e^{-x}(-1) + e^{-x} \\ &= e^{-x}[(x-8)(-1)+1] \\ &= e^{-x}(-x+8+1) \\ &= e^{-x}(-x+9) \end{aligned}$$

Now, we will find the partition numbers of f' such that $f'(x) = 0$:

$$\begin{aligned} f'(x) = 0 &\implies \\ e^{-x}(-x+9) = 0 & \end{aligned}$$

This gives two equations: $e^{-x} = 0$ and $-x+9 = 0$. Because $e^{-x} = \frac{1}{e^x} > 0$, there is no solution to the first equation. The second equation gives $x = 9$.

Checking the domain restrictions to determine for which x-values $f'(x)$ does not exist, we again see hidden division with the term $e^{-x} = \frac{1}{e^x}$. However, because $e^x > 0$, the denominator will never equal zero so $f'(x)$ exists everywhere.

Thus, $x = 9$ is the only partition number of f', and because the domain of f is $(-\infty, \infty)$, it is also the only critical value of f.

c. Again, we start by finding the domain of $f(x) = \frac{x}{(x-4)^2}$. This time the function includes division, so we must ensure the function in the denominator is nonzero:

$$\begin{aligned} (x-4)^2 &\neq 0 \\ \sqrt{(x-4)^2} &\neq \sqrt{0} \\ x-4 &\neq 0 \\ x &\neq 4 \end{aligned}$$

The domain of f is every x-value except $x = 4$, or using interval notation, $(-\infty, 4) \cup (4, \infty)$.

Next, we find the derivative of f using the Quotient Rule:

$$\begin{aligned} f'(x) &= \frac{(x-4)^2\left(\frac{d}{dx}(x)\right)-(x)\left(\frac{d}{dx}\left((x-4)^2\right)\right)}{((x-4)^2)^2} \\ &= \frac{(x-4)^2(1)-x\left(2(x-4)\left(\frac{d}{dx}(x-4)\right)\right)}{(x-4)^4} \\ &= \frac{(x-4)^2-2x(x-4)(1)}{(x-4)^4} \\ &= \frac{(x-4)^2-2x(x-4)}{(x-4)^4} \end{aligned}$$

Because we need to use this function to find the partition numbers of f', we will factor the $x-4$ term from the numerator (remember we factor the lowest power of the common term) and continue simplifying:

$$\begin{aligned} f'(x) &= \frac{(x-4)^2-2x(x-4)}{(x-4)^4} \\ &= \frac{(x-4)\left[(x-4)-2x\right]}{(x-4)^4} \\ &= \frac{\cancel{(x-4)}\left[(x-4)-2x\right]}{(x-4)^{\cancel{4}\,3}} \\ &= \frac{(x-4)-2x}{(x-4)^3} \\ &= \frac{-x-4}{(x-4)^3} \end{aligned}$$

To find the partition numbers of f', we find the x-values where $f'(x)=0$ or $f'(x)$ does not exist. $f'(x)=0$ when the numerator equals 0, and $f'(x)$ does not exist when the denominator equals zero (remember we are looking at the domain of f' when determining where $f'(x)$ does not exist). First, we will find the x-values where $f'(x)=0$:

$$\begin{aligned} f'(x) = 0 &\implies \\ -x-4 &= 0 \\ x &= -4 \end{aligned}$$

Next, to find the x-values where $f'(x)$ does not exist, we set the denominator of $f'(x)$ equal to zero:

$$\begin{aligned} f'(x) \text{ DNE } &\implies \\ \left((x-4)^3\right)^{\frac{1}{3}} &= (0)^{\frac{1}{3}} \\ x-4 &= 0 \\ x &= 4 \end{aligned}$$

Thus, the partition numbers of f' are $x=-4$ and $x=4$. Because the domain of f is $(-\infty,4)\cup(4,\infty)$, $x=-4$ is the only partition number of f' that is *in the domain* of f. Thus, $x=-4$ is the only critical value of f.

▪

Try It # 2:
Find the critical values of the following functions, if they exist.

a. $f(x) = x^5 - 5x^4 - 20x^3 + 100$

b. $f(x) = xe^{-0.5x^2}$

c. $f(x) = \dfrac{x^3}{x^2 - 9}$

■ **Example 4** Given a function f that is continuous on its domain of $(-\infty, \infty)$ and its derivative f' is defined by $f'(x) = -\frac{1}{2}(x+4)(x+2)(x-5)^2$, find the critical values of f, if they exist.

Solution:
The domain and derivative of f are given. Furthermore, because the derivative f' is a polynomial (of degree 4, which we would see if we multiplied all the terms), it exists everywhere. Thus, any critical values of f will occur at the x-values where $f'(x) = 0$:

$$f'(x) = 0 \implies$$
$$-\frac{1}{2}(x+4)(x+2)(x-5)^2 = 0$$

To find the x-values where this equation equals zero, we set each factor of the product on the left-hand side of the equation equal to zero. Thus, the partition numbers of f', and by extension the critical values of f, are $x = -4$, $x = -2$, and $x = 5$.

■

As mentioned previously, not all critical values will result in local extrema. In addition to the function g we discussed shown in **Figure 3.1.2** and **Figure 3.1.2**, the function f in the previous example also has a critical value, $x = 5$, in which there is no local extremum. The graph of f corresponding to the previous example is shown in **Figure 3.1.10**.

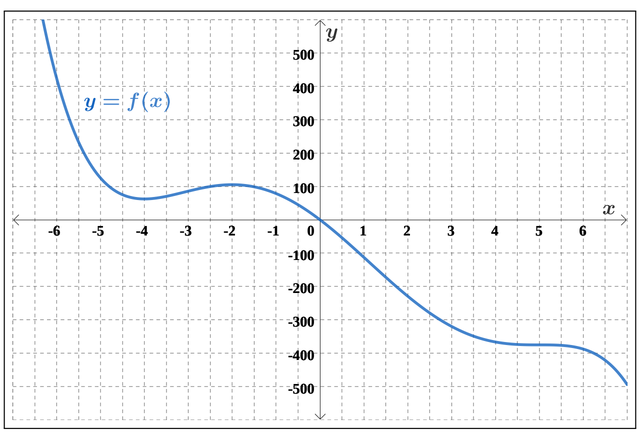


Figure 3.1.10: Graph of a function f with critical values $x = -4$, $x = -2$, and $x = 5$

It is important to remember that for a function to have a local extremum at a critical value, its derivative must change sign at the critical value.

First Derivative Test

To determine whether a function has a local extremum at a critical value $x = c$, we must determine the sign of $f'(x)$ on the left and right sides of $x = c$. This result is known as the **First Derivative Test**:

Theorem 3.2 First Derivative Test

Suppose f is continuous on an interval containing the critical value $x = c$ and f is differentiable near, and on both sides of, $x = c$. Then, $f(c)$ satisfies one of the following:

- If $f'(x)$ changes sign from positive to negative at $x = c$, then $f(c)$ is a local maximum of f.
- If $f'(x)$ changes sign from negative to positive at $x = c$, then $f(c)$ is a local minimum of f.
- If $f'(x)$ does not change sign at $x = c$, then $f(c)$ is neither a local maximum nor a local minimum of f.

To apply the First Derivative Test, as well as determine where a function is increasing/decreasing, we use the following four steps:

Finding Intervals Where f is Increasing/Decreasing and Local Extrema Using the First Derivative Test

1. Determine the domain of f.
2. Find the partition numbers of f'. Recall that these are the x-values where $f'(x) = 0$ or $f'(x)$ does not exist.
3. Determine which partition numbers of f' are in the domain of f. These are the critical values of f.
4. Create a sign chart of $f'(x)$ using the partition numbers of f' found in step 2 to divide the sign chart (number line) into intervals. Then,
   - Indicate whether or not each partition number is in the domain of f by drawing a solid dot or open circle on the sign chart.
   - Select any x-value in each interval, and evaluate the derivative f' at each x-value to determine the sign of $f'(x)$ and corresponding behavior of f on each interval.
   - Apply the First Derivative Test to find any local extrema of f.

Example 5 For each of the following functions, find the critical values of f, the intervals where f is increasing/decreasing, and the local extrema of f as well as where they occur.

a. $f(x) = 2x^3 - 3x^2 - 36x$

b. $f(x) = \dfrac{x^2 + 3}{x - 1}$

c. $f(x) = 7x\ln(x) - x$

Solution:

a. We will use the four steps outlined previously to apply the First Derivative Test:

1. Determine the domain of f:

The function f is a polynomial, so its domain is $(-\infty, \infty)$.

2. Find the partition numbers of f':

$$f'(x) = 6x^2 - 6x - 36$$

To find the partition numbers of f', we find the x-values where $f'(x) = 0$ or $f'(x)$ does not exist. Because f' is a polynomial and has a domain of all real numbers, it will exist everywhere. Thus, we only need to find the x-values where $f'(x) = 0$:

$$\begin{aligned} f'(x) &= 0 \implies \\ 6x^2 - 6x - 36 &= 0 \\ 6(x^2 - x - 6) &= 0 \\ 6(x+2)(x-3) &= 0 \\ &\implies x = -2 \text{ and } x = 3 \end{aligned}$$

Thus, the partition numbers of f' are $x = -2$ and $x = 3$.

3. Determine which partition numbers of f' are in the domain of f (i.e., find the critical values of f):

Because the domain of f is all real numbers, $x = -2$ and $x = 3$ are both critical values of f.

4. Create a sign chart of $f'(x)$:

We will place the partition numbers of f' on a number line, with $x = -2$ and $x = 3$ having solid dots to indicate that they are in the domain of f.

Now, we need to determine the sign of $f'(x)$ on the intervals $(-\infty, -2)$, $(-2, 3)$, and $(3, \infty)$. We will choose the values $x = -5,\ 0$, and 5 to test:

$$\begin{aligned} f'(x) &= 6x^2 - 6x - 36 \implies \\ f'(-5) &= 6(-5)^2 - 6(-5) - 36 = 144 > 0 \\ f'(0) &= 6(0)^2 - 6(0) - 36 = -36 < 0 \\ f'(5) &= 6(5)^2 - 6(5) - 36 = 84 > 0 \end{aligned}$$

Using this information, we can fill in the sign chart of $f'(x)$. Because we are also interested in the information this yields for f, we include that information below the number line. See **Figure 3.1.11**.

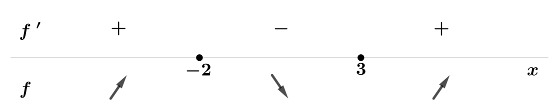


Figure 3.1.11: Sign chart of $f'(x)$ with the corresponding information for $f(x) = 2x^3 - 3x^2 - 36x$

Using **Theorem 3.2**, we see that a local maximum occurs at $x = -2$, and a local minimum occurs at $x = 3$ (note that we also double check on the number line that f is defined at both of these x-values). To find the local maximum and minimum values (i.e., y-values) associated with these x-values, we substitute the x-values into the *original function* f:

$$\begin{aligned} f(x) &= 2x^3 - 3x^2 - 36x \implies \\ f(-2) &= 2(-2)^3 - 3(-2)^2 - 36(-2) = 44 \\ f(3) &= 2(3)^3 - 3(3)^2 - 36(3) = -81 \end{aligned}$$

In conclusion, $x = -2$ and $x = 3$ are the critical values of f, f is increasing on $(-\infty, -2)$ and $(3, \infty)$, f is decreasing on $(-2, 3)$, and f has a local maximum of 44 at $x = -2$ and a local minimum of -81 at $x = 3$.

We can check our work by looking at the graph of f, which is shown in **Figure 3.1.12**, but using the First Derivative Test allows us to find *exactly* where the function is increasing/decreasing and has local extrema. While using technology to look at a graph of the function would give us roughly the same information, if we want exact answers, calculus is the way to go! Calculus provides more precision than technology allows.

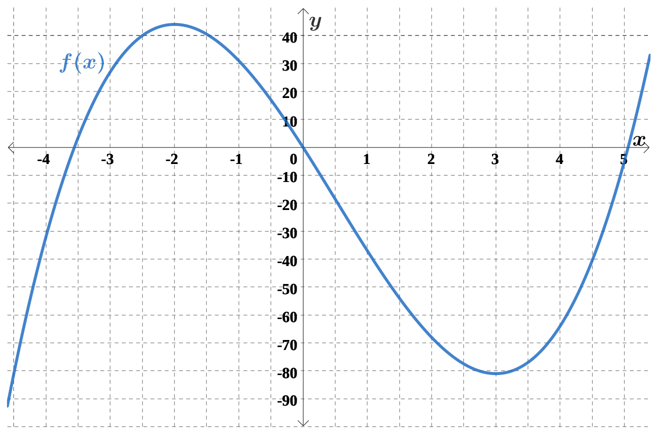


Figure 3.1.12: Graph of $f(x) = 2x^3 - 3x^2 - 36x$

b. Recall $f(x) = \dfrac{x^2+3}{x-1}$. Again, we follow the same four steps to apply the First Derivative Test:

1. Determine the domain of f:

The function f is a rational function, so we must ensure the denominator does not equal zero:

$$\begin{aligned} x - 1 &\neq 0 \\ x &\neq 1 \end{aligned}$$

Thus, the domain of f is $(-\infty, 1) \cup (1, \infty)$.

2. Find the partition numbers of f':

We start by finding $f'(x)$ using the Quotient Rule:

$$\begin{aligned} f'(x) &= \frac{(x-1)\left(\frac{d}{dx}\left(x^2+3\right)\right) - \left(x^2+3\right)\left(\frac{d}{dx}(x-1)\right)}{(x-1)^2} \\ &= \frac{(x-1)(2x) - \left(x^2+3\right)(1)}{(x-1)^2} \\ &= \frac{2x^2 - 2x - x^2 - 3}{(x-1)^2} \\ &= \frac{x^2 - 2x - 3}{(x-1)^2} \end{aligned}$$

To find the partition numbers of f', we find the x-values where $f'(x) = 0$ or $f'(x)$ does not exist. $f'(x) = 0$ when the numerator equals 0, and $f'(x)$ does not exist when the denominator equals zero (remember we are looking at the domain of f' when determining where $f'(x)$ does not exist). First, we find the x-values where $f'(x) = 0$:

$$f'(x) = 0 \implies$$
$$x^2 - 2x - 3 = 0$$
$$(x+1)(x-3) = 0$$
$$\implies x = -1 \text{ and } x = 3$$

Next, to find the x-values where $f'(x)$ does not exist, we set the denominator of $f'(x)$ equal to zero:

$$f'(x) \text{ DNE } \implies$$
$$(x-1)^2 = 0$$
$$\left((x-1)^2\right)^{\frac{1}{2}} = (0)^{\frac{1}{2}}$$
$$x - 1 = 0$$
$$x = 1$$

Therefore, the partition numbers of f' are $x = -1$, $x = 1$, and $x = 3$.

3. Determine which partition numbers of f' are in the domain of f (i.e., find the critical values of f):

The only x-value *not* in the domain of f is $x = 1$. Thus, the critical values of f are $x = -1$ and $x = 3$.

4. Create a sign chart of $f'(x)$:

We will place the partition numbers of f' on a number line, with $x = -1$ and $x = 3$ having solid dots and $x = 1$ having an open circle to indicate which are included in the domain of f.

Next, we determine the sign of $f'(x)$ on each of the four intervals created by the partition numbers of f': $(-\infty, -1)$, $(-1, 1)$, $(1, 3)$, and $(3, \infty)$. We select x-values to test in each of these intervals. We will choose $x = -3,\ 0,\ 2$, and 5, but remember, any x-values in the intervals will give the same *sign* of $f'(x)$:

$$f'(x) = \frac{x^2 - 2x - 3}{(x-1)^2} \implies$$
$$f'(-3) = \frac{(-3)^2 - 2(-3) - 3}{((-3)-1)^2} = \frac{3}{4} > 0$$
$$f'(0) = \frac{(0)^2 - 2(0) - 3}{((0)-1)^2} = -3 < 0$$
$$f'(2) = \frac{(2)^2 - 2(2) - 3}{((2)-1)^2} = -3 < 0$$
$$f'(5) = \frac{(5)^2 - 2(5) - 3}{((5)-1)^2} = \frac{3}{4} > 0$$

Using this information, we can fill in the sign chart of $f'(x)$. Again, we include the corresponding information for f below the number line. See **Figure 3.1.13**.

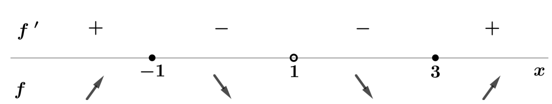


Figure 3.1.13: Sign chart of $f'(x)$ with the corresponding information for $f(x) = \dfrac{x^2+3}{x-1}$

Using **Theorem 3.2**, we see that a local maximum occurs at $x = -1$, and a local minimum occurs at $x = 3$. To find the local maximum and minimum values (i.e., y-values) associated with these x-values, we substitute the x-values into the *original function* f:

$$\begin{aligned} f(x) &= \frac{x^2+3}{x-1} \implies \\ f(-1) &= \frac{(-1)^2+3}{(-1)-1} = -2 \\ f(3) &= \frac{(3)^2+3}{(3)-1} = 6 \end{aligned}$$

In conclusion, $x = -1$ and $x = 3$ are the critical values of f, f is increasing on $(-\infty, -1)$ and $(3, \infty)$, f is decreasing on $(-1, 1)$ and $(1, 3)$, and f has a local maximum of -2 at $x = -1$ and a local minimum of 6 at $x = 3$.

N *Notice here that the local minimum is greater than the local maximum! This can happen because a local extremum only needs to be the largest/smallest y-value on some interval around the corresponding x-value.*

c. Recall $f(x) = 7x\ln(x) - x$. Again, we follow the same four steps to apply the First Derivative Test:

1. Determine the domain of f:

We need to ensure that the argument of the logarithm is positive, or in other words, $x > 0$. This means the domain of f is $(0, \infty)$.

2. Find the partition numbers of $f\,'$:

We start by finding $f\,'(x)$ using the Product Rule:

$$\begin{aligned} f\,'(x) &= 7\left(x\left(\frac{d}{dx}(\ln(x))\right) + (\ln(x))\left(\frac{d}{dx}(x)\right)\right) - 1 \\ &= 7\left(x\left(\frac{1}{x}\right) + (\ln(x))(1)\right) - 1 \\ &= 7(1 + \ln(x)) - 1 \\ &= 7 + 7\ln(x) - 1 \\ &= 7\ln(x) + 6 \end{aligned}$$

To find the partition numbers of $f\,'$, we find the x-values where $f\,'(x) = 0$ or $f\,'(x)$ does not exist. Let's start by finding the x-values where $f\,'(x) = 0$:

$$\begin{aligned} f\,'(x) &= 0 \implies \\ 7\ln(x) + 6 &= 0 \\ 7\ln(x) &= -6 \\ \ln(x) &= -\frac{6}{7} \\ e^{\ln(x)} &= e^{-\frac{6}{7}} \\ x &= e^{-\frac{6}{7}} \end{aligned}$$

When a function involves logarithms, it is important to check the answer by substituting it back into the function ($f\,'$ in this case):

$$\begin{aligned} f\,'\left(e^{-\frac{6}{7}}\right) &= 7\ln\left(e^{-\frac{6}{7}}\right) + 6 \\ &= 7\left(-\frac{6}{7}\right) + 6 \\ &= -6 + 6 = 0 \checkmark \end{aligned}$$

Next, let's consider the x-values where $f'(x)$ does not exist. Because the derivative function $f'(x) = 7\ln(x) + 6$ has a domain of $(0, \infty)$, it stands to reason that $f'(x)$ does not exist on the interval $(-\infty, 0]$. This would mean that every x-value in the interval $(-\infty, 0]$ would count as a partition number of f' (according to the definition of a partition number given).

However, having all of these x-values as partition numbers will not meaningfully partition a number line into discrete intervals that we can use to create a sign chart of $f'(x)$ (which is the goal). Although, because $f'(0)$ does not exist and there is a meaningful transition happening at $x = 0$ (the derivative changes from not existing to existing at $x = 0$), we consider $x = 0$ to be a partition number of f'. Also, $x = 0$ is needed to partition the number line into the proper intervals when creating the sign chart of $f'(x)$.

N *This example shows how sometimes it is necessary to be slightly imprecise with the definition of a partition number. We must use good judgement when determining the partition numbers of f' in cases like this one.*

Thus, there are two partition numbers of f': $x = 0$ and $x = e^{-6/7}$, which is approximately 0.4244.

3. Determine which partition numbers of f' are in the domain of f (i.e., find the critical values of f):

The only critical value of f is $x = e^{-6/7}$.

4. Create a sign chart of $f'(x)$:

Because f has a domain of $(0, \infty)$, we will indicate this on our number line (remember, we are considering $x = 0$ to also be a partition number of f'). Also, $x = e^{-6/7}$ will have a solid dot because it is in the domain of f.

There are only two intervals in the domain of f to check for the sign of the derivative: $\left(0, e^{-6/7}\right)$ and $\left(e^{-6/7}, \infty\right)$. We will select $x = 0.3$ and $x = 1$ to test:

$$\begin{aligned} f'(x) &= 7\ln(x) + 6 \implies \\ f'(0.3) &= 7\ln(0.3) + 6 \approx -2.43 < 0 \\ f'(1) &= 7\ln(1) + 6 = 6 > 0 \end{aligned}$$

Using this information, we can fill in the sign chart of $f'(x)$. See **Figure 3.1.14**.

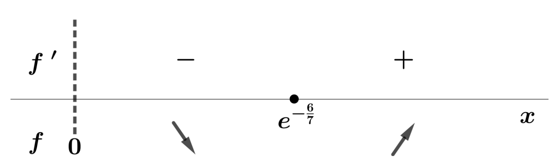


Figure 3.1.14: Sign chart of $f'(x)$ with the corresponding information for $f(x) = 7x\ln(x) - x$

Using **Theorem 3.2**, we see that a local minimum occurs at $x = e^{-6/7}$. To find the local minimum value (i.e., y-value) associated with this x-value, we substitute $x = e^{-6/7}$ into the *original function* f:

$$\begin{aligned} f(x) &= 7x\ln(x) - x \implies \\ f\left(e^{-\frac{6}{7}}\right) &= 7e^{-\frac{6}{7}}\ln\left(e^{-\frac{6}{7}}\right) - e^{-\frac{6}{7}} \\ &= 7e^{-\frac{6}{7}}\left(-\frac{6}{7}\right) - e^{-\frac{6}{7}} \\ &= -6e^{-\frac{6}{7}} - e^{-\frac{6}{7}} \\ &= e^{-\frac{6}{7}}(-6-1) \\ &= -7e^{-\frac{6}{7}} \end{aligned}$$

In conclusion, $x = e^{-6/7}$ is the only critical value of f, f is increasing on $(e^{-6/7}, \infty)$, f is decreasing on $(0, e^{-6/7})$, and f has a local minimum of $-7e^{-6/7}$ at $x = e^{-6/7}$.

▪

Try It # 3:
For each of the following functions, find the critical values of the function, the intervals where the function is increasing/decreasing, and the local extrema of the function as well as where they occur.

a. $f(x) = x^3 - 6x^2 + 9x + 2$

b. $g(x) = e^{-x}(x-8)$

Graphical Interpretation

Remember, there is more than one way a function may be given to us. Thus far, we have focused on finding partition numbers of f', critical values of f, intervals where f is increasing/decreasing, and local extrema of f algebraically using the rule of the function. Let's consider finding this information if we are given the graph of the function, or its derivative, instead.

▪ **Example 6** Given the graph of f shown in **Figure 3.1.15**, find each of the following.

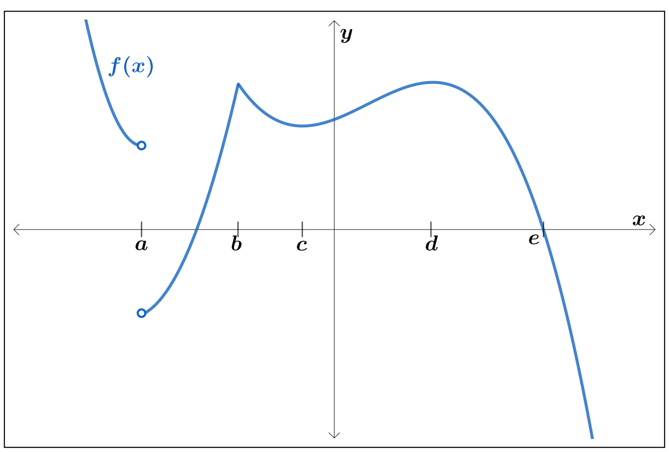


Figure 3.1.15: Graph of a function f

a. Partition numbers of f'

b. Critical values of f

c. Intervals where $f(x)$ is increasing/decreasing

d. x-values where any local extrema of f occur (specify the type)

Solution:

a. The partition numbers of f' are the x-values where $f'(x) = 0$ or $f'(x)$ does not exist. Recall from **Section 2.2** that the derivative f' does not exist when the graph of f has a cusp or corner, a discontinuity, or a vertical tangent line. This graph of f has a discontinuity at $x = a$ and a corner at $x = b$. That gives us two of the partition numbers of f'. We also need to find where $f'(x) = 0$, or where the graph of f has horizontal tangent lines. This occurs at $x = c$ and $x = d$. Thus, the partition numbers of f' are $x = a$, $x = b$, $x = c$, and $x = d$.

b. The critical values of f will be the subset of partition numbers of f' that are in the domain of f. Looking at the graph of f, we see its domain is $(-\infty, a) \cup (a, \infty)$. Thus, the only partition number of f' that is *not* in the domain of f is $x = a$. This means $x = a$ is not a critical value. All of the other partition numbers of f' are in the domain of f, so the critical values of f are $x = b$, $x = c$, and $x = d$.

c. The intervals we need to check and determine if f is increasing/decreasing on those intervals are $(-\infty, a)$, (a, b), (b, c), (c, d), and (d, ∞). Instead of selecting x-values in each of these intervals and algebraically checking the sign of the derivative like we have done previously, we will just look at the graph of f. We see that f is decreasing on $(-\infty, a)$, (b, c), and (d, ∞), and f is increasing on (a, b) and (c, d).

d. Local extrema are the largest (or smallest) y-values in a *localized* area of a function. We will first focus on finding the local maxima of the function f. By looking at the graph of f, we see local maxima occur at $x = b$ and $x = d$. Similarly, f has a local minimum at $x = c$. Notice that f does change from decreasing to increasing at $x = a$, but because $x = a$ is not in the domain of f, it cannot be a local extremum!

■

Now, suppose we want to find this same information for a function f, but we are only given the graph of its derivative, f'. Algebraically, we know that partition numbers of f' are the x-values where the derivative equals zero or does not exist. Correspondingly, we need to find the x-values where the graph of f' touches the x-axis or does not exist (i.e., is undefined). To find the critical values of f, we observe which of the partition numbers of f' are in the domain of f.

To determine where f is increasing when looking at the graph of f', we find where the graph of f' is above the x-axis because f is increasing where $f'(x) > 0$. Similarly, to determine where f is decreasing, we find where the graph of f' is below the x-axis. To find the x-values of any local extrema, we need to know where $f'(x)$ changes sign, or where the graph of f' *switches* from being positive (above the x-axis) to negative (below the x-axis), or vice versa. In addition, we need to check that such x-values are actually in the domain of f, which should be stated.

Example 7 Given the graph of f' shown in **Figure 3.1.16** and that f is continuous on its domain of $(-\infty, a) \cup (a, \infty)$, find each of the following.

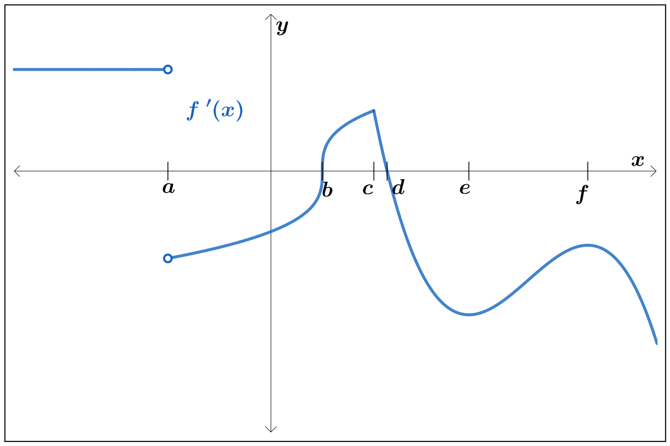


Figure 3.1.16: Graph of a derivative function f'

a. Partition numbers of f'
b. Critical values of f
c. Intervals where f is increasing/decreasing
d. x-values where any local extrema of f occur (specify the type)

Solution:
We will use the same ideas from the previous example, but we have to be more careful now because we are given the graph of f' instead of the graph of f.

a. The partition numbers of f' are the x-values where $f'(x) = 0$ or $f'(x)$ does not exist. Remember that we are given the graph of f', and the only x-value where $f'(x)$ does not exist is $x = a$ (the function is undefined at $x = a$). We may be tempted to include $x = b$ and $x = c$ because of the vertical tangent line and sharp turn, respectively, but these x-values would be where the derivative of f' does not exist, *not where f' itself does not exist!*

Next, we need to find the x-values where $f'(x) = 0$. Graphically, this means we need to find the x-values where the graph of f' touches (but not necessarily crosses) the x-axis. This occurs at $x = b$ and $x = d$.

Thus, the partition numbers of f' are $x = a$, $x = b$, and $x = d$.

b. The critical values of f are the partition numbers of f' that are in the domain of f. We are given that the domain of f is $(\infty, a) \cup (a, \infty)$. Therefore, the only partition number of f' outside the domain of f is $x = a$, so the critical values of f are $x = b$ and $x = d$.

c. To find the intervals where f is increasing, we need to determine where $f'(x) > 0$, or where the graph of f' is *above* the x-axis. This occurs on the intervals $(-\infty, a)$ and (b, d). Similarly, f decreases when $f'(x) < 0$, or where the graph of f' is *below* the x-axis. This occurs on the intervals (a, b) and (d, ∞).

In summary, f is increasing on $(-\infty, a)$ and (b, d), and f is decreasing on (a, b) and (d, ∞).

d. We have to be really careful when we are given the graph of f', but we need to find the local extrema of f. The local extrema of f will *not* be at the tops of hills or bottoms of valleys on the graph of f', which is the graph we are given here. To determine the local extrema of f, we must rely on the relationship between the sign of $f'(x)$ and the corresponding behavior of the graph of f.

To find the local maxima, we look at the critical values of f and see if $f'(x)$ changes from positive to negative, or in other words, we see if the graph of f' switches from being above the x-axis to below it. The only critical value of f where this occurs is $x = d$. For local minima, it is the opposite: we look at the critical values of f and see if $f'(x)$ changes from negative to positive, or in other words, we see if the graph of f' switches from being below the x-axis to above it. In this case, that happens at $x = b$.

In summary, f has a local maximum at $x = d$ and a local minimum at $x = b$.

N *Remember that even if the derivative changes sign at a particular x-value, that x-value must be in the domain of f for a local extremum to occur. Because $x = b$ and $x = d$ are critical values of f, they are in the domain of f.*

! *$x = c$ and $x = f$ are the x-values where the local maxima of f' occur, not the function f! Similarly, $x = e$ does not correspond to a local minimum of f, but it does correspond to a local minimum of f'. These x-values do provide important information about f, which we will discuss more in depth in the next section.*

■

Try It # 4:
Given the graph of f' shown in **Figure 3.1.17** and that f is continuous on its domain of $(-\infty,7)\cup(7,\infty)$, find each of the following.

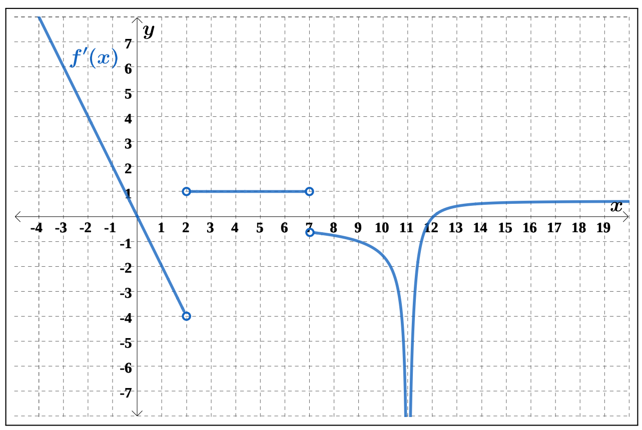


Figure 3.1.17: Graph of a derivative function f'

a. Partition numbers of f'
b. Critical values of f
c. Intervals where f is increasing/decreasing
d. x-values where any local extrema of f occur (specify the type)

Applications

We can apply what we have learned about intervals of increase/decrease and local extrema to analyze business functions such as profit, revenue, and cost, as well as their corresponding marginal, average, and marginal average functions.

Example 8 An office supply company that specializes in producing high-end office chairs with lumbar support has a cost function given by $C(x) = 0.04x^2 + 15x + 144$ dollars, where x is the number of chairs produced in one day. Find the intervals where the average cost function, $\overline{C}$, is increasing/decreasing and the local extrema of the average cost function.

Solution:
Before we can apply the First Derivative Test to the average cost function, $\overline{C}$, we must first find the average cost function:

$$\begin{aligned}\overline{C}(x) &= \frac{C(x)}{x} \\ &= \frac{0.04x^2 + 15x + 144}{x} \\ &= \frac{0.04x^2}{x} + \frac{15x}{x} + \frac{144}{x} \\ &= 0.04x + 15 + 144x^{-1}\end{aligned}$$

Now, we use the previous four steps to find where $\overline{C}$ is increasing/decreasing and apply the First Derivative Test:

1. Determine the domain of $\overline{C}$:

The domain of this function is $(-\infty, 0) \cup (0, \infty)$ because we cannot divide by zero. However, because this is a real-world application, we must restrict the domain to $(0, \infty)$. The company cannot produce a negative number of chairs!

2. Find the partition numbers of $\overline{C}\,'$:

First, we take the derivative:

$$\begin{aligned}\overline{C}\,'(x) &= 0.04 + 0 - 144x^{-2} \\ &= 0.04 - 144x^{-2} \\ &= 0.04 - \frac{144}{x^2}\end{aligned}$$

Now, we must find the x-values where $\overline{C}\,'(x) = 0$ or $\overline{C}\,'(x)$ does not exist. Let's start by finding the x-values where $\overline{C}\,'(x) = 0$:

$$\begin{aligned}\overline{C}\,'(x) &= 0 \implies \\ 0.04 - \frac{144}{x^2} &= 0 \\ 0.04 &= \frac{144}{x^2} \\ \left(x^2\right)(0.04) &= \frac{144}{x^2}\left(x^2\right) \\ 0.04x^2 &= 144 \\ x^2 &= \frac{144}{0.04} \\ x^2 &= 3600 \\ \implies x = -60 &\text{ and } x = 60\end{aligned}$$

Thus, the partition numbers of $\overline{C}\,'$ corresponding to where $\overline{C}\,'(x) = 0$ are $x = -60$ and $x = 60$. However, because we cannot have -60 high-end office chairs (remember we restricted the domain to be $(0, \infty)$), we will only consider the partition number $x = 60$.

Next, let's consider the x-values where $\overline{C}\,'(x)$ does not exist. We can see by looking at the function $\overline{C}\,'(x) = 0.04 - \dfrac{144}{x^2}$ that $\overline{C}\,'(x)$ does not exist at $x = 0$. However, because we restricted the domain of $\overline{C}$ to be $(0,\infty)$, it stands to reason that $\overline{C}\,'(x)$ actually does not exist on the interval $(-\infty, 0]$. This would mean that every x-value in the interval $(-\infty, 0]$ would count as a partition number of $\overline{C}\,'$ (according to the definition of a partition number given).

However, as we discussed previously, having all of these x-values as partition numbers will not meaningfully partition a number line into discrete intervals that we can use to create a sign chart of $\overline{C}\,'(x)$ (which is the goal). Although, because $\overline{C}\,'(0)$ does not exist and there is a meaningful transition happening at $x = 0$ (the derivative changes from not existing to existing at $x = 0$), we consider $x = 0$ to be a partition number of $\overline{C}\,'$. Also, $x = 0$ is needed to partition the number line into the proper intervals when creating the sign chart of $\overline{C}\,'(x)$.

In summary, the partition numbers of $\overline{C}\,'$ are $x = 0$ and $x = 60$.

3. Determine which partition numbers of $\overline{C}\,'$ are in the domain of $\overline{C}$ (i.e., find the critical values of $\overline{C}$):

The only partition number of $\overline{C}\,'$ in the restricted domain of $(0,\infty)$ is $x = 60$.

4. Create a sign chart of $\overline{C}\,'(x)$:

Because $\overline{C}$ has a restricted domain of $(0,\infty)$, we will indicate this on our number line (remember, we are considering $x = 0$ to also be a partition number of $\overline{C}\,'$). Also, $x = 60$ will have a solid dot because it is in the domain of the function.

We have two intervals on which to determine the sign of $\overline{C}\,'(x)$: $(0,60)$ and $(60,\infty)$. We will select the x-values $x = 30$ and $x = 100$ to test in order to determine the sign of $\overline{C}\,'(x)$:

$$\begin{aligned} \overline{C}\,'(x) &= 0.04 - \frac{144}{x^2} \implies \\ \overline{C}\,'(30) &= 0.04 - \frac{144}{(30)^2} = -0.12 < 0 \\ \overline{C}\,'(100) &= 0.04 - \frac{144}{(100)^2} = 0.0256 > 0 \end{aligned}$$

We will use this information to fill in the sign chart of $\overline{C}\,'(x)$. See **Figure 3.1.18**.

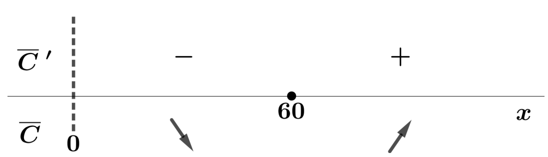


Figure 3.1.18: Sign chart of $\overline{C}\,'(x)$, where $\overline{C}(x) = 0.04x + 15 + 144x^{-1}$

This sign chart tells us that the average cost is decreasing when production is between 0 and 60 high-end office chairs, and the average cost is increasing when production is more than 60 chairs. A local extremum occurs at $x = 60$; **Theorem 3.2** tells us there is a local minimum at $x = 60$. To find the local minimum (value), we substitute $x = 60$ into $\overline{C}(x)$:

$$\begin{aligned} \overline{C}(x) &= 0.04x + 15 + 144x^{-1} \implies \\ \overline{C}(60) &= 0.04(60) + 15 + 144(60)^{-1} \\ &= \$19.80 \text{ per chair} \end{aligned}$$

Thus, when 60 high-end chairs are produced, the average cost function has a local minimum of \$19.80 per chair.

■

Try It # 5:
The aforementioned office supply company sells external web cameras. The price-demand function for the web cameras is $p(x) = -0.2x + 56$, where $p(x)$ is the price, in dollars, per camera when x web cameras are purchased. Find the intervals where the company's revenue function, R, is increasing/decreasing as well as the local extrema of the revenue function, assuming $R(x)$ is the company's revenue, in dollars, when x web cameras are sold.

Using Technology

Most calculators can numerically estimate, or approximate, the instantaneous rate of change (i.e., derivative) of a function at a certain x-value. Not only can we use the calculator to check our answers when we calculate rates of change, we can also use it to quickly find the sign of $f'(x)$ when creating sign charts. We can input the original function f into the calculator and have it find the value of $f'(x)$ at the x-value we are testing in an interval. This allows us to quickly see if the derivative is positive or negative on each interval of the sign chart of $f'(x)$.

The calculator used in this textbook is the TI-84 Plus CE. Any version of the TI-83 and TI-84 has this functionality, but your screen and inputs may look a little different from the ones we show here.

▪ **Example 9** Using technology, find $f'(4)$ if $f(x) = x^2 - 3$.

Solution:
Using the TI-84 Plus CE, we press the MATH button and then select command 8: nDeriv(. See **Figure 3.1.19**.

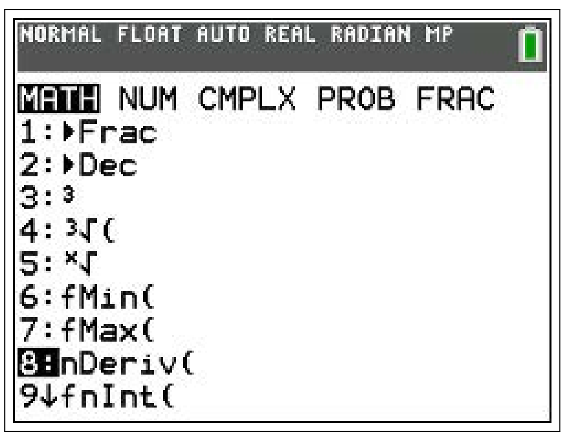


Figure 3.1.19: Location of the nDeriv(command in the TI-84 Plus CE

With newer calculators like this one, we can see the inputs needed: first is the variable, next is the function, and the third is the x-value where we want to find the value of the derivative. Inputting these three items in the correct location gives us the screen shown in **Figure 3.1.20**.

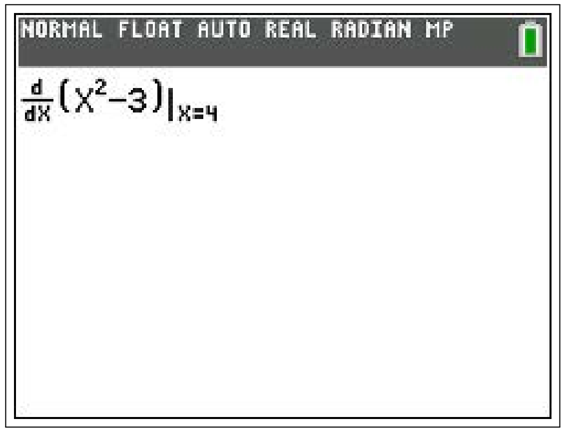


Figure 3.1.20: Inputs of the nDeriv(command: the variable, function, and x-value

Pressing ENTER will display the answer, 8. See **Figure 3.1.21**.

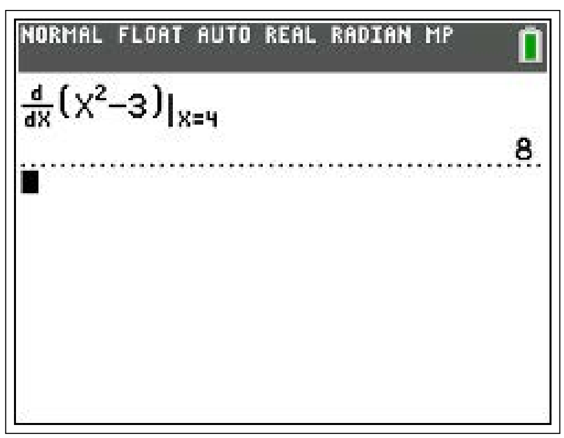


Figure 3.1.21: Approximation of the derivative after pressing ENTER

■

■ **Example 10** Using technology, find $f'(-3)$ if $f(x) = \dfrac{7}{5-9x}$. Round your answer to three decimal places, if necessary.

Solution:
Pressing the MATH button and then selecting command 8: nDeriv(will lead us to the same screen as before. Then, we input the correct variable, function, and x-value. Pressing ENTER gives the value shown in **Figure 3.1.22**:

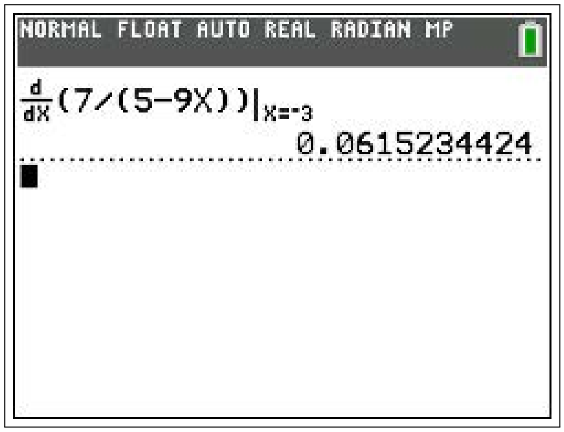


Figure 3.1.22: Using the nDeriv(command to find $f'(-3)$, where $f(x) = \dfrac{7}{5-9x}$

This answer, 0.062, is an *approximation* of the exact answer, which is $\frac{63}{1024}$. It is much preferred to have an exact answer (i.e., a decimal that terminates, a fraction, or an expression that can yield the exact number) rather than an approximation. Because we know the exact answer is a fraction, let's use the calculator to try and convert our approximation to a fraction.

Under the MATH menu, the command 1: Frac converts approximations to fractions (when possible). Selecting this command and pressing ENTER yields the same result. See **Figure 3.1.23**.

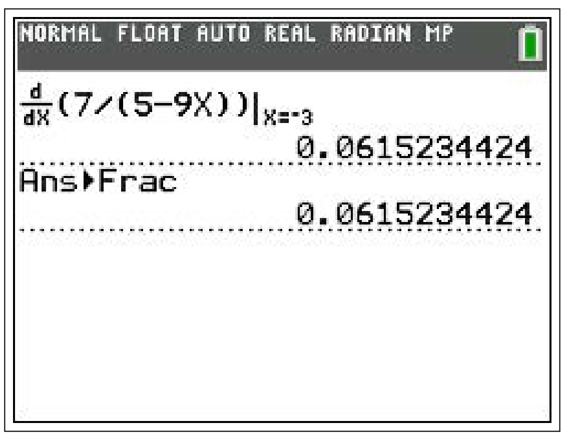


Figure 3.1.23: Attempting to use the Frac command to convert an approximate answer to a fraction (an exact answer)

It appears as though nothing happened! The calculator was unable to give us the exact answer. However, this answer is good enough for this problem because we can round to three decimal places. But, if we needed an exact answer, we would have to find $f'(-3)$ using the limit definition of the derivative (by hand).

■

In any case, using this calculator function to check your answers or quickly find the sign of $f'(x)$ on each of the intervals of your sign chart may prove quite useful!

! *Using a graphing calculator can help you check your answer. However, the calculator can only give an approximation. Even trying to convert the answer in the previous example to a fraction did not help. Using calculus is the only way to guarantee exact answers!*

Try It Answers

1. f is increasing on $(-\infty, 4)$, and f is decreasing on $(4, \infty)$.

2. **a.** $x = -2$; $x = 0$; $x = 6$

   b. $x = -1$; $x = 1$

   c. $x = -\sqrt{27}$; $x = 0$; $x = \sqrt{27}$

3. **a.** The critical values of f are $x = 1$ and $x = 3$; f is increasing on $(-\infty, 1)$ and $(3, \infty)$, and f is decreasing on $(1, 3)$; f has a local maximum of 6 at $x = 1$ and a local minimum of 2 at $x = 3$.

   b. The critical value of f is $x = 9$; f is increasing on $(-\infty, 9)$, and f is decreasing on $(9, \infty)$; f has a local maximum of $\frac{1}{e^9}$ at $x = 9$ and no local minima.

4. **a.** $x = 0$; $x = 2$; $x = 7$; $x = 11$; $x = 12$

   b. $x = 0$; $x = 2$; $x = 11$; $x = 12$

   c. f is increasing on $(-\infty, 0)$, $(2, 7)$, and $(12, \infty)$, and f is decreasing on $(0, 2)$, $(7, 11)$, and $(11, 12)$.

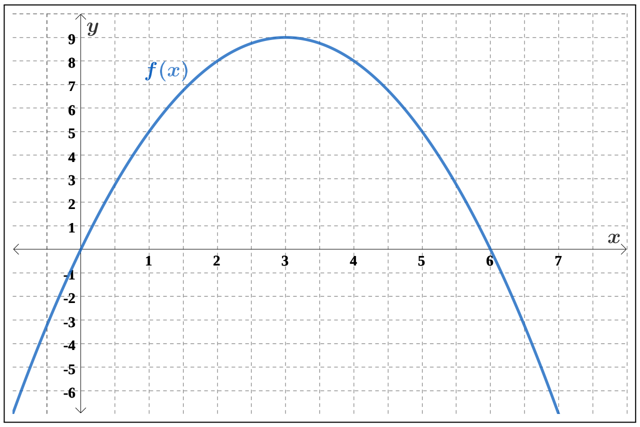
   d. f has a local maximum at $x = 0$ and local minima at $x = 2$ and $x = 12$.

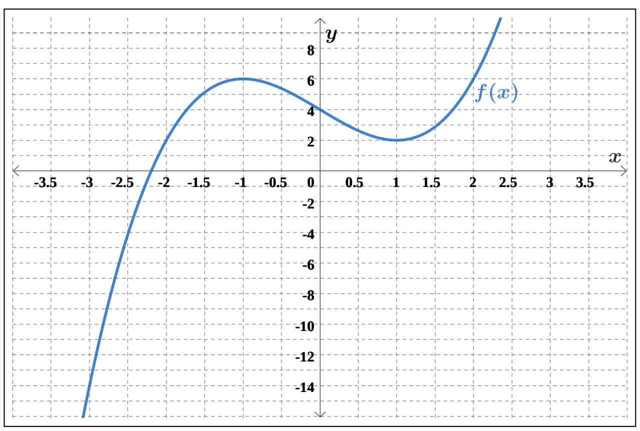
5. The revenue function, R, is increasing on $(0, 140)$ and decreasing on $(140, \infty)$, and it has a local maximum of \$3920 when $x = 140$ web cameras are sold.

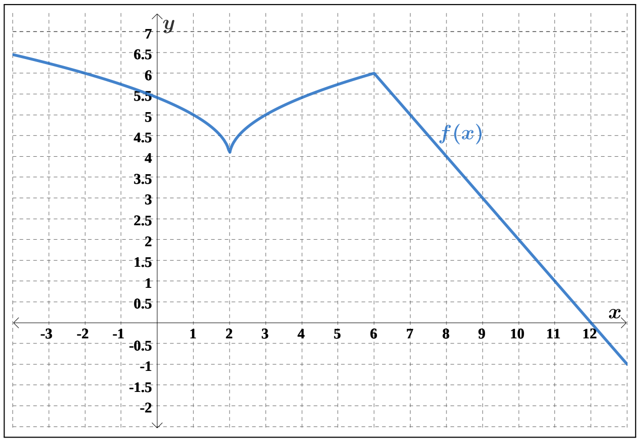
EXERCISES

Basic Skills Practice

For Exercises 1 - 3, the graph of f is shown. Find (a) the x-values where $f'(x) = 0$, (b) the x-values where $f'(x)$ does not exist, (c) the intervals where f is increasing/decreasing, and (d) the x-values where any local extrema of f occur (specify the type).

1. 

2. 

3. 

For Exercises 4 - 7, find (a) the partition numbers of f' and (b) the intervals where f is increasing/decreasing.

4. $f(x) = x^3 - 27x + 4$

5. $f(x) = 3x^4 - 40x^3 + 150x^2 - 25$

6. $f(x) = \frac{1}{4}x^4 - \frac{4}{3}x^3 - 16x^2$

7. $f(x) = -3x^4 - 14x^3 - 9x^2 + 6$

For Exercises 8 - 11, find the critical value(s) of the function.

8. $f(x) = x^3 + 15x^2 + 27x - 1$

9. $g(x) = x^5 + 5x^4 - 35x^3$

10. $h(t) = t^4 - 2t^2 + 9$

11. $f(x) = 9x^4 - 22x^3 - 30x^2 + 50$

For Exercises 12 - 16, f' and the domain of f are given. Assuming f is continuous on its domain, find (a) the partition numbers of f', (b) the critical values of f, (c) the intervals where f is increasing/decreasing, and (d) the x-values where any local extrema of f occur using the First Derivative Test (specify the type).

12. $f'(x) = (x+5)^2(x-2)(x-8)$; domain of f is $(-\infty, \infty)$

13. $f'(x) = 3x(x+4)(x-7)^3$; domain of f is $(-\infty, \infty)$

14. $f'(x) = \dfrac{(x-1)(x+2)}{(x-6)^4}$; domain of f is $(-\infty, 6) \cup (6, \infty)$

15. $f'(x) = \dfrac{-2x(x+8)^2}{(x+1)^6}$; domain of f is $(-\infty, -1) \cup (-1, \infty)$

16. $f'(x) = \dfrac{6(x-3)^2(x+9)^3}{(x-5)^7}$;   domain of f is $(-\infty, 5) \cup (5, \infty)$

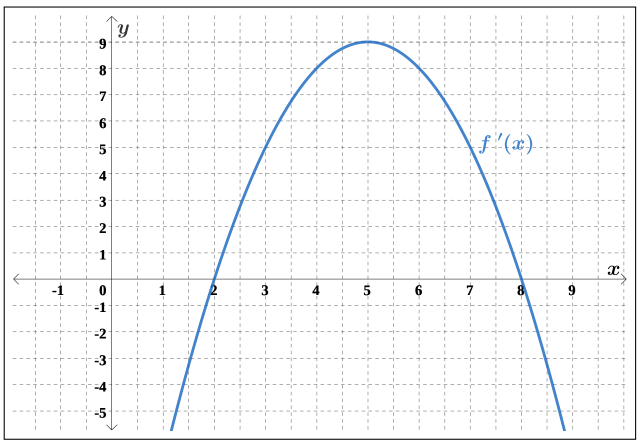
For Exercises 17 - 19, find (a) the partition numbers of f', (b) the critical values of f, (c) the intervals where f is increasing/decreasing, and (d) any local extrema of f as well as where they occur using the First Derivative Test (specify the type).

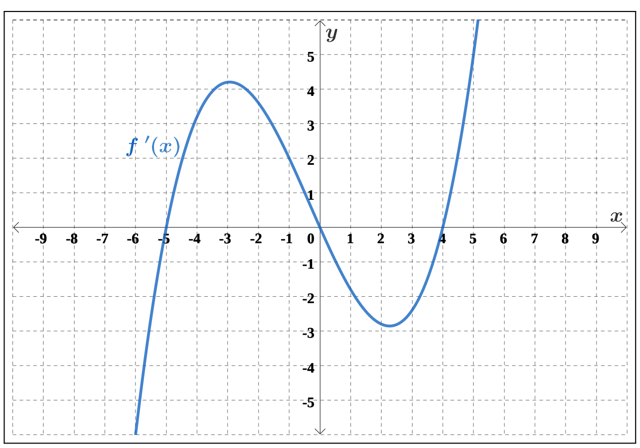
17. $f(x) = x^3 - 9x^2 + 24x - 5$

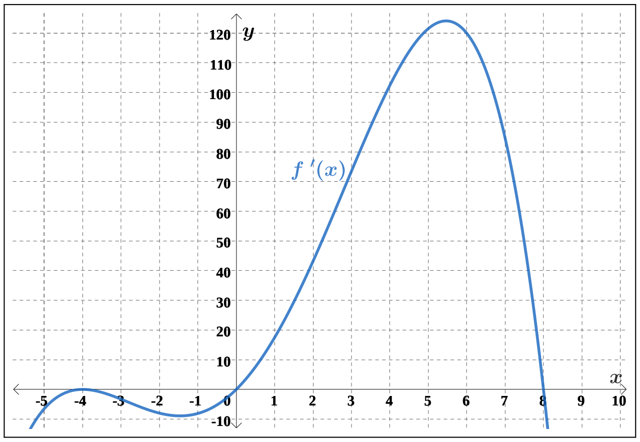
18. $f(x) = x^5 - 5x^4 - 20x^3 + 100$

19. $f(x) = 3x^4 - \dfrac{20}{3}x^3 - 16x^2$

For Exercises 20 - 22, the graph of f' is shown. Assuming f is continuous on its domain of $(-\infty, \infty)$, find (a) the partition numbers of f', (b) the critical values of f, (c) the intervals where f is increasing/decreasing, and (d) the x-values where any local extrema of f occur (specify the type).

20. 

21. 

22. 

23. Party Tune, a company that makes sing-a-long microphones for kids, has a revenue function given by $R(x) = 186x - 0.2x^2$ dollars, where x is the number of microphones sold.
    (a) Determine the intervals where the revenue is increasing and where it is decreasing.
    (b) Find the value of x in which revenue is maximum.

24. The profit function for a company that sells designer coats is given by $P(x) = -0.25x^2 + 460x - 10,500$ dollars, where x is the number of coats sold.
    (a) Determine the intervals where the profit is increasing and where it is decreasing.
    (b) Find the value of x in which profit is maximum.

25. A company that sells vacuum cleaners has a weekly cost function given by $C(x) = 4100 - 26x + 0.2x^2$ dollars, where x is the number of vacuum cleaners produced each week.
    (a) Determine the intervals where the cost is increasing and where it is decreasing.
    (b) Find the value of x in which cost is minimum.

Intermediate Skills Practice

For Exercises 26 - 29, find the intervals where f is increasing/decreasing.

26. $f(x) = xe^x$

27. $f(x) = \dfrac{x^2}{x+3}$

28. $f(x) = x^2 \ln(x)$

29. $f(x) = \left(x^2 - 25\right)^{2/3}$

For Exercises 30 - 35, find the critical value(s) of the function.

30. $f(x) = 2\ln(x) - 10x$

31. $g(x) = 3e^{0.2x^2}$

32. $h(t) = 2 + \frac{1}{t} + \frac{1}{t^2}$

33. $f(x) = x\ln(x)$

34. $g(t) = \frac{t^2 + 1}{t^2 - 4}$

35. $h(x) = \frac{e^x}{x^2}$

For Exercises 36 - 45, find (a) the critical values of f, (b) the intervals where f is increasing/decreasing, and (c) any local extrema of f as well as where they occur (specify the type).

36. $f(x) = (x-4)e^x$

37. $f(x) = x^3\ln(x)$

38. $f(x) = \frac{3x}{x^2 - 1}$

39. $f(x) = \sqrt[3]{\left(x^2 - 3x\right)^4}$

40. $f(x) = \frac{\ln(x)}{x^2}$

41. $f(x) = e^{x^3 - 27x}$

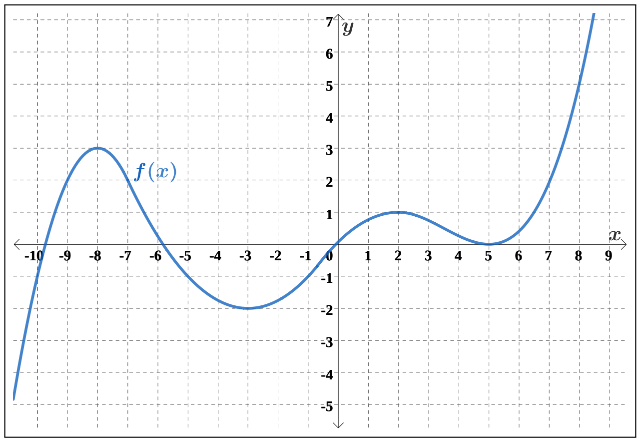
42. $f(x) = \frac{x^3}{x^2 - 9}$

43. $f(x) = \ln\left(x^2 + 4\right)$

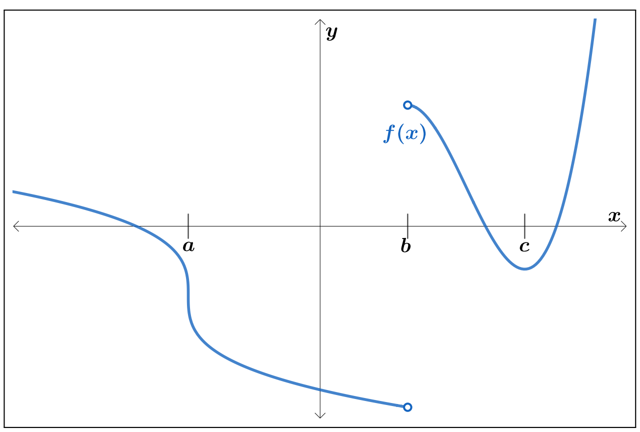
44. $f(x) = e^x + e^{-x}$

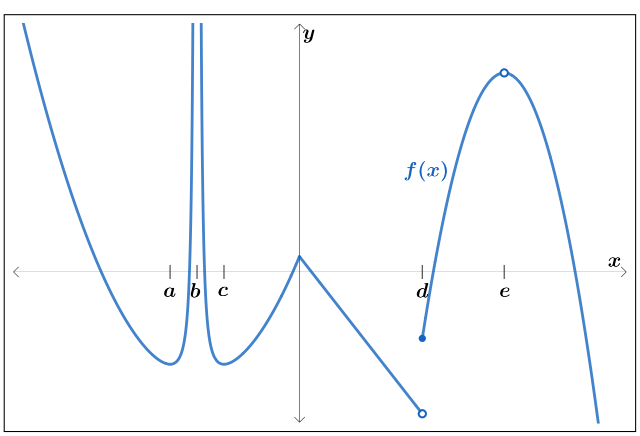
45. $f(x) = \frac{16}{x} + x$

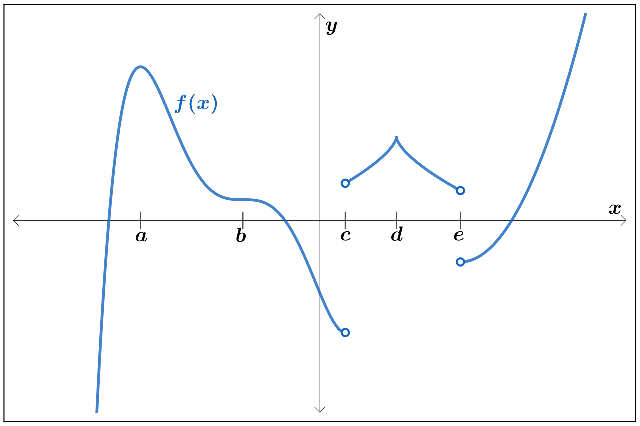
46. Given the graph of f shown below, find (a) the partition numbers of $f\,'$, (b) the critical values of f, (c) the intervals where f is increasing/decreasing, and (d) any local extrema of f as well as where they occur (specify the type).



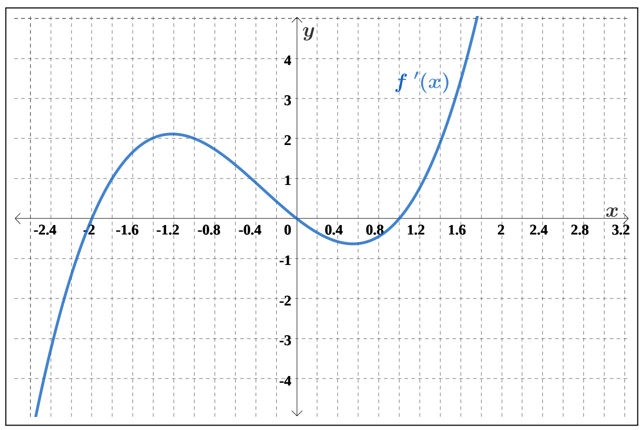
For Exercises 47 - 49, the graph of f is shown. Find (a) the partition numbers of $f\,'$, (b) the critical values of f, (c) the intervals where f is increasing/decreasing, and (d) the x-values where any local extrema of f occur (specify the type).

47. 

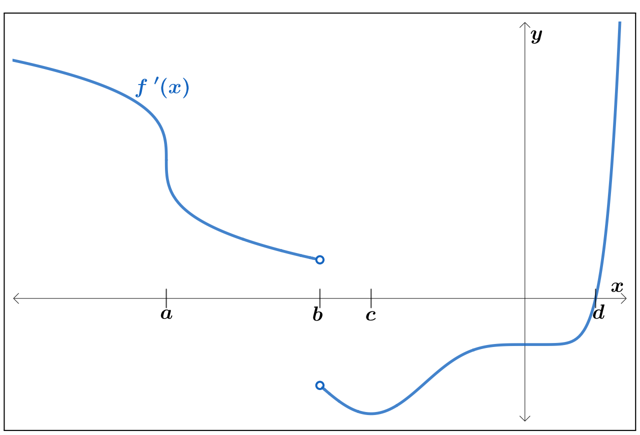
48. 

49. 

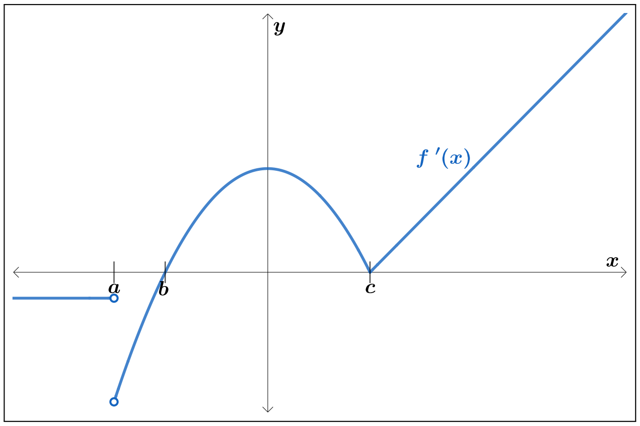
50. Given the graph of $f\,'$ shown below and that f is continuous on its domain of $(-\infty,\infty)$, find (a) the partition numbers of $f\,'$, (b) the critical values of f, (c) the intervals where f is increasing/decreasing, and (d) the x-values where any local extrema of f occur (specify the type).



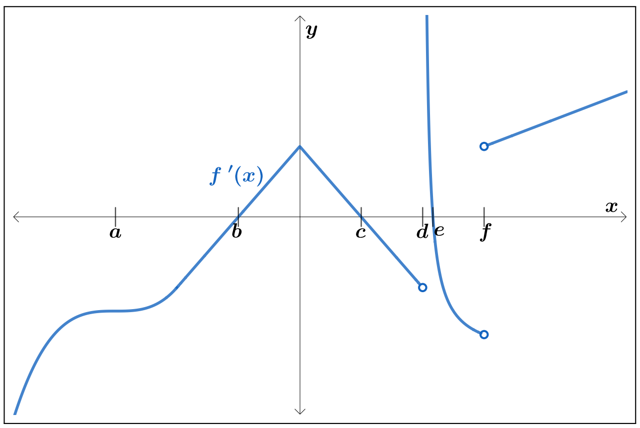
For Exercises 51 - 53, the graph of $f\,'$ and the domain of f are given. Assuming f is continuous on its domain, find (a) the partition numbers of $f\,'$, (b) the critical values of f, (c) the intervals where f is increasing/decreasing, and (d) the x-values where any local extrema of f occur (specify the type).

51. 

domain of f: $(-\infty,\infty)$

52. 

domain of f: $(-\infty, a) \cup (a, \infty)$

53. 

domain of f: $(-\infty, d) \cup (d, f) \cup (f, \infty)$

54. The profit function for a lamp manufacturer is given by $P(x) = 30x - 0.3x^2 - 240$ dollars, where x is the number of lamps sold. Find the intervals where P is increasing/decreasing, and interpret your answer.

55. A company specializing in custom cell phone cases has a weekly cost function given by $C(x) = 5000 - 32x + 0.4x^2$ dollars, where x is the number of cell phone cases produced each week. Find the intervals where C is increasing/decreasing, and interpret your answer.

56. Feeling Fabulous, a local spa, sells an all-inclusive spa package. It has determined its revenue function for the packages to be $R(x) = 200x - 0.8x^2$ dollars, where x is the number of spa packages sold. Find the intervals where R is increasing/decreasing, and interpret your answer.

57. A company has a revenue function given by $R(x) = -25x^3 + 225x^2$ dollars, where x is the number of items, in hundreds, sold.

    (a) Find the intervals where R is increasing/decreasing, and interpret your answer.

    (b) Find the company's average revenue function, $\overline{R}$.

    (c) How many items must the company sell so their average revenue per item is maximum?

Mastery Practice

For Exercises 58 - 60, find the intervals where f is increasing/decreasing.

58. $f(x) = \dfrac{\ln\left(x^2\right)}{x}$

59. $f(x) = xe^{-0.5x^2}$

60. $f(x) = \dfrac{4x^2}{(x-1)^3}$

For Exercises 61 - 63, find the critical value(s) of the function.

61. $f(x) = (x-3)^3(x-6)^2$

62. $g(x) = \dfrac{e^{2x}}{(x-4)^2}$

63. $h(t) = t\ln(2t)$

For Exercises 64 - 77, find (a) the critical values of f, (b) the intervals where f is increasing/decreasing, and (c) the x-values where any local extrema of f occur (specify the type).

64. $f(x) = \dfrac{e^{x^2}}{x^2}$

65. $f'(x) = e^{(x-1)}(x-7)(x+2)^5(x+6)^4$; f is continuous on its domain of $(-\infty, 7) \cup (7, \infty)$

66. $f(x) = 2\sqrt[3]{x} - \sqrt[3]{x^2}$

67. $f(x) = (x-3)^2 e^{-2x}$

68. $f'(x) = \dfrac{-3x^2(x-9)}{(x+4)^2}$; f is continuous on its domain of $(-\infty, -4) \cup (-4, \infty)$

69. $f(x) = \dfrac{\ln(3x)}{x^2}$

70. $f(x) = \dfrac{(x+5)^4}{(x-1)^3}$

71. $f'(x) = \dfrac{1-\ln(x)}{x^2}$; f is continuous on its domain of $(0, \infty)$

72. $f(x) = (x-1)^4(x+5)^6$

73. $f(x) = \dfrac{3}{e^{x^2}+2}$

74. $f'(x) = 12x^5 - 26x^4 - 16x^3$; f is continuous on its domain of $(-\infty, \infty)$

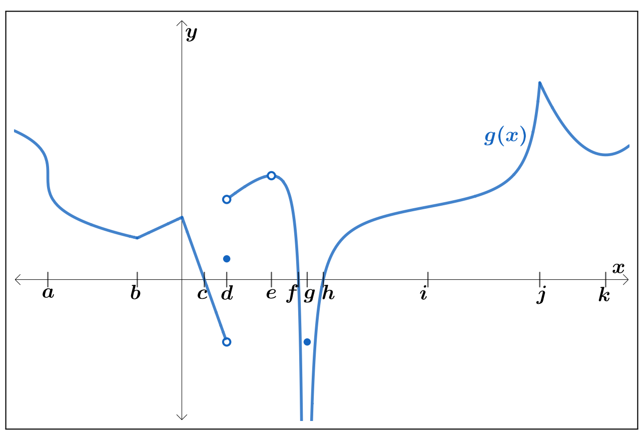
75. $f(x) = x\ln\left(x^2\right)$

76. $f(x) = \dfrac{x^2}{(x-7)^2}$

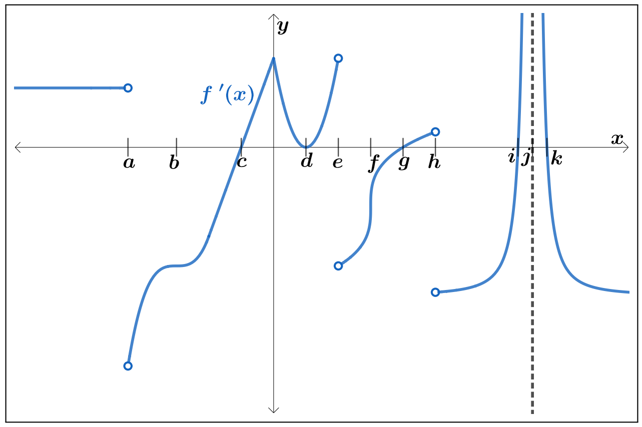
77. $f(x) = 2^{x^2}\left(x^2-4\right)$

78. Given $g'(t) = -t^3(t+a)^4(t-1)(t-b)^{15}$, where $1 < a < b$, and that g is continuous on its domain of all real numbers except $t = 1$, find (a) the partition numbers of g', (b) the critical values of g, (c) the intervals where g is increasing/decreasing, and (d) the t-values where any local extrema of g occur (specify the type).

79. Given the graph of g shown below, find (a) the partition numbers of g', (b) the critical values of g, (c) the intervals where g is increasing/decreasing, and (d) the x-values where any local extrema of g occur (specify the type).



80. Given the graph of $f\,'$ shown below and that f is continuous on its domain of $(-\infty, a) \cup (a, j) \cup (j, \infty)$, find (a) the partition numbers of $f\,'$, (b) the critical values of f, (c) the intervals where f is increasing/decreasing, and (d) the x-values where any local extrema of f occur (specify the type).



81. The price, in dollars, per flashlight when x flashlights are sold is given by the price-demand function $p(x) = 15 - 0.001x$.

    (a) Determine the intervals where the revenue is increasing and where it is decreasing.

    (b) What price should be charged for each flashlight so revenue is maximum?

82. Summer Splashin' can sell 100 pop-up pools when the price is \$141 per pool. If the price per pool decreases by \$1, they can sell an additional 50 pop-up pools. The company has fixed costs of \$8,000 and production costs of \$75 per pool. Find the intervals where the company's profit function is increasing/decreasing (assuming the company's price-demand function is linear), and interpret your answer.

83. Big-time Floats makes and sells floats shaped like unicorns. Their daily cost function is given by $C(x) = .05x^2 + 32x + 180$ dollars, where x is the number of unicorn floats produced each day. How many floats must be produced each day so the average cost per float is minimum?

84. A company has a revenue function given by $R(x) = -2x^3 + 96x$ dollars, where x is the number of items, in hundreds, sold. Determine the interval(s) where the marginal revenue is decreasing.

85. The weekly profit function for a bubble machine manufacturer, Pop It Like It's Hot, is given by $P(x) = 50x - 0.3x^2 - 270$ dollars, where x is the number of bubble machines made and sold each week. Find the manufacturer's profit when the average profit per bubble machine is maximum.

86. A company has a price-demand function given by $p(x) = 50e^{-0.002x}$ dollars per item when x items are sold. Determine how many items the company must sell so revenue is maximum.

Communication Practice

87. Describe the three-step process we can use to perform the Increasing/Decreasing Test to find the intervals where a function is increasing/decreasing.

88. If $f(x) > 0$ on the interval $(1,5)$, is f increasing on $(1,5)$? Explain.

89. Compare and contrast partition numbers of $f\,'$ and critical values of f.

90. If $f\,'(-2) = 0$, is $x = -2$ a critical value of f? Explain.

91. What does the First Derivative Test enable us to find?

92. Describe the three conditions necessary for a function to have a local extremum at $x = c$.

93. Describe the four-step process we can use to determine if a function has local extrema as well as the intervals where the function is increasing/decreasing.

94. If $f\,'(5) = 0$, does f have a local extremum at $x = 5$? Explain.

95. When using the four-step process for finding local extrema and the intervals where a function is increasing/decreasing, why is it important to indicate on the sign chart for the first derivative whether a function is defined at each partition number?

96. If a function does not have any critical values, is it possible for the function to have local extrema? Explain.

3.2 Analyzing Graphs with the Second Derivative

Thus far, we have learned that the derivative f' provides a great deal of information about the graph of f. In this section, we will investigate the information we can obtain about the graph of f from taking the derivative *of the derivative*. In other words, we will find $\frac{d}{dx}(f'(x))$, which is called the **second derivative** of f. It turns out that not only does the second derivative provide useful information about the graph of f, it also provides relevant information about the graph of the **first derivative**, f'. Why? Because the second derivative is the derivative of the first derivative!

Learning Objectives:

In this section, you will learn how to apply the Concavity Test to find where a function is concave up, concave down, and has inflection points as well as solve problems involving a real-world application: the point of diminishing returns. Also, you will learn how to apply the Second Derivative Test to find local extrema. Upon completion, you will be able to:

- Find the second derivative of a function.
- Examine the graph of a function to determine where its slopes are increasing and where they are decreasing.
- Define partition number of f''.
- Find the partition numbers of f'' for a function f given its rule.
- Specify the conditions necessary for a function to have an inflection point.
- Apply the Concavity Test to determine where a function f is concave up/down and has inflection points given its rule.
- Apply the Concavity Test to determine where a function f is concave up/down and has inflection points given the rule for its derivative f' and the domain of f.
- Apply the Concavity Test to determine where a function f is concave up/down and has inflection points given the rule for its second derivative f'' and the domain of f.
- Analyze the graph of f to determine the partition numbers of f'', the intervals where f is concave up/down, and the inflection points of f.
- Analyze the graph of f' to determine the partition numbers of f'', the intervals where f is concave up/down, and the x-values where f has inflection points.
- Analyze the graph of f'' to determine the partition numbers of f'', the intervals where f is concave up/down, and the x-values where f has inflection points.
- Investigate the relationships between f, f', and f'', and construct a table indicating these relationships.
- Analyze the graph of f, f', or f'' to determine pertinent information about the other two functions.
- Calculate the point of diminishing returns and explain its significance.
- Compare and contrast the Concavity Test and the Second Derivative Test.
- Compare and contrast the First Derivative Test and Second Derivative Test.
- Specify the type(s) of critical values we may attempt to use with the Second Derivative Test.
- Describe the conditions in which the Second Derivative Test fails.
- Perform the Second Derivative Test, if possible, to find and classify the local extrema of a function.

Finding the Second Derivative

We begin by formally defining the second derivative of a function f:

Definition
The **second derivative** of a function f is the derivative of f':

$$\frac{d}{dx}(f'(x)) = f''(x)$$

■

■ **Example 1** Find $f''(x)$, where $f(x) = 7x^4 - 3x^3 + 44x^2 + 22x - 83$.

Solution:
Before we can find $f''(x)$, we must find $f'(x)$:

$$\begin{aligned} f'(x) &= 7 \cdot 4x^3 - 3 \cdot 3x^2 + 44 \cdot 2x^1 + 22 - 0 \\ &= 28x^3 - 9x^2 + 88x + 22 \end{aligned}$$

Taking the derivative of this function to obtain $f''(x)$ gives

$$\begin{aligned} f''(x) &= 28 \cdot 3x^2 - 9 \cdot 2x^1 + 88 \\ &= 84x^2 - 18x + 88 \end{aligned}$$

■

There are several ways to write the second derivative:

Second Derivative Notation

Lagrange notation: $f''(x)$ and y''

Leibniz notation: $\frac{d^2 f}{dx^2}$ and $\frac{d^2 y}{dx^2}$

■ **Example 2** Find each of the following.

a. $f''(x)$ if $f(x) = 4^{8-2x^3}$

b. $\frac{d^2 g}{dx^2}$ if $g(x) = \frac{15\ln(x)}{-2x^7 + 14}$

c. $h''(x)$ if $h'(x) = \left(-9x^{11} - 8x^{31}\right)\sqrt{x^2 - 5x^{17}}$

Solution:

a. The first step when finding the second derivative is to find the first derivative of the function. Starting with the Chain Rule to obtain $f'(x)$ gives

$$\begin{aligned} f'(x) &= 4^{8-2x^3}(\ln(4))\frac{d}{dx}\left(8 - 2x^3\right) \\ &= 4^{8-2x^3}(\ln(4))\left(0 - 6x^2\right) \\ &= 4^{8-2x^3}(\ln(4))\left(-6x^2\right) \end{aligned}$$

Remember our rule on whether or not to algebraically manipulate our result: If we need to use the function to find another quantity or solve an equation, then we should algebraically manipulate the function. In this case, we are going to take the derivative of this function, so we should algebraically manipulate it, or in this case, rewrite it:

$$\begin{aligned} f'(x) &= 4^{8-2x^3}(\ln(4))\left(-6x^2\right) \\ &= -6(\ln(4))x^2 4^{8-2x^3} \end{aligned}$$

Now, we take the derivative of this function using the Product Rule:

$$\begin{aligned} f''(x) &= \frac{d}{dx}\left(-6(\ln(4))x^2 4^{8-2x^3}\right) \\ &= -6(\ln(4))\left(\frac{d}{dx}\left(x^2 4^{8-2x^3}\right)\right) \\ &= -6(\ln(4))\left(\left(x^2\right)\left(\frac{d}{dx}\left(4^{8-2x^3}\right)\right) + \left(4^{8-2x^3}\right)\left(\frac{d}{dx}\left(x^2\right)\right)\right) \\ &= -6(\ln(4))\left(\left(x^2\right)\left(4^{8-2x^3}(\ln(4))\left(\frac{d}{dx}\left(8-2x^3\right)\right)\right) + \left(4^{8-2x^3}\right)(2x)\right) \\ &= -6(\ln(4))\left(\left(x^2\right)\left(4^{8-2x^3}(\ln(4))\left(-6x^2\right)\right) + \left(4^{8-2x^3}\right)(2x)\right) \end{aligned}$$

When finding $f''(x)$, notice we had to find $\frac{d}{dx}\left(4^{8-2x^3}\right)$. We already found this derivative previously; it is $f'(x)$! Because we found it already, we could have substituted $f'(x) = -6(\ln(4))x^2 4^{8-2x^3}$ at that step instead of finding the derivative again.

b. Recall $g(x) = \dfrac{15\ln(x)}{-2x^7+14}$. To find the first derivative, we start with the Quotient Rule:

$$\begin{aligned} \frac{dg}{dx} &= \frac{\left(-2x^7+14\right)\left(\frac{d}{dx}(15\ln(x))\right) - (15\ln(x))\left(\frac{d}{dx}\left(-2x^7+14\right)\right)}{\left(-2x^7+14\right)^2} \\ &= \frac{\left(-2x^7+14\right)\left(\frac{15}{x}\right) - (15\ln(x))\left(-14x^6\right)}{\left(-2x^7+14\right)^2} \\ &= \frac{-30x^6+210x^{-1}+210x^6(\ln(x))}{\left(-2x^7+14\right)^2} \end{aligned}$$

Remember, the only reason we algebraically manipulated the function is so that taking the derivative of this function will be slightly easier. We take the derivative again starting with the Quotient Rule. Because this function is more complicated, we will start by finding the derivative of the numerator only:

$$\begin{aligned} \frac{d}{dx}\left(-30x^6+210x^{-1}+210x^6(\ln(x))\right) &= -180x^5 - 210x^{-2} + \frac{d}{dx}\left(210x^6\ln(x)\right) \\ &= -180x^5 - 210x^{-2} + \left(210x^6\right)\left(\frac{d}{dx}(\ln(x))\right) + (\ln(x))\left(\frac{d}{dx}\left(210x^6\right)\right) \\ &= -180x^5 - 210x^{-2} + \left(\left(210x^6\right)\frac{1}{x} + (\ln(x))\left(1260x^5\right)\right) \\ &= -180x^5 - 210x^{-2} + 210x^5 + 1260x^5(\ln(x)) \\ &= 30x^5 - 210x^{-2} + 1260x^5(\ln(x)) \end{aligned}$$

Next, we find the derivative of the denominator:

$$\begin{aligned}\frac{d}{dx}\left(\left(-2x^7+14)^2\right)\right) &= 2\left(-2x^7+14\right)\left(\frac{d}{dx}\left(-2x^7+14\right)\right)\\ &= 2\left(-2x^7+14\right)\left(-14x^6\right)\\ &= -28x^6\left(-2x^7+14\right)\\ &= 56x^{13}-392x^6\end{aligned}$$

Using the Quotient Rule and placing the two derivatives we found in the proper places gives

$$\begin{aligned}\frac{d^2g}{dx^2} &= \frac{d}{dx}\left(\frac{-30x^6+210x^{-1}+210x^6(\ln(x))}{\left(-2x^7+14\right)^2}\right)\\ &= \frac{\left(-2x^7+14\right)^2\left(30x^5-210x^{-2}+1260x^5(\ln(x))\right)-\left(-30x^6+210x^{-1}+210x^6(\ln(x))\right)\left(56x^{13}-392x^6\right)}{\left(\left(-2x^7+14\right)^2\right)^2}\end{aligned}$$

c. Recall that $h'(x)=\left(-9x^{11}-8x^{31}\right)\sqrt{x^2-5x^{17}}$. We must be careful! $h''(x)$ is the derivative of $h'(x)$. We are already given $h'(x)$, so we just need to take the derivative of this function. We start with the Product Rule and then incorporate the Chain Rule:

$$\begin{aligned}h''(x) &= \left(-9x^{11}-8x^{31}\right)\left(\frac{d}{dx}\left(\sqrt{x^2-5x^{17}}\right)\right)+\left(\sqrt{x^2-5x^{17}}\right)\left(\frac{d}{dx}\left(-9x^{11}-8x^{31}\right)\right)\\ &= \left(-9x^{11}-8x^{31}\right)\left(\frac{1}{2}\left(x^2-5x^{17}\right)^{-\frac{1}{2}}\left(\frac{d}{dx}\left(x^2-5x^{17}\right)\right)\right)+\left(\sqrt{x^2-5x^{17}}\right)\left(-99x^{10}-248x^{30}\right)\\ &= \left(-9x^{11}-8x^{31}\right)\left(\frac{1}{2}\left(x^2-5x^{17}\right)^{-\frac{1}{2}}\left(2x-85x^{16}\right)\right)+\left(\sqrt{x^2-5x^{17}}\right)\left(-99x^{10}-248x^{30}\right)\end{aligned}$$

▪

N *When finding the second derivative, all of our previous derivative rules apply. We will still use the Introductory Derivative Rules, the Product Rule, the Quotient Rule, and the Chain Rule (in all its forms).*

There is nothing stopping us from taking higher-order derivatives (third, fourth, and so on), except that the information we glean from these is beyond the scope of this textbook. The notations for higher-order derivatives are given below:

Higher Order Derivative Notation

Lagrange notation: $f^{(n)}(x)$ and $y^{(n)}$

Leibniz notation: $\dfrac{d^nf}{dx^n}$ and $\dfrac{d^ny}{dx^n}$

where n is the number of derivatives taken of the original function.

Try It # 1:

Find each of the following.

a. $f''(x)$ if $f(x)=\dfrac{4x^2-3}{10x^3-27x^2+15}$

b. y'' if $y=\left(-9x^{11}-8x^{31}\right)^{\frac{2}{3}}$

Concavity

Similar to the way the first derivative f' provides information about where the graph of a function f is increasing and where it is decreasing, as well as where it has local extrema, the second derivative also provides important information about the graph of f. The second derivative f'' informs us of the **concavity** of f.

Before we investigate this idea from a calculus standpoint, let's define concavity pictorially:

Definition
Pictorially, a graph or part of a graph is **concave up** if it looks like a bowl or part of a bowl. A graph or part of a graph is **concave down** if it looks like an upside-down bowl or part of an upside-down bowl. ▪

This is not a very precise definition, but we can use calculus to find a more precise definition. Let's look at the graph of a function that is concave up. In other words, the graph of the function "bends upward" like a bowl or part of a bowl. If we also look at lines tangent to the graph of the function, we see the slopes of the tangent lines are *increasing*. This will always be true for graphs that are concave up! See **Figures 3.2.1**, **3.2.2**, and **3.2.3**.

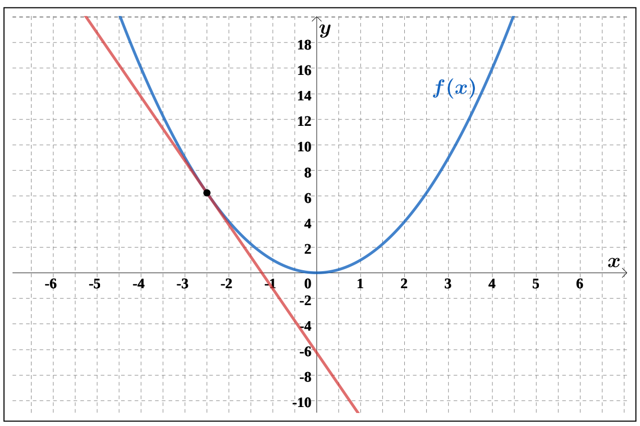


Figure 3.2.1: Graphs of a function that is concave up and a line tangent to the function at $x = -2.5$

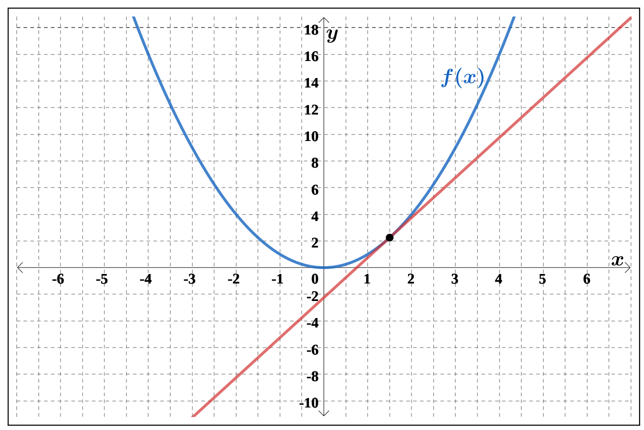


Figure 3.2.2: Graphs of a function that is concave up and a line tangent to the function at $x = 1.5$

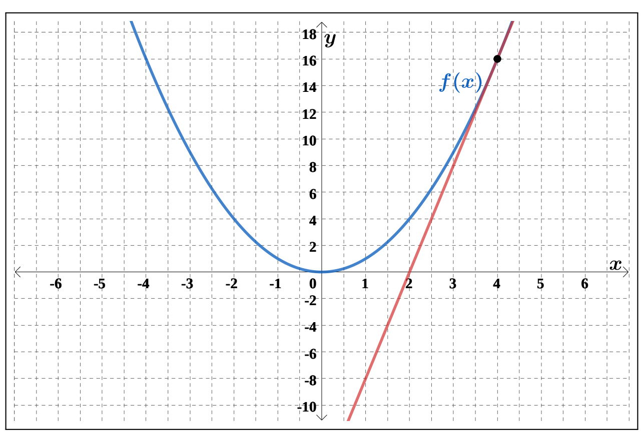


Figure 3.2.3: Graphs of a function that is concave up and a line tangent to the function at $x = 4$

Let's take a more in-depth look at why the slopes of the graph of the function shown in **Figures 3.2.1**, **3.2.2**, and **3.2.3** are always increasing. The slopes on the interval $(-\infty, 0)$ are negative because the function is decreasing. Because the function is getting less steep, its slopes are getting less negative. Thus, its slopes are increasing (numbers that are getting less negative are increasing numbers). This will always be true for graphs that are concave up!

Likewise, the slopes on the interval $(0,\infty)$ are positive because the function is increasing. Because the function is getting steeper (i.e., "more" steep), its slopes are getting more positive. Thus, its slopes are increasing (numbers that are getting more positive are increasing numbers). This will always be true for graphs that are concave up!

The same analysis holds for graphs of functions that are concave down (i.e., "bend downward" like an upside-down bowl or part of a bowl). The slopes of the graphs of such functions will be decreasing regardless of whether the function itself is increasing or decreasing. If the function is increasing and concave down, the slopes are getting less positive, and if the function is decreasing and concave down, the slopes are getting more negative. In either case, the slopes are decreasing. See **Figures 3.2.4**, **3.2.5**, and **3.2.6**.

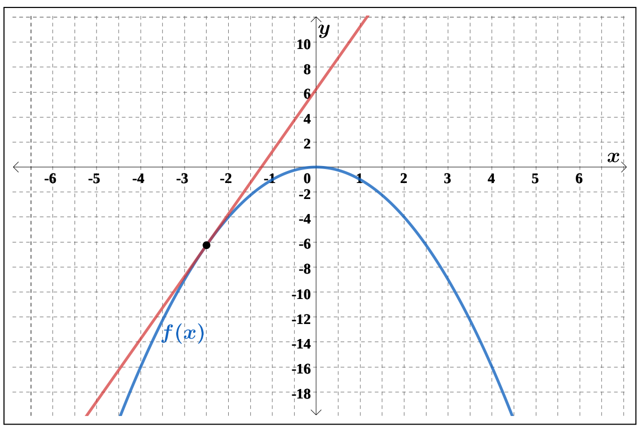


Figure 3.2.4: Graphs of a function that is concave down and a line tangent to the function at $x = -2.5$

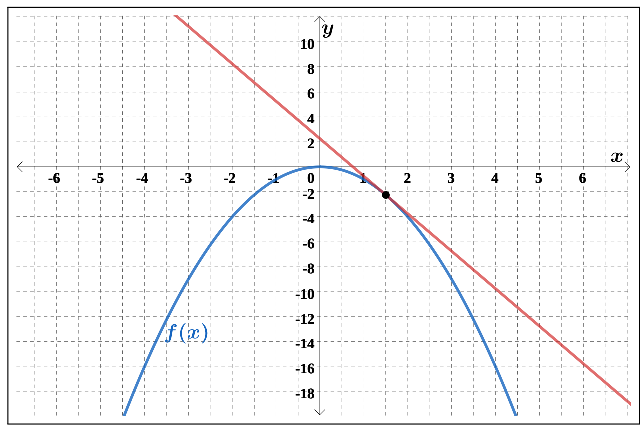


Figure 3.2.5: Graphs of a function that is concave down and a line tangent to the function at $x = 1.5$

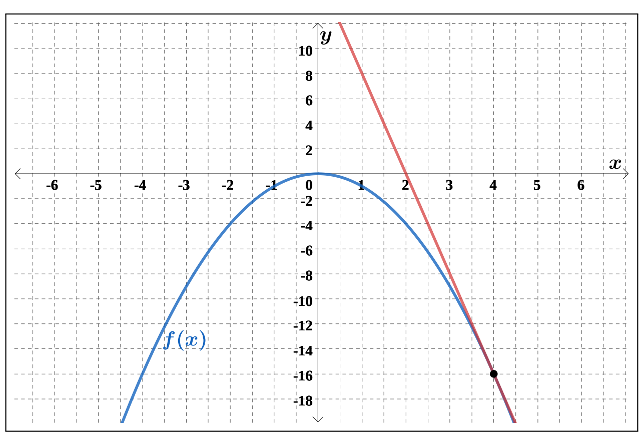


Figure 3.2.6: Graphs of a function that is concave down and a line tangent to the function at $x = 4$

Recall from **Section 3.1** that the slopes of the graph of a function f are given by its derivative function, f'. Thus, we can now state a formal definition of concavity:

Definition

Suppose the function f is differentiable on an interval.

- If f' is increasing on the interval, then ***f* is concave up** on the interval.
- If f' is decreasing on the interval, then ***f* is concave down** on the interval.

▪

Notice that for concavity, we are looking at the rate of change *of the rate of change (slope)*. In other words, we are looking at the second derivative! This allows us to interpret concavity from a calculus perspective:

Theorem 3.3 Concavity Test

Suppose f is twice-differentiable on an interval.

- If $f''(x) > 0$ on the interval, then ***f* is concave up** on the interval.
- If $f''(x) < 0$ on the interval, then ***f* is concave down** on the interval.

To help us remember the relationship between the concavity of f and the sign of $f''(x)$, think of smiling and frowning faces as shown in Figure **3.2.7**. The + eyes on the smiley face remind us that concave up (given by the smile) means the second derivative is positive, and the − eyes on the frowning face tell us the second derivative is negative when the function is concave down.

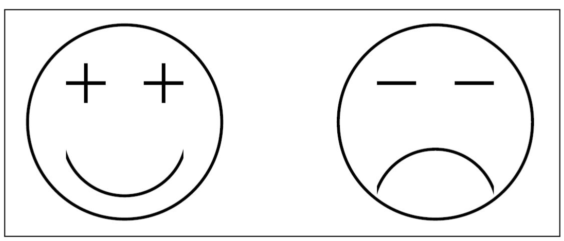


Figure 3.2.7: Smiling and frowning faces to show the relationship between the concavity of a function and the sign of its second derivative

If the graph of f changes concavity at a point, then that point is called an **inflection point**:

Definition

If f is continuous and changes concavity at a point, then the point is called an **inflection point**. ▪

In other words, $(c, f(c))$ is an inflection point of f if $f''(x)$ changes sign.

If we have the graph of a function, we can just look at the graph to find the intervals of concavity and identify inflection points. But, how can we determine the concavity and find inflection points if we are only give the rule of the function and not its graph? We will answer this question by asking another question: If f has an inflection point where it switches from concave up to concave down (or vice versa), what does that imply about f'' at the inflection point?

Well, if the concavity of f changes, then $f''(x)$ must change sign. There are only two possibilities for $f''(x)$ to change sign: $f''(x) = 0$ or if $f''(x)$ does not exist (for further explanation, see the analysis in **Section 3.1** about $f'(x)$ changing sign and there being only two ways to cross a river).

Each function whose graph is shown in **Figures 3.2.8** and **3.2.9** has an inflection point at $x = 4$ because the concavity of f changes:

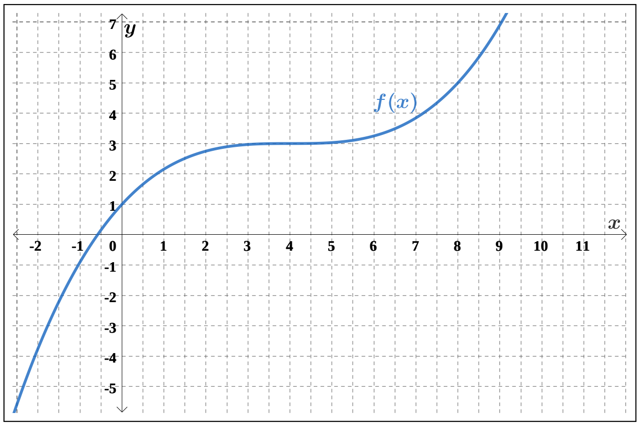


Figure 3.2.8: Graph of a function f that has an inflection point at $x = 4$

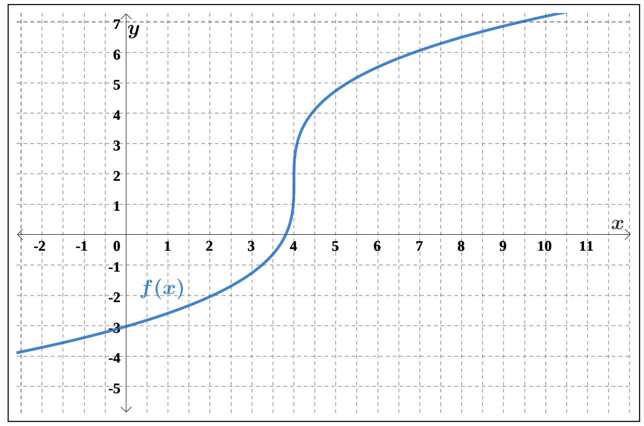


Figure 3.2.9: Graph of a function f that has an inflection point at $x = 4$

In **Figure 3.2.8**, we see the function f is concave down on the interval $(-\infty, 4)$, so $f''(x) < 0$ on this interval. On the interval $(4, \infty)$, the function is concave up, so $f''(x) > 0$ on this interval. And, in this case, $f''(4) = 0$.

In **Figure 3.2.9**, we see the function f is concave up on $(-\infty, 4)$, so $f''(x) > 0$ on this interval. On the interval $(4, \infty)$, f is concave down, so $f''(x) < 0$ on this interval. Looking at the behavior at $x = 4$, we see the function has a vertical tangent line. Thus, $f'(4)$ does not exist, and so $f''(4)$ does not exist either.

This may seem somewhat familiar. In **Section 3.1**, we learned that if a function has a local extremum, it must occur at an x-value in which $f'(x) = 0$ or $f'(x)$ does not exist (among other conditions). The analysis here is similar for inflection points, only we are dealing with the second derivative instead of the first. Thus, if a function has an inflection point at $x = c$, it must occur where $f''(c) = 0$ or $f''(c)$ does not exist. However, just like with local extrema, if there is an x-value such that $f''(x) = 0$ or $f''(x)$ does not exist, that does not necessarily mean there is an inflection point at that x-value.

For f to have an inflection point at $x = c$, the second derivative must also change sign (switching between positive/negative) as shown in **Figures 3.2.8** and **3.2.9**. Also, $x = c$ must be in the domain of f. There cannot be an inflection point if there is no point on the graph!

Thus, to find the intervals of concavity and any inflection points of a function f, we will create a sign chart of $f''(x)$. We will partition the sign chart again with "important" x-values. As discussed previously, these "important" x-values are where $f''(x) = 0$ and $f''(x)$ does not exist. We call these x-values **partition numbers of f''**. They will be used to *partition* the sign chart of $f''(x)$:

Definition

Partition numbers of f'' are the x-value(s) where $f''(x) = 0$ or $f''(x)$ does not exist. ▪

N *Similar to partition numbers of f', we will need to make judgement calls on which x-values that meet the conditions of the definition are "important enough" to be considered partition numbers of f'' based on how meaningfully the x-values partition the number line.*

Combining all of this information, we now have a process for finding where a function is concave up/down and has inflection points.

Concavity Test

To determine the intervals where a function is concave up and where it is concave down, which will determine if there are any inflection points, we will use the following three steps to apply the **Concavity Test**:

Finding Intervals of Concavity and Inflection Points of f Using the Concavity Test

1. Determine the domain of f.
2. Find the partition numbers of f''. Recall that these are the x-values where $f''(x) = 0$ or $f''(x)$ does not exist.
3. Create a sign chart of $f''(x)$ using the partition numbers of f'' to divide the sign chart (number line) into intervals. Then,
   - Indicate whether or not each partition number is in the domain of f by drawing a solid dot or open circle on the sign chart.
   - Select any x-value in each interval, and evaluate the second derivative f'' at each x-value to determine whether $f''(x)$ is positive or negative on each interval. Indicate whether $f''(x)$ is positive or negative on each interval by writing "+" or "-".
   - Apply the Concavity Test to find the intervals of concavity and any inflection points of f:
     - If $f''(x)$ is positive, then f is concave up.
     - If $f''(x)$ is negative, then f is concave down.

We will demonstrate using this process in the following example.

Example 3 For each of the following, find the intervals of concavity and any inflection points of f.

a. $f(x) = \frac{1}{2}x^4 + \frac{1}{3}x^3 - 14x^2$

b. $f(x) = \frac{x}{x^2 - 9}$

c. $f(x) = xe^{-x^2}$

Solution:

a. We will use the three steps outlined previously to apply the Concavity Test:

1. Determine the domain of f:

f is a polynomial, so its domain is $(-\infty, \infty)$.

2. Find the partition numbers of f'':

First, we must find $f'(x)$:

$$f'(x) = 2x^3 + x^2 - 28x$$

Now, we can find $f''(x)$:

$$f''(x) = 6x^2 + 2x - 28$$

To find the partition numbers of f'', we find the x-values where $f''(x) = 0$ or $f''(x)$ does not exist. Because f'' is a polynomial and has a domain of all real numbers, it will exist everywhere. Thus, we only need to find the x-values where $f''(x) = 0$:

$$\begin{aligned} f''(x) &= 0 \implies \\ 6x^2 + 2x - 28 &= 0 \\ 2(3x^2 + x - 14) &= 0 \\ 2(3x+7)(x-2) &= 0 \\ &\implies x = -\frac{7}{3} \text{ and } x = 2. \end{aligned}$$

Thus, the partition numbers of f'' are $x = -\frac{7}{3}$ and $x = 2$.

3. Create a sign chart of $f''(x)$:

We will place the partition numbers of f'' on a number line, with $x = -\frac{7}{3}$ and $x = 2$ both having solid dots to indicate they are in the domain of f.

Next, we need to determine the sign of $f''(x)$ on the intervals $\left(-\infty, -\frac{7}{3}\right)$, $\left(-\frac{7}{3}, 2\right)$, and $(2, \infty)$. We will choose the x-values $x = -4$, 0, and 3 to test:

$$\begin{aligned} f''(x) &= 6x^2 + 2x - 28 \implies \\ f''(-4) &= 6(-4)^2 + 2(-4) - 28 = 60 > 0 \\ f''(0) &= 6(0)^2 + 2(0) - 28 = -28 < 0 \\ f''(3) &= 6(3)^2 + 2(3) - 28 = 32 > 0 \end{aligned}$$

Using this information, we can fill in the sign chart of $f''(x)$. Because we are also interested in the information this yields for f, we include that information below the number line. See **Figure 3.2.10**.

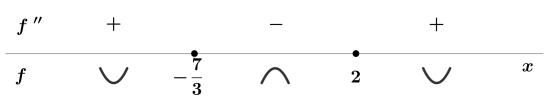


Figure 3.2.10: Sign chart of $f''(x)$ with the corresponding information for $f(x) = \frac{1}{2}x^4 + \frac{1}{3}x^3 - 14x^2$

Looking at the sign chart, we see that $f''(x)$ changes sign at both $x = -\frac{7}{3}$ and $x = 2$. Now, we must check to see if the function is defined at these x-values, which it is. Thus, f has inflection points at both of the partition numbers of f''. To find the y-values associated with the inflection points, we substitute the x-values into the *original function* f:

$$\begin{aligned} f(x) &= \frac{1}{2}x^4 + \frac{1}{3}x^3 - 14x^2 \implies \\ f\left(-\frac{7}{3}\right) &= \frac{1}{2}\left(-\frac{7}{3}\right)^4 + \frac{1}{3}\left(-\frac{7}{3}\right)^3 - 14\left(-\frac{7}{3}\right)^2 = -\frac{10,633}{162} \approx -65.6358 \\ f(2) &= \frac{1}{2}(2)^4 + \frac{1}{3}(2)^3 - 14(2)^2 = -\frac{136}{3} \approx -45.3333 \end{aligned}$$

In conclusion, $f(x)$ is concave up on $\left(-\infty, -\frac{7}{3}\right)$ and $(2, \infty)$. It is concave down on $\left(-\frac{7}{3}, 2\right)$ and has inflection points at $\left(-\frac{7}{3}, -\frac{10,633}{162}\right)$ and $\left(2, -\frac{136}{3}\right)$.

We can check our work by looking at the graph of f shown in **Figure 3.2.11**, but using the Concavity Test allows us to determine the concavity and find the inflection points *completely algebraically!* Again, this is why we study calculus: to calculate with more precision than technology allows, which is to say we can get *exact* answers! With this function, technology would give us roughly the same information we calculated, but if we want exact answers, then calculus is the way to go!

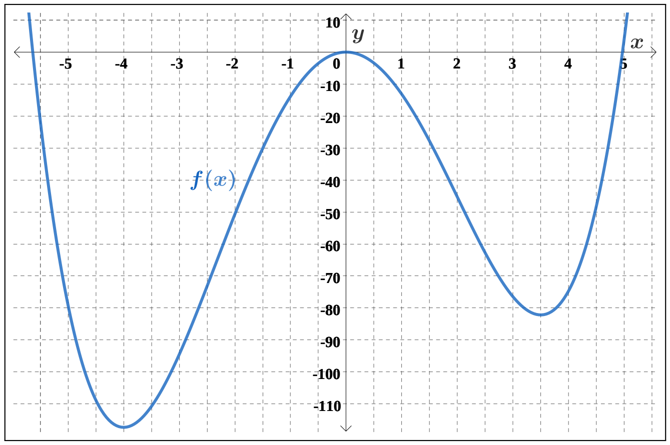


Figure 3.2.11: Graph of the function $f(x) = \frac{1}{2}x^4 + \frac{1}{3}x^3 - 14x^2$

b. Recall $f(x) = \dfrac{x}{x^2-9}$. Again, we will use the three steps outlined previously to apply the Concavity Test:

1. Determine the domain of f:

Considering the three domain restrictions listed previously, the function does not have any logarithms or even roots. However, there is division, so we need to make sure the denominator does not equal zero:

$$\begin{aligned} x^2 - 9 &\neq 0 \\ x^2 &\neq 9 \\ \implies x &\neq 3 \text{ and } x \neq -3 \end{aligned}$$

Thus, the domain of f is $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$.

2. Find the partition numbers of f'':

First, we must find $f'(x)$. Using the Quotient Rule gives

$$\begin{aligned} f'(x) &= \frac{(x^2-9)\left(\frac{d}{dx}(x)\right) - x\left(\frac{d}{dx}(x^2-9)\right)}{\left(x^2-9\right)^2} \\ &= \frac{(x^2-9)(1) - x(2x)}{\left(x^2-9\right)^2} \\ &= \frac{x^2-9-2x^2}{\left(x^2-9\right)^2} \\ &= \frac{-x^2-9}{\left(x^2-9\right)^2} \end{aligned}$$

Now, we can find $f''(x)$. We start with the Quotient Rule again and incorporate the Chain Rule:

$$\begin{aligned}
f''(x) &= \frac{\left(x^2-9\right)^2\left(\frac{d}{dx}\left(-x^2-9\right)\right)-\left(-x^2-9\right)\left(\frac{d}{dx}\left(\left(x^2-9\right)^2\right)\right)}{\left(\left(x^2-9\right)^2\right)^2} \\
&= \frac{\left(x^2-9\right)^2(-2x)-\left(-x^2-9\right)\left(2\left(x^2-9\right)\left(\frac{d}{dx}\left(x^2-9\right)\right)\right)}{\left(x^2-9\right)^4} \\
&= \frac{\left(x^2-9\right)^2(-2x)-\left(-x^2-9\right)\left(2\left(x^2-9\right)(2x)\right)}{\left(x^2-9\right)^4}
\end{aligned}$$

Because we need to use this function to find the partition numbers of f'', we need to algebraically manipulate it. We will factor the term x^2-9 from the numerator (remember we factor the lowest power of the term) and continue simplifying:

$$\begin{aligned}
f''(x) &= \frac{\left(x^2-9\right)^2(-2x)-\left(-x^2-9\right)\left(2\left(x^2-9\right)(2x)\right)}{\left(x^2-9\right)^4} \\
&= \frac{\left(x^2-9\right)\left[\left(x^2-9\right)(-2x)-\left(-x^2-9\right)(2)(2x)\right]}{\left(x^2-9\right)^4} \\
&= \frac{\cancel{\left(x^2-9\right)}\left[\left(x^2-9\right)(-2x)-\left(-x^2-9\right)(2)(2x)\right]}{\left(x^2-9\right)^{\cancel{4}\,3}} \\
&= \frac{\left(x^2-9\right)(-2x)-\left(-x^2-9\right)(2)(2x)}{\left(x^2-9\right)^3} \\
&= \frac{\left(-2x^3+18x\right)-\left(-x^2-9\right)(4x)}{\left(x^2-9\right)^3} \\
&= \frac{\left(-2x^3+18x\right)-\left(-4x^3-36x\right)}{\left(x^2-9\right)^3} \\
&= \frac{-2x^3+18x+4x^3+36x}{\left(x^2-9\right)^3} \\
&= \frac{2x^3+54x}{\left(x^2-9\right)^3}
\end{aligned}$$

To find the partition numbers of f'', we find the x-values where $f''(x)=0$ or $f''(x)$ does not exist. $f''(x)=0$ when the numerator equals 0, and $f''(x)$ does not exist when the denominator equals zero (remember we are looking at the domain of f'' when determining where it does not exist). First, we will find the x-values where $f''(x)=0$:

$$\begin{aligned} f''(x) = 0 &\implies \\ 2x^3 + 54x &= 0 \\ 2x\left(x^2+27\right) &= 0 \end{aligned}$$

This gives us two equations to solve: $2x = 0$ and $x^2 + 27 = 0$. $2x = 0$ gives $x = 0$, but $x^2 + 27 = 0$ has no solution because $x^2 \neq -27$ due to the fact that x^2 will be always be nonnegative.

Now, to find the x-values where $f''(x)$ does not exist, we set the denominator equal to zero:

$$\begin{aligned} f''(x) \text{ DNE } &\implies \\ \left(x^2-9\right)^3 &= 0 \\ \left(\left(x^2-9\right)^3\right)^{\frac{1}{3}} &= (0)^{\frac{1}{3}} \\ x^2 - 9 &= 0 \\ x^2 &= 9 \\ \implies x = 3 &\text{ and } x = -3 \end{aligned}$$

Thus, the partition numbers of f'' are $x = -3$, $x = 0$, and $x = 3$.

3. Create a sign chart of $f''(x)$:

We will place the partition numbers of f'' on a number line, with $x = 0$ having a solid dot and $x = -3$ and $x = 3$ having open circles to indicate which are included and which are not included in the domain of f, respectively.

Next, we need to determine the sign of $f''(x)$ on the intervals $(-\infty, -3)$, $(-3, 0)$, $(0, 3)$, and $(3, \infty)$. We will choose the x-values $x = -5,\ -1,\ 1$, and 5 to test:

$$\begin{aligned} f''(x) &= \frac{2x^3+54x}{\left(x^2-9\right)^3} \implies \\ f''(-5) &= \frac{2(-5)^3+54(-5)}{\left((-5)^2-9\right)^3} = -\frac{65}{512} < 0 \\ f''(-1) &= \frac{2(-1)^3+54(-1)}{\left((-1)^2-9\right)^3} = \frac{7}{64} > 0 \\ f''(1) &= \frac{2(1)^3+54(1)}{\left((1)^2-9\right)^3} = -\frac{7}{64} < 0 \\ f''(5) &= \frac{2(5)^3+54(5)}{\left((5)^2-9\right)^3} = \frac{65}{512} > 0 \end{aligned}$$

Using this information, we fill in the sign chart of $f''(x)$. Again, we include the corresponding information this yields for f below the number line. See **Figure 3.2.12**.

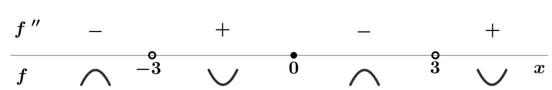


Figure 3.2.12: Sign chart of $f''(x)$ with the corresponding information for $f(x) = \dfrac{x}{x^2-9}$

Looking at the sign chart, we see that $f''(x)$ changes sign at all three partition numbers. We now check to see if the function is defined at these x-values. Because f is only defined at $x = 0$, there is only one inflection point. Remember, even if $f''(x)$ changes sign, that does not necessarily mean there is an inflection point. We must check to see if the x-value is in the domain of f. To find the y-value associated with the inflection point, we substitute $x = 0$ into the *original function* f:

$$f(x) = \frac{x}{x^2-9} \implies$$
$$f(0) = \frac{0}{(0)^2-9} = \frac{0}{-9} = 0$$

In conclusion, f is concave down on $(-\infty,-3)$ and $(0,3)$, and it is concave up on $(-3,0)$ and $(3,\infty)$. Also, f has an inflection point at $(0,0)$.

Again, we can check our work by looking at the graph of f. See **Figure 3.2.13**.

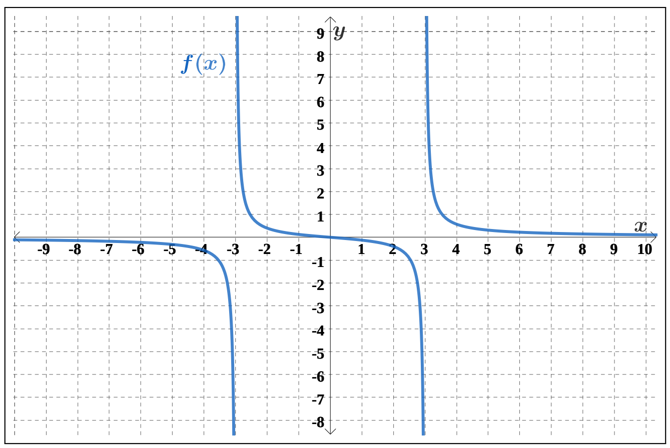


Figure 3.2.13: Graph of the function $f(x) = \dfrac{x}{x^2-9}$

c. Recall $f(x) = xe^{-x^2}$. Again, we will use the three steps outlined previously to apply the Concavity Test:

1. Determine the domain of f:

Considering the three domain restrictions listed previously, the function does not have any logarithms or even roots. There is division hidden in the function due to a negative exponent: $e^{-x^2} = \dfrac{1}{e^{x^2}}$. However, because $e^{x^2} > 0$, this denominator will never equal zero. Therefore, the domain of f is $(-\infty,\infty)$.

2. Find the partition numbers of f'':

First, we must find $f'(x)$. Using the Product Rule and then incorporating the Chain Rule gives

$$\begin{aligned} f'(x) &= x\left(\frac{d}{dx}\left(e^{-x^2}\right)\right) + e^{-x^2}\left(\frac{d}{dx}(x)\right) \\ &= xe^{-x^2}\left(\frac{d}{dx}\left(-x^2\right)\right) + e^{-x^2}(1) \\ &= xe^{-x^2}(-2x) + e^{-x^2} \end{aligned}$$

Because we are going to use this function to find $f''(x)$, we need to algebraically manipulate it by factoring the term e^{-x^2}:

$$\begin{aligned} f'(x) &= xe^{-x^2}(-2x) + e^{-x^2} \\ &= e^{-x^2}\left[x(-2x)+1\right] \\ &= e^{-x^2}\left(-2x^2+1\right) \end{aligned}$$

Now, we can find $f''(x)$. We start with the Product Rule again and incorporate the Chain Rule:

$$\begin{aligned} f''(x) &= e^{-x^2}\left(\frac{d}{dx}\left(-2x^2+1\right)\right) + \left(-2x^2+1\right)\left(\frac{d}{dx}\left(e^{-x^2}\right)\right) \\ &= e^{-x^2}(-4x) + \left(-2x^2+1\right)\left(e^{-x^2}\left(\frac{d}{dx}\left(-x^2\right)\right)\right) \\ &= -4xe^{-x^2} + \left(-2x^2+1\right)\left(e^{-x^2}(-2x)\right) \end{aligned}$$

Because we need to use this function to find the partition numbers of f'', we need to algebraically manipulate it. We will factor the term e^{-x^2} again:

$$\begin{aligned} f''(x) &= -4xe^{-x^2} + \left(-2x^2+1\right)\left(e^{-x^2}(-2x)\right) \\ &= e^{-x^2}\left[-4x + \left(-2x^2+1\right)(-2x)\right] \\ &= e^{-x^2}\left(-4x+4x^3-2x\right) \\ &= e^{-x^2}\left(4x^3-6x\right) \end{aligned}$$

To find the partition numbers of f'', we find the x-values where $f''(x) = 0$ or $f''(x)$ does not exist. $f''(x)$ does not have an even root or a logarithm, but it does have implied division as a result of the term $e^{-x^2} = \frac{1}{e^{x^2}}$. But, recall from our previous discussion that because $e^{x^2} > 0$, the denominator will never equal zero. Hence, there are no domain issues and $f''(x)$ will exist for all values of x. Thus, we only need to find the x-values where $f''(x) = 0$:

$$\begin{aligned} f''(x) = 0 &\implies \\ e^{-x^2}\left(4x^3-6x\right) &= 0 \end{aligned}$$

This gives us two equations to solve: $e^{-x^2} = 0$ and $4x^3 - 6x = 0$. Because $e^{-x^2} > 0$, there are no solutions to the first equation. Solving the second equation gives

$$\begin{aligned} 4x^3 - 6x &= 0 \\ 2x(2x^2-3) &= 0 \end{aligned}$$

Setting $2x = 0$ gives the partition number $x = 0$, and setting $2x^2 - 3 = 0$ gives

$$\begin{aligned} 2x^2 - 3 &= 0 \\ 2x^2 &= 3 \\ x^2 &= \frac{3}{2} \\ \implies x &= \sqrt{\frac{3}{2}} \text{ and } x = -\sqrt{\frac{3}{2}} \end{aligned}$$

Thus, the partition numbers of f'' are $x=-\sqrt{\frac{3}{2}}$, $x=0$, and $x=\sqrt{\frac{3}{2}}$.

3. Create a sign chart of $f''(x)$:

We will place the partition numbers of f'' on a number line. All of the partition numbers will have solid dots on the number line because they are all in the domain of f.

Now, we need to determine the sign of $f''(x)$ on the intervals $\left(-\infty,-\sqrt{\frac{3}{2}}\right)$, $\left(-\sqrt{\frac{3}{2}},0\right)$, $\left(0,\sqrt{\frac{3}{2}}\right)$, and $\left(\sqrt{\frac{3}{2}},\infty\right)$. We will choose the x-values $x=-2,\ -1,\ 1$, and 2 to test:

$$\begin{aligned} f''(x) &= e^{-x^2}\left(4x^3-6x\right) \implies \\ f''(-2) &= e^{-(-2)^2}\left(4(-2)^3-6(-2)\right) \approx -0.3663 < 0 \\ f''(-1) &= e^{-(-1)^2}\left(4(-1)^3-6(-1)\right) \approx 0.7358 > 0 \\ f''(1) &= e^{-(1)^2}\left(4(1)^3-6(1)\right) \approx -0.7358 < 0 \\ f''(2) &= e^{-(2)^2}\left(4(2)^3-6(2)\right) \approx 0.3663 > 0 \end{aligned}$$

Using this information, we can fill in the sign chart of $f''(x)$. Because we are also interested in the corresponding information this yields for f, we will include that below the number line. See **Figure 3.2.14**.

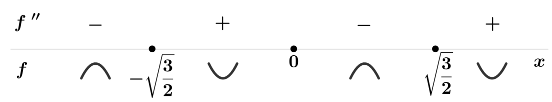


Figure 3.2.14: Sign chart of $f''(x)$ with the corresponding information for $f(x)=xe^{-x^2}$

Looking at the sign chart, we see that $f''(x)$ changes sign at all three partition numbers. We now check to see if the function is defined at these x-values, which it is. Thus, f has inflection points at all three partition numbers. To find the y-values associated with the inflection points, we substitute the x-values into the *original function* f:

$$f(x) = xe^{-x^2} \implies$$

$$f\left(-\sqrt{\frac{3}{2}}\right) = -\sqrt{\frac{3}{2}}e^{-\left(-\sqrt{\frac{3}{2}}\right)^2} = -\sqrt{\frac{3}{2}}e^{-\frac{3}{2}} \approx -0.2733$$

$$f(0) = (0)e^{-(0)^2} = 0$$

$$f\left(\sqrt{\frac{3}{2}}\right) = \sqrt{\frac{3}{2}}e^{-\left(\sqrt{\frac{3}{2}}\right)^2} = \sqrt{\frac{3}{2}}e^{-\frac{3}{2}} \approx 0.2733$$

In conclusion, f is concave down on $\left(-\infty,-\sqrt{\frac{3}{2}}\right)$ and $\left(0,\sqrt{\frac{3}{2}}\right)$, and it is concave up on $\left(-\sqrt{\frac{3}{2}},0\right)$ and $\left(\sqrt{\frac{3}{2}},\infty\right)$. The function has inflection points at $\left(-\sqrt{\frac{3}{2}},-\sqrt{\frac{3}{2}}e^{-\frac{3}{2}}\right)$, $(0,0)$, and $\left(\sqrt{\frac{3}{2}},\sqrt{\frac{3}{2}}e^{-\frac{3}{2}}\right)$.

Again, we can check our work by looking at the graph of f shown in **Figure 3.2.15**:

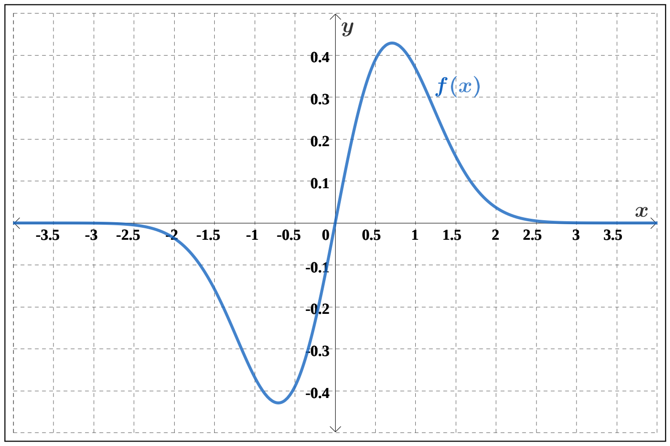


Figure 3.2.15: Graph of the function $f(x) = xe^{-x^2}$

■

Do not assume that the sign chart of $f''(x)$ will always have intervals with alternating signs. This just happened to be the case for the functions in the previous example. We will see that it is not always the case in the next example!

Try It # 2:
For each of the following, find the intervals of concavity and any inflection points of the function.

a. $f(x) = e^{2x} - e^{-2x}$

b. $g(x) = x^3 \ln(2x)$

■ **Example 4** Given f is continuous on its domain of $(-\infty, 2) \cup (2, \infty)$ and $f''(x) = -4x(x-2)^3(x+5)^2(x-7)$, find the intervals of concavity and the x-values of any inflection points of f.

Solution:
Even though we were not given the original function, f, we still proceed with the steps to apply the Concavity Test because we need to find the intervals of concavity and location of any inflection points:

1. Determine the domain of f:

We are given the domain of f is $(-\infty, 2) \cup (2, \infty)$.

2. Find the partition numbers of f'':

Because we are given $f''(x) = -4x(x-2)^3(x+5)^2(x-7)$, we can go straight to finding the x-values where $f''(x) = 0$ or $f''(x)$ does not exist.

Let's first consider the x-values where $f''(x)$ does not exist. f'' is a polynomial (it is just in factored form), so it will exist everywhere that f is defined. Thus, $f''(x)$ does not exist at $x = 2$, so there is a partition number of f'' at $x = 2$. Now, we need to find the x-values where $f''(x) = 0$:

$$\begin{aligned} f''(x) = 0 &\implies \\ -4x(x-2)^3(x+5)^2(x-7) = 0 & \\ &\implies x = 0,\ 2,\ -5, \text{ and } 7 \end{aligned}$$

Thus, the partition numbers of f'' are $x = -5$, $x = 0$, $x = 2$, and $x = 7$.

3. Create a sign chart of $f''(x)$:

We will place the partition numbers of f'' on a number line, with $x=-5$, $x=0$, and $x=7$ having solid dots and $x=2$ having an open circle to indicate which are included and which are not included in the domain of f, respectively.

Next, we need to determine the sign of $f''(x)$ on the intervals $(-\infty,-5)$, $(-5,0)$, $(0,2)$, $(2,7)$, and $(7,\infty)$. We will choose the x-values $x=-6,\ -1,\ 1,\ 3$, and 8 to test:

$$\begin{aligned}
f''(x) &= -4x(x-2)^3(x+5)^2(x-7) \implies \\
f''(-6) &= -4(-6)((-6)-2)^3((-6)+5)^2((-6)-7) = 159,744 > 0 \\
f''(-1) &= -4(-1)((-1)-2)^3((-1)+5)^2((-1)-7) = 13,824 > 0 \\
f''(1) &= -4(1)((1)-2)^3((1)+5)^2((1)-7) = -864 < 0 \\
f''(3) &= -4(3)((3)-2)^3((3)+5)^2((3)-7) = 3072 > 0 \\
f''(8) &= -4(8)((8)-2)^3((8)+5)^2((8)-7) = -1,168,128 < 0
\end{aligned}$$

Using this information, we can fill in the sign chart of $f''(x)$. Because we are also interested in the information this yields for f, we will include that information below the number line. See **Figure 3.2.16**.

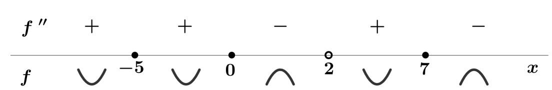


Figure 3.2.16: Sign chart of $f''(x)$ with the corresponding information for f

Looking at the sign chart, we see that $f''(x)$ changes sign at $x=0$, $x=2$, and $x=7$. We are given the domain of f is $(-\infty,2)\cup(2,\infty)$, so f has inflection points at $x=0$ and $x=7$. Note that we cannot determine the y-values of the inflection points because we were not given the original function f.

Furthermore, f is concave up on $(-\infty,-5)$, $(-5,0)$, and $(2,7)$. It is concave down on $(0,2)$ and $(7,\infty)$.

▪

Graphical Interpretation

Remember, there is more than one way a function may be given to us. Thus far, we have focused on finding partition numbers of f'', intervals where f is concave up/down, and inflection points of f algebraically using the rule of the function. Let's consider finding this information if we are given the graph of the function, the graph of its first derivative, or the graph of its second derivative instead.

■ **Example 5** Given the graph of f shown in **Figure 3.2.17**, find the intervals of concavity and the x-values of any inflection points of f.

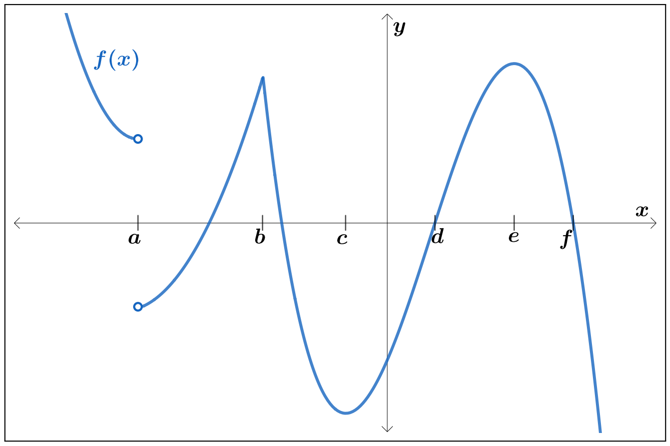


Figure 3.2.17: Graph of a function f

Solution:
Looking at the graph of f, we see that it is concave up on $(-\infty, a)$, (a, b), and (b, d), and it is concave down on (d, ∞). f has an inflection point at $x = d$ only.

■

Now, suppose we want to find this information, as well as the partition numbers of f'', for a function f, but we are only given the graph of its second derivative, f''. Algebraically, we know that the partition numbers of f'' are the x-values where the second derivative equals zero or does not exist. Correspondingly, we need to find the x-values where the graph of f'' touches the x-axis or does not exist (i.e., is undefined).

To determine where f is concave up when looking at the graph of f'', we find where the graph of f'' is above the x-axis because f is concave up where $f''(x) > 0$. Similarly, to determine where f is concave down, we find where the graph of f'' is below the x-axis. To find the x-values of any inflection points of f, we need to know where $f''(x)$ changes sign, or where the graph of f'' *switches* from above the x-axis (positive) to below the x-axis (negative), or vice versa. In addition, we need to check that such x-values are actually in the domain of f, which should be stated.

Example 6 Given the graph of f'' shown in **Figure 3.2.18** and that f is continuous on its domain of $(a,c)\cup(c,\infty)$, find each of the following.

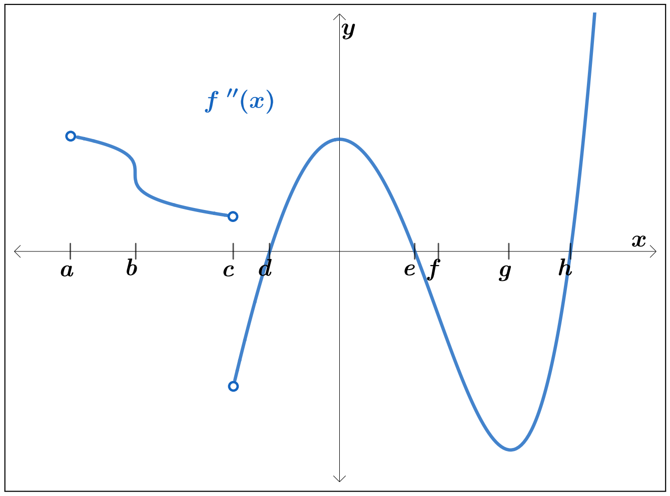


Figure 3.2.18: Graph of a second derivative function f''

a. Partition numbers of f''
b. Intervals of concavity of f
c. x-values of any inflection points of f

Solution:

We will use the same ideas discussed previously, but we have to be more careful now because we are given the graph of f'' instead of the graph of f.

a. The partition numbers of f'' are the x-values where $f''(x) = 0$ or $f''(x)$ does not exist. Let's start by finding the x-values where $f''(x)$ does not exist. Remember that we are given the graph of f'', and the only x-values where $f''(x)$ does not exist are $x = a$ and $x = c$ (the function is undefined at these x-values). Technically, $f''(x)$ does not exist for any $x \leq a$, in addition to $x = c$, but as discussed in **Section 3.1**, we would not include all of these x-values as partition numbers because they would not meaningfully partition a number line if we needed to create a sign chart of $f''(x)$. We do include $x = a$ as a partition number of f'' because there is a meaningful transition happening at $x = a$ (the derivative goes from not existing to existing). Also, $x = a$ would be needed to partition a number line into the appropriate, discrete intervals if we made a sign chart of $f''(x)$.

We may also be tempted to say $f''(x)$ does not exist at $x = b$ because there is a vertical tangent line, but $x = b$ would be where the derivative of f'' does not exist, *not where f'' itself does not exist!*

Next, we need to find the x-values where $f''(x) = 0$. Graphically, this means we need to find the x-values where the graph of f'' touches (but not necessarily crosses) the x-axis. This occurs at $x = d$, $x = e$, and $x = h$.

Thus, the partition numbers of f'' are $x = a$, $x = c$, $x = d$, $x = e$, and $x = h$.

b. To find the intervals where f is concave up, we need to determine where $f''(x) > 0$, or where the graph of f'' is above the x-axis. This occurs on the intervals (a,c), (d,e), and (h,∞). Similarly, to find the intervals where f is concave down, we need to determine where $f''(x) < 0$, or where the graph of f'' is below the x-axis. This occurs on the intervals (c,d) and (e,h).

In summary, f is concave up on (a,c), (d,e), and (h,∞), and f is concave down on (c,d) and (e,h).

c. Be careful! The inflection points of f are *not* where the concavity changes on the graph of f''. To find the inflection points of f, we must find the x-values where $f''(x)$ changes from positive to negative, or vice versa. Recall this amounts to finding where the graph of f'' switches from being above to below the x-axis, or vice versa. The partition numbers of f'' where this occurs are $x = c$, $x = d$, $x = e$, and $x = h$. Checking to see if these x-values are in the domain of f, which is $(a, c) \cup (c, \infty)$, we see that $x = c$ is *not* in the domain. Thus, the inflection points of f occur at $x = d$, $x = e$, and $x = h$.

The inflection points that occur on the graph of f'' at $x = b$ and $x = e$ are inflection points of f''!

■

Try It # 3:
Given the graph of f'' shown in **Figure 3.2.19** and that f is continuous on its domain of $(-\infty, 3) \cup (3, \infty)$, find each of the following.

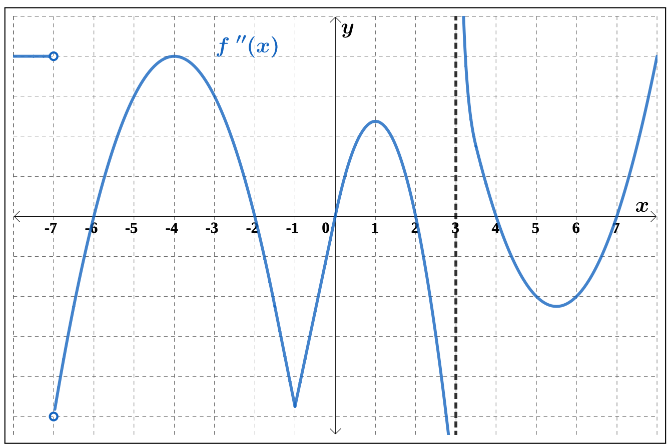


Figure 3.2.19: Graph of a second derivative function f''

a. Partition numbers of f''
b. Intervals of concavity of f
c. x-values of any inflection points of f

Now, let's consider how we might find this information (partition numbers of f'', intervals of concavity of f, and the x-values of any inflection points of f) when we are given the graph of f' and the domain of f.

The partition numbers of f'' are the x-values where $f''(x) = 0$ or $f''(x)$ does not exist. To find the x-values where $f''(x)$ does not exist, we need to look at the graph of f' and find where *its derivative* does not exist (because its derivative is f''). Recall from **Section 2.2** that the derivative of a function does not exist when the graph of the function has a cusp or corner, a discontinuity, or a vertical tangent line. To find where $f''(x) = 0$, we need to determine the x-values where the graph of f' has horizontal tangent lines because these are the x-values where its derivative equals zero (i.e., the x-values where $f''(x) = 0$).

To find where f is concave up given the graph of f', remember from our previous discussion that a function is concave up when the slopes of its graph are increasing (i.e., when f' is increasing). This is equivalent to finding where $f''(x) > 0$, or where the graph of f'' is above the x-axis, but because we are given the graph of f' and not the graph of f'', we must look for where the function f' is increasing. Likewise, we look for where f' is decreasing to find where f is concave down.

To find where f has inflection points given the graph of f', we must think about what the concavity of f changing means in terms of f'. If a function changes from concave up to concave down, then that means the slopes of its graph *switch* from increasing to decreasing. In other words, f' switches from increasing to decreasing. One way this can happen is if f' has a local maximum! Similarly, if a function changes from concave down to concave up, then that means the slopes of its graph *switch* from decreasing to increasing. In other words, f' switches from decreasing to increasing. Again, one way this could happen is if f' has a local minimum! So to find where f has inflection points when we are given the graph of f', we need to find the x-values where f' switches between increasing/decreasing and then check that these x-values are in the domain of f, which should be stated.

Example 7 Given the graph of f' shown in **Figure 3.2.20** and that f is continuous on its domain of $(-\infty, c) \cup (c, \infty)$, find each of the following.

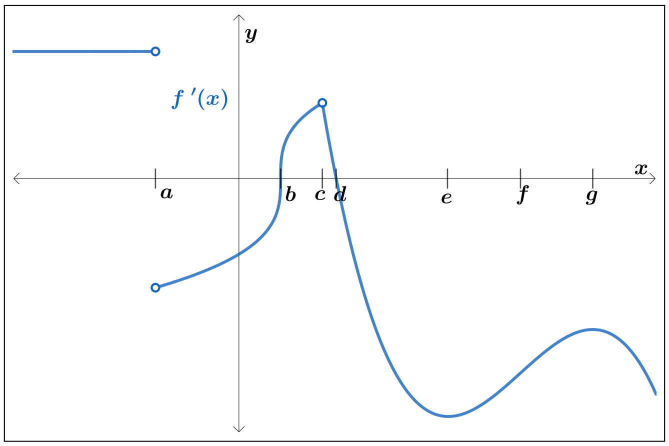


Figure 3.2.20: Graph of a first derivative function f'

a. Partition numbers of f''
b. Intervals of concavity of f
c. x-values of any inflection points of f

Solution:

a. To find the partition numbers of f'', we must find the x-values where $f''(x) = 0$ or $f''(x)$ does not exist. Let's start by finding the x-values where $f''(x)$ does not exist. Because f'' is the derivative of f', the x-values where $f''(x)$ does not exist correspond to the x-values where the derivative of f' does not exist. So we need to look at the graph of f' and see where its derivative does not exist. The derivative of f' does not exist at $x = a$ and $x = c$ because there are discontinuities. The derivative does not exist at $x = b$ because there is a vertical tangent line. Thus, $f''(x)$ does not exist at $x = a$, $x = b$, and $x = c$.

Now, we need to find the x-values where $f''(x) = 0$, which means the derivative of f' equals zero. The derivative equals zero where the function, f', has a horizontal tangent line. Looking at the graph of f', we see this occurs when $x < a$ and at $x = e$ and $x = g$. As discussed previously, we would not consider all of the x-values less than a to be partition numbers because they would not partition a number line in a meaningful, necessary way if we created a sign chart of $f''(x)$. Thus, we consider the "important" x-values where $f''(x) = 0$ to be $x = e$ and $x = g$, and so these are also partition numbers of f''.

In summary, the partition numbers of f'' are $x = a$, $x = b$, $x = c$, $x = e$, and $x = g$.

b. To find where f is concave up, we need to find where $f''(x) > 0$, or in this case, where f' is increasing. Looking at the graph of f', we see it is increasing on (a,b), (b,c), and (e,g). Thus, f is concave up on (a,b), (b,c), and (e,g).

To find where f is concave down, we need to find where $f''(x) < 0$, or in this case, where f' is decreasing. Looking at the graph of f', we see it is decreasing on (c,e) and (g,∞). Thus, f is concave down on (c,e) and (g,∞).

In summary, f is concave up on (a,b), (b,c), and (e,g), and f is concave down on (c,e) and (g,∞).

c. To find the inflection points of f, we need to find the x-values where the graph of f' switches between increasing/decreasing, as discussed previously, and then check to see if these x-values are in the domain of f. Looking at the graph of f', we see it switches between increasing/decreasing at $x = c$, $x = e$, and $x = g$. Because the domain of f is $(-\infty,c)\cup(c,\infty)$, which we were given, f has inflection points at $x = e$ and $x = g$ only.

The inflection points that occur on the graph of f' at $x = b$ and $x = f$ are inflection points of f' and not necessarily inflection points of f!

■

Try It # 4:
Given the graph of f' shown in **Figure 3.2.21** and that f is continuous on its domain of $(-\infty,7)\cup(7,\infty)$, find each of the following.

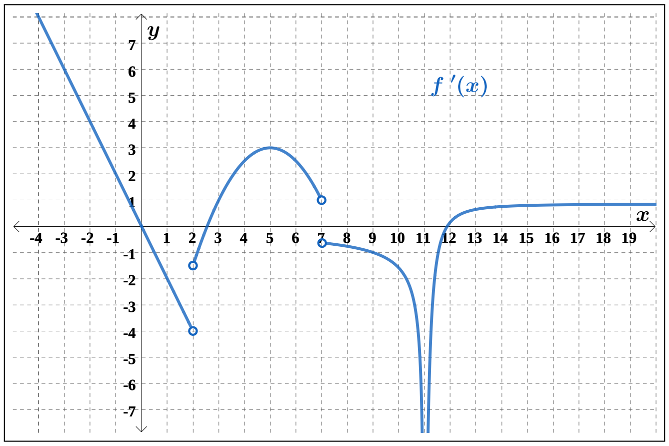


Figure 3.2.21: Graph of a first derivative function f'

a. Partition numbers of f''
b. Intervals of concavity of f
c. x-values of any inflection points of f

Relationships Between f, $f\,'$, and $f\,''$

Figure 3.2.22 summarizes the relationships between f, $f\,'$, and $f\,''$ we have discussed throughout this section and the last in a handy dandy chart. It may be helpful to use this chart when working problems similar to the last two examples where we are given the graph of the first or second derivative, as well as when we are given the graph of any one of the three functions and asked to find pertinent information about the other two.

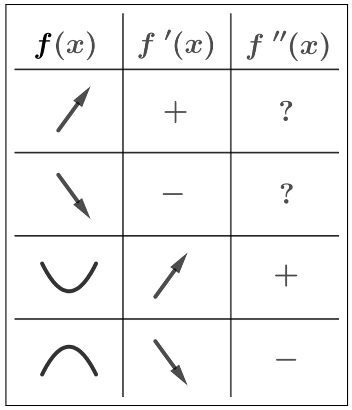


Figure 3.2.22: Chart showing relationships between f, $f\,'$, and $f\,''$

■ **Example 8** Use **Figure 3.2.22** and the graph shown in **Figure 3.2.23** to answer each of the following.

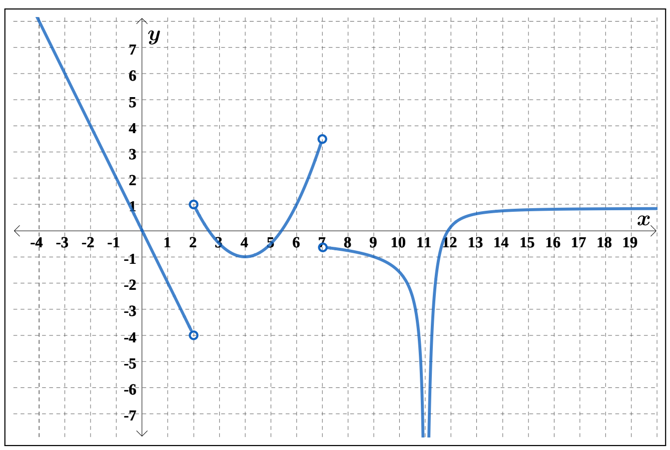


Figure 3.2.23: Graph of a function

a. If the graph shown is of f, find where $f\,''(x) < 0$.

b. If the graph shown is of f, find where $f\,'$ is increasing.

c. If the graph shown is of f, find where $f\,'(x) > 0$.

d. If the graph shown is of $f\,'$ and f is continuous on its domain of $(-\infty,2)\cup(2,7)\cup(7,11)\cup(11,\infty)$, find where $f\,''(x) > 0$.

e. If the graph shown is of $f\,'$ and f is continuous on its domain of $(-\infty,2)\cup(2,7)\cup(7,11)\cup(11,\infty)$, find where f is decreasing.

f. If the graph shown is of $f\,'$ and f is continuous on its domain of $(-\infty,2)\cup(2,7)\cup(7,11)\cup(11,\infty)$, find where the slopes of the graph of f are decreasing.

g. If the graph shown is of $f\,''$, f is continuous on its domain of $(-\infty,2)\cup(2,7)\cup(7,11)\cup(11,\infty)$, and $f\,'$ is continuous on its domain of $(-\infty,2)\cup(2,7)\cup(7,11)\cup(11,\infty)$, find where $f\,'$ is decreasing.

Solution:

a. To use the chart shown in **Figure 3.2.22**, we find the entry that states the question (i.e., we locate $f''(x) < 0$ in the chart). Then, we move across the row to the column corresponding to the function we are given, which is f in this case. Moving across the row to the column corresponding to f, we see that we need to find where the graph is concave down. Looking at the graph, we see that f is concave down on $(7,11)$ and $(11,\infty)$. Thus, $f''(x) < 0$ on $(7,11)$ and $(11,\infty)$.

b. We first locate f' increasing in the chart. Then, because we have the graph of f, we move across the row to the column corresponding to f and see that we need to look for where the graph is concave up. f is concave up on $(2,7)$, so f' is increasing on $(2,7)$.

c. We proceed as before and locate the question in the chart: $f'(x) > 0$. We move across the row to the column corresponding to f and see that we need to look for where the graph is increasing. f is increasing on $(4,7)$ and $(11,\infty)$. Therefore, $f'(x) > 0$ on $(4,7)$ and $(11,\infty)$.

d. Again, we locate the question in the chart: $f''(x) > 0$. Now, because we are assuming the graph is of f', we move across the row to the column corresponding to f'. We see $f''(x) > 0$ corresponds to where f' is increasing. Because we are assuming the graph is of f', we look for where the graph is increasing. This occurs on the intervals $(4,7)$ and $(11,\infty)$. Hence, $f''(x) > 0$ on $(4,7)$ and $(11,\infty)$.

e. We locate f decreasing in the chart and then move across the row to the column corresponding to f' because we are assuming the graph is of f'. We see that we need to look for where $f'(x) < 0$. The graph is negative on the intervals $(0,2)$, $(2.6,5.4)$, $(7,11)$, and $(11,11.8)$. Thus, f is decreasing on $(0,2)$, $(2.6,5.4)$, $(7,11)$, and $(11,11.8)$.

f. We want to know where the slopes of the graph of f are decreasing, and this translates to where the graph of f' is decreasing because $f'(x)$ gives the slope of the graph of f at x. While locating f' decreasing in the chart, we should notice that we do not need to move across the row. We are assuming the graph we have is of f', so all we need to do is look at the graph and see where it is decreasing. The graph is decreasing on $(-\infty,2)$, $(2,4)$, and $(7,11)$, so the slopes of the graph of f are decreasing on $(-\infty,2)$, $(2,4)$, and $(7,11)$.

g. We locate f' decreasing in the chart. Now, because we are assuming the graph is of f'', we move across the row to the column corresponding to f''. We see that we need to look for where $f''(x) < 0$. Because we are assuming the graph is of f'', we look for where the graph is below the x-axis. This occurs on the intervals $(0,2)$, $(2.6,5.4)$, $(7,11)$, and $(11,11.8)$. Therefore, f' is decreasing on $(0,2)$, $(2.6,5.4)$, $(7,11)$, and $(11,11.8)$.

■

Try It # 5:
Use the graph shown in **Figure 3.2.24** to answer each of the following.

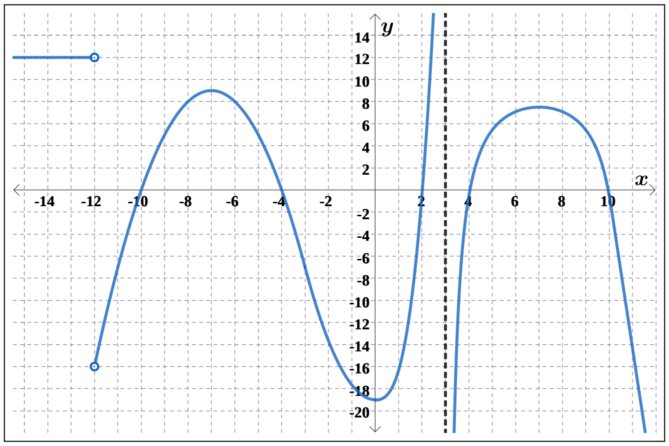


Figure 3.2.24: Graph of a function

a. If the graph shown is of f, find where $f''(x) < 0$.

b. If the graph shown is of f, find where f' is increasing.

c. If the graph shown is of f' and f is continuous on its domain of $(-\infty,-12)\cup(-12,3)\cup(3,\infty)$, find where $f''(x) > 0$

d. If the graph shown is of f' and f is continuous on its domain of $(-\infty,-12)\cup(-12,3)\cup(3,\infty)$, find where the slopes of the graph of f are decreasing.

e. If the graph shown is of f'', f is continuous on its domain of $(-\infty,-12)\cup(-12,3)\cup(3,\infty)$, and f' is continuous on its domain of $(-\infty,-12)\cup(-12,3)\cup(3,\infty)$, find where f' is decreasing.

Point of Diminishing Returns

One business application involving concavity and inflection points is the **point of diminishing returns**.

Let's say a company that sells a particular item is trying to increase their sales. The company's marketing team decides to increase the amount they spend on advertising. They purchase television and radio commercials, billboard advertisements, podcast advertisements, airplane fly-by banners and more. We would expect sales to increase when the company increases its advertising. At first, the company's sales will increase at an increasing rate. Eventually though, if the company continues its advertising campaign, sales will continue to increase, but they will increase *at a decreasing rate*.

Definition
The point where the rate of change of sales switches from increasing to decreasing is called the **point of diminishing returns**. ▪

In other words, the point where the first derivative of a sales function has a local maximum is the point of diminishing returns. As seen previously, this local maximum on the graph of the first derivative corresponds to an inflection point on the original sales function.

Example 9 Joseph Joeson owns a local coffee shop, A Cup of Joe, that specializes in making one-size flavored coffee drinks. Mr. Joeson expects that if he spends h hundred dollars per week in advertising, his shop will sell $J(h) = -0.2h^4 + 2h^3 + 250$ cups of Joe (coffee drinks), where $0 \leq h \leq 7.5$.

a. On what interval is the rate of change of sales increasing? On what interval is the rate of change of sales decreasing?

b. Find the point of diminishing returns.

c. Verify your answers by graphing the sales function, J, and the rate of change of sales function, J', on the same axes.

Solution:

a. To find where the rate of change of sales, J', is increasing and where it is decreasing, we must find its derivative, J''. Then, we can use the techniques discussed in **Section 3.1** to create a sign chart of $J''(h)$ to determine where $J''(h)$ is positive (i.e., where J' is increasing) and where $J''(h)$ is negative (i.e., where $J'(h)$ is decreasing). For $J(h) - 0.2h^4 + 2h^3 + 250$, we have

$$J'(h) = -0.8h^3 + 6h^2 \implies$$
$$J''(h) = -2.4h^2 + 12h$$

To create a sign chart of $J''(h)$, we need to partition a number line. Recall that the partition numbers of J'' are the h-values where $J''(h) = 0$ or $J''(h)$ does not exist. Because J'' is a polynomial, it will always exist. Thus, we only need to determine the h-values where $J''(h) = 0$:

$$J''(h) = 0 \implies$$
$$-2.4h^2 + 12h = 0$$
$$h(-2.4h + 12) = 0$$
$$\implies h = 0 \text{ and } h = \frac{12}{2.4} = 5$$

Next, we need to determine the sign of $J''(h)$ on the intervals $(0, 5)$ and $(5, 7.5)$; note that the function is defined for $0 \leq h \leq 7.5$. We must select h-values to test in each interval. We will test $h = 1$ and $h = 6$:

$$J''(h) = -2.4h^2 + 12h \implies$$
$$J''(1) = -2.4(1)^2 + 12(1) = 9.6 > 0$$
$$J''(6) = -2.4(6)^2 + 12(6) = -14.4 < 0$$

Now, we can create a sign chart with this information for J'', as well as the corresponding information for J' and J. Also, we will remember to mark that the original function, J, has a domain of $[0,\ 7.5]$ on the chart. See **Figure 3.2.25**.

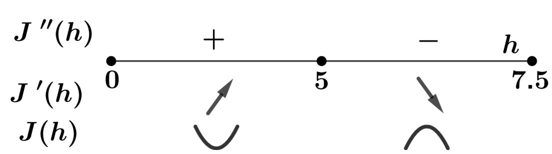


Figure 3.2.25: Sign chart of $J''(h)$ with the corresponding information for both J' and J

Thus, the rate of change of sales, J', is increasing on $(0, 5)$ and decreasing on $(5, 7.5)$. In other words, when the company spends up to \$500 on advertising, its coffee sales will increase at an increasing rate. If it spends more than \$500 (and less than \$750), sales will still increase, but at a decreasing rate.

b. Looking at the sign chart of $J''(h)$ in **Figure 3.2.25**, we see that the point of diminishing returns, which corresponds to where the rate of change of sales, J', switches from increasing to decreasing, occurs at $h = 5$. In other words, the point of diminishing returns occurs when the company spends \$500 per week on advertising.

> **N** *As seen in the sign chart of $J''(h)$, the point of diminishing returns of a function corresponds to an inflection point on the graph of the function and a local maximum on the graph of its first derivative.*

To find the y-value (number of cups of Joe sold) associated with the point of diminishing returns at $h = 5$, we calculate $J(5)$:

$$\begin{aligned} J(h) &= -0.2h^4 + 2h^3 + 250 \implies \\ J(5) &= -0.2(5)^4 + 2(5)^3 + 250 \\ &= 375 \end{aligned}$$

Therefore, the point of diminishing returns is $(5, 375)$, meaning \$500 spent on advertising and 375 cups of Joe sold.

c. **Figure 3.2.26** shows the graphs of both J and J' on the same axes:

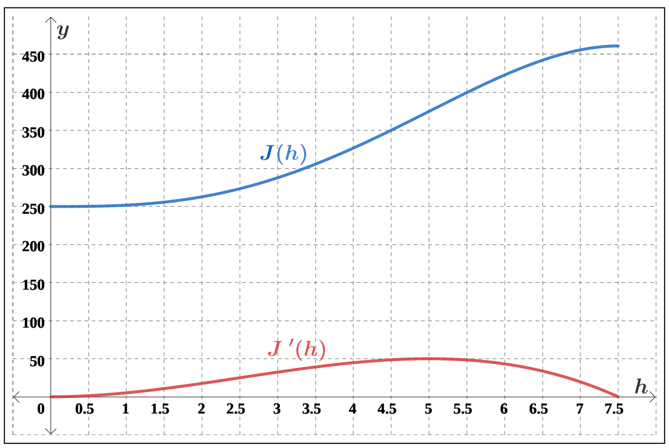


Figure 3.2.26: Graphs of J and J' showing the point of diminishing returns at $h = 5$

We can use these graphs to verify the information in the sign chart of $J''(h)$. We see that J' has a local maximum at $h = 5$, and J has an inflection point at $h = 5$.

■

The Second Derivative Test

Up until now, we have used the **First Derivative Test (Theorem 3.2)** to find local extrema. However, there is an alternative test we can *attempt* to use to find local extrema called the **Second Derivative Test**. The reason we say *attempt* is because there are circumstances in which this test cannot be applied (we will get to those shortly!). However, it may be helpful to try to use the Second Derivative Test (as opposed to the First Derivative Test) because using the Second Derivative Test is usually less computationally intensive, which saves us time.

We have already learned that local extrema occur at critical values. Recall that critical values of f are the x-values where $f'(x) = 0$ or $f'(x)$ does not exist, and they are in the domain of the function f. Now, consider the critical value $x = c$ of a function f such that $f'(c) = 0$, and the critical value is contained on an interval in which $f''(x) > 0$ (and therefore, $f''(c) > 0$). This means that f is concave up on the interval and has a horizontal tangent line at $x = c$.

Thus, the graph of f would have roughly the same shape as the graph shown in **Figure 3.2.27**. Notice that f has a local minimum at $x = c$!

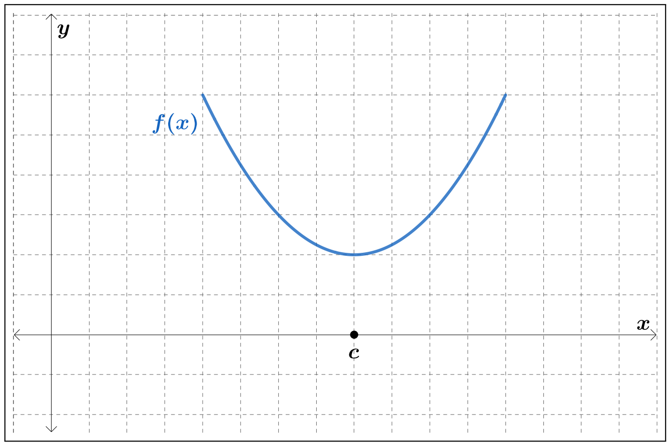


Figure 3.2.27: Graph of a function f that is concave up and has a horizontal tangent line at $x = c$

Likewise, if $x = c$ is a critical value of f such that $f'(c) = 0$ and $f''(x) < 0$ on an interval containing $x = c$ (and therefore, $f''(c) < 0$), then f is concave down on the interval and has a local maximum at $x = c$. See **Figure 3.2.28**.

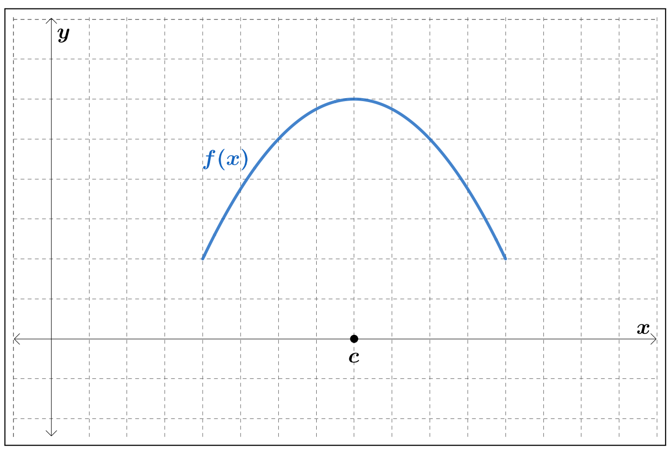


Figure 3.2.28: Graph of a function f that is concave down and has a horizontal tangent line at $x = c$

We now formally state the Second Derivative Test:

Theorem 3.4 Second Derivative Test

Suppose f is twice-differentiable at $x = c$ and $x = c$ is a critical value of f such that $f'(c) = 0$.

- If $f''(c) < 0$ then f is concave down and has a local maximum at $x = c$.
- If $f''(c) > 0$ then f is concave up and has a local minimum at $x = c$.
- If $f''(c) = 0$, then the Second Derivative Test fails, and f may have a local maximum, minimum, or neither at $x = c$.

One of the limitations regarding the Second Derivative Test is that we can only *attempt* to use it for critical values of f such that $f'(x) = 0$. If there are critical values of f such that $f'(x)$ does not exist, we cannot use the Second Derivative Test, and we must go back and use the First Derivative Test to determine if there are any local extrema at those critical values. Why? If f' does not exist at a particular x-value, then the derivative of f', which is f'', cannot possibly exist at that x-value either. If the second derivative does not exist, we cannot use the Second Derivative Test.

The other issue that can arise is if after taking the second derivative and evaluating it at a critical value $x = c$, we get that $f''(c) = 0$. In this case, we say the Second Derivative Test fails, meaning it is inconclusive. We would then have to go back and use the First Derivative Test to determine if a local extremum occurs at the critical value $x = c$.

Another similar possibility is that we apply the Second Derivative Test to a critical value $x = c$ in which $f'(c) = 0$ and the result is $f''(c)$ does not exist. Because the Second Derivative Test assumes f is twice-differentiable at $x = c$, we would not be able to conclude whether or not $f(c)$ is a local maximum, local minimum, or neither. We would, again, have to go back and use the First Derivative Test to determine if there is a local extremum at $x = c$.

! *If the Second Derivative Test fails at a critical value, that does not mean the function does not have a local extremum at that critical value. It just means we cannot determine if there is a local extremum using this test. Likewise, if there are critical values such that $f'(x)$ or $f''(x)$ does not exist, it does not mean there are no local extrema. It just means we cannot use the Second Derivative Test to determine the behavior. In any of these cases, we have to go back and use the First Derivative Test to determine if there are local extrema.*

Because the First Derivative Test (**Theorem 3.2**) and the Second Derivative Test (**Theorem 3.4**) can both be used to find local extrema, we will compare and contrast them here to help us learn which one to apply in which situation:

First Derivative Test	**Second Derivative Test**
Use with all critical values of f	Only use with critical values of f such that $f'(x) = 0$
Requires multiple test points	Requires only 1 test point (the critical value)
Never fails	Sometimes fails

Many students like the Second Derivative Test because it is often easier to use than the First Derivative Test. If the function is a polynomial, its second derivative will probably be a simpler function that its first derivative. However, if you need to use the Product, Quotient, or Chain Rules to find the first derivative, finding the second derivative could be a lot of work. In this case, you might choose to apply the First Derivative Test to find local extrema instead.

Remember that even if the second derivative is easy to find, the Second Derivative Test does not always give an answer. Also, it can only be used for critical values of f such that $f'(x) = 0$. These are reasons why some students prefer the First Derivative Test.

■ **Example 10** The domain of f is $(-\infty, \infty)$, and f is twice-differentiable on its domain. Use the Second Derivative Test, if possible, to classify whether f has a local minimum, local maximum, or neither at each of the following x-values. If it is not possible, explain why.

a. $f(6) = -22$, $f'(6) = 0$, and $f''(6) = -12$

b. $f(0) = 15$, $f'(0) = 20$, and $f''(0) = 10$

c. $f(-2) = 179$, $f'(-2) = 0$, and $f''(-2) = 0.07$

d. $f(13) = 13$, $f'(13) = 0$, and $f''(13) = 0$

Solution:

a. First, note that because f is defined at $x = 6$ and $f'(6) = 0$, $x = 6$ is a critical value of f. Furthermore, we can attempt to use the Second Derivative Test because $x = 6$ is a critical value of f in which $f'(x) = 0$. We are given $f''(6) = -12$, which means f is concave down at $x = 6$ because its second derivative is negative. Using **Theorem 3.4**, we see that f has a local maximum at the point $(6, -22)$.

b. Even though f is defined at $x = 0$, $f'(0) = 20$, which means that $x = 0$ is *not a critical value of f!* Recall that the critical values of f are the x-values where $f'(x) = 0$ or $f'(x)$ does not exist (although we cannot use the

Second Derivative Test with critical values such that $f'(x)$ does not exist). Therefore, because $x = 0$ is not a critical value of f, there cannot possibly be a local extremum at $x = 0$.

c. Unlike in part **b**, $x = -2$ is a critical value of f. Also, because $f'(-2) = 0$, we can attempt to use the Second Derivative Test. Because $f''(-2) = 0.07$, which is positive, f is concave up at $x = -2$. Using **Theorem 3.4**, we see that f has a local minimum at the point $(-2, 179)$.

d. Again, $x = 13$ is a critical value of f. However, because $f''(13) = 0$, the Second Derivative Test fails. This means we cannot determine whether or not there is a local extremum at $x = 13$. Without more information to conduct the First Derivative Test, we cannot conclude anything more than $x = 13$ is a critical value of f.

▪

Try It # 6:
The domain of f is $(-\infty, \infty)$, and f is twice-differentiable on its domain. Use the Second Derivative Test, if possible, to classify whether f has a local minimum, local maximum, or neither at each of the following x-values. If it is not possible, explain why.

a. $f(8) = 0,\ f'(8) = 0$, and $f''(8) = 120$

b. $f(-15) = 23,\ f'(-15) = 0$, and $f''(-15) = 0$

To apply the Second Derivative Test when we are given a function f, we use the following three steps:

Finding Local Extrema Using the Second Derivative Test

1. Determine the domain of f.
2. Find the critical values of f. Recall that these are the x-values where $f'(x) = 0$ or $f'(x)$ does not exist, and they are in the domain of f.
3. Find $f''(x)$ and evaluate it at each critical value where $f'(x) = 0$. Then, determine whether f has local extrema at these critical values using the Second Derivative Test.

▪ **Example 11** Use the Second Derivative Test, if possible, to find any local extrema of the following functions. If it is not possible, explain why.

a. $f(x) = 2x^3 - 15x^2 + 24x - 7$

b. $f(x) = 3x^5 - 8x^4$

Solution:

a. We will use the three steps outlined previously to *attempt* to use the Second Derivative Test:

1. Determine the domain of f:

f is a polynomial, so its domain is $(-\infty, \infty)$.

2. Find the critical values of f:

$$f'(x) = 6x^2 - 30x + 24$$

f' is a polynomial, so it exists everywhere. Thus, there are no critical values such that $f'(x)$ does not exist. Now, we find the x-values where $f'(x) = 0$:

$$\begin{aligned} f'(x) = 0 &\implies \\ 6x^2 - 30x + 24 &= 0 \\ 6(x^2 - 5x + 4) &= 0 \\ 6(x-1)(x-4) &= 0 \\ &\implies x = 1 \text{ and } x = 4 \end{aligned}$$

Because both of these x-values are in the domain of f, they are both critical values of f. Furthermore, because they are both critical values such that $f'(x) = 0$, we can continue and attempt to use the Second Derivative Test with each critical value.

3. Find $f''(x)$ and evaluate it at the critical values:

$$\begin{aligned} f''(x) &= 12x - 30 \implies \\ f''(1) &= 12(1) - 30 = -18 < 0 \\ f''(4) &= 12(4) - 30 = 18 > 0 \end{aligned}$$

Using **Theorem 3.4**, we see that f is concave down at $x = 1$, so there is a local maximum at $x = 1$. At $x = 4$, f is concave up, so there is a local minimum at $x = 4$.

To find the y-values associated with these x-values, we substitute the $x = 1$ and $x = 4$ into the *original function* f:

$$\begin{aligned} f(x) &= 2x^3 - 15x^2 + 24x - 7 \implies \\ f(1) &= 2(1)^3 - 15(1)^2 + 24(1) - 7 = 4 \\ f(4) &= 2(4)^3 - 15(4)^2 + 24(4) - 7 = -23 \end{aligned}$$

Thus, f has a local maximum of 4 at $x = 1$ and a local minimum of -23 at $x = 4$.

We can check our work by looking at the graph of f shown in **Figure 3.2.29**, but applying the Second Derivative Test allows us to find the local extrema *completely algebraically* and get *exact* answers!

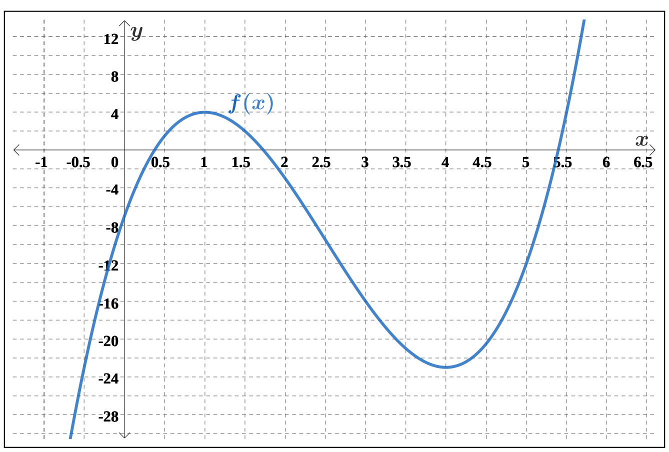


Figure 3.2.29: Graph of the function $f(x) = 2x^3 - 15x^2 + 24x - 7$

b. Recall $f(x) = 3x^5 - 8x^4$. We will use the three steps outlined previously to *attempt* to use the Second Derivative Test:

1. Determine the domain of f:

f is a polynomial, so its domain is $(-\infty, \infty)$.

2. Find the critical values of f:

$$f'(x) = 15x^4 - 32x^3$$

f' is a polynomial, so it exists everywhere. Thus, there are no critical values such that $f'(x)$ does not exist. Now, we find the x-values where $f'(x) = 0$:

$$\begin{aligned} f'(x) = 0 &\implies \\ 15x^4 - 32x^3 &= 0 \\ x^3(15x - 32) &= 0 \\ &\implies x = 0 \text{ and } x = \frac{32}{15} \end{aligned}$$

Because both of these x-values are in the domain of f, they are both critical values of f. Furthermore, because they are both critical values such that $f'(x) = 0$, we can continue and attempt to use the Second Derivative Test with each critical value.

3. Find $f''(x)$ and evaluate it at the critical values:

$$\begin{aligned} f''(x) &= 60x^3 - 96x^2 \implies \\ f''(0) &= 60(0)^3 - 96(0)^2 = 0 \\ f''\left(\frac{32}{15}\right) &= 60\left(\frac{32}{15}\right)^3 - 96\left(\frac{32}{15}\right)^2 = \frac{32768}{225} \approx 145.6356 > 0 \end{aligned}$$

Using **Theorem 3.4**, we see that the Second Derivative Test fails at $x = 0$ because $f''(0) = 0$. Thus, we cannot draw any conclusions about a local extremum at $x = 0$ (there may or may not be one). At $x = \frac{32}{15}$, f is concave up, so there is a local minimum at $x = \frac{32}{15}$.

To find the y-value associated with $x = \frac{32}{15}$, we substitute this x-value into the *original function* f:

$$\begin{aligned} f(x) &= 3x^5 - 8x^4 \implies \\ f\left(\frac{32}{15}\right) &= 3\left(\frac{32}{15}\right)^5 - 8\left(\frac{32}{15}\right)^4 \approx -33.1402 \end{aligned}$$

Thus, f has a local minimum of -33.1402 at $x = \frac{32}{15}$.

We can check our work by looking at the graph of f shown in **Figure 3.2.30**:

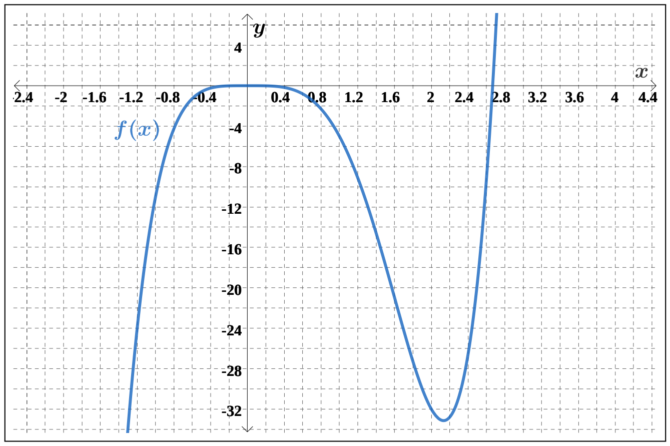


Figure 3.2.30: Graph of the function $f(x) = 3x^5 - 8x^4$

*Even though the Second Derivative Test was inconclusive regarding a local extremum at $x = 0$, we can see from the graph of f in **Figure 3.2.30** that it appears as though the function does have a local maximum at $x = 0$. To be sure, however, we would need to use the First Derivative Test.*

Try It # 7:
Use the Second Derivative Test, if possible, to find any local extrema of the following functions. If it is not possible, explain why.

a. $f(x) = x\ln(x)$

b. $f(x) = \dfrac{100}{x} + x$

Try It Answers

1. **a.** $f''(x) = \dfrac{\left(10x^3 - 27x^2 + 15\right)^2\left(-160x^3 + 180x - 42\right) - \left(-40x^4 + 90x^2 - 42x\right)\left(2\left(10x^3 - 27x^2 + 15\right)\left(30x^2 - 54x\right)\right)}{\left(10x^3 - 27x^2 + 15\right)^4}$

   b. $y'' = \frac{2}{3}\left(-9x^{11} - 8x^{31}\right)^{-1/3}\left(-990x^9 - 7440x^{29}\right) + \left(-99x^{10} - 248x^{30}\right)\left(-\frac{2}{9}\left(-9x^{11} - 8x^{31}\right)^{-4/3}\left(-99x^{10} - 248x^{30}\right)\right)$

2. **a.** f is concave up on $(0, \infty)$, and f is concave down on $(-\infty, 0)$; f has an inflection point at $(0, 0)$.

   b. g is concave up on $\left(\frac{1}{2}e^{-5/6}, \infty\right)$, and g is concave down on $\left(0, \frac{1}{2}e^{-5/6}, \infty\right)$; g has an inflection point at $\left(\frac{1}{2}e^{-5/6}, -\frac{5}{48}e^{-5/2}\right)$.

3. **a.** $x = -7$; $x = -6$; $x = -2$; $x = 0$; $x = 2$; $x = 3$; $x = 4$; $x = 7$

   b. f is concave up on $(-\infty, -7)$, $(-6, -2)$, $(0, 2)$, $(3, 4)$, and $(7, \infty)$, and f is concave down on $(-7, -6)$, $(-2, 0)$, $(2, 3)$, and $(4, 7)$.

   c. $x = -7$; $x = -6$; $x = -2$; $x = 0$; $x = 2$; $x = 4$; $x = 7$

4. **a.** $x = 2$; $x = 5$; $x = 7$; $x = 11$

   b. f is concave up on $(2,5)$ and $(11,\infty)$, and f is concave down on $(-\infty,2)$, $(5,7)$, and $(7,11)$.

   c. $x = 2$; $x = 5$; $x = 11$

5. **a.** $(-12,-3)$ and $(3,\infty)$

   b. $(-3,3)$

   c. $(-12,-7)$, $(0,3)$, and $(3,7)$

   d. $(-7,0)$ and $(7,\infty)$

   e. $(-12,-10)$, $(-4,2)$, $(3,4)$, and $(10,\infty)$

6. **a.** f has a local minimum at $x = 8$.

   b. The Second Derivative Test fails at $x = -15$ because $f''(-15) = 0$, so we cannot determine whether or not the function has a local extremum at $x = -15$.

7. **a.** f has a local minimum of $-e^{-1}$ at $x = e^{-1}$.

   b. f has a local maximum of -20 at $x = -10$ and a local minimum of 20 at $x = 10$.

EXERCISES

Basic Skills Practice

For Exercises 1 - 6, find $f''(x)$.

1. $f(x) = 7x^3 + 8\ln(x) - x - x^{-5}$

2. $f(x) = (2x+13)^9$

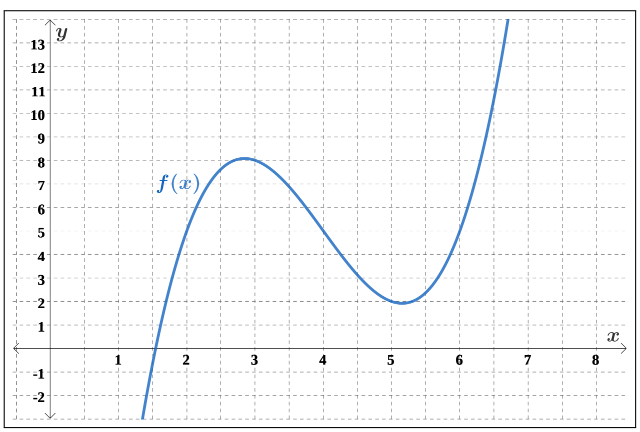
3. $f(x) = e^x + \dfrac{4}{x^2} - 21$

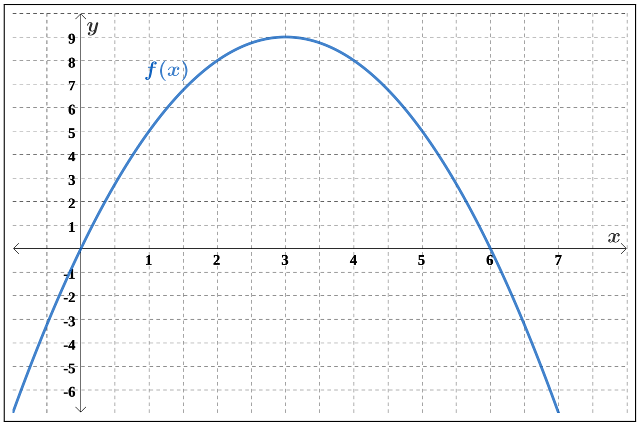
4. $f'(x) = e^{5x+9}$

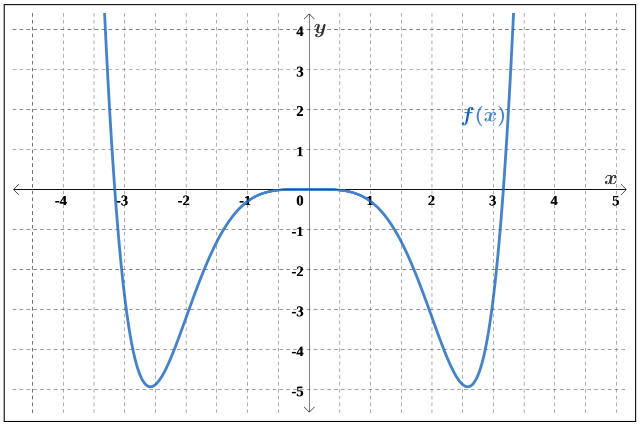
5. $f'(x) = x^{44} - 0.3x^{12} + \sqrt{x} - x^{-4/5}$

6. $f'(x) = \ln\left(8x^3 - 22\right)$

For Exercises 7 - 9, the graph of f is shown. Find (a) the intervals of concavity and (b) the x-values of any inflection points of f.

7. 

8. 

9. 

For Exercises 10 - 12, f'' and the domain of f are given. Assuming f is continuous on its domain, find (a) the partition numbers of f'', (b) the intervals of concavity of f, and (c) the x-values of any inflection points of f using the Concavity Test.

10. $f''(x) = (x+6)^4(x-3)(x-7)$; domain of f is $(-\infty, \infty)$

11. $f''(x) = 2x(x+4)(x-8)^5$; domain of f is $(-\infty, \infty)$

12. $f''(x) = \dfrac{3(x-10)(x+2)}{(x+5)^7}$; domain of f is $(-\infty, -5) \cup (-5, \infty)$

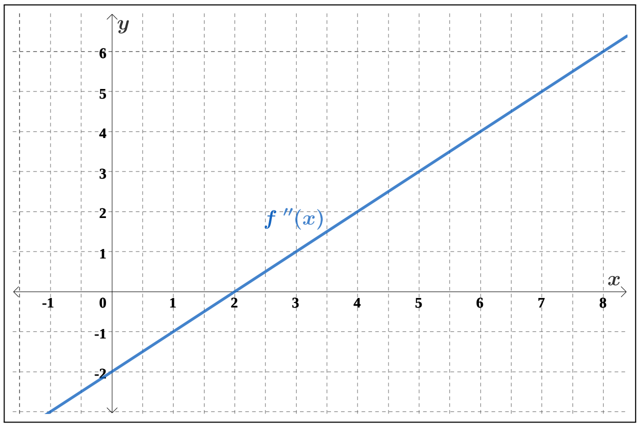
For Exercises 13 - 15, find (a) the partition numbers of f'', (b) the intervals of concavity of f, and (c) any inflection points of f using the Concavity Test.

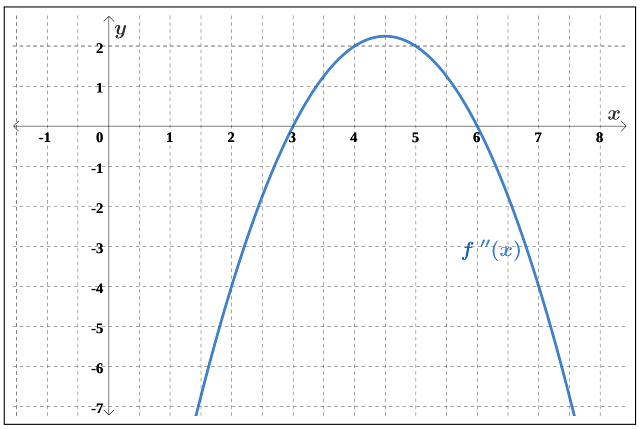
13. $f(x) = x^2 - 1$

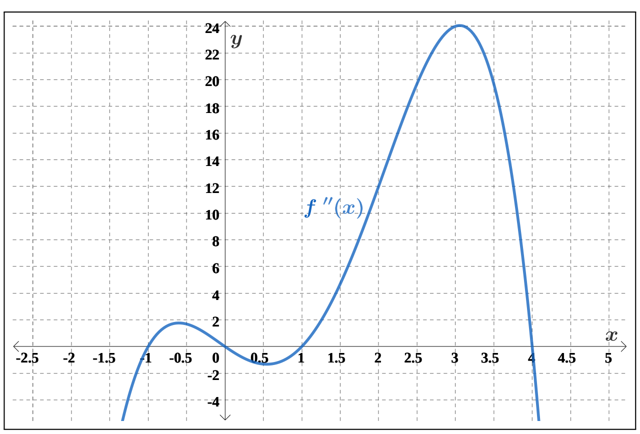
14. $f(x) = 4x^3 + 24x^2 - 384x + 3$

15. $f(x) = x^4 - 54x^2$

For Exercises 16 - 18, the graph of f'' is shown, and f is continuous on its domain of $(-\infty, \infty)$. Find (a) the partition numbers of f'', (b) the intervals of concavity of f, and (c) the x-values of any inflection points of f.

16. 

17. 

18. 

19. Dark Owl Records sells vinyl records. Michelle Nguyen, the owner, has determined if she spends x hundred dollars per quarter in advertising, she can sell $V(x) = -0.002x^5 + 0.4x^3 + 0.75x + 120$ hundred records in one quarter, where $0 \le x \le 15.5$. Find the amount of money spent by Michelle at the point of diminishing returns. Round your answer to the nearest cent.

20. Innocent Until Proven Quilt-y makes quilts. Rebekah Shapiro, the owner, has found that she can sell $Q(x) = -0.009x^3 + 0.25x^2 + 0.34x + 22$ quilts when she spends x dollars in advertising per week. Find the amount of money spent by Rebekah at the point of diminishing returns.

21. Angels and Angles is a company that sells protractors blessed by holy people of various religions. The marketing team has found that if they spend x dollars on advertising each day, they sell $A(x) = -0.0003x^3 + 0.04x^2 + 2.36x + 35.7$ protractors. Find the amount of money spent by the company at the point of diminishing returns.

For Exercises 22 - 26, the domain of f is $(-\infty, \infty)$, and f is twice-differentiable on its domain. Use the Second Derivative Test, if possible, to classify whether f has a local minimum, local maximum, or neither at the indicated x-value. If it is not possible, explain why.

22. $x = 1$, $f'(1) = 0$, and $f''(1) = 9$

23. $x = -8$, $f'(-8) = 0$, and $f''(-8) = -13$

24. $x = 14$, $f'(14) = 0$, and $f''(14) = -7.93$

25. $x = 12.5$, $f'(12.5) = 0$, and $f''(12.5) = 827$

26. $x = -127$, $f'(-127) = 0$, and $f''(-127) = -e$

Intermediate Skills Practice

For Exercises 27 - 31, find $\dfrac{d^2y}{dx^2}$.

27. $y = x^7 - x^{1/9} + \dfrac{1}{x^3}$

28. $y = xe^x$

29. $y = \dfrac{x^4}{x+14}$

30. $y = \log_7(8x+5) - 10x^3$

31. $y = 13^{x^5+4x^2}$

For Exercises 32 - 37, find (a) the intervals of concavity and (b) any inflection points of f.

32. $f(x) = \dfrac{2x-18}{3x+18}$

33. $y = \dfrac{x^2-3}{x-1}$

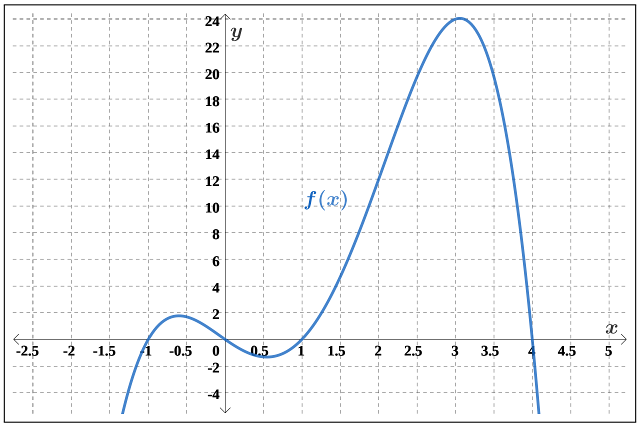
34. $f(x) = \dfrac{2x+3}{x-1}$

35. $f(x) = (8x-16)^7$

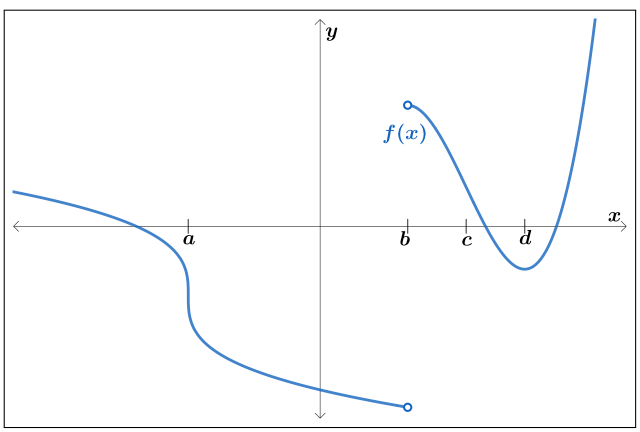
36. $f(x) = e^x\left(x^2-4x-94\right)$

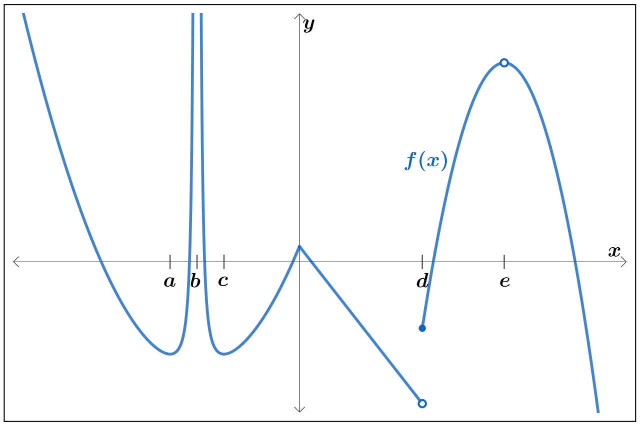
37. $f(x) = e^{64-2x^2}$

38. Given the graph of f shown below, find (a) the intervals of concavity and (b) any inflection points of f.

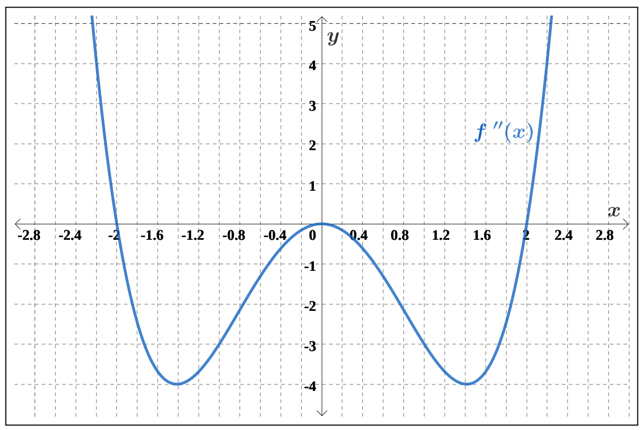


For Exercises 39 and 40, the graph of f is shown. Find (a) the intervals of concavity and (b) the x-values of any inflection points of f.

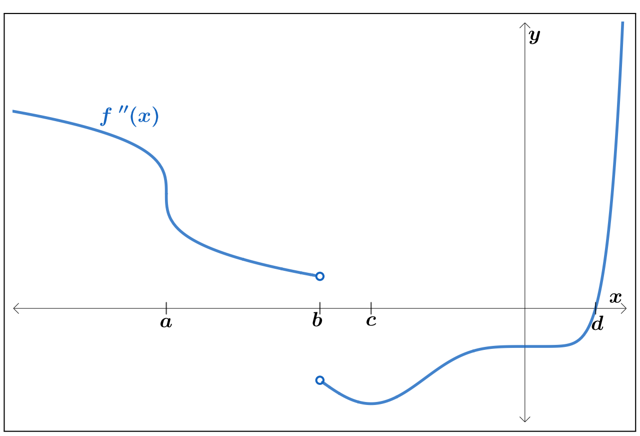
39. 

40. 

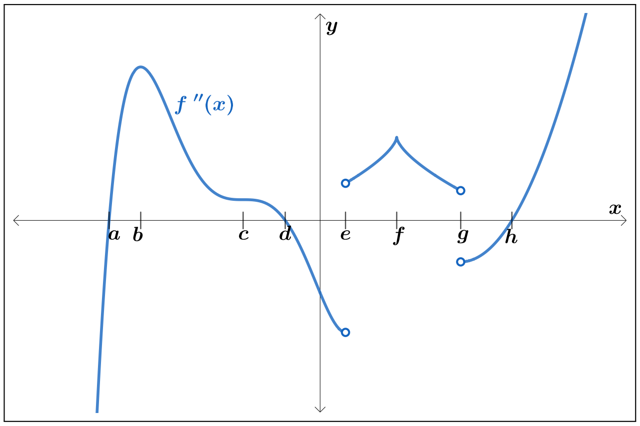
41. Given the graph of f'' shown below and that f is continuous on its domain of $(-\infty, \infty)$, find (a) the partition numbers of f'', (b) the intervals of concavity of f, and (c) the x-values of any inflection points of f.



For Exercises 42 and 43, the graph of f'' and the domain of f are given. Assuming f is continuous on its domain, find (a) the partition numbers of f'', (b) the intervals of concavity of f, and (c) the x-values of any inflection points of f.

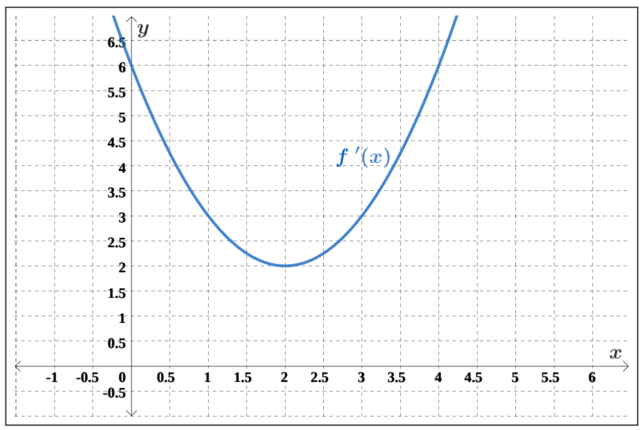
42. 

domain of f: $(-\infty, \infty)$

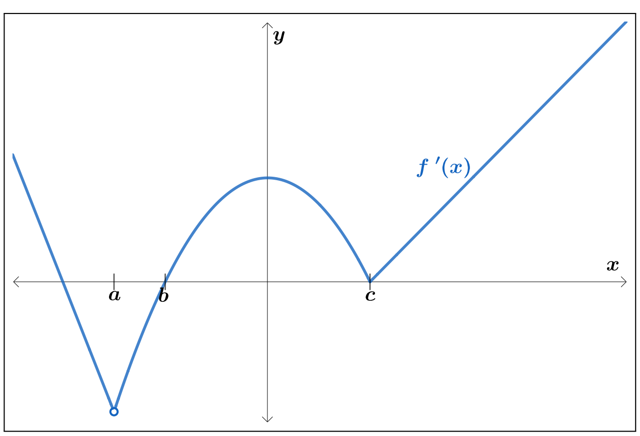
43. 

domain of f: $(-\infty, e) \cup (e, \infty)$

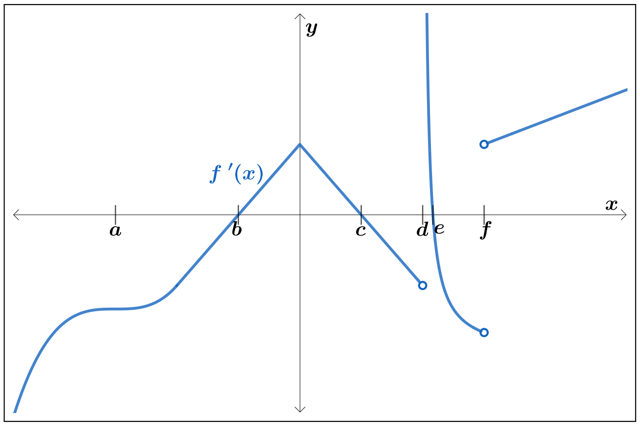
44. Given the graph of $f\,'$ shown below and that f is continuous on its domain of $(-\infty, \infty)$, find (a) the partition numbers of $f\,''$, (b) the intervals of concavity of f, and (c) the x-values of any inflection points of f.



For Exercises 45 and 46, the graph of $f\,'$ and the domain of f are given. Assuming f is continuous on its domain, find (a) the partition numbers of $f\,''$, (b) the intervals of concavity of f, and (c) the x-values of any inflection points of f.

45. 

domain of f: $(-\infty, a) \cup (a, \infty)$

46. 

domain of f: $(-\infty, d) \cup (d, f) \cup (f, \infty)$

47. Tom Nook has an island getaway package. He has determined that if he spends x thousand dollars on advertising each month, he sells $A(x) = -\frac{1}{6}x^4 + 8x^3 - 25x^2 + 75x + 300$ getaway packages. Find the x-coordinate of the inflection point of this function to five decimal places, and interpret the meaning of the value.

48. Greyt Toys, Inc. sells toys tailored specifically to greyhound dogs. The marketing team found that they sell $T(x) = -0.1x^3 + 9.8x^2 + 2.5x + 1755$ toys when they spend x thousand dollars on advertisements each quarter. Find the x-coordinate of the inflection point of this function to five decimal places, and interpret the meaning of the value.

49. Sister Elga's House of Curses offers haunted house tours in the month of October. The owner, Elga, spends x dollars in advertising in September and sells $H(x) = -0.0002x^3 + 0.055x^2 + 0.025x + 300$ tours in October. Find the x-coordinate of the inflection point of this function to two decimal places, and interpret the meaning of the value.

For Exercises 50 - 52, the domain of f is $(-\infty, \infty)$, and f is twice-differentiable on its domain. Use the Second Derivative Test, if possible, to classify whether f has a local minimum, local maximum, or neither at the indicated x-value. If it is not possible, explain why.

50. $x = 7$, $f'(7) = 0$, and $f''(7) = 9$

51. $x = 100$, $f'(100) = 0$, and $f''(100) = 0$

52. $x = -37$, $f'(-37) = 0$, and $f''(-37) = -7$

For Exercises 53 - 55, use the Second Derivative Test, if possible, to find any local extrema of the function. If it is not possible, explain why.

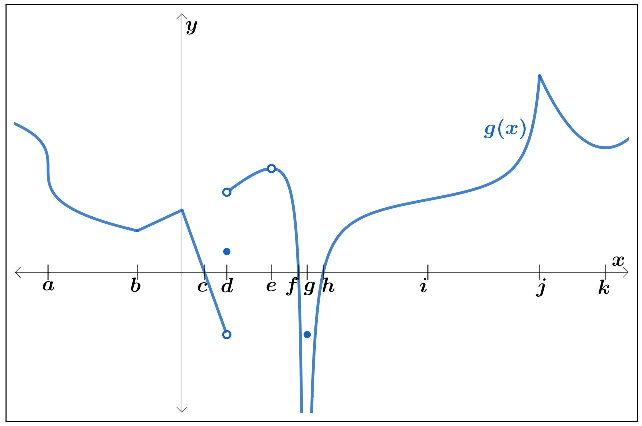
53. $f(x) = 2x^3 + 3x^2 - 36x + 16$

54. $f(x) = e^x\left(x^3 + 5x^2 - 3x + 3\right)$

55. $f(x) = e^{x^4 - 16}$

Mastery Practice

56. Given the graph of g shown below, find (a) the intervals of concavity and (b) the x-values of any inflection points of g.



For Exercises 57 - 62, find $f''(x)$.

57. $f(x) = x^{12} - 3x^{10} + 4x^6 + 19$

58. $f(x) = \log\left(7 + x^2\right)$

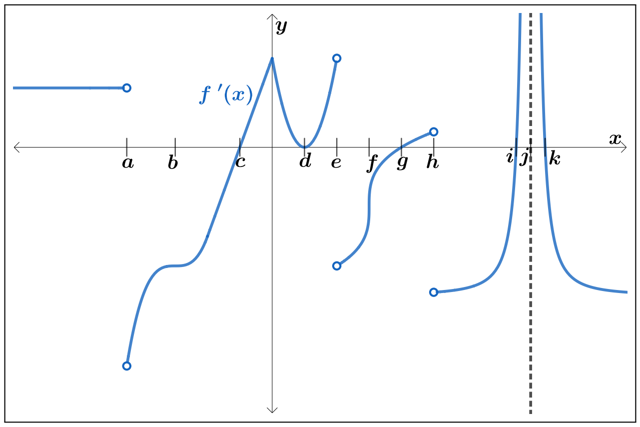
59. $f(x) = \dfrac{8x + 14}{(x-3)^4}$

60. $f(x) = 9^{7x^8 - 33x}$

61. $f(x) = \sqrt{44x^3 - 8x^2 + 1}$

62. $f(x) = \ln\left(\dfrac{x^2\left(4x^2+7\right)^9}{(13-2x)^8}\right)$

63. Given the graph of f'' shown below and that f is continuous on its domain of $(-\infty, a) \cup (a, j) \cup (j, \infty)$, find (a) the partition numbers of f'', (b) the intervals of concavity of f, and (c) the x-values of any inflection points of f.



64. Big Box Store sells cardboard boxes in bulk to moving companies. The moving companies buy $B(x) = -0.05x^3 + 2.2x^2 + 3.8x + 828$ thousand boxes from Big Box Store when it spends x thousand dollars advertising during a two month period. Find the amount of money spent by Big Box Store at the point of diminishing returns. Round your answer to the nearest cent.

For Exercises 65 - 75, find (a) the intervals of concavity and (b) the x-values of any inflection points of f.

65. $f(x) = 5x^4 - 9x^3$

66. $f''(x) = (x-3)(x+2)^3(x+7)^4$; f is continuous on its domain of $(-\infty, 3) \cup (3, \infty)$

67. $f(x) = (0.5)^x$

68. $f'(x) = x^3 - 3x^2 + 1$; f is continuous on its domain of $(-\infty, \infty)$

69. $f(x) = (x-16)e^{-2x}$

70. $f''(x) = \dfrac{(x-1)(x+6)}{(x+2)^4}$; f is continuous on its domain of $(-\infty, -2) \cup (-2, \infty)$

71. $f(x) = \dfrac{\ln(x)}{x}$

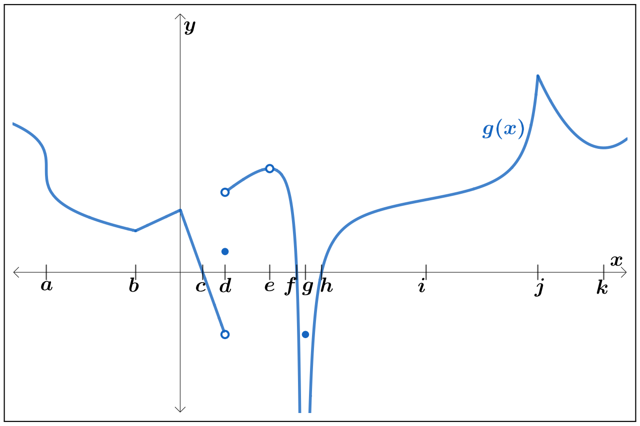
72. $f'(x) = \dfrac{2x}{(x^2+3)^2}$; f is continuous on its domain of $(-\infty, \infty)$

73. $f(x) = \dfrac{x-2}{(x-4)^2}$

74. $f(x) = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

75. $f''(x) = 3x^4 - 2x^3 - 5x^2$; f is continuous on its domain of $(-\infty, \infty)$

76. Given the graph of f' shown below and that f is continuous on its domain of $(-\infty, d) \cup (d, g) \cup (g, \infty)$, find (a) the partition numbers of f'', (b) the intervals of concavity of f, and (c) the x-values of any inflection points of f.



77. Given $g''(t) = (t-a)(t+b)^4(t-c)^3$, where $0 < a < b < c$, and that g is continuous on its domain of all real numbers except $t = c$, find (a) the partition numbers of g'', (b) the intervals of concavity of g, and (c) the t-values of any inflection points of g.

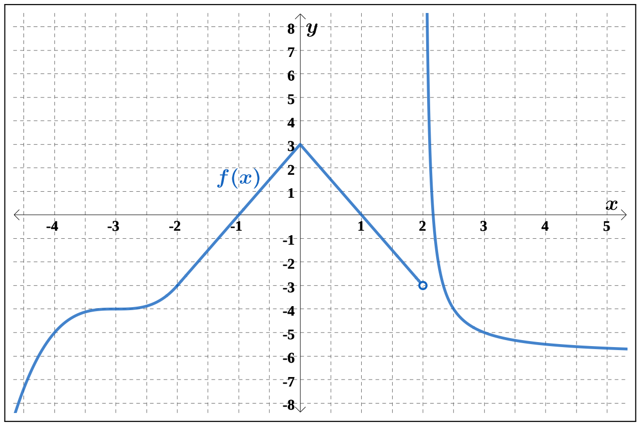
For Exercises 78 - 80, use the Second Derivative Test, if possible, to find any local extrema of the function. If it is not possible, explain why.

78. $f(x) = 1 + 9x + 3x^2 - x^3$

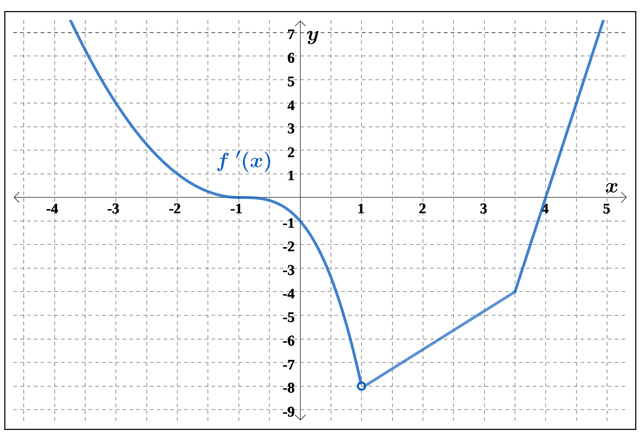
79. $f(x) = x^3 e^{-x/2}$

80. $f(x) = \dfrac{x^2}{4x+8}$

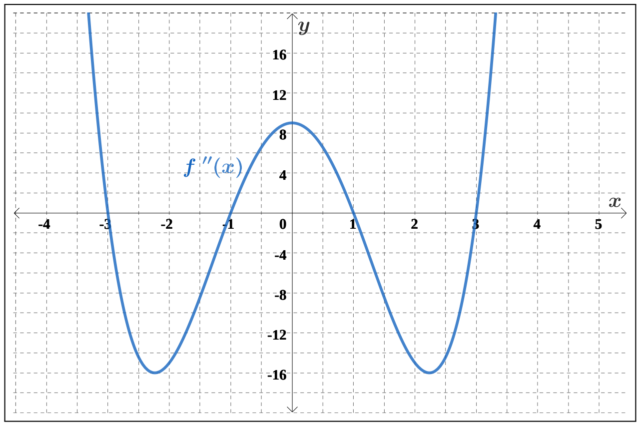
81. The graph of f is shown below. Use the graph to find (a) the x-values where $f'(x) = 0$, (b) the x-values where $f'(x)$ does not exist, (c) the intervals where $f'(x)$ is positive and the intervals where $f'(x)$ is negative, (d) the intervals where $f''(x)$ is positive and the intervals where $f''(x)$ is negative, and (e) the x-values where f' has any local extrema (specify the type).



82. The graph of f' is shown below, and f is continuous on its domain of $(-\infty, \infty)$. Use the graph to find (a) the intervals where f is increasing/decreasing, (b) the x-values where f has any local extrema (specify the type), (c) the intervals where $f''(x)$ is positive and the intervals where $f''(x)$ is negative, (d) the x-values where $f''(x) = 0$, (e) the x-values where $f''(x)$ does not exist, and (f) the x-values of any inflection points of f.



83. The graph of f'' is shown below, f is continuous on its domain of $(-\infty, \infty)$, and f' is continuous on its domain of $(-\infty, \infty)$. Use the graph to find (a) the intervals where f' is increasing/decreasing, (b) the x-values where f' has any local extrema (specify the type), (c) the intervals of concavity of f, (d) the x-values where f' has any inflection points, and (e) the x-values where $f''(x) = 0$.



84. Staples Depot sells office furniture. The marketing team determines that they sell $C(x) = -0.033x^3 + 19x^2 + 0.12x + 4096$ chairs when they spend x thousand dollars marketing them each year. Find the x-coordinate of the inflection point of this function to five decimal places, and interpret the meaning of the value.

85. Civet Coffee, Inc sells high end coffee beans. Their marketing team has determined that when they spend x thousand dollars in advertising, $C(x) = -0.03x^3 + 5x^2 + x + 120$ pounds of coffee beans are sold in a year. Find the x-coordinate of the inflection point of this function to five decimal places, and interpret the meaning of the value.

For Exercises 86 - 89, the domain of f is $(-\infty, \infty)$, and f is twice-differentiable on its domain. Use the Second Derivative Test, if possible, to classify whether f has a local minimum, local maximum, or neither at the indicated x-value. If it is not possible, explain why.

86. $x = -24$, $f'(-24) = 0$, and $f''(-24) = -2.887$

87. $x = 0$, $f'(0) = 5$, and $f''(0) = 19$

88. $x = 3$, $f'(3) = 0$, and $f''(3) = 0$

89. $x = 11$, $f'(11) = 0$, and $f''(11) = -8$

Communication Practice

90. Describe how to find the partition numbers of f''.

91. What does the Concavity Test enable us to find?

92. Describe the three-step process used to find the intervals of concavity of a function and where the function has inflection points.

93. Describe the three conditions necessary for a function to have an inflection point at $x = c$.

94. If $f''(-3) = 0$, does f have an inflection point at $x = -3$? Explain.

95. Write three statements equivalent to "f is concave up" in which each statement involves either f, f', or f''.

96. Although there are restrictions on its use, what does the Second Derivative Test enable us to find?

97. We can only attempt to use the Second Derivative Test to find local extrema of a function f with which type(s) of critical values?

98. Under what conditions will the Second Derivative Test fail?

99. If the Second Derivative Test fails, does that mean the function does not have a local extremum at the critical value in question? Explain.

100. If the Second Derivative Test fails, what can we do to determine if the function has a local extremum at the critical value in question?

101. If $f'(x) > 0$ on an interval (a,b), what, if anything, can we conclude about (a) f and (b) f''?

102. If $f(x) > 0$ on an interval (a,b), what, if anything, can we conclude about (a) f' and (b) f''?

103. If $f''(x) > 0$ on an interval (a,b), what, if anything, can we conclude about (a) f and (b) f'?

3.3 The Graphing Strategy

When developing the ideas of calculus, Newton and Leibniz did not have access to technology to graph functions. Plotting points to graph a function can be tedious and time consuming, but calculus provides a way to avoid that tedium!

We can use information about the functions f (including information about holes and asymptotes that we learned in **Section 1.3**), f' (including information about where a function is increasing/decreasing and has local extrema that we learned in **Section 3.1**), and f'' (including information about concavity and inflection points that we learned in **Section 3.2**) to construct a sketch (if not a perfect graph!) of the function f, *all without modern technology*.

Learning Objectives:
In this section, you will learn how to construct the graph of a function without the aid of technology using information obtained from the rule of the function as well as its first and second derivatives. Upon completion you will be able to:

- Implement the graphing strategy by analyzing a function and categorizing information about its first and second derivatives and then graphing the function without technology.
- Construct the graph of a function given information about its first and second derivatives as well as basic information about the function such as its domain, asymptotes, and intercepts.

Graphing Without Technology

Before learning how to combine all the information we have learned thus far to create the graph of a function without using a graphing calculator, let's review the terms for the features and significant points of a function with an example!

▪ **Example 1** Given the graph of f shown in **Figure 3.3.1**, answer each of the following.

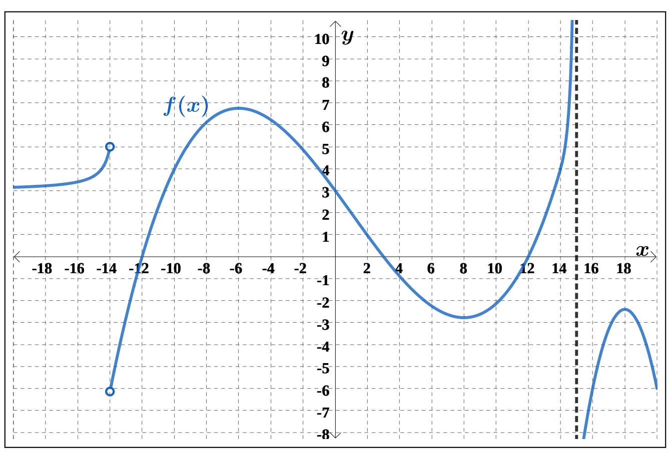


Figure 3.3.1: Graph of a function f

a. Find the domain and x- and y-intercepts of f.
b. Find any holes and vertical or horizontal asymptotes of f.
c. Find the intervals where f is increasing/decreasing and the x-values of any local extrema.
d. Find the intervals of concavity and any inflection points of f.

Solution:

a. Looking at the graph of f, we see it is defined on $(-\infty,14)\cup(-14,15)\cup(15,\infty)$. f has x-intercepts where the graph intersects, or touches, the x-axis, which occurs at $x=-12,\ 3,$ and 12. Thus, the x-intercepts are $(-12,0)$, $(3,0)$, and $(12,0)$. The y-intercept of f is where the graph of the function intersects the y-axis: $(0,3)$.

b. The graph of f has holes at $x=-14$, and it has a vertical asymptote at $x=15$. Furthermore, we can describe the infinite behavior near the vertical asymptote using limits: $\lim\limits_{x\to15^-} f(x)\to\infty$ and $\lim\limits_{x\to15^+} f(x)\to-\infty$. Looking at the end behavior, we see the graph of f has a horizontal asymptote, $y=3$, as $x\to-\infty$.

c. f is increasing on $(-\infty,-14)$, $(-14,-6)$, $(8,15)$, and $(15,18)$. It is decreasing on $(-6,8)$ and $(18,\infty)$. The function has local maxima at $x=-6$ and $x=18$, and it has a local minimum at $x=8$.

d. f is concave up on $(-\infty,-14)$ and $(1,15)$, and it is concave down on $(-14,1)$ and $(15,\infty)$. The function has an inflection point at $(1,2)$.

∎

Now that we have reviewed the pertinent terms that help us describe the graph of a function, we will learn how to construct the graph of a function without technology. To do so, we will implement the **graphing strategy** using the following steps:

Graphing Strategy

I. Find the domain, x- and y-intercepts, holes, and any vertical or horizontal asymptotes of f.

II. Find the intervals where f is increasing/decreasing and any local extrema using the First Derivative Test.

III. Find the intervals of concavity and any inflection points of f using the Concavity Test.

IV. Create a *shape chart* for f by combining the sign charts from steps II and III and drawing the resulting "shape" in each interval.

V. Create a rough sketch of f using the shapes from step IV as well as the information from step I.

While creating the shape chart in step IV, there will only be four possible combinations of increasing/decreasing and concavity that can occur in each interval: increasing and concave up, increasing and concave down, decreasing and concave up, and decreasing and concave down. If we can get an idea of what each of these four possibilities look like, we will be in pretty good *shape*!

We can see what all four of these shapes look like by dividing a circle into fourths. See **Figure 3.3.2**.

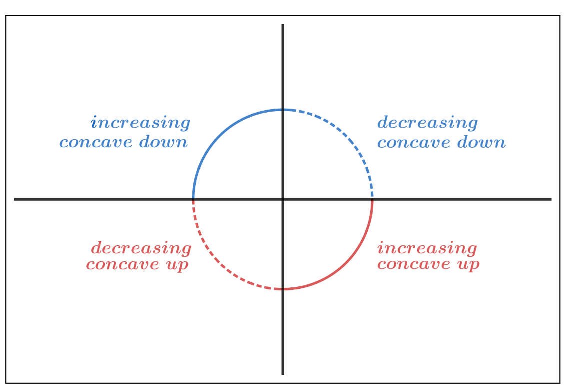


Figure 3.3.2: Four possible shapes of a function based on whether it is increasing/decreasing and its concavity

We will now demonstrate how to use the graphing strategy with examples.

Example 2 Use the graphing strategy to sketch the graph of $f(x) = \dfrac{9x}{x^3+1}$.

Solution:
To sketch a graph of f, we must work through all five steps of the graphing strategy:

I. Find the domain, x- and y-intercepts, holes, and any vertical or horizontal asymptotes of f:

When finding the domain of f, we need to ensure the denominator of the function does not equal zero:

$$\begin{aligned} x^3+1 &\neq 0 \\ x^3 &\neq -1 \\ \implies x &\neq -1 \end{aligned}$$

Thus, the domain of f is $(-\infty,-1)\cup(-1,\infty)$.

To find the x-intercept(s), we need to find the x-value(s) where the graph of f intersects (or touches) the x-axis. In other words, we need to find the x-value(s) where $f(x) = 0$. So we set the function equal to zero, and recall that a rational function equals zero when its numerator equals zero:

$$\begin{aligned} \frac{9x}{x^3+1} &= 0 \implies \\ 9x &= 0 \\ x &= 0 \end{aligned}$$

Thus, f has only one x-intercept: $(0,0)$.

To find where the graph of f intersects the y-axis (i.e., the point where $x = 0$), we need to calculate $f(0)$. Because $x = 0$ is in the domain of f, we can substitute and find the corresponding y-value:

$$\begin{aligned} f(x) &= \frac{9x}{x^3+1} \implies \\ f(0) &= \frac{9(0)}{(0)^3+1} \\ &= \frac{0}{1} \\ &= 0 \end{aligned}$$

Thus, the y-intercept of f is $(0,0)$.

N *Because we found that the point $(0,0)$ is the x-intercept of f, we could have deduced that the y-intercept is also $(0,0)$ and avoided substituting $x = 0$ into the function to find the y-intercept (because we already found that when $x = 0$, $y = 0$).*

To find any holes, we start by looking for the x-values where the limit of the function would be of the indeterminate form $\frac{0}{0}$. Because the numerator is already factored, we can just observe what happens when the limit of the numerator equals zero. The limit of the numerator only equals 0 when $x \to 0$, but the limit of the denominator equals 1 when $x \to 0$. So the function does not have any holes because the limit of the function will never be of the indeterminate form $\frac{0}{0}$. See **Section 1.2** for further explanation.

To find any vertical asymptotes, we look for x-values where the limit of the denominator equals zero but the limit of the numerator does not. Setting the denominator equal to zero gives $x = -1$ (we discovered this when finding the domain of f). Therefore, the limit of the denominator as $x \to -1$ equals zero. The limit of the numerator as $x \to -1$ equals -9, so f has a vertical asymptote at $x = -1$. See **Section 1.2** for further explanation.

To find any horizontal asymptotes, we need to find both $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$. Remember that for rational functions, if the limit as either $x \to \infty$ or $x \to -\infty$ is a finite number, then the other limit at infinity will equal the same finite number.

Recall that regardless of which limit at infinity we are calculating, for rational functions we divide both the numerator and the denominator by the term in the denominator with the highest power of x, which in this case is x^3.

Let's start by finding $\lim_{x\to\infty} f(x)$. Dividing both the numerator and denominator by x^3 gives

$$\begin{aligned}\lim_{x\to\infty} f(x) &= \lim_{x\to\infty} \frac{9x}{x^3+1} \\ &= \lim_{x\to\infty} \frac{\frac{9x}{x^3}}{\frac{x^3+1}{x^3}} \\ &= \lim_{x\to\infty} \frac{\frac{9x}{x^3}}{\frac{x^3}{x^3}+\frac{1}{x^3}} \\ &= \lim_{x\to\infty} \frac{\frac{9}{x^2}}{1+\frac{1}{x^3}}\end{aligned}$$

Recall that we have to be careful when finding the limit of this resulting quotient. We must investigate the behavior of the numerator and denominator separately.

Let's start by determining the behavior of the numerator. We will apply **Theorem 1.3**:

$$\begin{aligned}\lim_{x\to\infty} \frac{9}{x^2} &= \lim_{x\to\infty} \overbrace{\frac{9}{x^2}}^{0} \\ &= 0\end{aligned}$$

Next, we will determine the behavior of the denominator:

$$\begin{aligned}\lim_{x\to\infty}\left(1+\frac{1}{x^3}\right) &= \lim_{x\to\infty}\left(\overbrace{1}^{1}+\overbrace{\frac{1}{x^3}}^{0}\right) \\ &= 1\end{aligned}$$

We see both limits exist and the limit of the function in the denominator is nonzero. Now, we can finish finding the limit using the Properties of Limits and dividing the limits of the numerator and denominator:

$$\lim_{x\to\infty} \frac{\frac{9}{x^2}}{1+\frac{1}{x^3}} = \frac{\lim_{x\to\infty}\left(\frac{9}{x^2}\right)}{\lim_{x\to\infty}\left(1+\frac{1}{x^3}\right)}$$

$$= \frac{0}{1}$$
$$= 0$$

Hence,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{9x}{x^3+1} = \lim_{x\to\infty} \frac{\frac{9}{x^2}}{1+\frac{1}{x^3}} = 0$$

Therefore, as $x \to \infty$, the graph of the function has a horizontal asymptote: $y = 0$.

Due to the fact that f is a rational function and the limit as $x \to \infty$ equals a finite number, 0, we know the limit as $x \to -\infty$ is also 0. In other words, $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$. So the function has one horizontal asymptote, $y = 0$.

II. Find the intervals where f is increasing/decreasing and any local extrema using the First Derivative Test:

Recall $f(x) = \dfrac{9x}{x^3+1}$. We will proceed with the four steps we learned in **Section 3.1** to apply the First Derivative Test:

1. Determine the domain of f:

We found the domain in step I: $(-\infty, -1) \cup (-1, \infty)$.

2. Find the partition numbers of f':

We must use the Quotient Rule to find $f'(x)$:

$$f'(x) = \frac{\left(x^3+1\right)\left(\frac{d}{dx}(9x)\right) - (9x)\left(\frac{d}{dx}\left(x^3+1\right)\right)}{\left(x^3+1\right)^2}$$

$$= \frac{\left(x^3+1\right)(9) - (9x)\left(3x^2\right)}{\left(x^3+1\right)^2}$$

$$= \frac{9x^3+9-27x^3}{\left(x^3+1\right)^2}$$

$$= \frac{-18x^3+9}{\left(x^3+1\right)^2}$$

To find the partition numbers of f', we find the x-values where $f'(x) = 0$ or $f'(x)$ does not exist. Because f' is a rational function, $f'(x)$ equals zero when the numerator equals zero, and it does not exist when the denominator equals zero (remember we are looking at the domain of f' when determining where it does not exist). First, let's find the x-values such that $f'(x) = 0$:

$$\begin{aligned} f'(x) = 0 &\implies \\ -18x^3 + 9 &= 0 \\ -18x^3 &= -9 \\ x^3 &= \frac{1}{2} \\ x &= \sqrt[3]{\frac{1}{2}} \end{aligned}$$

Now, to find the x-values where $f'(x)$ does not exist, we set the denominator equal to zero:

$$\begin{aligned} f'(x) \text{ DNE } &\implies \\ \left(x^3+1\right)^2 &= 0 \\ \left(\left(x^3+1\right)^2\right)^{\frac{1}{2}} &= (0)^{\frac{1}{2}} \\ x^3 + 1 &= 0 \\ x^3 &= -1 \\ x &= -1 \end{aligned}$$

This means the partition numbers of f' are $x = -1$ and $x = \sqrt[3]{\frac{1}{2}}$.

3. Determine which partition numbers of f' are in the domain of f (i.e., find the critical values of f):

Recall the domain of f is $(-\infty, -1) \cup (-1, \infty)$. Thus, the only partition number of f' *not* in the domain of f is $x = -1$. Therefore, $x = \sqrt[3]{\frac{1}{2}}$ is the only critical value of f.

4. Create a sign chart of $f'(x)$:

We will place the partition numbers of f' on a number line, with $x = \sqrt[3]{\frac{1}{2}}$ having a solid dot and $x = -1$ having an open circle to indicate which is in the domain of f and which is not, respectively.

Next, we determine the sign of $f'(x)$ on the intervals $(-\infty, -1)$, $\left(-1, \sqrt[3]{\frac{1}{2}}\right)$, and $\left(\sqrt[3]{\frac{1}{2}}, \infty\right)$. We must select x-values to test in each interval. We will select $x = -2,\ 0,$ and 2:

$$\begin{aligned} f'(x) = \frac{-18x^3+9}{\left(x^3+1\right)^2} &\implies \\ f'(-2) = \frac{-18(-2)^3+9}{((-2)^3+1)^2} &= \frac{153}{49} > 0 \\ f'(0) = \frac{-18(0)^3+9}{((0)^3+1)^2} &= 9 > 0 \\ f'(2) = \frac{-18(2)^3+9}{((2)^3+1)^2} &= -\frac{135}{81} < 0 \end{aligned}$$

Using this information, we can fill in the sign chart of $f'(x)$. We will also include the information this yields for f below the number line. See **Figure 3.3.3**.

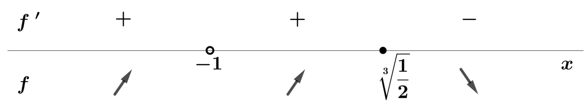


Figure 3.3.3: Sign chart of $f'(x)$ with the corresponding information for $f(x) = \dfrac{9x}{x^3+1}$

Using the First Derivative Test, we see that a local maximum occurs at $x = \sqrt[3]{\frac{1}{2}}$. To find the local maximum (y-value) associated with this x-value, we substitute the x-value into the *original function* f:

$$
\begin{aligned}
f(x) &= \frac{9x}{x^3+1} \implies \\
f\left(\sqrt[3]{\frac{1}{2}}\right) &= \frac{9\sqrt[3]{\frac{1}{2}}}{\left(\sqrt[3]{\frac{1}{2}}\right)^3 + 1} \\
&= \frac{9\sqrt[3]{\frac{1}{2}}}{\frac{1}{2}+1} \\
&= \frac{9\sqrt[3]{\frac{1}{2}}}{\frac{3}{2}} \\
&= 6\sqrt[3]{\frac{1}{2}} \approx 4.7622
\end{aligned}
$$

In conclusion, f is increasing on $(-\infty, -1)$ and $\left(-1, \sqrt[3]{\frac{1}{2}}\right)$, and it is decreasing on $\left(\sqrt[3]{\frac{1}{2}}, \infty\right)$. f has a local maximum of $6\sqrt[3]{\frac{1}{2}}$ at $x = \sqrt[3]{\frac{1}{2}}$.

III. Find the intervals of concavity and any inflection points of f using the Concavity Test:

Recall $f(x) = \dfrac{9x}{x^3+1}$. We will proceed with the three steps we learned in **Section 3.2** to apply the Concavity Test:

1. Determine the domain of f:

Again, we know the domain of f is $(-\infty, -1) \cup (-1, \infty)$.

2. Find the partition numbers of f'':

We found $f'(x)$ in step II, so we can take the derivative of this function to find $f''(x)$:

$$
\begin{aligned}
f'(x) &= \frac{-18x^3+9}{\left(x^3+1\right)^2} \implies \\
f''(x) &= \frac{\left(x^3+1\right)^2\left(\frac{d}{dx}\left(-18x^3+9\right)\right)-\left(-18x^3+9\right)\left(\frac{d}{dx}\left(\left(x^3+1\right)^2\right)\right)}{\left(\left(x^3+1\right)^2\right)^2} \\
&= \frac{\left(x^3+1\right)^2\left(-54x^2\right)-\left(-18x^3+9\right)\left(2\left(x^3+1\right)\left(\frac{d}{dx}\left(x^3+1\right)\right)\right)}{\left(x^3+1\right)^4} \\
&= \frac{\left(x^3+1\right)^2\left(-54x^2\right)-\left(-18x^3+9\right)\left(2\left(x^3+1\right)\left(3x^2\right)\right)}{\left(x^3+1\right)^4}
\end{aligned}
$$

Because we need to use this function to find the partition numbers of f'', we need to algebraically manipulate it. We will factor the term x^3+1 term from the numerator (remember we factor the lowest power of the term) and continue simplifying:

$$
\begin{aligned}
f''(x) &= \frac{\left(x^3+1\right)^2\left(-54x^2\right)-\left(-18x^3+9\right)\left(2\left(x^3+1\right)\left(3x^2\right)\right)}{\left(x^3+1\right)^4} \\
&= \frac{\left(x^3+1\right)\left[\left(x^3+1\right)\left(-54x^2\right)-\left(-18x^3+9\right)(2)\left(3x^2\right)\right]}{\left(x^3+1\right)^4} \\
&= \frac{\cancel{\left(x^3+1\right)}\left[\left(x^3+1\right)\left(-54x^2\right)-\left(-18x^3+9\right)(2)\left(3x^2\right)\right]}{\left(x^3+1\right)^{\cancel{4}\,3}} \\
&= \frac{\left(x^3+1\right)\left(-54x^2\right)-\left(-18x^3+9\right)(2)\left(3x^2\right)}{\left(x^3+1\right)^3} \\
&= \frac{\left(-54x^5-54x^2\right)-\left(-18x^3+9\right)\left(6x^2\right)}{\left(x^3+1\right)^3} \\
&= \frac{\left(-54x^5-54x^2\right)-\left(-108x^5+54x^2\right)}{(x^3+1)^3} \\
&= \frac{-54x^5-54x^2+108x^5-54x^2}{\left(x^3+1\right)^3} \\
&= \frac{54x^5-108x^2}{\left(x^3+1\right)^3}
\end{aligned}
$$

To find the partition numbers of f'', we find the x-values where $f''(x) = 0$ or $f''(x)$ does not exist. $f''(x) = 0$ when the numerator equals zero, and $f''(x)$ does not exist when the denominator equals zero (remember we are looking at the domain of f'' when determining where it does not exist). First, we will find the x-values where $f''(x) = 0$:

$$\begin{aligned} f''(x) = 0 &\implies \\ 54x^5 - 108x^2 &= 0 \\ 54x^2\left(x^3 - 2\right) &= 0 \end{aligned}$$

This gives us two equations to solve: $54x^2 = 0$ and $x^3 - 2 = 0$. The first equation, $54x^2 = 0$, gives $x = 0$. The second equation, $x^3 - 2 = 0$, gives $x = \sqrt[3]{2}$.

Now, to find the x-values where $f''(x)$ does not exist, we set the denominator equal to zero:

$$\begin{aligned} f''(x) \text{ DNE } &\implies \\ \left(x^3 + 1\right)^3 &= 0 \\ \left(\left(x^3 + 1\right)^3\right)^{\frac{1}{3}} &= (0)^{\frac{1}{3}} \\ x^3 + 1 &= 0 \\ x^3 &= -1 \\ x &= -1 \end{aligned}$$

Thus, the partition numbers of f'' are $x = -1$, $x = 0$, and $x = \sqrt[3]{2}$.

3. Create a sign chart of $f''(x)$:

We will place the partition numbers of f'' on a number line, with $x = 0$ and $x = \sqrt[3]{2}$ having solid dots and $x = -1$ having an open circle to indicate which are included in the domain of f and which are not, respectively.

Next, we need to determine the sign of $f''(x)$ on the intervals $(-\infty, -1)$, $(-1, 0)$, $\left(0, \sqrt[3]{2}\right)$, and $\left(\sqrt[3]{2}, \infty\right)$. We will choose the x-values $x = -2$, -0.5, 1, and 2 to test:

$$\begin{aligned} f''(x) &= \frac{54x^5 - 108x^2}{\left(x^3 + 1\right)^3} \implies \\ f''(-2) &= \frac{54(-2)^5 - 108(-2)^2}{((-2)^3 + 1)^3} = \frac{2160}{343} > 0 \\ f''(-0.5) &= \frac{54(-0.5)^5 - 108(-0.5)^2}{((-0.5)^3 + 1)^3} = -\frac{14,688}{343} < 0 \\ f''(1) &= \frac{54(1)^5 - 108(1)^2}{((1)^3 + 1)^3} = -\frac{27}{4} < 0 \\ f''(2) &= \frac{54(2)^5 - 108(2)^2}{((2)^3 + 1)^3} = \frac{16}{9} > 0 \end{aligned}$$

Using this information, we can fill in the sign chart of $f''(x)$. Again, we include the information this yields for f below the number line. See **Figure 3.3.4**.

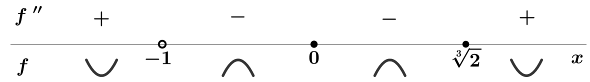


Figure 3.3.4: Sign chart of $f''(x)$ with the corresponding information for $f(x) = \dfrac{9x}{x^3+1}$

Looking at the sign chart, we see that $f''(x)$ changes sign at the partition numbers $x = -1$ and $x = \sqrt[3]{2}$. Of these two x-values, f is only defined at $x = \sqrt[3]{2}$, so there is only one inflection point. Remember, even if $f''(x)$ changes sign, it does not necessarily mean there is an inflection point. We must check that the x-value is in the domain of the function.

To find the y-value associated with the inflection point, we substitute $x = \sqrt[3]{2}$ into the *original function* f:

$$\begin{aligned} f(x) &= \frac{9x}{x^3+1} \implies \\ f\left(\sqrt[3]{2}\right) &= \frac{9\sqrt[3]{2}}{\left(\sqrt[3]{2}\right)^3+1} \\ &= \frac{9\sqrt[3]{2}}{2+1} \\ &= \frac{9\sqrt[3]{2}}{3} \\ &= 3\sqrt[3]{2} \approx 3.7798 \end{aligned}$$

In conclusion, f is concave down on $(-1,0)$ and $\left(0, \sqrt[3]{2}\right)$, and it is concave up on $(-\infty,-1)$ and $\left(\sqrt[3]{2},\infty\right)$. The function has an inflection point at $\left(\sqrt[3]{2},\ 3\sqrt[3]{2}\right)$.

IV. Create a shape chart for f by combining the sign charts from steps II and III and drawing the resulting "shape" in each interval:

We will now combine the sign charts of $f'(x)$ and $f''(x)$ from steps II and III, respectively, into one *combined* sign chart with both the partition numbers of f' and the partition numbers of f''. In each interval on the combined sign chart, we will mark whether the function is increasing/decreasing as well as whether it is concave up/down. Then, we will draw the resulting shape for each interval (see **Figure 3.3.2** for a reminder of the four possible shapes for each interval).

The partition numbers of both sign charts from steps II and III include $x = -1$, $x = 0$, $x = \sqrt[3]{\frac{1}{2}}$, and $x = \sqrt[3]{2}$. We will place all of these x-values on a number line and use the previous sign charts to record the increasing/decreasing and concavity information for each of the resulting intervals.

On the interval $(-\infty,-1)$, f is increasing and concave up. f is increasing and concave down on both $(-1,0)$ and $\left(0, \sqrt[3]{\frac{1}{2}}\right)$. On $\left(\sqrt[3]{\frac{1}{2}}, \sqrt[3]{2}\right)$, the function is decreasing and concave down, and on the interval $\left(\sqrt[3]{2},\infty\right)$, f is decreasing and concave up.

The shape chart for f shown in **Figure 3.3.5** contains the relevant information and corresponding shape for each interval:

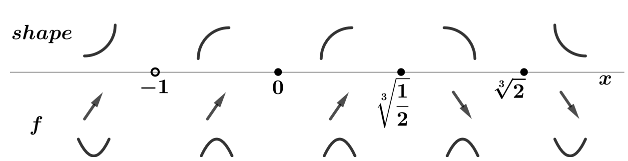


Figure 3.3.5: Shape chart for $f(x) = \dfrac{9x}{x^3+1}$

V. Create a rough sketch of f using the shapes from step IV as well as the information from step I:

Now, we are ready to sketch the graph of f! We start by plotting the points we know: the x- and y-intercepts, local maximum, and inflection point. We can also sketch the vertical asymptote at $x = -1$.

Next, we sketch the first shape in the shape chart, and keep in mind that $\lim_{x\to-\infty} f(x) = 0$ (i.e., $y = 0$ is a horizontal asymptote). We continue sketching the shapes on the axes while making sure to incorporate the intercepts, local maximum, and inflection point. We also need to remember that our last shape should incorporate the horizontal asymptote $y = 0$ as $x \to \infty$. Our resulting sketch is shown in **Figure 3.3.6**:

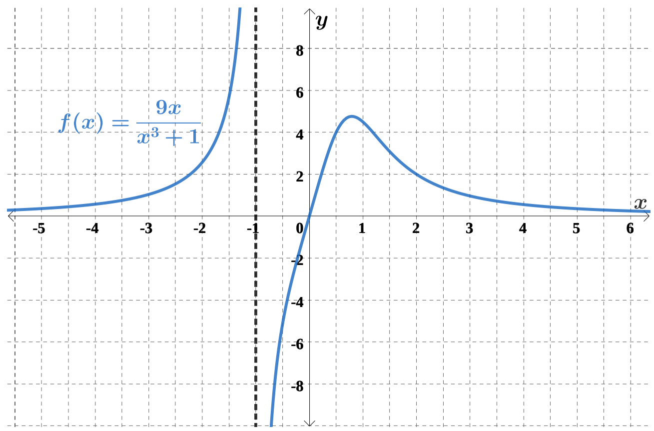


Figure 3.3.6: Sketch of the graph of the function $f(x) = \dfrac{9x}{x^3+1}$

■

When using the graphing strategy, you should obtain a rough sketch of the graph of f. Meaning, there may be variety in the way the graph is drawn. However, the intercepts, asymptotes, local extrema, inflection points, and basic shapes of the graph should all be the same.

■ **Example 3** Given $\lim_{x\to\infty} xe^{-2x^2} = 0$ and $\lim_{x\to-\infty} xe^{-2x^2} = 0$, use the graphing strategy to sketch the graph of $f(x) = xe^{-2x^2}$.

Solution:
To sketch a graph of f, we must work through all five steps of the graphing strategy:

I. Find the domain, x- and y-intercepts, holes, and any vertical or horizontal asymptotes of f:

The function does not have any even roots or logarithms, but there is division: $xe^{-2x^2} = \dfrac{x}{e^{2x^2}}$. However, because $e^{2x^2} > 0$, the denominator will never equal zero. Thus, the domain of f is $(-\infty, \infty)$.

To find the x-intercept(s), we need to find the x-value(s) where the graph of f intersects (or touches) the x-axis. In other words, we need to find the x-value(s) where $f(x) = 0$. So we set the function equal to zero:

$$xe^{-2x^2} = 0$$

This gives us two equations to solve: $x = 0$ and $e^{-2x^2} = 0$. The first equation has the solution $x = 0$. Because $e^{-2x^2} = \dfrac{1}{e^{2x^2}}$ and $e^{2x^2} > 0$, $e^{-2x^2} > 0$. Thus, there is no solution to the second equation. Therefore, f has only one x-intercept: $(0, 0)$.

To find where the graph of f intersects the y-axis (i.e., the point where $x = 0$), we need to calculate $f(0)$. Because $x = 0$ is in the domain of f, we can substitute and find the corresponding y-value:

$$\begin{aligned} f(x) &= xe^{-2x^2} \implies \\ f(0) &= (0)\left(e^{-2(0)^2}\right) \\ &= (0)(1) \\ &= 0 \end{aligned}$$

Thus, the y-intercept of f is $(0,0)$.

N *Because we found that the point $(0,0)$ is the x-intercept of f, we could have deduced that the y-intercept is also $(0,0)$ and avoided substituting $x = 0$ into the function to find the y-intercept (because we already found that when $x = 0$, $y = 0$).*

To find any holes, we start by looking for the x-values where the limit of the function would be of the indeterminate form $\frac{0}{0}$. Recalling $f(x) = xe^{-2x^2} = \frac{x}{e^{2x^2}}$, we see the limit of the numerator only equals 0 when $x \to 0$. The limit of the denominator equals 1 when $x \to 0$, so the function does not have any holes because the limit of the function will never be of the indeterminate form $\frac{0}{0}$.

To find any vertical asymptotes, we look for x-values where the limit of the denominator equals zero but the limit of the numerator does not. Setting the denominator equal to zero gives the equation $e^{2x^2} = 0$ again. However, as stated previously, e^{2x^2} will never equal zero because $e^{2x^2} > 0$. Thus, the function does not have any vertical asymptotes.

The method required to find any horizontal asymptotes of the function $f(x) = xe^{-2x^2}$ is beyond the scope of this textbook, but that is okay because we were given $\lim\limits_{x\to\infty} xe^{-2x^2} = 0$ and $\lim\limits_{x\to-\infty} xe^{-2x^2} = 0$. Thus, the graph of f has a horizontal asymptote: $y = 0$.

II. Find the intervals where f is increasing/decreasing and any local extrema using the First Derivative Test:

Recall $f(x) = xe^{-2x^2}$. We will proceed with the four steps we learned in **Section 3.1** to apply the First Derivative Test:

1. Determine the domain of f:

We found the domain in step I: $(-\infty, \infty)$.

2. Find the partition numbers of $f\,'$:

We must use the Product Rule to find $f\,'(x)$:

$$\begin{aligned} f\,'(x) &= x\left(\frac{d}{dx}\left(e^{-2x^2}\right)\right) + e^{-2x^2}\left(\frac{d}{dx}(x)\right) \\ &= xe^{-2x^2}\left(\frac{d}{dx}\left(-2x^2\right)\right) + e^{-2x^2}(1) \\ &= xe^{-2x^2}(-4x) + e^{-2x^2} \end{aligned}$$

Because we must continue and find where this derivative equals zero or does not exist, we will algebraically manipulate it by factoring the term e^{-2x^2}:

$$\begin{aligned} f'(x) &= xe^{-2x^2}(-4x) + e^{-2x^2} \\ &= e^{-2x^2}\left[x(-4x) + 1\right] \\ &= e^{-2x^2}\left(-4x^2 + 1\right) \end{aligned}$$

To find the partition numbers of f', we find the x-values where $f'(x) = 0$ or $f'(x)$ does not exist. Let's start by finding the x-values such that $f'(x) = 0$:

$$\begin{aligned} f'(x) = 0 &\implies \\ e^{-2x^2}\left(-4x^2 + 1\right) &= 0 \end{aligned}$$

This gives us two equations solve: $e^{-2x^2} = 0$ and $-4x^2 + 1 = 0$. Recall from our previous discussions that $e^{-2x^2} > 0$, so there is no solution to the first equation. Solving the second equation gives

$$\begin{aligned} -4x^2 + 1 &= 0 \\ -4x^2 &= -1 \\ x^2 &= \frac{1}{4} \\ \implies x = \sqrt{\frac{1}{4}} &= \frac{1}{2} \text{ and } x = -\sqrt{\frac{1}{4}} = -\frac{1}{2} \end{aligned}$$

Now, to find the x-values where $f'(x)$ does not exist, we look at the domain of f' and see if there are any x-values that are excluded. Again, there are no even roots or logarithms, but there is division:

$$f'(x) = e^{-2x^2}\left(-4x^2 + 1\right) = \frac{-4x^2 + 1}{e^{2x^2}}$$

Because $e^{2x^2} > 0$, the denominator will never equal zero.

Thus, the only partition numbers of f' are $x = -\frac{1}{2}$ and $x = \frac{1}{2}$.

3. Determine which partition numbers of f' are in the domain of f (i.e., find the critical values of f):

Because the domain of f is $(-\infty, \infty)$, $x = -\frac{1}{2}$ and $x = \frac{1}{2}$ are both critical values of f.

4. Create a sign chart of $f'(x)$:

We will place the partition numbers of f' on a number line, and both will have solid dots to indicate that they are in the domain of f.

Next, we determine the sign of $f'(x)$ on the intervals $\left(-\infty, -\frac{1}{2}\right)$, $\left(-\frac{1}{2}, \frac{1}{2}\right)$, and $\left(\frac{1}{2}, \infty\right)$. We will select $x = -1,\ 0,$ and 1 to test:

$$\begin{aligned} f'(x) &= e^{-2x^2}\left(-4x^2 + 1\right) \implies \\ f'(-1) &= e^{-2(-1)^2}\left(-4(-1)^2 + 1\right) = e^{-2}(-3) = \frac{-3}{e^2} < 0 \\ f'(0) &= e^{-2(0)^2}\left(-4(0)^2 + 1\right) = 1(0 + 1) = 1 > 0 \\ f'(1) &= e^{-2(1)^2}\left(-4(1)^2 + 1\right) = e^{-2}(-3) = \frac{-3}{e^2} < 0 \end{aligned}$$

Using this information, we can fill in the sign chart of $f'(x)$. We will also include the information this yields for f below the number line. See **Figure 3.3.7**.

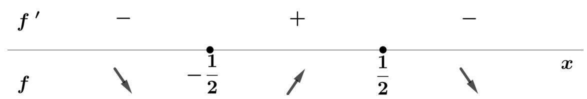


Figure 3.3.7: Sign chart of $f'(x)$ with the corresponding information for $f(x) = xe^{-2x^2}$

Using the First Derivative Test, we see that a local minimum occurs at $x = -\frac{1}{2}$. To find the local minimum (y-value), we substitute $x = -\frac{1}{2}$ into the *original function* f:

$$\begin{aligned} f(x) &= xe^{-2x^2} \implies \\ f\left(-\frac{1}{2}\right) &= -\frac{1}{2}e^{-2(-\frac{1}{2})^2} \\ &= -\frac{1}{2}e^{-2(\frac{1}{4})} \\ &= -\frac{1}{2}e^{-\frac{1}{2}} \\ &\approx -0.3033 \end{aligned}$$

We also see that a local maximum occurs at $x = \frac{1}{2}$. To find the local maximum, we substitute $x = \frac{1}{2}$ into the *original function* f:

$$\begin{aligned} f(x) &= xe^{-2x^2} \implies \\ f\left(\frac{1}{2}\right) &= \frac{1}{2}e^{-2(\frac{1}{2})^2} \\ &= \frac{1}{2}e^{-2(\frac{1}{4})} \\ &= \frac{1}{2}e^{-\frac{1}{2}} \\ &\approx 0.3033 \end{aligned}$$

In conclusion, f is increasing on $\left(-\frac{1}{2}, \frac{1}{2}\right)$, and it is decreasing on $\left(-\infty, -\frac{1}{2}\right)$ and $\left(\frac{1}{2}, \infty\right)$. f has a local minimum of $-\frac{1}{2}e^{-\frac{1}{2}}$ at $x = -\frac{1}{2}$ and a local maximum of $\frac{1}{2}e^{-\frac{1}{2}}$ at $x = \frac{1}{2}$.

III. Find the intervals of concavity and any inflection points of f using the Concavity Test:

Recall $f(x) = xe^{-2x^2}$. We will proceed with the three steps we learned in **Section 3.2** to apply the Concavity Test:

1. Determine the domain of f:

Again, we know the domain of f is $(-\infty, \infty)$.

2. Find the partition numbers of f'':

We found $f'(x)$ in step II, so we can take the derivative of this function to find $f''(x)$:

$$\begin{aligned}
f'(x) &= e^{-2x^2}\left(-4x^2+1\right) \implies \\
f''(x) &= e^{-2x^2}\left(\frac{d}{dx}\left(-4x^2+1\right)\right)+\left(-4x^2+1\right)\left(\frac{d}{dx}\left(e^{-2x^2}\right)\right) \\
&= e^{-2x^2}(-8x)+\left(-4x^2+1\right)\left(e^{-2x^2}\right)\left(\frac{d}{dx}\left(-2x^2\right)\right) \\
&= e^{-2x^2}(-8x)+\left(-4x^2+1\right)\left(e^{-2x^2}\right)(-4x)
\end{aligned}$$

Because we need to use this function to find the partition numbers of f'', we need to algebraically manipulate it. We will factor both the terms x and e^{-x^2}:

$$\begin{aligned}
f''(x) &= e^{-2x^2}(-8x)+\left(-4x^2+1\right)\left(e^{-2x^2}\right)(-4x) \\
&= xe^{-2x^2}\left[-8+\left(-4x^2+1\right)(-4)\right] \\
&= xe^{-2x^2}\left(-8+16x^2-4\right) \\
&= xe^{-2x^2}\left(16x^2-12\right)
\end{aligned}$$

To find the partition numbers of f'', we find the x-values where $f''(x)=0$ or $f''(x)$ does not exist. f'' has a domain of all real numbers because it does not have an even root or a logarithm and the implied division will not result in a zero in the denominator (as explained previously). Hence, $f''(x)$ will exist for all values of x. Thus, we only need to find the x-values where $f''(x)=0$:

$$\begin{aligned}
f''(x) &= 0 \implies \\
xe^{-2x^2}\left(16x^2-12\right) &= 0
\end{aligned}$$

This gives us three equations to solve: $x=0$, $e^{-2x^2}=0$, and $16x^2-12=0$. The first equation has the solution $x=0$. Because $e^{-2x^2}>0$, there is no solution to the second equation. Solving the third equation gives

$$\begin{aligned}
16x^2-12 &= 0 \\
16x^2 &= 12 \\
x^2 &= \frac{12}{16}=\frac{3}{4} \\
\implies x = -\sqrt{\frac{3}{4}} &\text{ and } x=\sqrt{\frac{3}{4}}
\end{aligned}$$

Thus, the partition numbers of f'' are $x=-\sqrt{\frac{3}{4}}$, $x=0$, and $x=\sqrt{\frac{3}{4}}$.

3. Create a sign chart of $f''(x)$:

We will place the partition numbers of f'' on a number line. All of the partition numbers will have solid dots because they are all in the domain of f.

Next, we determine the sign of $f''(x)$ on the intervals $\left(-\infty,-\sqrt{\frac{3}{4}}\right)$, $\left(-\sqrt{\frac{3}{4}},0\right)$, $\left(0,\sqrt{\frac{3}{4}}\right)$ and $\left(\sqrt{\frac{3}{4}},\infty\right)$. We will choose the x-values $x=-2$, $-\frac{1}{2}$, $\frac{1}{2}$, and 2 to test:

$$f''(x) = xe^{-2x^2}\left(16x^2 - 12\right) \implies$$

$$\begin{aligned} f''(-2) &= (-2)e^{-2(-2)^2}\left(16(-2)^2 - 12\right) \\ &= -2e^{-8}(52) = -\frac{104}{e^8} < 0 \end{aligned}$$

$$\begin{aligned} f''\left(-\frac{1}{2}\right) &= \left(-\frac{1}{2}\right)e^{-2(-\frac{1}{2})^2}\left(16\left(-\frac{1}{2}\right)^2 - 12\right) \\ &= -\frac{1}{2}e^{-\frac{1}{2}}(-8) = \frac{4}{e^{\frac{1}{2}}} > 0 \end{aligned}$$

$$\begin{aligned} f''\left(\frac{1}{2}\right) &= \left(\frac{1}{2}\right)e^{-2(\frac{1}{2})^2}\left(16\left(\frac{1}{2}\right)^2 - 12\right) \\ &= \frac{1}{2}e^{-\frac{1}{2}}(-8) = -\frac{4}{e^{\frac{1}{2}}} < 0 \end{aligned}$$

$$\begin{aligned} f''(2) &= (2)e^{-2(2)^2}\left(16(2)^2 - 12\right) \\ &= 2e^{-8}(52) = \frac{104}{e^8} > 0 \end{aligned}$$

Using this information, we can fill in the sign chart of $f''(x)$. Because we are also interested in the information this yields for f, we will include that information below the number line. See **Figure 3.3.8**.

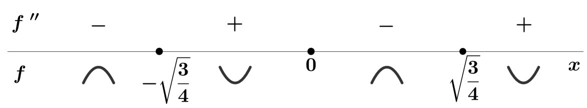


Figure 3.3.8: Sign chart of $f''(x)$ with the corresponding information for $f(x) = xe^{-2x^2}$

Looking at the sign chart, we see that $f''(x)$ changes sign at all three partition numbers. Now, we must check to see if the function is defined at these x-values, which it is. Thus, f has inflection points at all three partition numbers of f''. To find the y-values associated with the inflection points, we substitute the x-values into the *original function* f:

$$f(x) = xe^{-2x^2} \implies$$

$$f\left(-\sqrt{\frac{3}{4}}\right) = -\sqrt{\frac{3}{4}}e^{-2\left(-\sqrt{\frac{3}{4}}\right)^2} = -\sqrt{\frac{3}{4}}e^{-2(\frac{3}{4})} = -\sqrt{\frac{3}{4}}e^{-\frac{3}{2}} \approx -0.1932$$

$$f(0) = (0)e^{-2(0)^2} = 0$$

$$f\left(\sqrt{\frac{3}{4}}\right) = \sqrt{\frac{3}{4}}e^{-2\left(\sqrt{\frac{3}{4}}\right)^2} = \sqrt{\frac{3}{4}}e^{-2(\frac{3}{4})} = \sqrt{\frac{3}{4}}e^{-\frac{3}{2}} \approx 0.1932$$

In conclusion, f is concave down on $\left(-\infty, -\sqrt{\frac{3}{4}}\right)$ and $\left(0, \sqrt{\frac{3}{4}}\right)$, and it is concave up on $\left(-\sqrt{\frac{3}{4}}, 0\right)$ and $\left(\sqrt{\frac{3}{4}}, \infty\right)$. The function has inflection points at $\left(-\sqrt{\frac{3}{4}}, -\sqrt{\frac{3}{4}}e^{-\frac{3}{2}}\right)$, $(0,0)$, and $\left(\sqrt{\frac{3}{4}}, \sqrt{\frac{3}{4}}e^{-\frac{3}{2}}\right)$.

IV. Create a shape chart for f by combining the sign charts from steps II and III and drawing the resulting "shape" in each interval:

We will now combine the sign charts of $f'(x)$ and $f''(x)$ from steps II and III, respectively, into one *combined* sign chart with both the partition numbers of f' and the partition numbers of f''. In each interval on the combined sign chart, we will mark whether the function is increasing/decreasing as well as whether it is concave up/down. Then, we will draw the resulting shape for each interval (see **Figure 3.3.2** for a reminder of the four possible shapes for each interval).

The partition numbers of both sign charts include $x = -\sqrt{\frac{3}{4}}$, $x = -\frac{1}{2}$, $x = 0$, $x = \frac{1}{2}$, and $x = \sqrt{\frac{3}{4}}$. We will place all of these x-values on a number line and use the previous sign charts to record the increasing/decreasing and concavity information for each of the resulting intervals.

On the interval $\left(-\infty, -\sqrt{\frac{3}{4}}\right)$, f is decreasing and concave down. f is decreasing and concave up on $\left(-\sqrt{\frac{3}{4}}, -\frac{1}{2}\right)$. On $\left(-\frac{1}{2}, 0\right)$, f is increasing and concave up, but on $\left(0, \frac{1}{2}\right)$, it is increasing and concave down. The function is decreasing and concave down on $\left(\frac{1}{2}, \sqrt{\frac{3}{4}}\right)$, but it is decreasing and concave up on $\left(\sqrt{\frac{3}{4}}, \infty\right)$.

The shape chart for f shown in **Figure 3.3.9** contains the relevant information and corresponding shape for each interval:

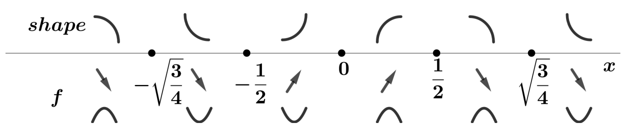


Figure 3.3.9: Shape chart for $f(x) = xe^{-2x^2}$

V. Create a rough sketch of f using the shapes from step IV as well as the information from step I:

Now, we are ready to sketch the graph of f! We start by plotting the points we know: the x- and y-intercepts, local extrema, and inflection points.

Next, we sketch the first shape in the shape chart, and keep in mind that $\lim_{x\to-\infty} f(x) = 0$ (i.e., $y = 0$ is a horizontal asymptote). We continue sketching the shapes on the axes while making sure to incorporate the intercepts, local extrema, and inflection points. We also need to remember that our last shape should incorporate the horizontal asymptote $y = 0$ as $x \to \infty$. Our resulting sketch is shown in **Figure 3.3.10**:

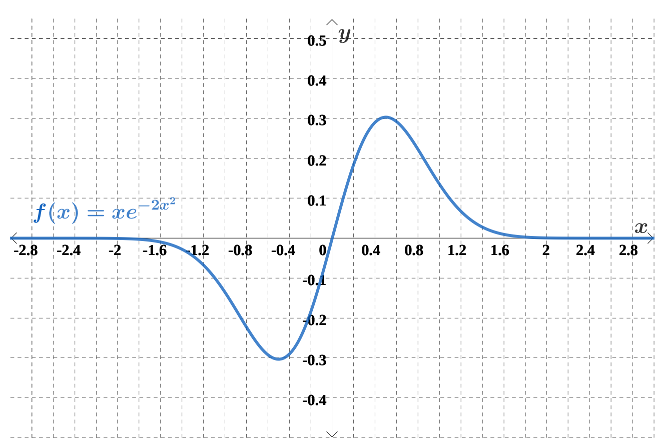


Figure 3.3.10: Sketch of the graph of the function $f(x) = xe^{-2x^2}$

■

Try It # 1:
Use the graphing strategy to sketch the graph of each of the following functions.

a. $f(x) = \dfrac{2x^2+5}{x^2-4}$

b. $f(x) = 4x^4 - 16x^3$

Graphing Given Derivative Information

We can also sketch a rough graph of a function even if we are not given the function itself, as long as we are given information about its first and second derivatives (namely, the intervals where each is positive/negative). Also, the more information we are given about the function we are attempting to graph, the more accurate our sketch will be. For example, if we are given specific points on the graph, such as intercepts, we will be able to position our "shapes" better.

If we are given a list of such information about a function and its first and second derivatives, we can use the information to work "backward" and create sign charts of both $f'(x)$ and $f''(x)$ (and include increasing/decreasing and concavity information on the appropriate chart). Then, we can use these sign charts like we did previously to create a shape chart for f.

■ **Example 4** Use the information below to sketch a possible graph of f:

- domain of f: $(-\infty,-2)\cup(-2,2)\cup(2,\infty)$
- Vertical asymptotes at $x=-2$ and $x=2$
- $\lim\limits_{x\to\infty} f(x) = 7$ and $\lim\limits_{x\to-\infty} f(x) \to -\infty$
- $f(-6) = 4$ and $f(4) = 6.5$
- $f'(-6) = 0$
- $f'(x) > 0$ on $(-\infty,-6)$ and $(2,\infty)$
- $f'(x) < 0$ on $(-6,-2)$ and $(-2,2)$
- $f''(x) > 0$ on $(-2,0)$
- $f''(x) < 0$ on $(-\infty,-2)$, $(0,2)$, and $(2,\infty)$

Solution:
We are given some preliminary information about the function f including its domain, vertical asymptotes, and horizontal asymptote. Note that the horizontal asymptote, $y=7$, only occurs when $x\to\infty$. For the end behavior as $x\to-\infty$, we are told $f(x)\to-\infty$. We are also given two points on the graph of f which will help us orient our graph better. We note that $f'(-6)=0$, which indicates a horizontal tangent line on the graph at $x=-6$.

Now, we can create a sign chart of $f'(x)$. Looking at the intervals pertaining to the sign of $f'(x)$ (where it is positive as well as where it is negative), we see the important x-values listed in the intervals (in order) include $x=-6$, $x=-2$, and $x=2$. Thus, we can create a sign chart of $f'(x)$ with the partition numbers $x=-6$, $x=-2$, and $x=2$. Because $x=-6$ is the only partition number in the domain of the function, we will draw a solid dot on the sign chart at $x=-6$ and open circles at $x=-2$ and $x=2$.

We can use the information in the problem to record the sign of $f'(x)$ on each interval, and then we can denote whether f is increasing/decreasing on each interval. $f'(x)$ is positive on $(-\infty,-6)$ and $(2,\infty)$, which means f is increasing on these intervals. $f'(x)$ is negative on $(-6,-2)$ and $(-2,2)$, so f is decreasing on these intervals. The sign chart of $f'(x)$ is shown in **Figure 3.3.11**:

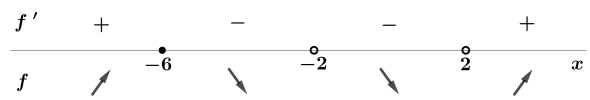


Figure 3.3.11: Sign chart of $f'(x)$

Now, we can use the same process to create the sign chart of $f''(x)$. Working backward, we see the important x-values pertaining to the sign of $f''(x)$ include $x = -2$, $x = 0$, and $x = 2$. Thus, we can create a sign chart of $f''(x)$ with the partition numbers $x = -2$, $x = 0$, and $x = 2$. Because $x = 0$ is the only partition number in the domain of the function, we will draw a solid dot on the sign chart at $x = 0$ and open circles at $x = -2$ and $x = 2$.

We can use the information in the problem to record the sign of $f''(x)$ on each interval, and then we can denote whether f is concave up/down on each interval. $f''(x)$ is positive only on $(-2,0)$, so f is concave up on this interval. $f''(x)$ is negative on $(-\infty,-2)$, $(0,2)$, and $(2,\infty)$, so f is concave down on these intervals. The sign chart of $f''(x)$ is shown in **Figure 3.3.12**.

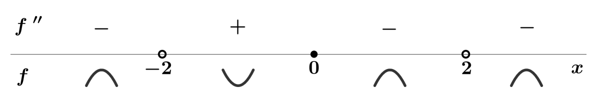


Figure 3.3.12: Sign chart of $f''(x)$

Because we have all the increasing/decreasing and concavity information organized on our sign charts, we can now create a shape chart with all the partition numbers and information from both sign charts (and resulting shapes!). Our shape chart will have all of the following partition numbers: $x = -6$, $x = -2$, $x = 0$, and $x = 2$. We will transfer the increasing/decreasing and concavity information from the individual sign charts to the combined sign chart. Then, we will draw the corresponding shape for each interval.

f is increasing and concave down on the interval $(-\infty,-6)$, but it is decreasing and concave down on $(-6,-2)$. On $(-2,0)$, the function is decreasing and concave up, but then it is decreasing and concave down on $(0,2)$. Finally, the function is increasing and concave down on $(2,\infty)$. See **Figure 3.3.13**.

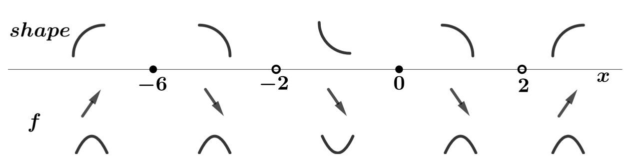


Figure 3.3.13: Shape chart for f

Now, we can use our shape chart to create a sketch of the graph of f. When you draw the shapes for each interval, remember to incorporate the points that were given as well as the asymptotes (see **Figure 3.3.14**). We can also see the horizontal tangent line at $x = -6$ on the graph:

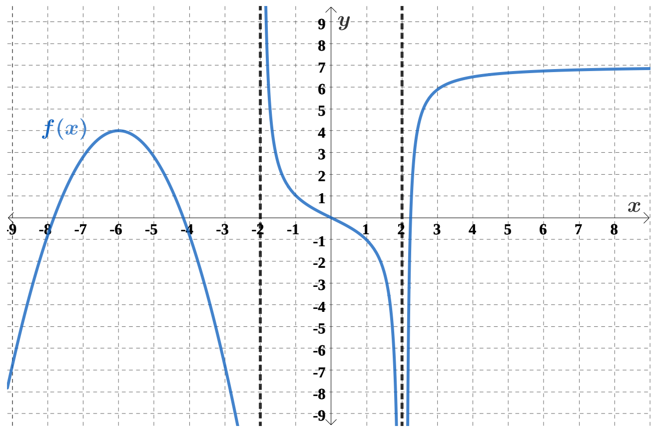


Figure 3.3.14: Sketch of the graph of f

■

Remember that when you are using the graphing strategy to graph f, whether you are given the function itself or information about the function's derivatives, your graph may look different than someone else's. We are just sketching a rough idea of what the function looks like (namely, in terms of increasing/decreasing and concavity). For instance, in the previous example we drew our inflection point at $x = 0$ at the point $(0,0)$. Because that point was not specified, we could have just as easily have drawn it at the point $(0,3)$. The important thing is that we are showing that f has an inflection point at $x = 0$.

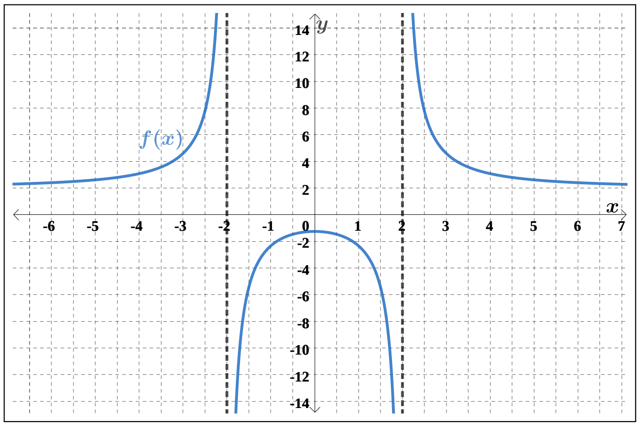
Try It # 2:

Use the information below to sketch a possible graph of f:

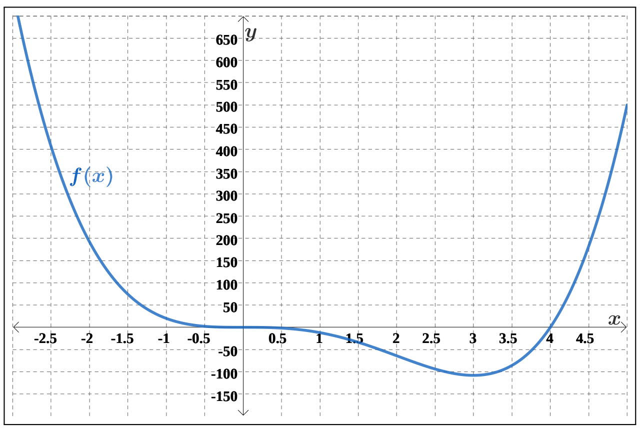
- domain of f: $(-\infty,7)\cup(7,\infty)$
- Vertical asymptote at $x = 7$ such that $\lim\limits_{x\to 7^-} f(x) \to -\infty$ and $\lim\limits_{x\to 7^+} f(x) = 8$
- x-intercepts: $(-9,0)$, $(-3,0)$, and $(6,0)$
- y-intercept: $(0,4)$
- $f'(10)$ does not exist
- $f'(x) > 0$ on $(-6,0)$
- $f'(x) < 0$ on $(-\infty,-6)$, $(0,7)$, $(7,10)$, and $(10,\infty)$
- $f''(x) > 0$ on $(-\infty,-3)$
- $f''(x) < 0$ on $(-3,7)$, $(7,10)$, and $(10,\infty)$

Try It Answers

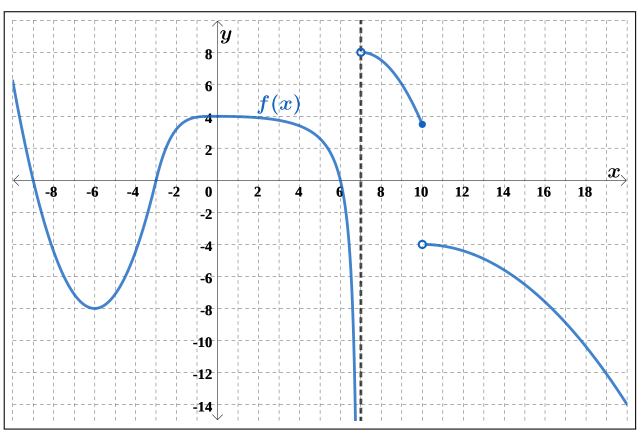
1. **a.**



b.



2.

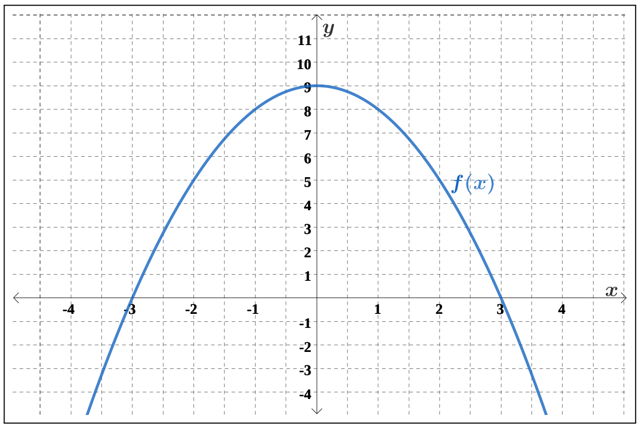


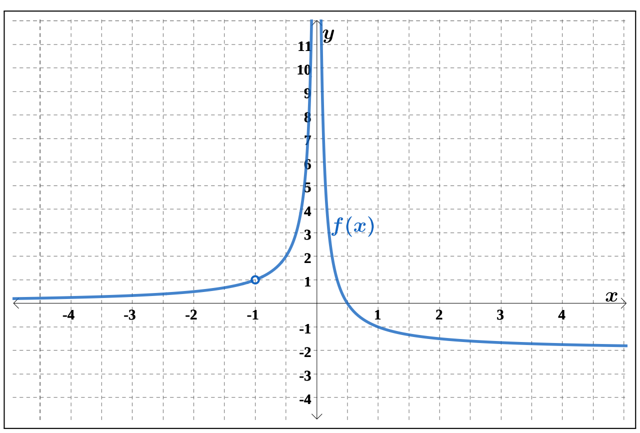
Note: This is just one of many possible graphs that meets all the requirements.

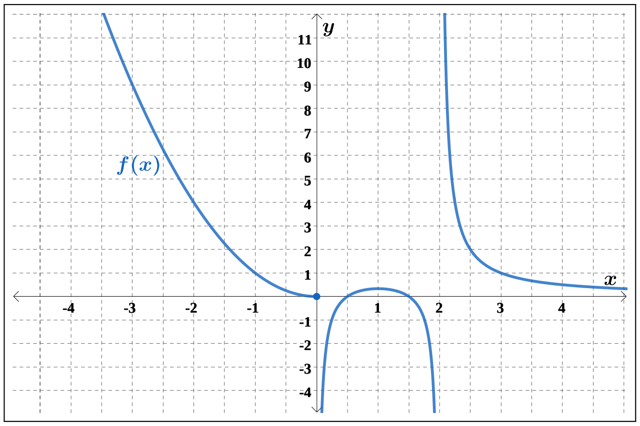
EXERCISES

Basic Skills Practice

For Exercises 1 - 3, the graph of f is shown. Use the graph to find (a) the domain of f, (b) the x- and y-intercepts of f, (c) any holes and vertical or horizontal asymptotes of f, (d) the intervals where f is increasing/decreasing, (e) the x-values where any local extrema of f occur (specify the type), (f) the intervals of concavity of f, and (g) the x-values where any inflection points of f occur.

1. 

2. 

3. 

For Exercises 4 - 7, use the graphing strategy to sketch the graph of the given function. Verify your answer using a graphing calculator.

4. $f(x) = 2x^2 - 3$

5. $R(x) = -0.03x^2 + 9x$

6. $g(x) = x^2 + 1$

7. $f(x) = x^3 - 8$

For Exercises 8 - 11, use the information provided to sketch a possible graph of f.

8.
- Domain of f: $(-\infty, \infty)$
- $f(0) = 4$
- $f'(0) = 0$
- $f'(x) > 0$ on $(-\infty, 0)$
- $f'(x) < 0$ on $(0, \infty)$
- $f''(x) < 0$ on $(-\infty, \infty)$

9.
- Domain of f: $(-\infty, \infty)$
- $\lim_{x\to-\infty} f(x) \to \infty$ and $\lim_{x\to\infty} f(x) \to -\infty$
- $f(0) = 6$ and $f(3) = -21$
- $f'(3) = 0$
- $f'(x) < 0$ on $(-\infty, 3)$ and $(3, \infty)$
- $f''(x) > 0$ on $(-\infty, 3)$
- $f''(x) < 0$ on $(3, \infty)$

10.
- Domain of f: $(-\infty, \infty)$
- $\lim_{x\to-\infty} f(x) \to -\infty$ and $\lim_{x\to\infty} f(x) \to \infty$
- $f(0) = 7$
- $f'(-1) = 0$ and $f'(1) = 0$
- $f'(x) > 0$ on $(-\infty, -1)$ and $(1, \infty)$
- $f'(x) < 0$ on $(-1, 1)$
- $f''(x) > 0$ on $(0, \infty)$
- $f''(x) < 0$ on $(-\infty, 0)$

11.
- Domain of f: $(-\infty, \infty)$
- $\lim_{x\to-\infty} f(x) \to \infty$ and $\lim_{x\to\infty} f(x) \to \infty$
- $f(-4) = 0$, $f(0) = 100$, and $f(10) = 0$
- $f(x) \geq 0$ on $(-\infty, \infty)$
- $f'(-4) = 0$, $f'(3) = 0$, and $f'(10) = 0$
- $f'(x) > 0$ on $(-4, 3)$ and $(10, \infty)$
- $f'(x) < 0$ on $(-\infty, -4)$ and $(3, 10)$
- $f''(x) > 0$ on $(-\infty, -1)$ and $(7, \infty)$
- $f''(x) < 0$ on $(-1, 7)$

Intermediate Skills Practice

For Exercises 12 - 17, use the graphing strategy to sketch the graph of the given function.

12. $f(x) = \dfrac{8x}{4x+12}$

13. $k(x) = \dfrac{9x-18}{x^2-4}$

14. $j(x) = \dfrac{3x^2+6x}{x+1}$

15. $h(x) = e^x + e^{-x}$

16. $l(x) = 6x^2e^x$

17. $g(x) = 10(x-5)(4x-2)^{1/2}$

For Exercises 18 - 22, use the information provided to sketch a possible graph of f.

18.
- Domain of f: $(-\infty,0)\cup(0,\infty)$
- Vertical asymptote at $x=0$
- $\lim_{x\to-\infty} f(x)=0$ and $\lim_{x\to\infty} f(x)=0$
- $f(0.5)=2$ and $f(-5)=-0.2$
- $f'(x)<0$ on $(-\infty,0)$ and $(0,\infty)$
- $f''(x)>0$ on $(0,\infty)$
- $f''(x)<0$ on $(-\infty,0)$

19.
- Domain of f: $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$
- Vertical asymptotes at $x=-3$ and $x=3$
- $\lim_{x\to-\infty} f(x)=0$ and $\lim_{x\to\infty} f(x)=0$
- $f(0)=0.8$, $f(-2)=0$, and $f(2)=0$
- $f'(x)>0$ on $(-\infty,-3)$ and $(-3,0)$
- $f'(x)<0$ on $(0,3)$ and $(3,\infty)$
- $f''(x)>0$ on $(-\infty,-3)$ and $(3,\infty)$
- $f''(x)<0$ on $(-3,3)$

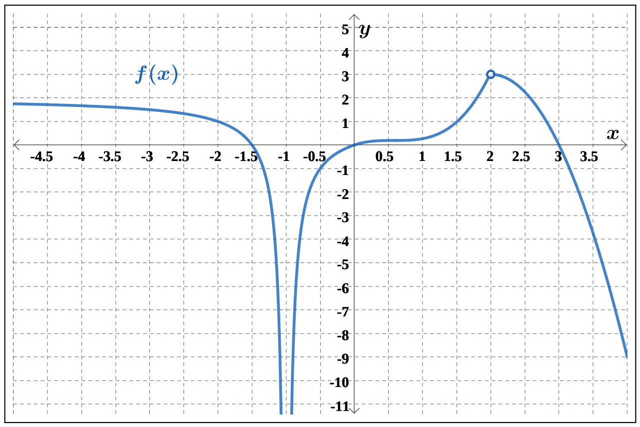
20.
- Domain of f: $(-\infty,\infty)$
- $f(-3)=1$, $f(0)=9$, and $f(3)=0$
- $f'(0)=0$
- $f'(-3)$ does not exist
- $f'(x)>0$ on $(-3,0)$
- $f'(x)<0$ on $(-\infty,-3)$ and $(0,\infty)$
- $f''(x)<0$ on $(-3,\infty)$

21.
- f is continuous on $(-\infty,\infty)$
- $f(0)=0$, $f(2)=8$, and $f(4)=0$
- $f'(0)$ and $f'(4)$ do not exist
- $f'(x)>0$ on $(0,2)$ and $(4,\infty)$
- $f'(x)<0$ on $(-\infty,0)$ and $(2,4)$
- $f''(x)<0$ on $(-\infty,0)$, $(0,4)$, and $(4,\infty)$

22.
- Domain of f: $(-\infty,2)\cup(2,\infty)$
- Vertical asymptote at $x=2$
- $\lim_{x\to-\infty} f(x)\to-\infty$ and $\lim_{x\to\infty} f(x)=0$
- $f(0)=0$
- $f'(0)$ does not exist
- $f'(x)>0$ on $(-\infty,0)$
- $f'(x)<0$ on $(-\infty,2)$ and $(2,\infty)$
- $f''(x)>0$ on $(-\infty,0)$ and $(2,\infty)$
- $f''(x)<0$ on $(0,2)$

Mastery Practice

23. Given the graph of f shown below, find (a) the domain of f, (b) the x- and y-intercepts of f, (c) any holes and vertical or horizontal asymptotes of f, (d) the intervals where f is increasing/decreasing, (e) the x-values where any local extrema of f occur (specify the type), (f) the intervals of concavity of f, and (g) the x-values where any inflection points of f occur.



For Exercises 24 - 30, use the graphing strategy to sketch the graph of the given function.

24. $f(x) = 2x^3 - 16$

25. $f(x) = \dfrac{x}{5x^2 - 15}$

26. $f(x) = \dfrac{x}{2e^x}$

27. $f(x) = x\ln(x)$

28. $f(x) = x^5 \ln(x)$

29. $f(x) = \dfrac{8x + 14}{12(x-3)^2}$

30. $f(x) = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

For Exercises 31 - 35, use the information provided to sketch a possible graph of f.

31.
- Domain of f: $(-\infty, \infty)$
- $\lim\limits_{x\to-\infty} f(x) \to -\infty$ and $\lim\limits_{x\to\infty} f(x) \to \infty$
- $f(0) = -140$
- $f'(3) = 0$
- $f'(x) > 0$ on $(-\infty, 3)$ and $(3, \infty)$
- $f''(x) > 0$ on $(3, \infty)$
- $f''(x) < 0$ on $(-\infty, 3)$

32.
- f is continuous on $(-\infty, \infty)$
- $\lim_{x\to-\infty} f(x) \to -\infty$ and $\lim_{x\to\infty} f(x) \to \infty$
- $f(0) = 10$
- f has a vertical tangent line at $x = 0$
- $f'(x) > 0$ on $(-\infty, 0)$ and $(0, \infty)$
- $f''(x) > 0$ on $(-\infty, 0)$
- $f''(x) < 0$ on $(0, \infty)$

33.
- Domain of f: $(0, \infty)$
- $\lim_{x\to0^+} f(x) = 0$ and $\lim_{x\to\infty} f(x) \to \infty$
- $f(e) = 0$
- $f'(1) = 0$
- $f'(x) > 0$ on $(1, \infty)$
- $f'(x) < 0$ on $(0, 1)$
- $f''(x) > 0$ on $(0, \infty)$

34.
- Domain of f: $[-9, 9]$
- $f(-9) = 0$, $f(0) = 3$, and $f(9) = 0$
- $f'(0)$ does not exist
- $f'(x) > 0$ on $(-9, 0)$
- $f'(x) < 0$ on $(0, 9)$
- $f''(x) < 0$ on $(-9, 0)$ and $(0, 9)$

35.
- Domain of f: $(-\infty, -3) \cup (-3, \infty)$
- f is continuous on $(-\infty, -3) \cup (-3, 1) \cup (1, \infty)$
- Vertical asymptote at $x = -3$
- $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) \to -\infty$
- $f(0) = -\dfrac{1}{3}$
- $f'(4) = 0$
- $f'(x) > 0$ on $(-\infty, -3)$, $(-3, 1)$, and $(1, 4)$
- $f'(x) < 0$ on $(4, \infty)$
- $f''(x) > 0$ on $(-\infty, -3)$
- $f''(x) < 0$ on $(-3, 1)$ and $(1, \infty)$

Communication Practice

36. Briefly describe the five steps of the graphing strategy.

37. If, while performing the graphing strategy, we discover $\lim_{x\to-\infty} f(x) = 5$ and $\lim_{x\to\infty} f(x) = 5$, what, if anything, can we conclude about the graph of f?

38. If, while performing the graphing strategy, we discover $\lim_{x \to -2} f(x) \to \infty$, what, if anything, can we conclude about the graph of f?

39. When creating the shape chart for f, describe the possible shapes.

3.4 Absolute Extrema

Until now, we have only discussed one type of extrema: local extrema. Recall that a local extremum occurs only in a neighborhood (or localized area) around it. Another way to think of this is that the function must exist on both sides of a local extremum. However, a function need not be continuous at a local extremum (we will explore this more soon!).

In this section, we will learn how to find another type of extrema: **absolute extrema**. Absolute extrema are the **absolute maximum** and **absolute minimum** (values) of a function. You can think of the word "absolute" as meaning exactly as it sounds: Absolute extrema are the absolute highest (maximum) and lowest (minimum) values of a function.

Figure 3.4.1 below shows the graph of a function f that is defined on the interval $[-10, 8]$ and has an absolute maximum of 5 occurring at $x = 8$ and an absolute minimum of -2 occurring at $x = -3$:

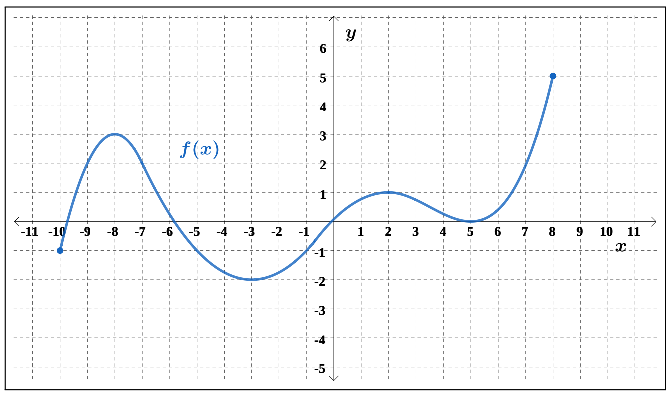


Figure 3.4.1: Graph of a function f that is defined on the interval $[-10, 8]$

Notice f has local maxima at $x = -8$ and $x = 2$ and local minima at $x = -3$ and $x = 5$; local and absolute extrema can occur simultaneously!

Learning Objectives:

In this section, you will learn how to identify absolute extrema graphically as well as how to use calculus to find the absolute extrema of a function on a closed interval on which the function is continuous. Upon completion you will be able to:

- Identify any local and absolute extrema of a function by examining the graph of the function.
- State the Extreme Value Theorem.
- Identify any absolute extrema of a function on a given interval (open or closed) by examining the graph of the function.
- Describe the conditions in which the Closed Interval Method can be used to calculate the absolute extrema of a function on a given interval.
- Perform the Closed Interval Method to determine the absolute maximum and minimum values of a function on a closed interval on which the function is continuous.
- Determine the absolute maximum and minimum values of a function on an interval in which the Closed Interval Method cannot be performed by using alternative strategies, such as graphing the function.

Identifying Absolute Extrema Graphically

As stated previously, the absolute maximum and absolute minimum of a function are the highest (largest) and lowest (smallest) y-values of the function. Together, the absolute maximum and minimum are called the absolute extrema of a function. We refer to either an absolute maximum or absolute minimum as an **absolute extremum**.

We now formally define absolute extrema:

Definition

- $f(c)$ is an **absolute maximum** of f if $f(c) \geq f(x)$ for all x in the domain of f.
- $f(c)$ is an **absolute minimum** of f if $f(c) \leq f(x)$ for all x in the domain of f.
- $f(c)$ is an **absolute extremum** of f if $f(c)$ is an absolute maximum or minimum.

Absolute extrema can occur simultaneously with local extrema inside a function or at an endpoint, as seen in **Figure 3.4.1**. Because local extrema must occur in a neighborhood of a function, they cannot occur at endpoints (remember, the function must exist on both sides of a local extremum, even if the function is not continuous at the local extremum). However, absolute extrema can occur anywhere!

In contrast to local extrema, a function can have only one absolute maximum and one absolute minimum (y-values). Although, the absolute maximum and minimum can occur at infinitely many x-values. This type of behavior can be seen in the function defined on $[-9, 9]$ shown in **Figure 3.4.2**. There is only one absolute maximum, $y = 5$, and one absolute minimum, $y = 1$, but they both occur at multiple x-values.

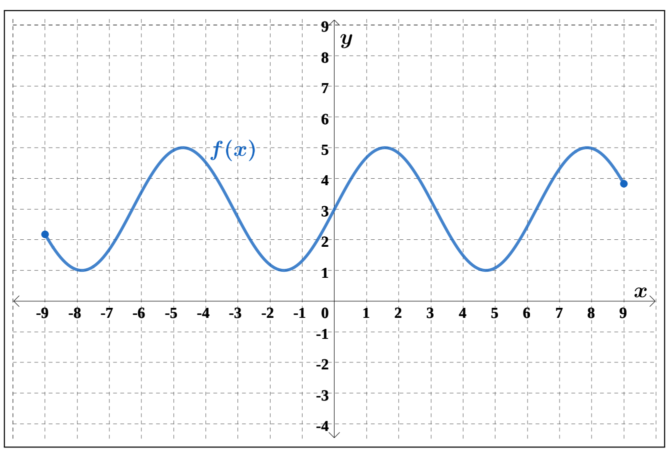


Figure 3.4.2: Graph of a function f that has absolute extrema occurring at multiple x-values

The following table summarizes the similarities and differences of local and absolute extrema:

Local Extrema	Absolute Extrema
Occur inside the function	Occur inside the function at local extrema or at an endpoint
Multiple local extrema (y-values)	At most one absolute max. and one absolute min. (y-values)

Before we get into the details of using calculus to find the absolute extrema of a function, let's practice identifying absolute extrema and review finding local extrema graphically!

■ **Example 1** Given the graph of f shown in **Figure 3.4.3**, find any local and absolute extrema of f as well as where they occur.

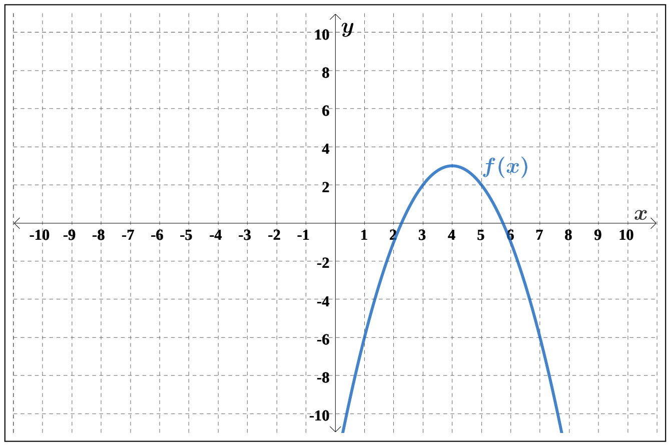


Figure 3.4.3: Graph of a continuous function f

Solution:
First, let's start by finding any absolute extrema. To find the absolute maximum, we look for the highest (largest) y-value on the graph. The absolute maximum is 3, and it occurs at $x = 4$. To find the absolute minimum, we look for the lowest (smallest) y-value on the graph. Because $f(x) \to -\infty$ as $x \to -\infty$ and $x \to \infty$, the function continues toward negative infinity on both ends and never reaches a lowest y-value (i.e., it never "stops").

In terms of local extrema, f also has a local maximum of 3 at $x = 4$. Recall that local and absolute extrema can occur simultaneously. However, the function does not have a local minimum.

In summary, the absolute maximum of f is 3, and it occurs at $x = 4$. There is no absolute minimum. f also has a local maximum of 3 occurring at $x = 4$, but there is no local minimum.

■

Try It # 1:
Given the graph of f shown in **Figure 3.4.4**, find any local and absolute extrema of f as well as where they occur.

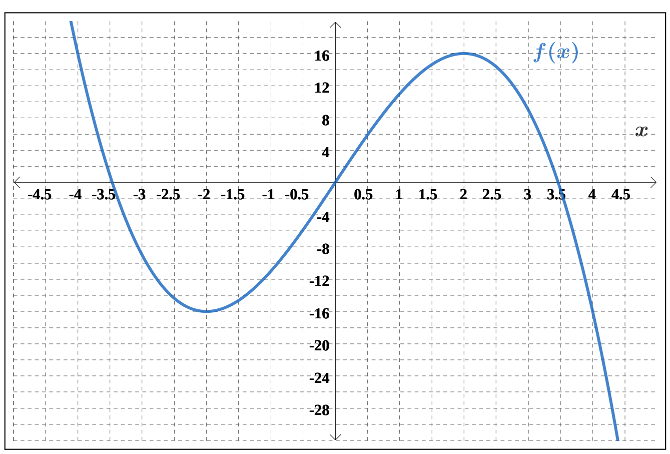


Figure 3.4.4: Graph of a continuous function f

Example 2 Given the graph of f shown in **Figure 3.4.5**, find any local and absolute extrema of f as well as where they occur.

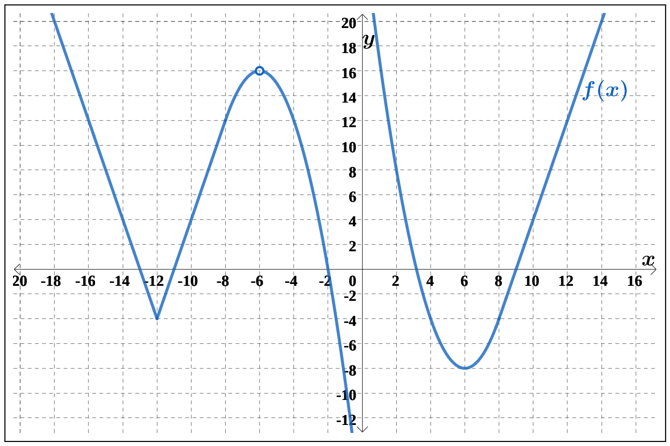


Figure 3.4.5: Graph of a discontinuous function f

Solution:
Let's start with finding the absolute extrema. We see f does not have an absolute maximum nor an absolute minimum because of the behavior of the function near the vertical asymptote, $x = 0$. Specifically, because $f(x) \to -\infty$ as $x \to 0^-$, the function continues to decrease and never reaches a lowest y-value. Thus, there is no absolute minimum.

Likewise, because $f(x) \to \infty$ as $x \to 0^+$, the function continues to increase and never reaches a highest y-value. Thus, there is no absolute maximum. Note that there is also no absolute maximum because of the end behavior of the function (both ends tend to positive infinity).

Looking for local extrema, we see f has local minima of -4 at $x = -12$ and -8 at $x = 6$. While we might be tempted to say f has a local maximum at $x = -6$, the function is undefined at $x = -6$, so there cannot be any type of extrema.

In summary, f does not have any absolute extrema. f has local minima of -4 and -8 occurring at $x = -12$ and $x = 6$, respectively, but it does not have any local maxima.

■

Example 3 Given the graph of f shown in **Figure 3.4.6**, find any local and absolute extrema of f as well as where they occur.

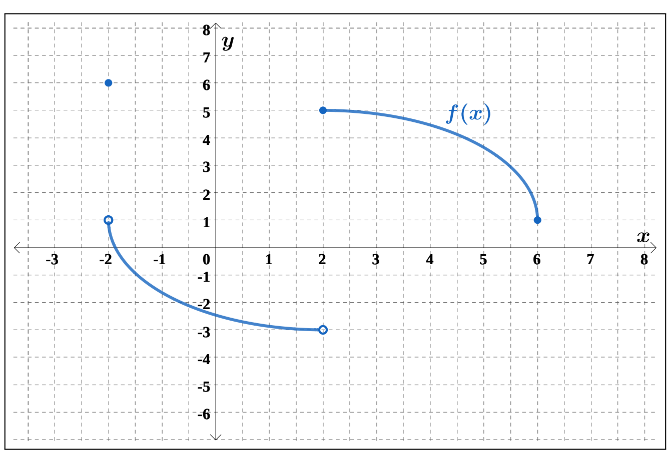


Figure 3.4.6: Graph of a discontinuous function f

Solution:
First, let's look for absolute extrema, if they exist. The absolute highest y-value of the function is 6, and it occurs at $x = -2$. Again, we may be tempted to say there is an absolute minimum at $x = 2$, but the function is undefined at $x = 2$, so there cannot be any type of extrema. We can also think of the function as never "reaching" the point at $x = 2$ (hence, the hole in the graph). Therefore, the function has no absolute minimum.

In terms of local extrema, f has a local maximum of 5 at $x = 2$. It may seem surprising that there is a local maximum at the point $(2,5)$. However, remember that a function does not have to be continuous at a local extremum; the function just has to exist on both sides of the local extremum (i.e., a local extremum must occur in a neighborhood of the function). In this case, the y-value 5 is greater than the function values on either side of $x = 2$, so 5 is a local maximum, even though the function is not continuous at $x = 2$.

f does not have any local minima. There is no local minimum at $x = 6$ because local extrema cannot occur at endpoints (local extrema must occur in neighborhoods in which there is function on both sides of the extrema). There is no local minimum at $x = 2$ because, again, f is undefined at $x = 2$.

In summary, f has an absolute maximum of 6 occurring at $x = -2$, but it does not have an absolute minimum. f has a local maximum of 5 occurring at $x = 2$, but it does not have any local minima.

■

Try It # 2:
Given the graph of f shown in **Figure 3.4.7**, find any local and absolute extrema of f as well as where they occur.

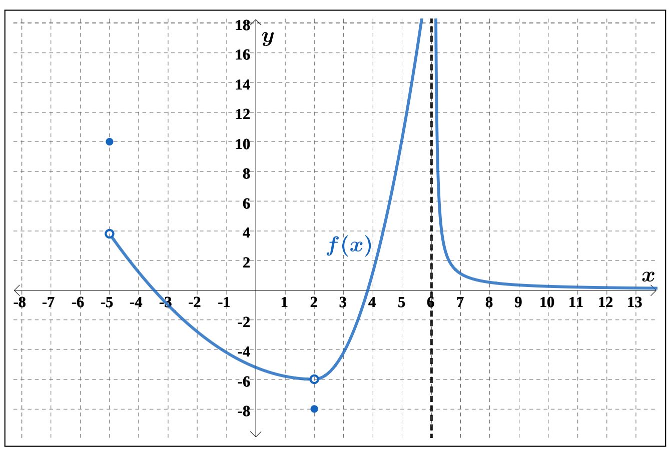


Figure 3.4.7: Graph of a discontinuous function f

Absolute Extrema on an Interval

In the previous examples, we were looking for the extrema of a function on its entire domain. We will now learn how to find absolute extrema on a specified interval (either open, half-open, or closed). If the interval given is open, the corresponding endpoints (x-values) are not included and the interval will have parentheses around both endpoints. If the interval is half-open, the endpoint with a parenthesis is not included, but the endpoint with a bracket is included. If the interval is closed, then both endpoints are included and both will have brackets.

There is a very important theorem called the **Extreme Value Theorem** which states that if a function is *continuous* on a *closed* interval $[a,b]$, then the function must have both an absolute maximum and absolute minimum on the interval:

Theorem 3.5 Extreme Value Theorem

If a function is continuous on a closed interval $[a,b]$, then it must have both an absolute maximum and absolute minimum on the interval.

Example 4 Given the graph of f shown in **Figure 3.4.8**, find the absolute maximum and minimum of f on each of the following intervals.

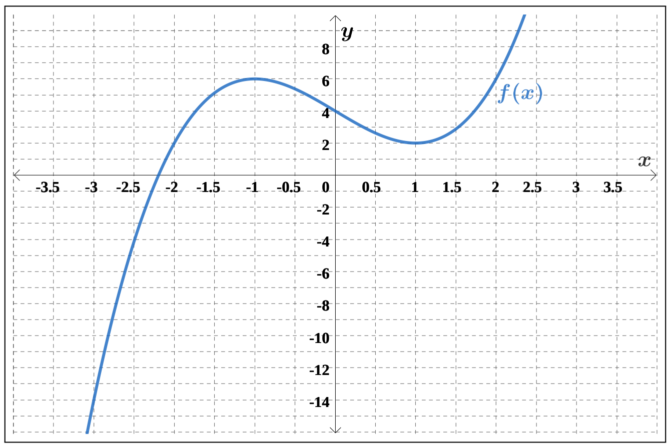


Figure 3.4.8: Graph of a continuous function f

a. $[0,2]$
b. $[-2,1]$

Solution:

a. First, note that $[0,2]$ is a closed interval and that the function f is continuous on the interval (in fact, it is continuous everywhere!). Thus, the Extreme Value Theorem says the function must have an absolute maximum and absolute minimum. Only looking at and considering the graph of the function from $x = 0$ to $x = 2$ (including these endpoints), we see the absolute maximum is 6, and it occurs at $x = 2$. The absolute minimum is 2, and it occurs at $x = 1$.

b. Again, the interval $[-2,1]$ is closed, and we know f is continuous on the interval. Only looking at and considering the graph of the function from $x = -2$ to $x = 1$ (including these endpoints), we see the absolute maximum is 6, and it occurs at $x = -1$. The absolute minimum is 2, and it occurs at $x = -2$ *and* $x = 1$. Remember, a function can have only one absolute maximum and one absolute minimum (y-values), but they can occur at multiple x-values.

■

Try It # 3:
Given the graph of f shown in **Figure 3.4.9**, find the absolute maximum and minimum of f on each of the following intervals.

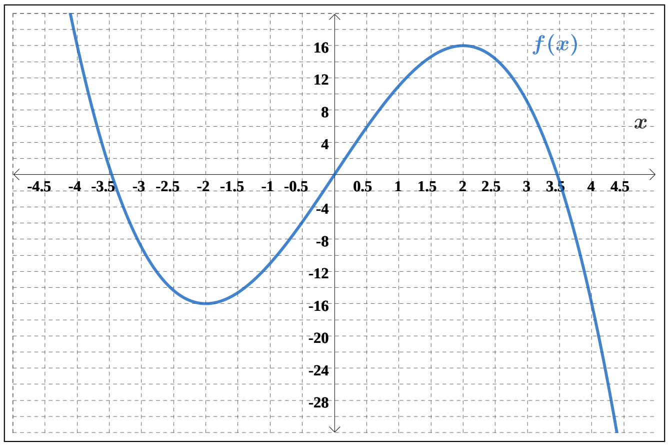


Figure 3.4.9: Graph of a continuous function f

a. $[-3,1]$
b. $[-2,4]$

Now, let's practice working examples in which we find where absolute extrema occur on a variety of types of intervals! Remember, the Extreme Value Theorem only guarantees there are absolute extrema if the function is continuous on a closed interval.

■ **Example 5** Given the graph of f shown in **Figure 3.4.10**, determine where any absolute extrema occur on each of the following intervals.

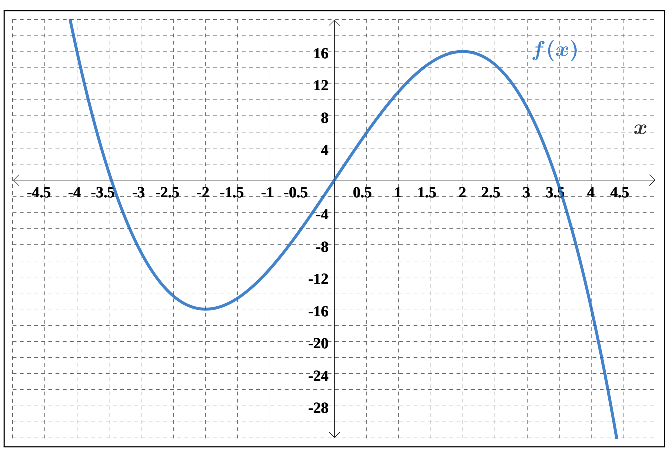


Figure 3.4.10: Graph of a continuous function f

a. $(-2,3)$
b. $[-1,2)$
c. $(-\infty,\infty)$
d. $[0,4]$

Solution:

a. Notice first that the interval we are given, $(-2,3)$, is an open interval (parentheses around both endpoints). This means that we *do not* include the endpoints in our consideration for absolute extrema. It also means the Extreme Value Theorem does not apply because the interval is not closed.

To just focus on the part of the graph on the interval $(-2,3)$ and remember the endpoints are not included, it may be helpful to mark (using your pencil) an open circle, ○, on the graph at the points at $x=-2$ and $x=3$. This may help you remember *not* to consider these points as absolute extrema. Remember, a hole on the graph of a function means the function never "reaches" the point. See **Figure 3.4.11**.

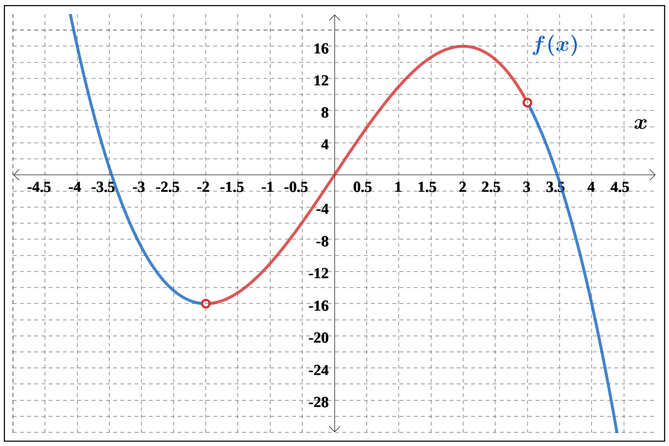


Figure 3.4.11: Graph of f on the interval $(-2,3)$

Looking only at the interval $(-2,3)$, we see f has an absolute maximum at $x=2$. Although there would be an absolute minimum at $x=-2$ if the endpoint was included, there is no absolute minimum because the interval is open so the point at $x=-2$ is not included. Again, we can think of the function as never "reaching" the point at $x=-2$ because it is not included (as indicated by the hole at the point if you drew an open circle on the graph).

In summary, on the interval $(-2,3)$, f has an absolute maximum at $x=2$, but it does not have an absolute minimum. Because the question asked us to find "where" any absolute extrema occur, we only need to identify the x-value(s) of the absolute extrema.

b. The interval $[-1,2)$ is said to be half-open (or half-closed). This means the point at $x=-1$ is included and can be considered when finding absolute extrema, but the point at $x=2$ is not included and cannot be considered.

Using the technique above, we will draw a solid dot, ●, on the graph at $x=-1$ and an open circle, ○, on the graph at $x=2$ to help us remember which endpoint is included and which is not. See **Figure 3.4.12**.

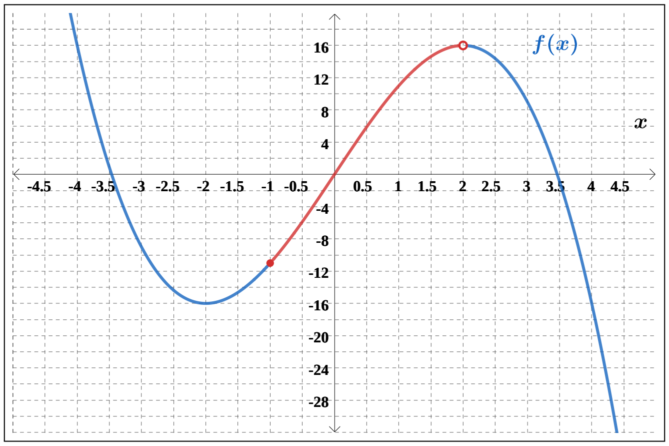


Figure 3.4.12: Graph of f on the interval $[-1, 2)$

Looking at the interval $[-1, 2)$, we see f does not have an absolute maximum because the function never "reaches" the point at $x = 2$ (as indicated by the hole at the point if you drew an open circle on the graph). The smallest y-value on the interval occurs at $x = -1$. As indicated by the solid dot on the graph, this endpoint is included. Thus, the function has an absolute minimum at $x = -1$.

c. The interval $(-\infty, \infty)$ is considered an open interval; it is the real number line. So we must find the absolute extrema of the entire function (not just on a finite interval). Looking at the entire graph, we see there are no absolute extrema because of the end behavior of the function. As $x \to -\infty$, $f(x) \to \infty$, so there is no absolute maximum. As $x \to \infty$, $f(x) \to -\infty$, so there is no absolute minimum.

d. The interval $[0, 4]$ is closed because there are brackets around both endpoints. This means the points at both $x = 0$ and $x = 4$ are included, and we can consider them both when determining where the absolute extrema occur. To help us focus on this part of the graph and remember both endpoints are included, we will draw solid dots, ●, on the graph at both endpoints. See **Figure 3.4.13**.

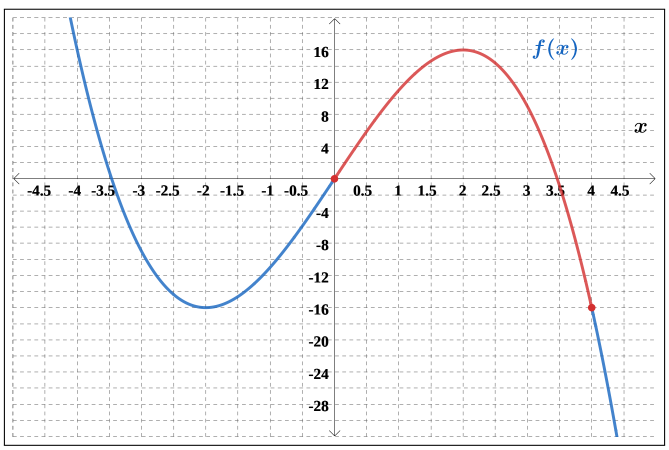


Figure 3.4.13: Graph of f on the interval $[0, 4]$

We see that f has an absolute maximum at $x = 2$ and an absolute minimum at $x = 4$ on the interval $[0, 4]$. Note that because this interval is closed, and f is continuous on the interval, the Extreme Value Theorem guarantees the function will have both an absolute maximum and absolute minimum.

▪

■ **Example 6** Given the graph of f shown in **Figure 3.4.14**, determine where any absolute extrema occur on each of the following intervals.

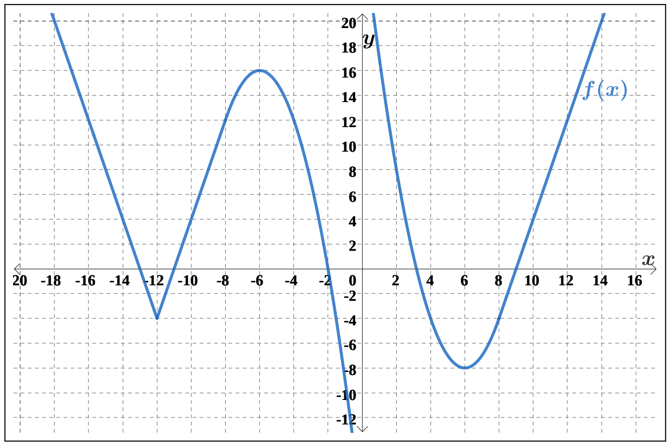


Figure 3.4.14: Graph of a discontinuous function f

a. $[-16, -6)$
b. $(2, 4)$
c. $[-6, 6]$

Solution:
a. The interval $[-16, -6)$ is half-open (or half-closed). This means that the point at $x = -16$ is included and can be considered when finding the absolute extrema, but the point at $x = -6$ is not included and cannot be considered.

Using the technique from the previous example, we will draw a solid dot, ●, on the graph at $x = -16$ and an open circle, ○, on the graph at $x = -6$ to help us remember which endpoint is included and which is not. See **Figure 3.4.15**.

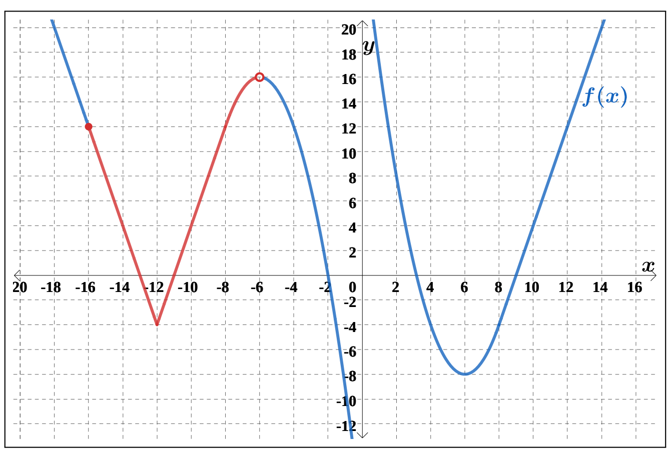


Figure 3.4.15: Graph of f on the interval $[-16, 6)$

Looking at the interval $[-16, -6)$, we see f does not have an absolute maximum because it never "reaches" the point at $x = -6$ (as indicated by the hole at the point if you drew an open circle on the graph). The smallest y-value on the interval occurs at $x = -12$. Thus, the function has an absolute minimum at $x = -12$.

b. The interval $(2, 4)$ is open, so neither endpoint is included. To help us focus on this part of the graph, we will draw open circles, ○, at the points at both $x = 2$ and $x = 4$. See **Figure 3.4.16**.

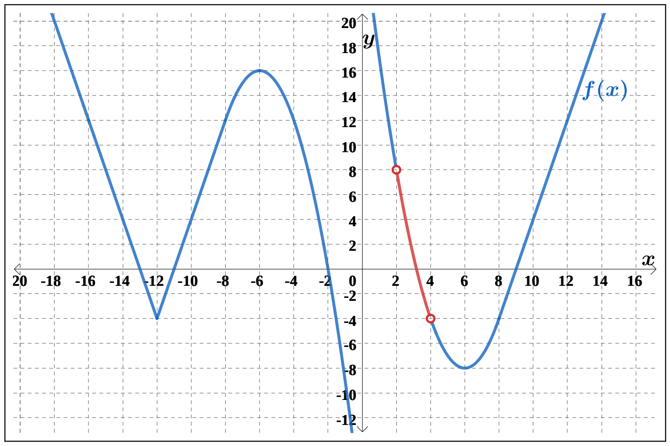


Figure 3.4.16: Graph of f on the interval $(2,4)$

We see that f does not have any absolute extrema on the interval $(2,4)$. The function never "reaches" either of the endpoints (as indicated by the holes at the endpoints), so it does not have an absolute maximum nor an absolute minimum.

c. Although the interval $[-6,6]$ is closed, the function is not continuous on the interval, so it is not guaranteed to have absolute extrema.

To help us focus on this part of the graph and remember that both endpoints are included, we will draw solid dots, ●, on the graph at both endpoints. See **Figure 3.4.17**.

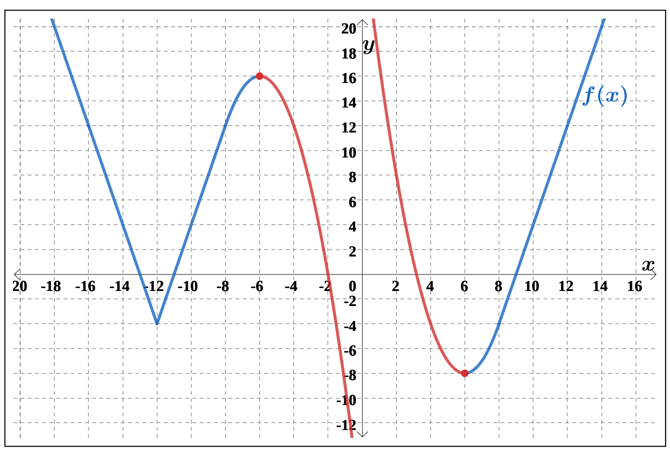


Figure 3.4.17: Graph of f on the interval $[-6,6]$

f does not have absolute extrema on the interval $[-6,6]$ due to the behavior of the function near the vertical asymptote, $x=0$. Specifically, because $f(x) \to -\infty$ as $x \to 0^-$, there is no absolute minimum. Likewise, because $f(x) \to \infty$ as $x \to 0^+$, there is no absolute maximum.

■

Try It # 4:
Given the graph of f shown in **Figure 3.4.18**, determine where any absolute extrema occur on each of the following intervals.

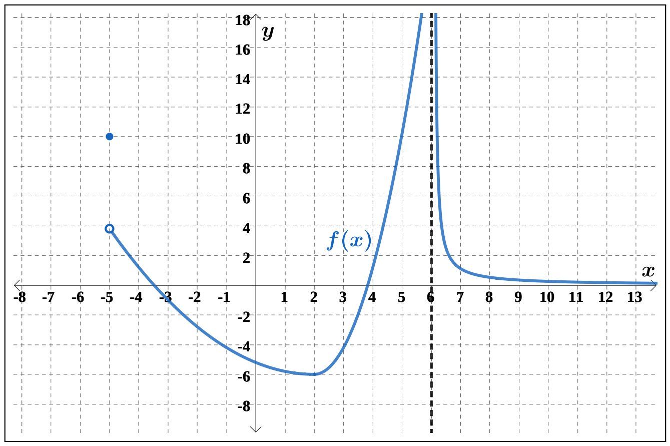


Figure 3.4.18: Graph of a discontinuous function f

a. $[-5,2)$
b. $(2,7]$
c. $(-3,5]$

Determining Absolute Extrema Using Calculus

Now that we have learned how to find absolute extrema graphically, let's learn how to find the absolute extrema of a function on an interval when we are only given the rule of the function (and not the graph of the function). The Extreme Value Theorem guarantees that as long as a function is continuous on a closed interval, it will have an absolute maximum and absolute minimum on the interval. So we will start by learning how to find the absolute extrema of a function on a closed interval on which the function is continuous.

To establish the calculus method for finding the absolute extrema of a function on a closed interval on which the function is continuous, we will use an important fact established previously: Absolute extrema either occur simultaneously with local extrema inside a function or at an endpoint. Recall that local extrema occur at the critical values of a function. Thus, as a result, absolute extrema will either occur at the critical values of a function or at an endpoint:

Theorem 3.6 If a function has an absolute extremum, it must occur at a critical value or at an endpoint.

To find the absolute extrema of a function on a closed interval on which the function is continuous, we need a test that compares the values of the function at the critical values in the interval and the endpoints of the interval. This test is called the **Closed Interval Method**. It consists of three steps:

Finding Absolute Extrema Using the Closed Interval Method

1. Determine the domain of f, and check that the function is continuous on the closed interval. If f is not continuous on the interval, we cannot use this method.
2. Find the critical values of f. Recall that these are the x-values where $f'(x) = 0$ or $f'(x)$ does not exist, and they are in the domain of f.
3. Evaluate f at the critical values in the interval and the endpoints of the interval, and determine the largest (absolute maximum) and smallest (absolute minimum) values.

,

N *Later in this section, we will use technology to help us find absolute extrema if we cannot use the Closed Interval Method because the interval is not closed or the function is not continuous on the interval, or both.*

▪ **Example 7** Find the absolute maximum and minimum of the function $f(x) = x^3 - 2x^2 - 4x + 6$ on each of the following intervals.

a. $[-2, 3]$
b. $[0, 4]$
c. $[-3, -1]$

Solution:

a. Because the interval $[-2, 3]$ is closed (as indicated by the brackets around both endpoints), we can attempt to use the Closed Interval Method to find the absolute extrema:

1. Determine the domain of f, and check that the function is continuous on the closed interval:

f is a polynomial, so its domain is $(-\infty, \infty)$. Hence, the function is continuous on $(-\infty, \infty)$; recall from **section 1.4** that polynomials, rational functions, power functions, exponential functions, logarithmic functions, and combinations of these are continuous on their domain. Thus, the function is continuous on the closed interval $[-2, 3]$, so we can proceed with the Closed Interval Method.

2. Find the critical values of f:

Recall that the critical values of f are the x-values where $f'(x) = 0$ or $f'(x)$ does not exist, and they are in the domain of f. To find the critical values of f, we must first find $f'(x)$:

$$f(x) = x^3 - 2x^2 - 4x + 6 \implies$$
$$f'(x) = 3x^2 - 4x - 4$$

Next, we find the x-values where $f'(x) = 0$ or $f'(x)$ does not exist. Because f' is a polynomial and has a domain of all real numbers, it will exist everywhere. Thus, we only need to find the x-values where $f'(x) = 0$:

$$f'(x) = 0 \implies$$
$$3x^2 - 4x - 4 = 0$$
$$(3x + 2)(x - 2) = 0$$
$$\implies x = -\frac{2}{3} \text{ and } x = 2$$

Because the domain of f is all real numbers, $x = -\frac{2}{3}$ and $x = 2$ are both in the domain. Thus, both are critical values of f.

3. Evaluate f at the critical values in the interval and the endpoints of the interval:

Because both critical values ($x = -\frac{2}{3}$ and $x = 2$) are in the interval $[-2, 3]$, we will find the value of the function at each critical value and each endpoint of the interval ($x = -2$ and $x = 3$):

$$f(x) = x^3 - 2x^2 - 4x + 6 \implies$$
$$f\left(-\frac{2}{3}\right) = \left(-\frac{2}{3}\right)^3 - 2\left(-\frac{2}{3}\right)^2 - 4\left(-\frac{2}{3}\right) + 6 = \frac{202}{27} \approx 7.4815$$
$$f(2) = (2)^3 - 2(2)^2 - 4(2) + 6 = -2$$
$$f(-2) = (-2)^3 - 2(-2)^2 - 4(-2) + 6 = -2$$
$$f(3) = (3)^3 - 2(3)^2 - 4(3) + 6 = 3$$

Looking at the function values, we see the largest is $\frac{202}{27}$ at $x = -\frac{2}{3}$. The smallest function value is -2, and it occurs at both $x = -2$ and $x = 2$. Thus, on the interval $[-2, 3]$, the absolute maximum of f is $\frac{202}{27}$, and the absolute minimum of f is -2.

We can check our work by looking at the graph of f shown in **Figure 3.4.19**, but using the Closed Interval Method allows us to find the absolute extrema *completely algebraically* in order to get *exact* answers.

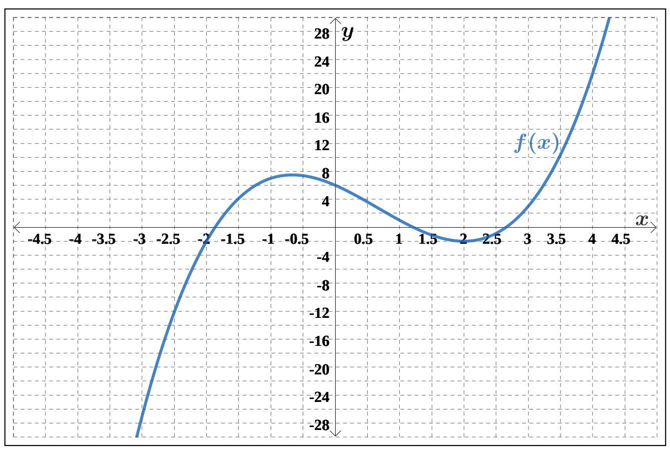


Figure 3.4.19: Graph of the function $f(x) = x^3 - 2x^2 - 4x + 6$

Notice when performing the Closed Interval Method, we use the original function f to "test" the x-values to find the largest and smallest y-values (i.e., the absolute maximum and minimum). All of the previous tests we have learned in Chapter 3 require using either the first or second derivative to perform the respective "test".

b. Because the interval we are given, $[0, 4]$, is closed and we already know the function $f(x) = x^3 - 2x^2 - 4x + 6$ is continuous everywhere, the Closed Interval Method applies.

We found the critical values of f in part **a** ($x = -\frac{2}{3}$ and $x = 2$), so we can go straight to step 3 of the Closed Interval Method:

3. Evaluate f at the critical values in the interval and the endpoints of the interval:

Because only one of the critical values, $x = 2$, is in the interval $[0, 4]$, we will only find the value of the function at this critical value as well as each endpoint of the interval:

$$\begin{aligned}
f(x) &= x^3 - 2x^2 - 4x + 6 \implies \\
f(2) &= (2)^3 - 2(2)^2 - 4(2) + 6 = -2 \\
f(0) &= (0)^3 - 2(0)^2 - 4(0) + 6 = 6 \\
f(4) &= (4)^3 - 2(4)^2 - 4(4) + 6 = 22
\end{aligned}$$

Looking at the function values, we see the largest is 22 at $x = 4$. The smallest function value is -2, and it occurs at $x = 2$. Thus, on the interval $[0,4]$, the absolute maximum of f is 22, and the absolute minimum of f is -2.

We can verify our answer by looking at the graph of f again in **Figure 3.4.19**.

c. Again, because we are given a closed interval, $[-3,-1]$, and we already know the function $f(x) = x^3 - 2x^2 - 4x + 6$ is continuous everywhere, the Closed Interval Method applies.

We found the critical values of f in part **a** ($x = -\frac{2}{3}$ and $x = 2$), so we can go straight to step 3 of the Closed Interval Method again:

3. Evaluate f at the critical values in the interval and the endpoints of the interval:

Neither of the critical values is in the interval $[-3,-1]$, so we will only find the value of the function at each endpoint of the interval:

$$\begin{aligned}
f(x) &= x^3 - 2x^2 - 4x + 6 \implies \\
f(-3) &= (-3)^3 - 2(-3)^2 - 4(-3) + 6 = -27 \\
f(-1) &= (-1)^3 - 2(-1)^2 - 4(-1) + 6 = 7
\end{aligned}$$

Looking at the function values, we see the largest is 7 at $x = -1$. The smallest function value is -27, and it occurs at $x = -3$. Thus, on the interval $[-3,-1]$, the absolute maximum of f is 7, and the absolute minimum of f is -27.

We can verify our answer by looking at the graph of f again in **Figure 3.4.19**.

■

Try It # 5:

Find the absolute maximum and minimum of the function $f(x) = \dfrac{1}{4}x^4 - \dfrac{9}{2}x^2 + 6$ on each of the following intervals.

a. $[-5,4]$
b. $[1,6]$

■ **Example 8** Find the absolute maximum and minimum of the function $f(x) = \dfrac{2x^2 - 10}{x - 3}$ on each of the following intervals.

a. $[4,8]$
b. $[-5,0]$

Solution:

a. Because the interval $[4,8]$ is closed, we can attempt to use the Closed Interval Method to find the absolute extrema:

1. Determine the domain of f, and check that the function is continuous on the closed interval:

$f(x) = \dfrac{2x^2 - 10}{x-3}$ is a rational function, so the denominator must be nonzero. There are no other domain issues (there is no even root or logarithm). Thus, we just need to ensure $x - 3 \neq 0$, which gives $x \neq 3$. Therefore, the domain of f is $(-\infty, 3) \cup (3, \infty)$, so f is continuous on $(-\infty, 3) \cup (3, \infty)$.

Because $[4, 8]$ is a closed interval and f is continuous on this interval (note that $x = 3$ is not in the interval), we can proceed with the Closed Interval Method.

2. Find the critical values of f:

The critical values of f are the x-values where $f'(x) = 0$ or $f'(x)$ does not exist, and they are in the domain of f. To find the critical values of f, we must first find $f'(x)$ using the Quotient Rule:

$$\begin{aligned}
f(x) &= \frac{2x^2 - 10}{x-3} \implies \\
f'(x) &= \frac{(x-3)\left(\dfrac{d}{dx}\left(2x^2 - 10\right)\right) - \left(2x^2 - 10\right)\left(\dfrac{d}{dx}(x-3)\right)}{(x-3)^2} \\
&= \frac{(x-3)(4x) - \left(2x^2 - 10\right)(1)}{(x-3)^2} \\
&= \frac{4x^2 - 12x - 2x^2 + 10}{(x-3)^2} \\
&= \frac{2x^2 - 12x + 10}{(x-3)^2}
\end{aligned}$$

Now, we find the x-values where $f'(x) = 0$ or $f'(x)$ does not exist. $f'(x) = 0$ when the numerator equals 0, and $f'(x)$ does not exist when the denominator equals zero (remember we are looking at the domain of f' when determining where it does not exist). First, let's find the x-values where $f'(x) = 0$:

$$\begin{aligned}
f'(x) = 0 &\implies \\
2x^2 - 12x + 10 &= 0 \\
2(x^2 - 6x + 5) &= 0 \\
2(x-1)(x-5) &= 0 \\
&\implies x = 1 \text{ and } x = 5
\end{aligned}$$

Next, to find the x-values where $f'(x)$ does not exist, we set the denominator equal to zero:

$$\begin{aligned}
f'(x) \text{ DNE} &\implies \\
(x-3)^2 &= 0 \\
\left((x-3)^2\right)^{\frac{1}{2}} &= (0)^{\frac{1}{2}} \\
x - 3 &= 0 \\
x &= 3
\end{aligned}$$

Because the domain of f is $(-\infty, 3) \cup (3, \infty)$, only $x = 1$ and $x = 5$ are in the domain. Thus, these are the only critical values of f.

3. Evaluate f at the critical values in the interval and the endpoints of the interval:

Because only one critical value, $x = 5$, is in the interval $[4, 8]$, we will only find the value of the function at this critical value as well as each endpoint of the interval:

$$
\begin{aligned}
f(x) &= \frac{2x^2-10}{x-3} \implies \\
f(5) &= \frac{2(5)^2-10}{5-3} = 20 \\
f(4) &= \frac{2(4)^2-10}{4-3} = 22 \\
f(8) &= \frac{2(8)^2-10}{8-3} = \frac{118}{5} = 23.6
\end{aligned}
$$

Looking at the function values, we see the largest is $\frac{118}{5}$ at $x = 8$. The smallest function value is 20, and it occurs at $x = 5$. Thus, on the interval $[4,8]$, the absolute maximum of f is $\frac{118}{5}$, and the absolute minimum of f is 20.

We can verify our answer by looking at the graph of f shown in **Figure 3.4.20**.

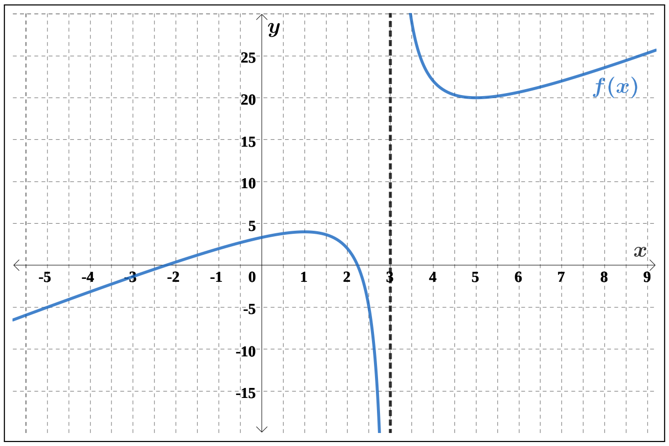


Figure 3.4.20: Graph of the function $f(x) = \dfrac{2x^2-10}{x-3}$

b. Because the interval $[-5,0]$ is closed and we already know $f(x) = \dfrac{2x^2-10}{x-3}$ is continuous on $(-\infty,3)\cup(3,\infty)$, the Closed Interval Method applies because the function is continuous on the interval $[-5,0]$.

We found the critical values of f in part **a** ($x = 1$ and $x = 5$), so we can go straight to step 3 of the Closed Interval Method:

3. Evaluate f at the critical values in the interval and the endpoints of the interval:

Neither of the critical values is in the interval $[-5,0]$, so we will only find the value of the function at each endpoint of the interval:

$$
\begin{aligned}
f(x) &= \frac{2x^2-10}{x-3} \implies \\
f(-5) &= \frac{2(-5)^2-10}{-5-3} = -5 \\
f(0) &= \frac{2(0)^2-10}{0-3} = \frac{10}{3} \approx 3.3333
\end{aligned}
$$

Looking at the function values, we see the largest is $\frac{10}{3}$ at $x = 0$. The smallest function value is -5, and it occurs at $x = -5$. Thus, on the interval $[-5, 0]$, the absolute maximum of f is $\frac{10}{3}$, and the absolute minimum of f is -5.

We can verify our answer by looking at the graph of f again in **Figure 3.4.20**.

■

Try It # 6:

Find the absolute maximum and minimum of the function $f(x) = \dfrac{x^3}{x^2 - 1}$ on each of the following intervals.

a. $[-0.5, 0.5]$
b. $[4/3, 6]$

What if we cannot use the Closed Interval Method?

In the previous examples, the function given was continuous on a closed interval. What do we do if the function is not continuous on the interval or the interval is not closed, or both? Although there are calculus techniques we can use to find the absolute extrema in such cases, for the purposes of this textbook, we will calculate the critical values and then graph the function using technology to determine if there are any absolute extrema.

We must also remember to consider the information about the endpoints of the interval when we look at the graph of the function on our calculator. If the interval is open, that means the endpoints are not included, which could affect the answer in some situations.

■ **Example 9** Find the absolute maximum and minimum of the function $f(x) = \dfrac{x^2 + 9}{x + 4}$ on each of the following intervals, if they exist.

a. $[-10, 3]$
b. $(-2, 4)$
c. $(-13, -10]$

Solution:

a. Because the interval $[-10, 3]$ is closed, we can attempt to use the Closed Interval Method to find any absolute extrema:

1. Determine the domain of f, and check that the function is continuous on the closed interval:

f is a rational function, so the denominator must be nonzero. There are no other domain issues (there is no even root or logarithm). Thus, we just need to ensure $x + 4 \neq 0$, which gives $x \neq -4$. Therefore, the domain of f is $(-\infty, -4) \cup (-4, \infty)$, so f is continuous on $(-\infty, -4) \cup (-4, \infty)$.

Even though $[-10, 3]$ is a closed interval, f is *not* continuous on this interval because $x = -4$ is in the interval. Therefore, we cannot proceed with the Closed Interval Method.

Before attempting to find any absolute extrema by graphing the function on our calculator, we will calculate the critical values of f. Knowing the critical values will help us adjust our viewing window so we can see the pertinent information on the graph, as well as help us verify the *exact* values of potential x-coordinates where the extrema occur.

The critical values of f are the x-values where $f'(x) = 0$ or $f'(x)$ does not exist, and they are in the domain of f. To find the critical value sof f, we will start by finding $f'(x)$ using the Quotient Rule:

$$\begin{aligned} f(x) &= \frac{x^2+9}{x+4} \implies \\ f'(x) &= \frac{(x+4)\left(\frac{d}{dx}\left(x^2+9\right)\right) - \left(x^2+9\right)\left(\frac{d}{dx}(x+4)\right)}{(x+4)^2} \\ &= \frac{(x+4)(2x) - \left(x^2+9\right)(1)}{(x+4)^2} \\ &= \frac{2x^2+8x-x^2-9}{(x+4)^2} \\ &= \frac{x^2+8x-9}{(x+4)^2} \end{aligned}$$

Now, we find the x-values where $f'(x) = 0$ or $f'(x)$ does not exist. $f'(x) = 0$ when the numerator equals 0, and $f'(x)$ does not exist when the denominator equals zero (remember we are looking at the domain of f' when determining where it does not exist). First, we will find the x-values where $f'(x) = 0$:

$$\begin{aligned} f'(x) = 0 &\implies \\ x^2+8x-9 &= 0 \\ (x+9)(x-1) &= 0 \\ &\implies x = -9 \text{ and } x = 1 \end{aligned}$$

Next, to find the x-values where $f'(x)$ does not exist, we set the denominator equal to zero:

$$\begin{aligned} f'(x) \text{ DNE } &\implies \\ (x+4)^2 &= 0 \\ \left((x+4)^2\right)^{\frac{1}{2}} &= (0)^{\frac{1}{2}} \\ x+4 &= 0 \\ x &= -4 \end{aligned}$$

Because the domain of f is $(-\infty,-4)\cup(4,\infty)$, only $x=-9$ and $x=1$ are in the domain. Thus, these are the only critical values of f.

Using this information, we can adjust our calculator's viewing window to be sure we can see the behavior of the function on the interval $[-10,3]$, as well as at the critical values. See **Figure 3.4.21**.

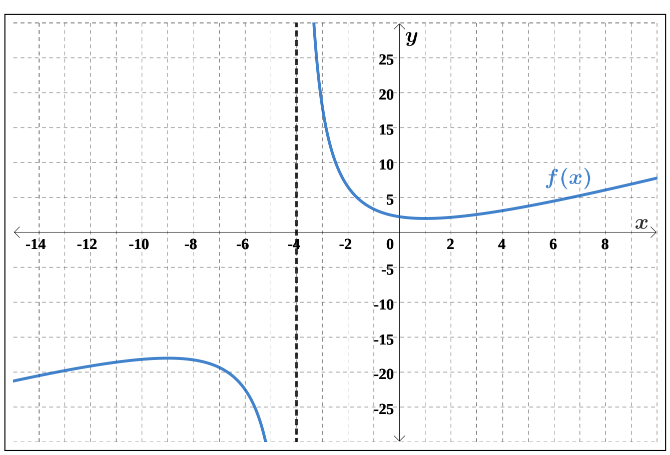


Figure 3.4.21: Graph of the function $f(x) = \dfrac{x^2+9}{x+4}$

Regardless of whether or not the endpoints of the interval $[-10,3]$ are included in our consideration (which they both are), the behavior of the function near the vertical asymptote at $x=-4$ shows there is no absolute maximum nor absolute minimum on the interval.

Specifically, because $f(x)\to-\infty$ as $x\to-4^-$, there is no absolute minimum. Likewise, because $f(x)\to\infty$ as $x\to-4^+$, there is no absolute maximum.

b. Because the interval $(-2,4)$ is open, we know immediately that we cannot use the Closed Interval Method (regardless of whether or not the function is continuous on the interval, which it is).

Again, we will look at the graph of f shown in **Figure 3.4.21** to locate any absolute extrema on the interval $(-2,4)$. When looking at the graph on the interval $(-2,4)$, we must remember that the endpoints are not included because the interval is open. Recall that you can draw open circles at each endpoint on your own paper to help remind you that the endpoints are not included.

Looking at the graph, we see that the function never "reaches" an absolute maximum at $x=-2$ because that endpoint is not included. However, it *appears* as though the function has an absolute minimum near $x=1$. Because we know from part **a** the function has a critical value at $x=1$, we can say with confidence that there is an absolute minimum at $x=1$.

Even though we could use the First Derivative Test or the Second Derivative Test to verify there is a minimum at $x=1$, this is not necessary because we can see from the graph there is a minimum. Knowing the critical values of f helps us determine the exact location of the absolute minimum. Now, we can find the absolute minimum exactly by evaluating f at $x=1$:

$$f(x)=\frac{x^2+9}{x+4}\implies$$
$$f(1)=\frac{(1)^2+9}{1+4}=2$$

Thus, on the interval $(-2,4)$, the absolute minimum of f) is 2, but there is no absolute maximum.

c. We are given a half-open (or half-closed) interval, $(-13,-10]$, so we know immediately we cannot use the Closed Interval Method because we do not have a closed interval (regardless of whether or not the function is continuous on the interval, which it is).

Again, we look at the graph of f shown in **Figure 3.4.21** to locate any absolute extrema on the interval $(-13,-10]$. When looking at the graph on the interval $(-13,-10]$, we must remember that the endpoint $x=-13$ is not included, but the endpoint $x=-10$ is included. Remember, you can draw an open circle at the point at $x=-13$ and a solid dot at the point at $x=-10$ on your own paper to help remind you which endpoint is included and which is not.

Looking at the graph, we see that the absolute maximum on the interval $(-13,-10]$ occurs at $x=-10$, but the function never "reaches" an absolute minimum at $x=-13$ because that endpoint is not included. We can find the absolute maximum exactly by evaluating f at $x=-10$:

$$f(x)=\frac{x^2+9}{x+4}\implies$$
$$f(-10)=\frac{(-10)^2+9}{-10+4}=\frac{109}{-6}\approx-18.1667$$

Thus, on the interval $(-13,-10]$, the absolute maximum of f is $-\frac{109}{6}$, but there is no absolute minimum.

■

■ **Example 10** Find the absolute maximum and minimum of the function $f(x) = \dfrac{x-5}{(x+2)^2}$ on the interval $(-4, 15)$, if they exist.

Solution:
The interval $(-4, 15)$ is open, so we know immediately we cannot use the Closed Interval Method (regardless of whether or not the function is continuous on the interval, which it is not).

Before attempting to find any absolute extrema by graphing the function on our calculator, we will calculate the critical values of f so we know *exactly* where there may be absolute extrema and can adjust the viewing window on our calculator accordingly.

The critical values of f are the x-values where $f'(x) = 0$ or $f'(x)$ does not exist, and they are in the domain of f. To find the critical values of f, we will start by finding $f'(x)$ using the Quotient Rule:

$$\begin{aligned}
f'(x) &= \frac{(x+2)^2\left(\frac{d}{dx}(x-5)\right) - (x-5)\left(\frac{d}{dx}\left((x+2)^2\right)\right)}{\left((x+2)^2\right)^2} \\
&= \frac{(x+2)^2(1) - (x-5)\left(2(x+2)\left(\frac{d}{dx}(x+2)\right)\right)}{(x+2)^4} \\
&= \frac{(x+2)^2 - (x-5)(2(x+2)(1))}{(x+2)^4} \\
&= \frac{(x+2)^2 - (x-5)(2x+4)}{(x+2)^4} \\
&= \frac{x^2+4x+4-(2x^2-6x-20)}{(x+2)^4} \\
&= \frac{x^2+4x+4-2x^2+6x+20}{(x+2)^4} \\
&= \frac{-x^2+10x+24}{(x+2)^4} \\
&= \frac{-\left(x^2-10x-24\right)}{(x+2)^4} \\
&= \frac{-(x+2)(x-12)}{(x+2)^4} \\
&= \frac{-\cancel{(x+2)}(x-12)}{(x+2)^{\cancel{4}\,3}} \\
&= \frac{-(x-12)}{(x+2)^3}
\end{aligned}$$

N *Alternatively, we could have factored the term $x+2$ from both the numerator and denominator immediately after using the Quotient Rule like we have done previously to simplify the derivative. There are a variety of techniques you can use to algebraically manipulate a function correctly!*

Now, we find the x-values where $f'(x) = 0$ or $f'(x)$ does not exist. $f'(x) = 0$ when the numerator equals 0, and $f'(x)$ does not exist when the denominator equals zero (remember we are looking at the domain of f' when determining where it does not exist). First, we will find the x-values where $f'(x) = 0$:

$$f'(x) = 0 \implies$$
$$-(x-12) = 0$$
$$\implies x = 12$$

Next, to find the x-values where $f'(x)$ does not exist, we set the denominator equal to zero:

$$f'(x) \text{ DNE } \implies$$
$$(x+2)^3 = 0$$
$$\left((x+2)^3\right)^{\frac{1}{3}} = (0)^{\frac{1}{3}}$$
$$x+2 = 0$$
$$x = -2$$

Because $f(x) = \dfrac{x-5}{(x+2)^2}$ is a rational function and its denominator must be nonzero, its domain is $(-\infty,-2)\cup(-2,\infty)$. Thus, only $x = 12$ is in the domain, so it is the only critical value of f.

Graphing the function initially to observe its behavior on the interval $(-4, 15)$, we may see a graph that looks similar to the one shown in **Figure 3.4.22**:

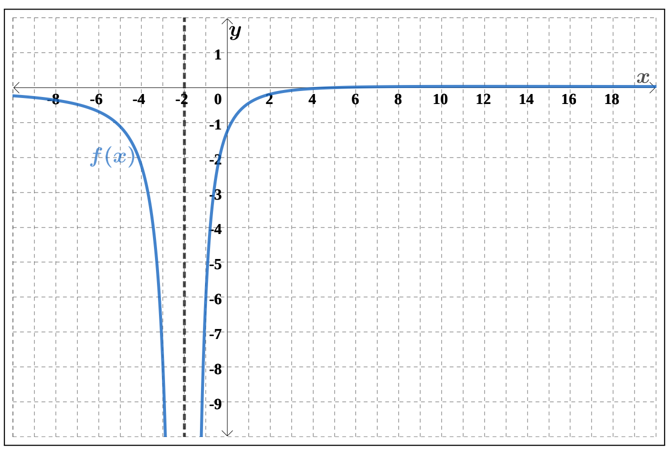


Figure 3.4.22: Graph of the function $f(x) = \dfrac{x-5}{(x+2)^2}$

Looking at this graph, it appears as though there is no absolute minimum on the interval $(-4, 15)$ because of the vertical asymptote at $x = -2$. It also looks like there may not be an absolute maximum due to the function appearing to get closer and closer to the horizontal asymptote $y = 0$.

Because we know there may be some type of extrema at $x = 12$ because it is a critical value of f, we need to adjust our calculator's viewing window so we can see the behavior of the function near this critical value. See **Figure 3.4.23**.

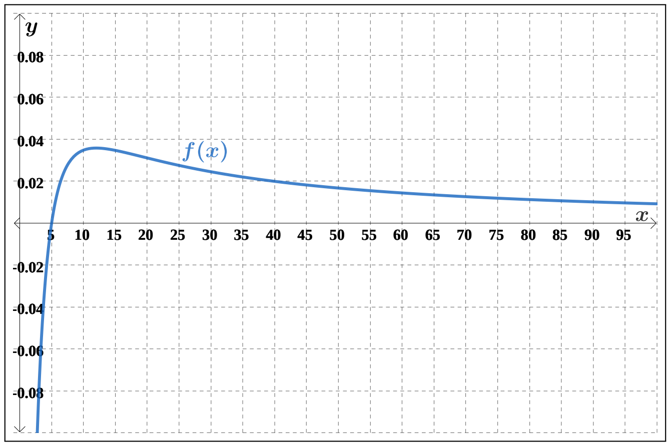


Figure 3.4.23: Graph of the function $f(x) = \dfrac{x-5}{(x+2)^2}$ near $x = 12$

We can see from this graph that the function crosses the x-axis and there is an absolute maximum. Because we know $x = 12$ is the only critical value of f, the absolute maximum must occur at $x = 12$. Recall that we could see the vertical asymptote at $x = -2$ when we first graphed f (see **Figure 3.4.22** again), so we know there is no absolute minimum.

To find the absolute maximum of f on the interval $(-4, 15)$ exactly, we calculate $f(12)$:

$$f(x) = \frac{x-5}{(x+2)^2} \implies$$
$$f(12) = \frac{12-5}{(12+2)^2} = \frac{7}{196} \approx 0.0357$$

Thus, the absolute maximum of f on the interval $(-4, 15)$ is $\frac{7}{196}$, but there is no absolute minimum.

N *f does have a horizontal asymptote, $y = 0$, but it reaches its absolute maximum at $x = 12$ before approaching the line $y = 0$.*

■

Try It # 7:

Find the absolute maximum and minimum of the function $f(x) = \dfrac{8}{x^2-4}$ on each of the following intervals, if they exist.

a. $[0, 5]$
b. $(-1, 1)$
c. $[-6, -3)$

Other Tests for Absolute Extrema

Recall that we learned two tests for finding local extrema: the First Derivative Test (**Section 3.1**) and the Second Derivative Test (**Section 3.2**). In certain circumstances, these tests can also be used to find absolute extrema.

If a function, f, has only one critical value in some interval on which f is continuous and there is a local extremum at that critical value, then the local extremum must also be an absolute extremum. Why should this make sense intuitively?

If a continuous function has a local maximum at $x = 2$, for example, and it has no other critical values other than $x = 2$, then the graph of the function would never be able to "turn around" and increase so that it reaches a value greater (higher) than that at $x = 2$. Thus, the local maximum would also be the absolute maximum.

Figure 3.4.24 shows the graph of a continuous function with only one critical value, $x = 2$, that has a local maximum at the critical value. Because $x = 2$ is the only critical value of f, the graph of the function will never "turn around" again. So the local maximum is also the absolute maximum:

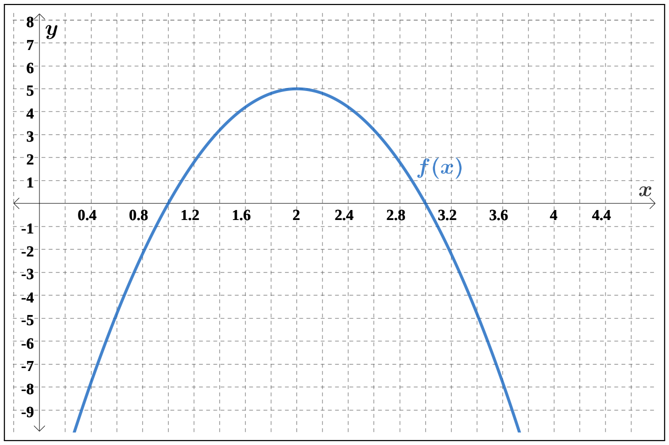


Figure 3.4.24: Graph of a function f that has both a local and absolute maximum at $x = 2$

We will use this result in the next section when solving optimization problems!

Try It Answers

1. Local maximum of 16 at $x = 2$; Local minimum of -16 at $x = -2$; No absolute maximum; No absolute minimum

2. No local maxima; Local minimum of -8 at $x = 2$; No absolute maximum; Absolute minimum is -8 at $x = 2$

3. **a.** Absolute maximum: 11; Absolute minimum: -16

   b. Absolute maximum: 16; Absolute minimum: -16

4. **a.** Absolute maximum at $x = -5$; No absolute minimum

   b. No absolute maximum; No absolute minimum

   c. Absolute maximum at $x = 5$; Absolute minimum at $x = 2$

5. **a.** Absolute maximum: $\frac{199}{4}$; Absolute minimum: $-\frac{57}{4}$

   b. Absolute maximum: 168; Absolute minimum: $-\frac{57}{4}$

6. **a.** Absolute maximum: $\frac{1}{6}$; Absolute minimum: $-\frac{1}{6}$

   b. Absolute maximum: $\frac{216}{35}$; Absolute minimum: $\frac{3\sqrt{3}}{2}$

7. **a.** Absolute maximum: None; Absolute minimum: None

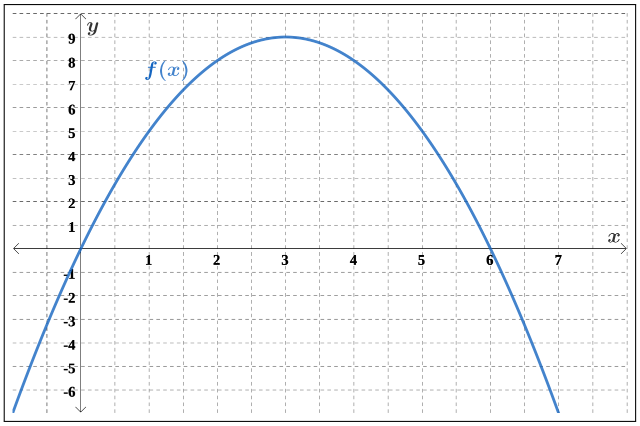
   b. Absolute maximum: -2; Absolute minimum: None

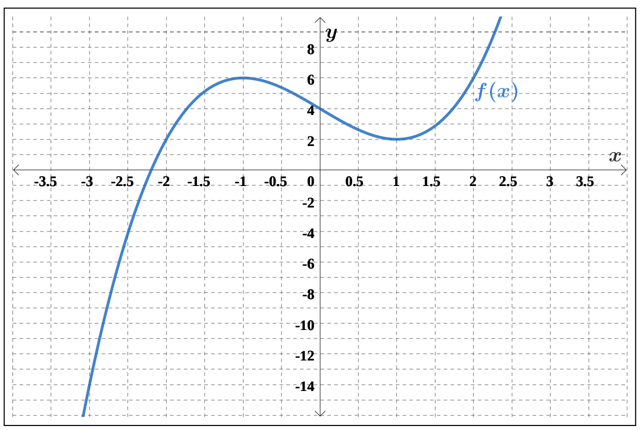
   c. Absolute maximum: None; Absolute minimum: $\frac{1}{4}$

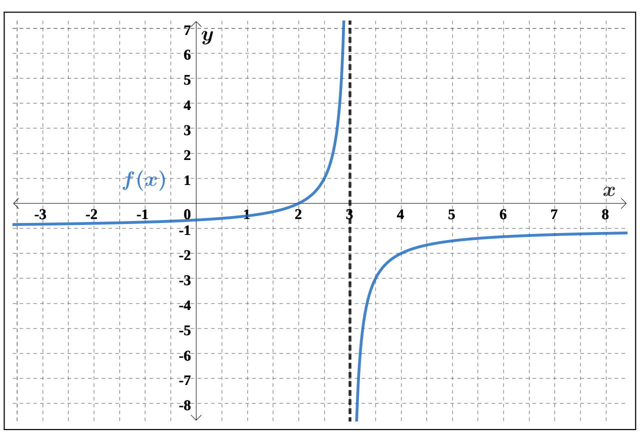
EXERCISES

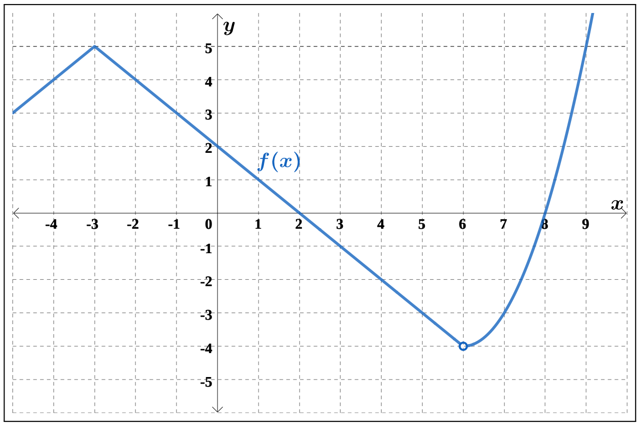
Basic Skills Practice

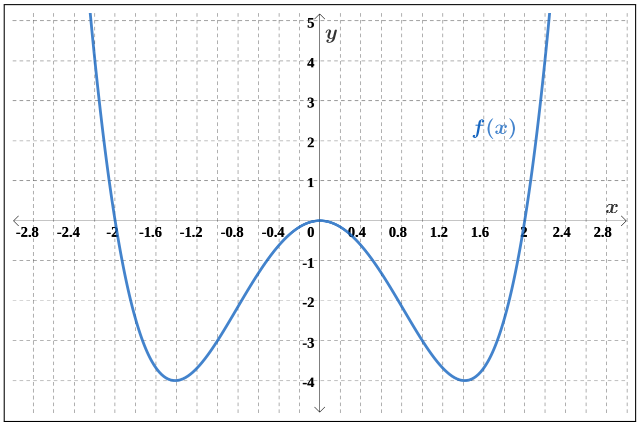
For Exercises 1 - 5, the graph of f is shown. Find (a) the x-values of any local extrema and (b) the x-values of any absolute extrema of f. Specify whether an extremum is a minimum or maximum.

1. 

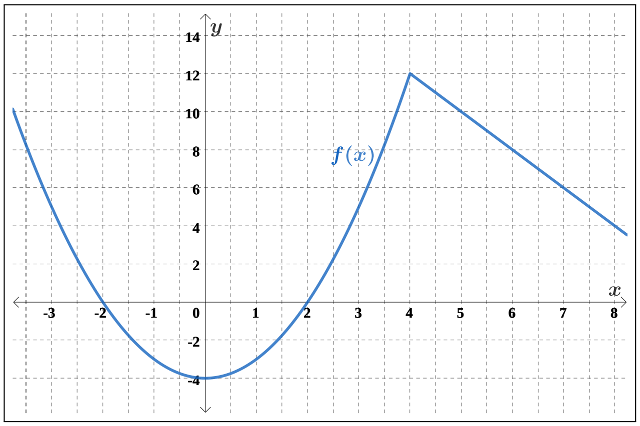
2. 

3. 

4. 

5. 

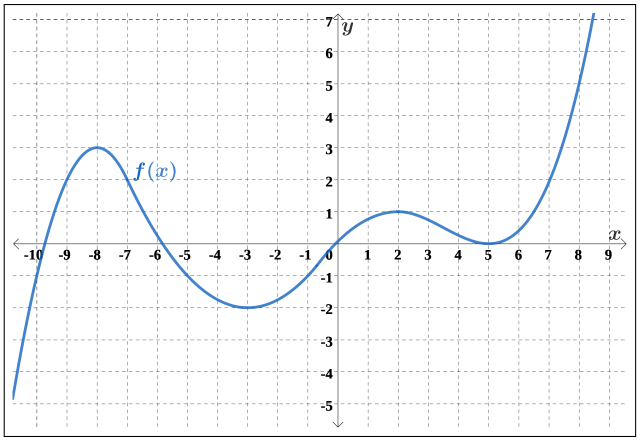
For Exercises 6 - 11, the graph of f is shown. Find the absolute maximum and minimum of f on each of the given intervals, if they exist.

6. 

(a) $[2,6]$

(b) $[-3,4.5]$

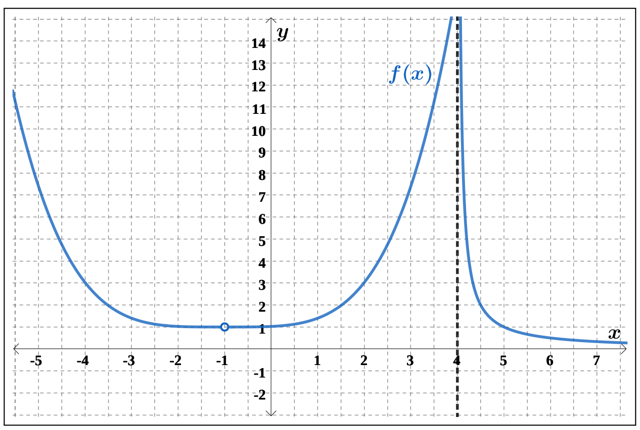
(c) $[5,7]$

7. 

(a) $[-1,2]$

(b) $[-9,8]$

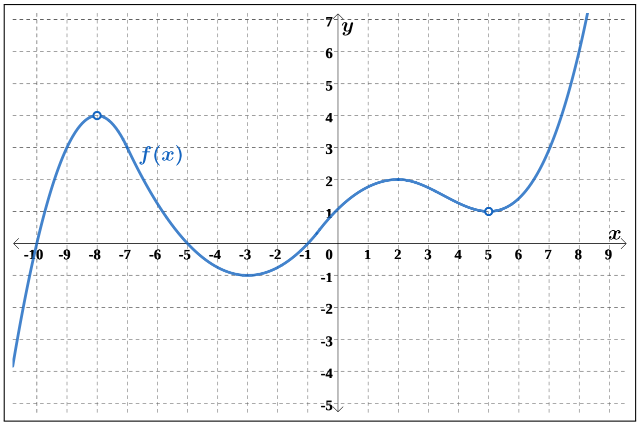
(c) $[-7,5]$

8. 

(a) $[-4,2]$

(b) $[-5,5]$

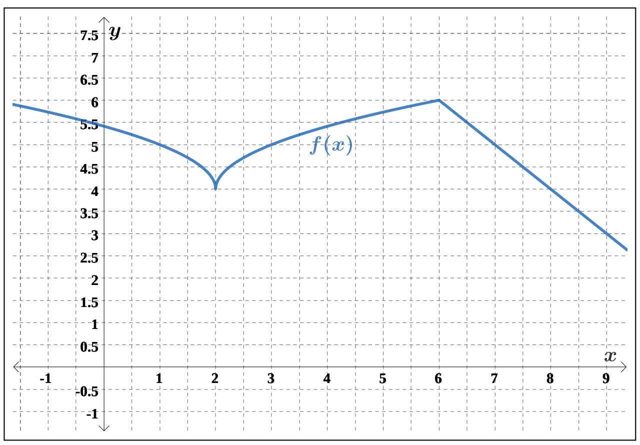
(c) $[-4,4.5]$

9. 

(a) $[-3,2]$

(b) $[-9,7]$

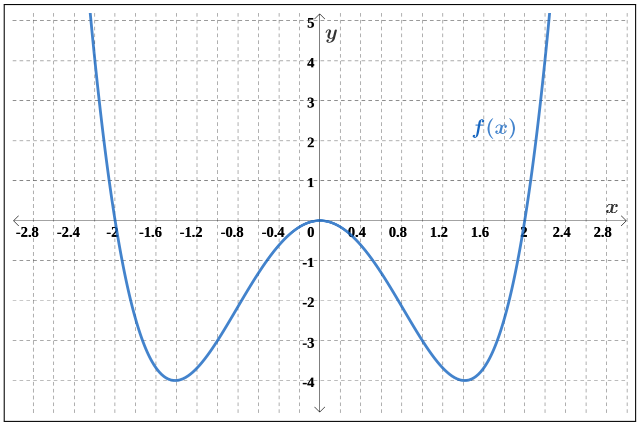
(c) $[2,8]$

10. 

(a) $(2,6)$

(b) $[3,8)$

(c) $(1,6]$

11. 

(a) $(-1.4, 1.4)$

(b) $[0, 1)$

(c) $(-2, -1]$

For Exercises 12 - 16, use the Closed Interval Method to find the absolute maximum and minimum of f on each of the given intervals.

12. $f(x) = x^5 + 5x^4 - 35x^3$

    (a) $[-9, 7]$

    (b) $[-1, 4]$

    (c) $[-8, 6]$

13. $f(x) = x^3 - 9x^2 + 24x - 5$

    (a) $[3, 7]$

    (b) $[-4, 1]$

    (c) $[0, 5]$

14. $f(x) = \dfrac{2x^2}{x+5}$

    (a) $[-4, 2]$

    (b) $[0, 9]$

    (c) $[-12, -6]$

15. $f(x) = \dfrac{8}{x-10}$

    (a) $[-6, 2]$

    (b) $[12, 20]$

    (c) $[-10, 0]$

16. $f(x) = \dfrac{-x^2}{3-x}$
    (a) $[7,12]$
    (b) $[-3,2]$
    (c) $[4,9]$

For Exercises 17 - 21, find the absolute maximum and minimum of f on each of the given intervals, if they exist.

17. $f(x) = x^3 + 15x^2 + 27x - 1$
    (a) $(-11,0)$
    (b) $[-14,2)$
    (c) $(-4,3]$

18. $f(x) = x^5 - 5x^4 - 20x^3 + 100$
    (a) $(-1,8]$
    (b) $(-3,7)$
    (c) $[-4,-1)$

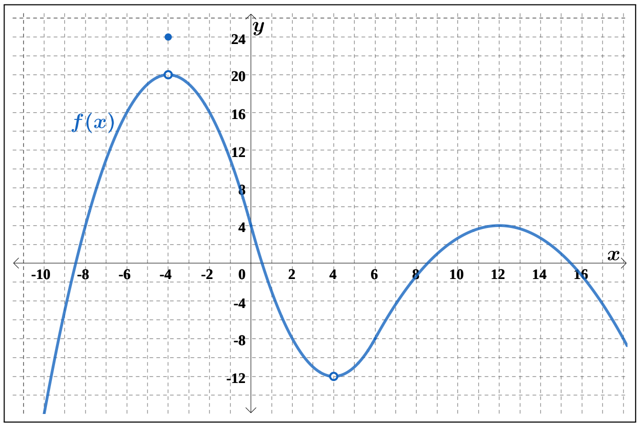
19. $f(x) = 3x^4 - 44x^3 - 360x^2 + 50$
    (a) $[-6,0)$
    (b) $(12,18]$
    (c) $(-1,20)$

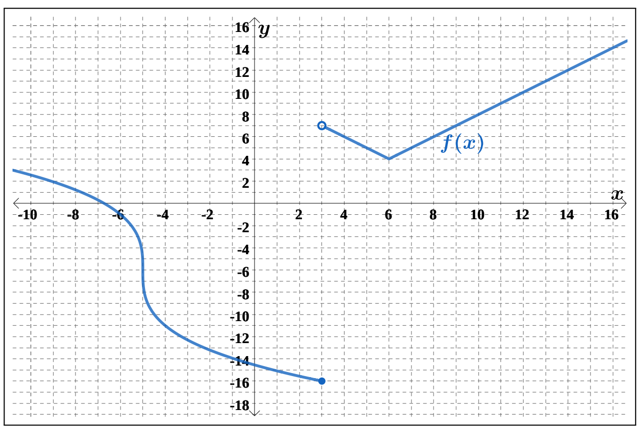
20. $f(x) = \dfrac{x^2 - 4x - 12}{x^2}$
    (a) $[-9,5]$
    (b) $[-3,6]$
    (c) $[-1,6]$

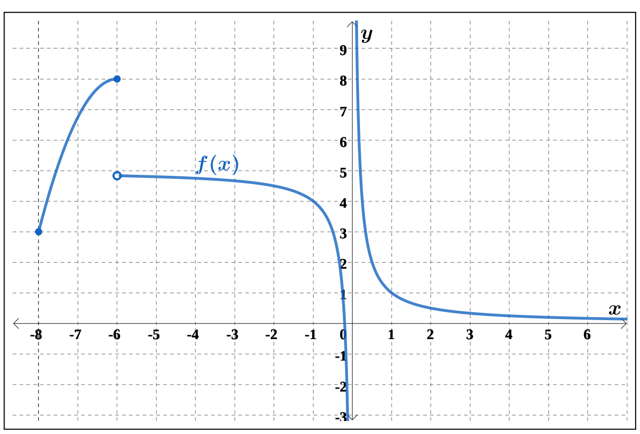
21. $f(x) = \dfrac{x^2}{x+3}$
    (a) $[-5,-1]$
    (b) $[-8,4]$
    (c) $[-6,0]$

Intermediate Skills Practice

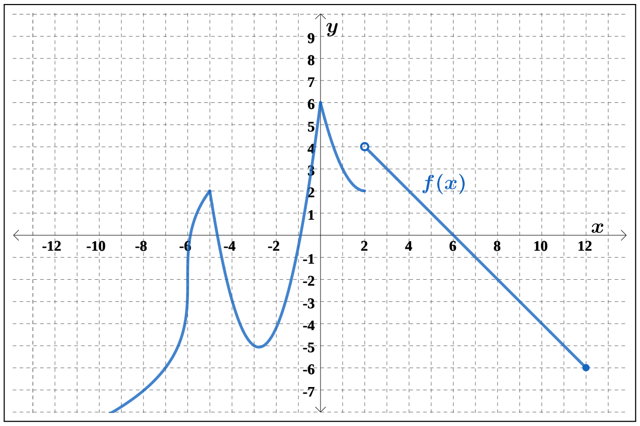
For Exercises 22 - 24, the graph of f is shown. Find (a) any local extrema and (b) any absolute extrema, as well as where they occur, of f. Specify whether an extremum is a minimum or maximum.

22. 

23. 

24. 

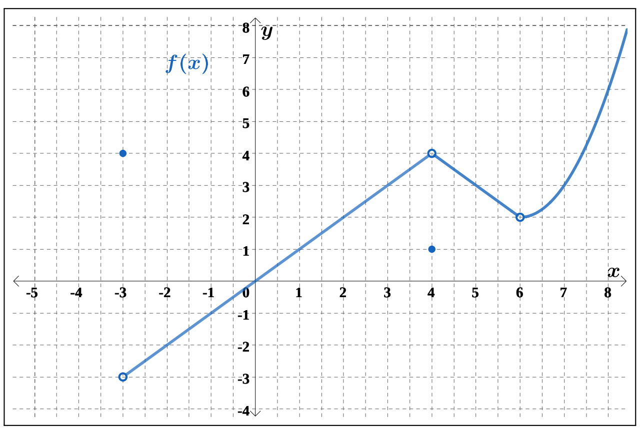
For Exercises 25 and 26, the graph of f is shown. Find the absolute extrema of f on each of the given intervals, if they exist.

25. 

(a) $(-8,-4)$

(b) $[0,6)$

(c) $[1,3]$

26. 

(a) $[-3,2]$

(b) $(-1,5)$

(c) $[2,8)$

For Exercises 27 - 34, find the absolute maximum and minimum of f on each of the given intervals.

27. $f(x) = (x-2)e^x$

(a) $[0,2]$

(b) $[-5,0]$

(c) $[1,3]$

28. $f(x) = \left(x^2-4\right)^5$

(a) $[-4,-1]$

(b) $[-1,3]$

(c) $[-5,5]$

29. $f(x) = \dfrac{\ln(x)}{x^2}$

(a) $[1,3]$

(b) $\left[1/2, \sqrt{e}\right]$

(c) $[2,4]$

30. $f(x) = \sqrt[3]{\left(x^2-1\right)^2}$

(a) $[-2,3]$

(b) $[-1,4]$

(c) $[-1,1]$

31. $f(x) = e^{-x^2}$

(a) $[-7,-3]$

(b) $[-1,1]$

(c) $[0,3/2]$

32. $f(x) = 4x^2\ln(x)$

(a) $[1,3]$

(b) $[2,5]$

(c) $[1/10,3/4]$

33. $f(x) = \dfrac{x^3}{x^2-9}$

(a) $[-15,-10]$

(b) $[-2,1]$

(c) $[4,6]$

34. $f(x) = x - 5\ln(x)$

(a) $[1,6]$

(b) $[2,4]$

(c) $[15,20]$

For Exercises 35 - 40, find the absolute maximum and minimum of f on each of the given intervals, if they exist.

35. $f(x) = \dfrac{4x}{x^2 - 4x + 4}$
    (a) $[1, 6]$
    (b) $[-8, 0)$
    (c) $(-2, 4]$

36. $f(x) = \sqrt[3]{\left(x^2 - 3x\right)^4}$
    (a) $[-4, 1)$
    (b) $(2, 5)$
    (c) $(0, 3]$

37. $f(x) = \dfrac{3x}{x^2 + 4}$
    (a) $(-8, 4)$
    (b) $(-2, 2]$
    (c) $[0, 6)$

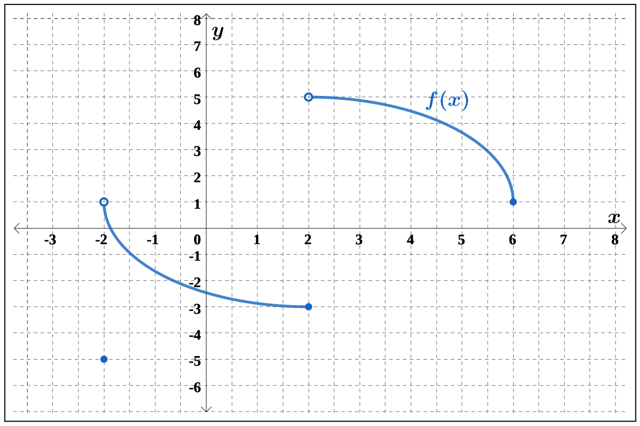
38. $f(x) = \dfrac{-5e^x}{x^2}$
    (a) $(-1, 3)$
    (b) $(1, 4]$
    (c) $(5, 8]$

39. $f(x) = \ln(x) - 2x^2$
    (a) $[1/3, 8/9)$
    (b) $(1/2, 4)$
    (c) $(0, 2]$

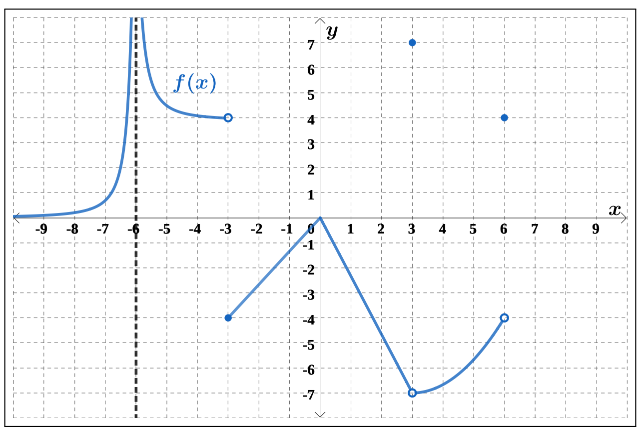
40. $f(x) = \dfrac{x^2 + 1}{x^2 - 25}$
    (a) $[-4, 3)$
    (b) $[-7, 2]$
    (c) $(1, 7]$

Mastery Practice

41. Given the graph of f shown below, find (a) any local extrema and (b) any absolute extrema, as well as where they occur, of f. Specify whether an extremum is a minimum or maximum.



42. Given the graph of f shown below, find the absolute extrema of f on each of the given intervals, if they exist.



(a) $(-6, 0]$

(b) $[-3, 6]$

(c) $[0, 3)$

(d) $(-\infty, \infty)$

(e) $(3, 6]$

(f) $(-3, 0)$

For Exercises 43 - 52, find the absolute maximum and minimum of f on each of the given intervals, if they exist.

43. $f(x) = x^2 e^{-0.04x}$
    (a) $(10, 60)$
    (b) $[-1, 52]$
    (c) $(-25, -15)$

44. $f(x) = \dfrac{5x}{(x+6)^2}$
    (a) $[-4, 8]$
    (b) $[-2, 3)$
    (c) $(-8, -5]$

45. $f(x) = 2x\ln(x)$
    (a) $[1/4, 3/4)$
    (b) $(0, 1]$
    (c) $(1, 5)$

46. $f(x) = \dfrac{e^{0.25x^2}}{x}$
    (a) $[-4, -2]$
    (b) $[1/2, 2]$
    (c) $[-2, 2]$

47. $f(x) = 3x^4 - 40x^3 + 150x^2 - 25$
    (a) $(-2, 5)$
    (b) $[1, 6]$
    (c) $(-4, -1]$

48. $f(x) = \dfrac{(x+5)^4}{(x-1)^3}$
    (a) $[-6, 20]$
    (b) $[2, 15)$
    (c) $[-7, 0]$

49. $f(x) = (x-1)^{10}(x-5)^{10}$
    (a) $[1/2, 2]$
    (b) $(3/2, 11/2)$
    (c) $[1, 3]$

50. $f(x) = \dfrac{16 - x^2}{x^2 + 9}$

    (a) $[-3, 3]$

    (b) $(-1, 2)$

    (c) $(0, 4]$

51. $f(x) = \dfrac{4\ln\left(x^2\right)}{x}$

    (a) $(1, 5)$

    (b) $[-4, 2]$

    (c) $[1, e)$

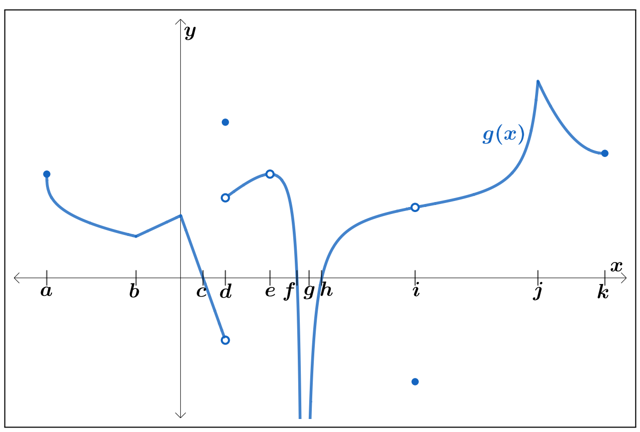
52. $f(x) = \sqrt[5]{\left(x^3 - 3x\right)^4}$

    (a) $[-3, 2]$

    (b) $(2, 5)$

    (c) $[-3/2, 0)$

53. Given the graph of g shown below, find (a) where any local extrema and (b) where any absolute extrema of f occur. Specify whether an extremum is a minimum or maximum.



54. Determine where each of the following functions has an absolute minimum, if it exists, on the interval $(0, \infty)$.

    (a) $f(x) = \dfrac{-5x^{24}}{e^{3x}}$

    (b) $g(x) = \dfrac{2x^{14}}{e^{2x}}$

For Exercises 55 and 56, use the given information to sketch a possible graph of f.

55. - The domain of f is $(-\infty, \infty)$.
    - f is discontinuous at $x = 7$ only.
    - f has a local and absolute minimum at $x = 7$.
    - f does not have local or absolute maxima.

56. - The domain of f is $[0, 4]$.
    - f is discontinuous at $x = 4$ only.
    - f has an absolute minimum at $x = 0$.
    - f has an absolute maximum at $x = 4$.
    - f does not have any local extrema.

Communication Practice

57. Is it possible for a function to have a local maximum at $x = 6$, but not have any absolute extrema? Explain.

58. Is it possible for f to have a local minimum at $x = 3$ if f is discontinuous at $x = 3$? Explain.

59. Explain, in general terms, the Extreme Value Theorem.

60. Where on the graph of a function can absolute extrema occur?

61. Briefly describe the three steps of the Closed Interval Method.

62. To use the Closed Interval Method, why does the function need to be continuous and the interval closed?

63. Can a function have more than one absolute maximum? Explain.

64. Are we able to use the Closed Interval Method to find the absolute minimum of $f(x) = \dfrac{x^2}{x+5}$ on the interval $[-7, -1]$? Explain.

65. If we are unable to use the Closed Interval Method to find the absolute extrema of a function on an interval because the function is not continuous or the interval is not closed (or both), how should we proceed in order to find any absolute extrema?

66. Compare and contrast local and absolute extrema.

3.5 Optimization

In this section, we will apply the calculus techniques we have learned thus far to find the **optimal solution** of real-world problems. For instance, a manufacturing company may want to maximize its profit or minimize its cost. A farmer may need to build a pen for his cattle with maximum area, but he may have a limited amount of fencing with which to build the pen.

An optimal solution is the absolute maximum (if we are maximizing a quantity) or the absolute minimum (if we are minimizing a quantity) of a function. The function we are trying to maximize or minimize is called the **objective function**. Because we will be working with mostly real-world problems, there will be **constraints** (due to limited resources, etc.) associated with the objective function. Thus, we will have to consider these constraints when trying to find the optimal solution.

Rather than graphing the objective function on our calculator and approximating the absolute maximum or minimum, we will use calculus techniques developed throughout Chapter 3 so we can find the *exact*, optimal solution!

Learning Objectives:
In this section, you will learn how to solve real-world optimization problems using calculus techniques. Upon completion you will be able to:

- Create variables and formulate equations to develop an objective function that must be maximized or minimized and represents a “numbers" optimization problem.
- Create variables and formulate equations to develop an objective function that must be maximized or minimized and represents a business optimization problem.
- Create variables and formulate equations to develop an objective function that must be maximized or minimized and represents a geometric optimization problem.
- Create realistic intervals corresponding to the variable used during the optimization process.
- Apply the appropriate calculus technique (Closed Interval Method, First Derivative Test, or Second Derivative Test) while performing the optimization process to calculate an optimal solution.

Optimization Process

The process of finding an absolute maximum or minimum is called **optimization**. Again, the function we are maximizing or minimizing (i.e., optimizing) is called the objective function. The quantity we are optimizing, which is given by the objective function, can be recognized by its proximity to "est" words (greatest, smallest, largest, least, most, farthest, highest, etc.), or the problem may explicitly say to maximize or minimize a certain quantity. For example, we may need to find the dimensions of a cardboard box with the *largest* possible volume given some constraints pertaining to the amount of cardboard available. In this case, our objective function would be a function representing the volume of the box because this is the quantity we would need to maximize.

Often, the most challenging part of solving an optimization problem is understanding the situation at hand and translating it into mathematical form. To translate the problem into mathematical form, we have to create variables and understand the relationships between the variables to formulate the objective function (it can be helpful to draw a picture, when applicable). Also, if there are constraints given in the problem, we have to translate those into mathematical form as well. We will create **constraint equations** that will relate the variables in the objective function. The constraint equations are always equations, so they should have equal signs.

Usually, an objective function will consist of two (or more) variables. If an objective function has two variables and there is a constraint equation associated with the problem, we can solve the constraint equation for one of the variables and then rewrite the objective function as a function of just one variable. This will be the goal in most of our problems!

Because we will be solving real-world optimization problems, we will also have to find an **interval** on which the objective function makes sense in terms of the context of the problem (i.e., find the domain of the objective function). For example, a company cannot sell a negative number of items, but it is possible that it will not sell any items. The same with the price per item: The price cannot be negative, but it is possible the company may be giving items away for free in order to advertise their product. Another example would be as mentioned above in the introduction: If a farmer is building a pen to hold cattle, the dimensions of the pen must be positive. Creating an interval on which the objective function is defined ensures we are considering the realistic aspects of the problem.

After setting up the optimization problem by finding the objective function and corresponding interval, we will use the Closed Interval Method we learned in **Section 3.4** to find the optimal solution if we are maximizing or minimizing an objective function on a closed interval and the function is continuous on the interval. If, instead, we are maximizing or minimizing an objective function on an open interval in which the function is continuous and it has only one critical value in the interval, we can use either the First Derivative Test (**Section 3.1**) or the Second Derivative Test (**Section 3.2**) to determine the optimal solution (assuming the necessary conditions for each test are met, which in this textbook, they will be). You may be wondering how we can use these tests for local extrema to find absolute extrema. We mentioned this briefly at the end of **Section 3.4**: If a function has only one critical value in some interval on which the function is continuous and there is a local extremum at that critical value, then the local extremum must also be an absolute extremum because the graph of the function will not be able to "turn around" and reach a point that is higher (or lower) than the local extremum.

After solving an optimization problem using calculus, we need to go back and reread the problem to be sure that we answer the question that is asked. It is possible that we may forget the original question in the problem after performing the entire optimization process!

We now summarize the steps for solving an optimization problem:

Optimization Process for Solving Optimization Problems

1. Translate the word problem into mathematical form:
    - Identify the quantity you *want* to maximize or minimize as well as the quantity you *know* (typically given by a constraint).
    - Introduce variables and draw a picture with labels, if applicable.
    - Determine the relationships between the variables to create the objective function and, possibly, constraint equation. Remember a constraint equation should have an equal sign and be equal to a number!
2. State the objective function in terms of one variable. If there is more than one variable in the objective function, solve the constraint equation for one variable and substitute the result into the objective function. This will give an objective function with only one variable.
3. Find the interval on which the objective function must be optimized. To do this, consider the context of the problem and restrict the variables to determine a realistic domain for the objective function.

4. Use calculus to find the optimal solution:
   - Find the critical values of the objective function. Recall these are the x-values where $f'(x) = 0$ or $f'(x)$ does not exist, and they are in the domain (i.e., in the interval found above in step 3).
   - If the interval in step 3 is closed and the objective function is continuous on the interval, use the Closed Interval Method to find the absolute maximum or minimum. Remember, this means to evaluate the objective function at the critical values in the interval as well as the endpoints of the interval.
   - If the interval in step 3 is open and there is only one critical value in the interval, perform one of the following tests to find the absolute maximum or minimum:
     - the First Derivative Test (assuming the objective function is continuous on the interval and is differentiable near, and on both sides of, the critical value)
     - the Second Derivative Test (assuming the objective function is twice-differentiable at the critical value and the first derivative equals zero at the critical value)

     Remember, if a function has only one critical value in an interval, the function is continuous on the interval, and a local extremum occurs at the critical value, then the local extremum must also be an absolute extremum.
5. Reread the problem to be sure you answer the original question.

N *If the objective function is twice differentiable on the interval, then all the necessary conditions to use either the First Derivative Test or the Second Derivative test are satisfied. Recall from* **Section 2.2** *that if a function is differentiable on an interval, then it must be continuous on the interval. And, if a function is twice differentiable on an interval, then its first derivative certainly exists on the interval. In this textbook, the objective functions we formulate will be twice differentiable on their respective intervals. Hence, we can use either the First Derivative Test or the Second Derivative Test to find the absolute extremum when the objective function has only one critical value in an open interval.*

In this textbook, we will focus on solving three types of optimization problems: "numbers", business, and geometric problems.

"Numbers" Problems

We will start with a "numbers" optimization problem to help demonstrate the optimization process. A "numbers" problem usually consists of a question that asks us to find two numbers that meet certain criteria. We will see this in our first example.

■ **Example 1** Find two nonnegative numbers x and y such that $3x + y = 30$ and their product is a maximum.

Solution:

The word "maximum" tells us this is an optimization problem, so we proceed with the optimization process outlined previously:

1. Translate the word problem into mathematical form:

To translate the problem, we express the quantities mathematically we *want* (the objective function) and *know* (in this case, a constraint equation). Because we *want* the product of the numbers to be a maximum, the objective function will represent the product of the numbers. We will call the function P to represent the product:

$$want: \; P = xy$$

The equation given in the problem is what we *know*, so it is the constraint equation. Recall constraint equations always have equal signs:

$$\textit{know: } 3x + y = 30$$

2. State the objective function in terms of one variable:

Because the objective function has two variables, x and y, we must solve the constraint equation for one variable and substitute the result into the objective function. We can choose either x or y to solve for in the constraint equation, but we will choose y because it will be slightly easier than solving for x:

$$\begin{aligned} 3x + y &= 30 \\ y &= 30 - 3x \end{aligned}$$

Substituting y into the objective function gives

$$\begin{aligned} P &= xy \\ &= x(30 - 3x) \\ &= 30x - 3x^2 \end{aligned}$$

Thus, the objective function in terms of one variable, x, is

$$P(x) = 30x - 3x^2$$

3. Find the interval on which the objective function must be optimized:

To find the interval, we must consider the context of the problem and restrict the variables to determine a realistic domain for the objective function. However, in this particular problem, we are explicitly given restrictions: both x and y are nonnegative (meaning they must equal zero or be positive). So we have

$$x \geq 0 \text{ and } y \geq 0$$

However, because the objective function is in terms of x, we need to substitute for y in the second inequality to ensure our restrictions lead us to an interval consisting of x-values. Recall that $y = 30 - 3x$ from step 2. Substituting for y in the second inequality and solving for x gives

$$\begin{aligned} y &\geq 0 \implies \\ 30 - 3x &\geq 0 \\ 30 &\geq 3x \\ 10 &\geq x, \text{ or } x \leq 10 \end{aligned}$$

Thus, we must maximize the objective function, $P(x) = 30x - 3x^2$, on the interval $[0, 10]$. Note that the interval has brackets because the inequalities were inclusive, and therefore the interval is closed.

4. Use calculus to find the optimal solution:

To find the optimal solution, we need to find the critical values of the objective function, P, we found in step 2 and use an appropriate test that we learned previously to find the absolute maximum.

The critical values of P are the x-values where $P'(x) = 0$ or $P'(x)$ does not exist, and they are in the domain of P (i.e., they are in the interval we found in step 3). To find the critical values of P, we must find $P'(x)$ first:

$$\begin{aligned} P(x) &= 30x - 3x^2 \implies \\ P'(x) &= 30 - 6x \end{aligned}$$

Now, we find the x-values where $P'(x) = 0$ or $P'(x)$ does not exist. Because P' is a polynomial and has a domain of all real numbers, it will exist everywhere. Thus, we only need to find the x-values where $P'(x) = 0$:

$$\begin{aligned} P'(x) &= 0 \implies \\ 30 - 6x &= 0 \\ 30 &= 6x \\ 5 &= x \end{aligned}$$

Notice that $x = 5$ is in the interval $[0, 10]$ we found in step 3. Therefore, $x = 2$ is a critical value of P.

Because the interval $[0, 10]$ is closed and the objective function, P, is continuous on the interval, we will use the Closed Interval Method we learned in **section 3.4** to find the absolute maximum. Because we already know the critical value, $x = 5$, we can go straight to evaluating the objective function at the critical value and the endpoints of the interval:

$$\begin{aligned} P(x) &= 30x - 3x^2 \implies \\ P(5) &= 30(5) - 3(5)^2 = 75 \\ P(0) &= 30(0) - 3(0)^2 = 0 \\ P(10) &= 30(10) - 3(10)^2 = 0 \end{aligned}$$

Looking at the function values, we see the largest is 75 at $x = 5$. Thus, the absolute maximum of P is 75, and it occurs at $x = 5$.

5. Reread the problem to be sure you answer the original question:

After working through the optimization process, it is easy to forget what the original question asked us to find! Looking back at the problem, we see that we need to find two numbers, x and y, whose product is a maximum. We found that the maximum product is 75, and it occurs at $x = 5$. So we have one number: $x = 5$. Thus, we just need to find the other number, y.

To find y, we substitute $x = 5$ into the constraint equation we solved for y in step 2:

$$\begin{aligned} y &= 30 - 3x \implies \\ y &= 30 - 3(5) \\ &= 15 \end{aligned}$$

Therefore, the numbers whose product is a maximum are $x = 5$ and $y = 15$.

■

Try It # 1:
Find two nonnegative numbers x and y whose sum is 100 and their product is a maximum.

Business Problems

We will now turn our attention to business optimization problems. These types of optimization problems usually consist of a question in which we are trying to maximize revenue or profit.

Example 2 A concert promoter has found that if she sells tickets for \$50 each, she can sell 1200 tickets. For each \$5 she raises the price, 50 less people attend. For what price should she sell each ticket to maximize her revenue? What is her maximum revenue?

Solution:
We need to *maximize* revenue, so we know this is an optimization problem. We will proceed with the optimization process:

1. Translate the word problem into mathematical form:

To translate the problem, we express the quantities mathematically we *want* (the objective function) and *know*. Because we *want* the revenue to be a maximum, the objective function will be the revenue function.

Although we are not given a constraint equation we *know* in this example, there are several pieces of information we *know* that will help us create the objective function in terms of one variable (we are given information about ticket price and demand). Because the objective function will represent revenue, we will call the function R and apply the general formula for revenue:

$$\textit{want: } R = x \cdot p$$

where x is the number of tickets sold at a price of \$$p$ per ticket.

Again, we do not have a constraint equation given, but we *know* that when price is \$50 per ticket, 1200 tickets will be sold. Also, we know that for every \$5 increase in price, 50 fewer tickets are sold. We can use this information to find the price-demand function, p.

We know p is a linear function because the problem says for every \$5 increase in price, 50 fewer tickets are sold. We can find the equation of the line by finding the slope of the line and then using the point-slope formula to find the equation.

To find the slope of the line, we need two points. We can use the information in the problem to get two points of the form (x, p). We know one point is $(1200, 50)$, and we can create another point, $(1150, 55)$, based on the information about price and demand. Now, we can calculate the slope, m:

$$\begin{aligned} m &= \frac{p_2 - p_1}{x_2 - x_1} \implies \\ m &= \frac{55 - 50}{1150 - 1200} \\ &= \frac{5}{-50} \\ &= -\frac{1}{10} \end{aligned}$$

N *Recall that you can let either point be (x_1, p_1) or (x_2, p_2) when calculating the slope, m. You will get the same slope no matter which points you consider to be (x_1, p_1) and (x_2, p_2).*

Now, we will use the point-slope formula to find the equation of the line:

$$\begin{aligned} p - p_1 &= m(x - x_1) \implies \\ p - 50 &= -\frac{1}{10}(x - 1200) \\ p - 50 &= -\frac{1}{10}x + 120 \\ p &= -\frac{1}{10}x + 170 \end{aligned}$$

N *Again, you can use either point in the point-slope formula above, and you will get the same answer.*

Thus, we can now say we *know* the price-demand function, p:

$$\textit{know:}\ \ p(x) = -\frac{1}{10}x + 170$$

2. State the objective function in terms of one variable:

Substituting the function we *know*, p, into the objective function we *want*, $R = x \cdot p$, we get

$$\begin{aligned} R(x) &= x \cdot p(x) \\ &= x\left(-\frac{1}{10}x + 170\right) \\ &= -\frac{1}{10}x^2 + 170x \end{aligned}$$

Thus, the objective function in terms of one variable, x, is

$$R(x) = -\frac{1}{10}x^2 + 170x$$

3. Find the interval on which the objective function must be optimized:

To find the interval when we are working business optimization problems, we have to remember that the objective function is based on the variables x and p (quantity and price, respectively). Both of these quantities must be nonnegative (meaning they must equal zero or be positive). Remember, there may not be any items sold (concert tickets in this case), or the items may be given away for free for advertising purposes. So in terms of restrictions, we have

$$x \geq 0 \ \text{ and } \ p(x) \geq 0$$

Recall that $p(x) = -\frac{1}{10}x + 170$ from step 1. Substituting for $p(x)$ in the second inequality and solving for x gives

$$\begin{aligned} p(x) \geq 0 &\implies \\ -\frac{1}{10}x + 170 &\geq 0 \\ 170 &\geq \frac{1}{10}x \\ 1700 &\geq x, \ \text{ or } \ x \leq 1700 \end{aligned}$$

Thus, we must maximize the objective function, $R(x) = -\frac{1}{10}x^2 + 170x$, on the interval $[0, 1700]$. Note that the interval is closed, as it should be for business optimization problems!

4. Use calculus to find the optimal solution:

To find the optimal solution, we need to find the critical values of the objective function, R, we found in step 2 and use an appropriate test that we learned previously to find the absolute maximum.

The critical values of R are the x-values where $R'(x) = 0$ or $R'(x)$ does not exist, and they are in the domain of $R(x)$ (i.e., they are in the interval we found in step 3). To find the critical value of R, we must find $R'(x)$ first:

$$\begin{aligned} R(x) &= -\frac{1}{10}x^2 + 170x \implies \\ R'(x) &= -\frac{2}{10}x + 170 \end{aligned}$$

Now, we find the x-values where $R'(x) = 0$ or $R'(x)$ does not exist. Because R' is a polynomial and has a domain of all real numbers, it will exist everywhere. Thus, we only need to find the x-values where $R'(x) = 0$:

$$\begin{aligned} R'(x) = 0 &\implies \\ -\frac{2}{10}x + 170 &= 0 \\ 170 &= \frac{2}{10}x \\ 1700 &= 2x \\ 850 &= x \end{aligned}$$

Notice that $x = 850$ is in the interval $[0, 1700]$ we found in step 3. Therefore, $x = 850$ is a critical value of R.

Because the interval $[0, 1700]$ is closed and the objective function, R, is continuous on the interval, we will use the Closed Interval Method we learned in **Section 3.4** to find the absolute maximum. Because we already know the critical value, $x = 850$, we can go straight to evaluating the revenue function at the critical value and the endpoints of the interval:

$$\begin{aligned} R(x) &= -\frac{1}{10}x^2 + 170x \implies \\ R(850) &= -\frac{1}{10}(850)^2 + 170(850) = \$72,250 \\ R(0) &= -\frac{1}{10}(0)^2 + 170(0) = \$0 \\ R(1700) &= -\frac{1}{10}(1700)^2 + 170(1700) = \$0 \end{aligned}$$

Looking at the function values, we see the largest is \$72,250 at $x = 850$. Thus, the maximum revenue is \$72,250, and it occurs when 850 concert tickets are sold.

5. Reread the problem to be sure you answer the original question:

Looking back at the problem, we see that we need to find the selling price of each ticket that will maximize the concert promoter's revenue. We found that the maximum revenue is \$72,250, and it occurs at $x = 850$ (i.e., when 850 tickets are sold). However, we need to find the selling price of each ticket, which is given by the price-demand function p we found in step 1. We will calculate $p(850)$ to determine the selling price:

$$\begin{aligned} p(x) &= -\frac{1}{10}x + 170 \implies \\ p(850) &= -\frac{1}{10}(850) + 170 \\ &= \$85 \end{aligned}$$

Therefore, the concert promoter must charge \$85 per ticket to maximize her revenue. The maximum revenue is \$72,250.

■

Example 3 A company sells x ribbon winders per year at a price of $\$p$ per ribbon winder. The company's price-demand function for the ribbon winders is given by $p(x) = 300 - 0.02x$. The ribbon winders cost \$30 each to manufacture, and the company has fixed costs of \$9000 per year. How many ribbon winders must the company sell to maximize its profit? What is the company's maximum profit?

Solution:
We need to *maximize* profit, so we proceed with the optimization process:

1. Translate the word problem into mathematical form:

To translate the problem, we express the quantities mathematically we *want* (the objective function) and *know*. Because we *want* the profit to be a maximum, our objective function will be the profit function.

Although we are not given a constraint equation we *know* in this example, there are several pieces of information we *know* that will help us create the objective function in terms of one variable (we are given the price-demand function and information about cost). Because the objective function will represent profit, we will call the function P and apply the general formula for profit:

$$\textit{want:}\;\; P = R - C$$

where R and C represent the revenue and cost functions, respectively.

Again, we do not have a constraint equation given, but we do *know* the price-demand function, p, and we can use it to find the revenue function, R (remember $R = x \cdot p$, where x is the number of items sold at a price of $\$p$ per item). We can also use the information about cost to find the cost function, C. Then, we can find the objective function, P.

First, let's find the revenue function, R, by substituting the price-demand function, p, given to us in the problem:

$$\begin{aligned} R(x) &= x \cdot p(x) \\ &= x(300 - 0.02x) \\ &= 300x - 0.02x^2 \end{aligned}$$

Thus, we now *know* the revenue function $R(x) = 300x - 0.02x^2$, where $R(x)$ is the revenue, in dollars, from selling x ribbon winders:

$$\textit{know:}\;\; R(x) = 300x - 0.02x^2$$

Next, we must find the cost function, C. The general formula for cost is given by $C = Vx + F$, where x is the number of items made, V represents the cost of producing each item, and F represents the company's fixed costs. The problem states that it costs 30 to make each ribbon winder and that the company has fixed costs of \$9000 per year. Thus, we *know* the cost function is given by $C(x) = 30x + 9000$, where $C(x)$ is the cost, in dollars, of making x ribbon winders:

$$\textit{know:}\;\; C(x) = 30x + 9000$$

2. State the objective function in terms of one variable:

Substituting the functions we *know*, R and C, into the objective function we *want*, $P = R - C$, gives

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= \left(300x - 0.02x^2\right) - (30x + 9000) \\ &= \left(300x - 0.02x^2\right) - 30x - 9000) \\ &= -0.02x^2 + 270x - 9000 \end{aligned}$$

Thus, the objective function in terms of one variable, x, is

$$P(x) = -0.02x^2 + 270x - 9000$$

3. Find the interval on which the objective function must be optimized:

Like the previous business example, the objective function is based on the variables x and p (quantity and price, respectively), and they must both be nonnegative. So in terms of restrictions, we have

$$x \geq 0 \text{ and } p(x) \geq 0$$

Recall that $p(x) = 300 - 0.02x$. Substituting for $p(x)$ in the second inequality and solving for x gives

$$\begin{aligned} p(x) &\geq 0 \implies \\ 300 - 0.02x &\geq 0 \\ 300 &\geq 0.02x \\ 15,000 &\geq x, \text{ or } x \leq 15,000 \end{aligned}$$

Thus, we must maximize the objective function, $P(x) = -0.02x^2 + 270x - 9000$, on the interval $[0, 15,000]$. Again, note that the interval is closed.

4. Use calculus to find the optimal solution:

To find the optimal solution, we need to find the critical values of the objective function, P, we found in step 2 and use an appropriate test that we learned previously to find the absolute maximum.

The critical values of P are the x-values where $P'(x) = 0$ or $P'(x)$ does not exist, and they are in the domain of P (i.e., they are in the interval we found in step 3). To find the critical values of P, we must find $P'(x)$ first:

$$\begin{aligned} P(x) &= -0.02x^2 + 270x - 9000 \implies \\ P'(x) &= -0.04x + 270 \end{aligned}$$

Now, we find the x-values where $P'(x) = 0$ or $P'(x)$ does not exist. Because P' is a polynomial and has a domain of all real numbers, it will exist everywhere. Thus, we only need to find the x-values where $P'(x) = 0$:

$$\begin{aligned} P'(x) &= 0 \implies \\ -0.04x + 270 &= 0 \\ 270 &= 0.04x \\ 6750 &= x \end{aligned}$$

Notice that $x = 6750$ is in the interval $[0, 15,000]$ we found in step 3. Therefore, $x = 6750$ is a critical value of P.

Because the interval $[0, 15,000]$ is closed and the objective function, P, is continuous on the interval, we will use the Closed Interval Method we learned in **Section 3.4** to find the absolute maximum. Because we already know the critical value, $x = 6750$, we can go straight to evaluating the profit function at the critical value and the endpoints of the interval:

$$\begin{aligned} P(x) &= -0.02x^2 + 270x - 9000 \implies \\ P(6750) &= -0.02(6750)^2 + 270(6750) - 9000 = \$902,250 \\ P(0) &= -0.02(0)^2 + 270(0) - 9000 = -\$9000 \\ P(15,000) &= -0.02(15,000)^2 + 270(15,000) - 9000 = -\$459,000 \end{aligned}$$

Looking at the function values, we see the largest is $\$902,250$ at $x = 6750$. Thus, the maximum profit is $\$902,250$, and it occurs when 6750 ribbon winders are sold.

5. Reread the problem to be sure you answer the original question:

Looking back at the problem, we see that we need to find how many ribbon winders the company must sell to maximize its profit. In other words, we need the quantity, which is x. We found that the absolute maximum occurs at $x = 6750$. Thus, the company must sell 6750 ribbon winders to maximize its profit. We also found the maximum profit in step 4: $\$902,250$.

■

Notice in each of the examples above, the optimal solution occurred at the critical value and not at an endpoint of the interval. It is possible that an optimal solution may occur at an endpoint, so we must always check the function values at the endpoints as well as the critical value(s).

Try It # 2:
A company that makes toy firetrucks has a price-demand function given by $p(x) = -0.02x + 45$, where x is the number of trucks sold at a price of $\$p$ each. The company has a weekly cost function, in dollars, given by $C(x) = 12x + 1000$, where x is the number of trucks manufactured each week. Determine the selling price of each truck that will maximize the company's profit.

Geometric Problems

Finally, we will learn how to use the optimization process to find optimal solutions to problems involving geometry. We will work several examples to demonstrate this type of optimization problem because they tend to be the most challenging. And, unlike the business optimization problems, the intervals associated with geometric optimization problems are typically open. We will see why in the next example!

■ **Example 4** A rectangular garden is to be constructed using an existing rock wall for one side and wire fencing for the other three sides. If 100 feet of wire fencing is to be used, determine the dimensions of the garden so it has the largest area possible. What is the largest area?

Solution:
We need to find the largest area possible, which means we need to *maximize* area. So we proceed with the optimization process:

1. Translate the word problem into mathematical form:

To translate the problem, we express the quantities mathematically we *want* (the objective function) and *know* (in this case, a constraint equation). Because we *want* the area to be a maximum, the objective function will give the area of the garden.

Although we are not explicitly given a constraint equation we *know*, we do *know* there is only 100 feet of wire fencing available. We will be able to use this known value to create a constraint equation.

Often when we have a geometric optimization problem, it is helpful to draw a picture, define variables, and label the picture in order to formulate the objective function and constraint equation. The garden is rectangular, so we will draw a rectangle and designate one of the walls to be the rock wall (it does not matter which one!). Next, we will define y to be the length, in feet, of the side opposite the rock wall. Finally, we will let x be the length, in feet, of each of the remaining two sides. See **Figure 3.5.1**.

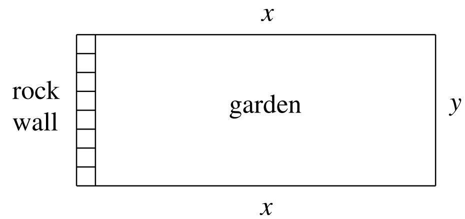


Figure 3.5.1: Rectangular garden with a rock wall on one side

The objective function must represent the area of the garden because we *want* to maximize its area. The area of a rectangle is equal to its width times its length. We will create the objective function using our variables, and we will call the objective function A because it gives the area of the garden:

$$\textit{want: } A = xy$$

We can also create the constraint equation using our picture. We know 100 feet of wire fencing will be used, so we can create an equation we *know* representing the amount of fencing used. We will add the lengths of all the sides, except for the rock wall, and set the sum equal to the total amount of wire fencing (recall constraint equations have equal signs):

$$\begin{aligned} \textit{know: } x+y+x &= 100 \implies \\ 2x+y &= 100 \end{aligned}$$

2. State the objective function in terms of one variable:

Because the objective function has two variables, x and y, we must solve the constraint equation for one variable and substitute the result into the objective function. We can choose either x or y to solve for in the constraint equation, but we will choose y because it will be slightly easier than solving for x:

$$\begin{aligned} 2x+y &= 100 \\ y &= 100-2x \end{aligned}$$

Substituting y into the objective function gives

$$\begin{aligned} A &= xy \\ &= x(100-2x) \\ &= 100x-2x^2 \end{aligned}$$

Thus, the objective function in terms of one variable, x, is

$$A(x) = 100x-2x^2$$

3. Find the interval on which the objective function must be optimized:

To find the interval, we must consider the context of the problem and determine a realistic domain for the objective function. When solving geometric optimization problems, we need to remember that dimensions must be positive. Why? They certainly cannot be negative, and if they equal zero there would be no geometric shape (in this case, there would be no garden!). In this example, the dimensions are given by x and y, so we have

$$x > 0 \text{ and } y > 0$$

However, because the objective function is in terms of x, we need to substitute for y in the second inequality to ensure our restrictions lead us to an interval consisting of x-values. Recall that $y = 100-2x$ from step 2.

Substituting for y in the second inequality and solving for x gives

$$\begin{aligned} y > 0 &\implies \\ 100 - 2x > 0 & \\ 100 > 2x & \\ 50 > x, &\text{ or } x < 50 \end{aligned}$$

Thus, we must maximize the objective function, $A(x) = 100x - 2x^2$, on the interval $(0,50)$. Note that the interval has parentheses because the inequalities were strict inequalities. Therefore, the interval is open, which it typically is for geometric optimization problems!

4. Use calculus to find the optimal solution:

To find the optimal solution, we need to find the critical values of the objective function, A, we found in step 2 and use an appropriate test to find the absolute maximum.

The critical values of A are the x-values where $A'(x) = 0$ or $A'(x)$ does not exist, and they are in the domain of A (i.e., they are in the interval we found in step 3). To find the critical values of A, we must find $A'(x)$ first:

$$\begin{aligned} A(x) = 100x - 2x^2 &\implies \\ A'(x) = 100 - 4x & \end{aligned}$$

Now, we find the x-values where $A'(x) = 0$ or $A'(x)$ does not exist. Because A' is a polynomial and has a domain of all real numbers, it will exist everywhere. Thus, we only need to find the x-values where $A'(x) = 0$:

$$\begin{aligned} A'(x) = 0 &\implies \\ 100 - 4x = 0 & \\ 100 = 4x & \\ 25 = x & \end{aligned}$$

Notice that $x = 25$ is in the interval $(0,50)$ we found in step 3. Therefore, $x = 15$ is a critical value of A (and the only critical value in the interval).

Because the interval $(0,50)$ is open, we cannot use the Closed Interval Method to find the absolute maximum. However, because the objective function only has one critical value in the interval and is twice differentiable on the interval, we can use either the First Derivative Test we learned in **Section 3.1** or the Second Derivative Test we learned in **Section 3.2** (because the critical value occurs when $A'(x) = 0$) to find the absolute extremum.

Recall from **Section 3.4** that if a function has only one critical value in some interval on which the function is continuous and there is a local extremum at that critical value, then the local extremum must also be an absolute extremum because the graph of the function will not be able to "turn around" and reach a point that is higher (or lower) than the local extremum. As stated previously, either the First Derivative Test or the Second Derivative Test for local extrema can be used to find absolute extrema when working geometric optimization problems in this textbook because the objective functions will only have one critical value in and be twice differentiable on their corresponding (open) intervals.

In this example, we will demonstrate using the First Derivative Test. Because we already know the critical value, $x = 25$, we can go straight to creating the sign chart of $A'(x)$ to determine the behavior of the objective function at the critical value.

Recall from **Section 3.1**, we place the critical value (also a partition number) on a number line and indicate that it is in the interval with a solid dot. We will also place a dotted line at $x = 0$ and $x = 50$ on the number line so we remember not to select x-values to test that are outside of the interval $(0,50)$.

Now, we need to determine the sign of $A'(x)$ on the intervals $(0,25)$ and $(25,50)$. We will choose the x-values $x = 10$ and $x = 30$ to test:

$$A'(x) = 100 - 4x \implies$$
$$A'(10) = 100 - 4(10) = 60 > 0$$
$$A'(30) = 100 - 4(30) = -20 < 0$$

Using this information, we can fill in the sign chart of $A'(x)$. Because we are also interested in the information this yields for A, we include that information below the number line. See **Figure 3.5.2**.

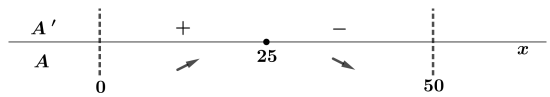


Figure 3.5.2: Sign chart of $A'(x)$ with the corresponding information for $A(x) = 100x - 2x^2$

Using **Theorem 3.1**, we see that a local maximum occurs at $x = 25$ (note that we also double check that A is defined at $x = 25$, which it is). Because $x = 25$ is the only critical value in the interval $(0,50)$ and A is continuous on the interval, we know this local maximum is also the absolute maximum. Now, we need to double check that we are in fact looking for an absolute maximum, which we are!

5. Reread the problem to be sure you answer the original question:

Looking back at the problem, we see that we need to find the dimensions that will maximize the area of the garden as well as the maximum area. The dimensions are given by x and y. We found that the absolute maximum occurs at $x = 25$ feet, and we can find y by substituting $x = 25$ into the constraint equation we solved for y in step 2:

$$y = 100 - 2x \implies$$
$$y = 100 - 2(25)$$
$$= 50 \text{ feet}$$

To find the maximum area, we substitute $x = 25$ into the objective function:

$$A(x) = 100x - 2x^2 \implies$$
$$A(25) = 100(25) - 2(25)^2 = 1250 \text{ square feet}$$

Therefore, the dimensions that maximize the area of the garden are 25 feet by 50 feet. The maximum area is 1250 square feet. Note that the length of the rock wall is 50 feet.

■

■ **Example 5** A box with a square base and an open top must have a volume of 32,000 cubic centimeters. Find the dimensions of the box so that the least amount of material is used to make the box.

Solution:

This problem asks us to find the least amount of material used to make the box, which means we need to *minimize* the surface area of the box. So we proceed with the optimization process:

1. Translate the word problem into mathematical form:

To translate the problem, we express the quantities mathematically we *want* (the objective function) and *know* (in this case, a constraint equation). Because we *want* the surface area to be a minimum, the objective function will give the surface area of the box.

Although we are not explicitly given a constraint equation we *know*, we do *know* that the box must have a volume of 32,000 cubic centimeters. We will be able to use this known value to create a constraint equation.

Because we have a geometric optimization problem, we will draw a picture, define variables, and label the picture to help us create the objective function and constraint equation. We will define x to be the length and width, in centimeters, of the bottom of the box. Remember, the box has a square base. We will let y be the height, in centimeters, of the box. See **Figure 3.5.3**.

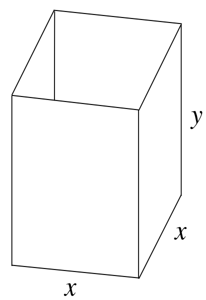


Figure 3.5.3: Box with a square base and open top

The objective function must represent the surface area of the box (i.e., the amount of material used to make the box) because we *want* to minimize the surface area. The surface area of the box is equal to the sum of the areas of all its surfaces. We can create the objective function using our variables, and we will call the objective function S because it gives the surface area. Adding the areas of all the surfaces, except for the top of the box because it is open, we can formulate the objective function we *want*. We start by finding the area of the bottom of the box and then adding the areas of all four sides:

$$\begin{aligned} \textit{want:}\ S &= xx + xy + xy + xy + x \implies \\ S &= x^2 + 4xy \end{aligned}$$

We can also create the constraint equation using our picture. We know the volume is 32,000 cubic centimeters, so we can create an equation we *know* representing the volume of the box. The volume of the box is equal to its length times its width times its height:

$$\begin{aligned} \textit{know:}\ x \cdot x \cdot y = 32,000 \implies \\ x^2 y = 32,000 \end{aligned}$$

2. State the objective function in terms of one variable:

Because the objective function has two variables, x and y, we must solve the constraint equation for one variable and substitute the result into the objective function. We can choose either x or y to solve for in the constraint equation, but we will choose y because it will be easier than solving for x:

$$\begin{aligned} x^2 y &= 32,000 \\ y &= \frac{32,000}{x^2} \end{aligned}$$

Substituting y into the objective function gives

$$\begin{aligned} S &= x^2 + 4xy \\ &= x^2 + 4x\left(\frac{32,000}{x^2}\right) \\ &= x^2 + \frac{128,000}{x} \end{aligned}$$

Thus, the objective function in terms of one variable, x, is

$$S(x) = x^2 + \frac{128,000}{x}$$

3. Find the interval on which the objective function must be optimized:

To find the interval, we must consider the context of the problem and determine a realistic domain for the objective function. We know the dimensions of the box must be positive. In this example, the dimensions are given by x and y, so we have

$$x > 0 \text{ and } y > 0$$

However, because the objective function is in terms of x, we need to substitute for y in the second inequality to ensure our restrictions lead us to an interval consisting of x-values. Recall that $y = \frac{32{,}000}{x^2}$ from step 2. Substituting for y in the second inequality gives

$$\begin{aligned} y > 0 &\implies \\ \frac{32{,}000}{x^2} &> 0 \end{aligned}$$

Unlike our previous examples, we cannot solve this inequality algebraically for x (if we multiply both sides by x^2, we lose the variable!). Thus, we have to simply observe this quotient and note that the numerator is positive and the denominator will always be positive as long as $x \neq 0$ because x is squared. Hence, this quotient will always be positive provided $x \neq 0$.

Thus, we have $x > 0$ and $x \neq 0$, which means the only restriction is $x > 0$.

Therefore, we must maximize the objective function, $S(x) = x^2 + \frac{128{,}000}{x}$, on the interval $(0, \infty)$. Note that the interval is open.

How can we have an interval in which the x-value can be infinitely large when there is a restricted volume of 32,000 *cubic centimeters? Remember, this interval applies to x-values only. So if x is a really large number, the box can still have a volume of* 32,000 *cubic centimeters as long as y is a really small number!*

4. Use calculus to find the optimal solution:

To find the optimal solution, we need to find the critical values of the objective function, S, we found in step 2 and use an appropriate test to find the absolute minimum.

The critical values of S are the x-values where $S'(x) = 0$ or $S'(x)$ does not exist, and they are in the domain of S (i.e., they are in the interval we found in step 3). To find the critical values of S, we must find $S'(x)$ first. Before doing so, we will rewrite the function S to avoid using the Quotient Rule:

$$\begin{aligned} S(x) &= x^2 + \frac{128{,}000}{x} \\ &= x^2 + 128{,}000x^{-1} \end{aligned}$$

Finding the derivative and simplifying gives

$$\begin{aligned} S'(x) &= 2x - 128{,}000x^{-2} \\ &= 2x - \frac{128{,}000}{x^2} \end{aligned}$$

To find the critical values of S, we find the x-values where $S'(x) = 0$ or $S'(x)$ does not exist. First, let's find the x-values where $S'(x) = 0$:

$$\begin{aligned}
S'(x) &= 0 \implies \\
2x - \frac{128,000}{x^2} &= 0 \\
2x &= \frac{128,000}{x^2} \\
\left(x^2\right)(2x) &= \frac{128,000}{x^2}\left(x^2\right) \\
2x^3 &= 128,000 \\
x^3 &= 64,000 \\
x &= 40
\end{aligned}$$

Now, we find the x-values where $S'(x)$ does not exist. $S'(x) = 2x - \frac{128,000}{x^2}$ does not exist when $x^2 = 0$ (remember we are looking at the domain of S' when determining where it does not exist). Solving the equation $x^2 = 0$ for x gives $x = 0$. Hence, $S'(x)$ does not exist when $x = 0$.

N *Recall that, technically, $S'(x)$ also does not exist when $x < 0$ because we restricted the domain (i.e., interval) of S to be $(0, \infty)$. However, as discussed previously, we would not consider these to be "important" x-values when finding where $S'(x)$ does not exist. Furthermore, we are looking for the critical values of S, and these x-values would not be critical values because they are not in the restricted domain (i.e., interval) of S. Therefore, the only "important" x-value where $S'(x)$ does not exist is $x = 0$.*

Of the x-values $x = 40$ and $x = 0$, only $x = 40$ is in the interval $(0, \infty)$. Thus, it is the only critical value of S.

Because the interval $(0, \infty)$ is open, we cannot use the Closed Interval Method to find the absolute minimum. However, because the objective function only has one critical value in the interval and is twice differentiable on the interval, we can use either the First Derivative Test or the Second Derivative Test (because the critical value occurs when $S'(x) = 0$) to find the absolute extremum. Because we reviewed the First Derivative Test in the last example, we will use the Second Derivative Test in this example.

We already know the critical value, $x = 40$, so we can go straight to finding $S''(x)$ and evaluating it at the critical value. To avoid using the Quotient Rule, we will use the original form of $S'(x)$ to find $S''(x)$:

$$\begin{aligned}
S'(x) &= 2x - 128,000x^{-2} \implies \\
S''(x) &= 2 + 256,000x^{-3}
\end{aligned}$$

Because we only have to find the value of $S''(x)$ when $x = 40$ to determine if there is an absolute minimum, there is no need to continue simplifying the second derivative. We now find $S''(40)$:

$$\begin{aligned}
S''(x) &= 2 + 256,000x^{-3} \implies \\
S''(40) &= 2 + 256,000(40)^{-3} = 6 > 0
\end{aligned}$$

Using **Theorem 3.4**, we see that S is concave up at $x = 40$, so there is a local minimum at $x = 40$. Because $x = 40$ is the only critical value in the interval $(0, \infty)$ and S is continuous on the interval, this local minimum is also the absolute minimum of the objective function on the interval (which is good because our goal was to minimize the surface area!).

5. Reread the problem to be sure you answer the original question:

Looking back at the problem, we see that we need to find the dimensions that will minimize the surface area of the box. In other words, we need both x and y. We found that the absolute minimum occurs at $x = 40$ centimeters, and we can find y by substituting $x = 40$ into the constraint equation we solved for y in step 2:

$$\begin{aligned} y &= \frac{32,000}{x^2} \implies \\ y &= \frac{32,000}{(40)^2} \\ &= 20 \text{ centimeters} \end{aligned}$$

Therefore, the dimensions that minimize the surface area of the box (and, hence, the amount of material used to make the box) are 40 centimeters by 40 centimeters by 20 centimeters. Note that the length and width of the box are each 40 centimeters, and the height of the box is 20 centimeters.

■

Try It # 3:
Kirby has 1728 square inches of material, and he needs to make a box with a square base and an open top that has the largest volume possible. Find the dimensions and volume of his box.

In some cases, we may have a geometric optimization problem in which we are asked to minimize cost (so in a sense, this is a business application). However, the solution will follow the ideas we have established for solving a geometric optimization problem, as we will see in the next example.

■ **Example 6** Farmer Ben needs to enclose a rectangular pen that has a total area of 2560 square feet and a partition dividing the pen in order to separate his chickens and pigs (see **Figure 3.5.4**). The fencing to build the partition, which needs to be reinforced to keep the animals in their respective regions, costs \$24 per foot. The rest of the fencing needed to build the pen costs \$8 per foot. Find the length of the partition that will minimize Farmer Ben's cost of building the pen.

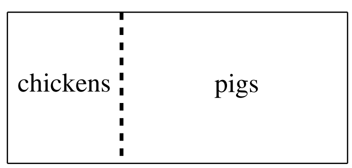


Figure 3.5.4: Rectangular pen with a partition

Solution:
This problem asks us to find the length of the partition that will *minimize* Farmer Ben's cost. We proceed with the optimization process:

1. Translate the word problem into mathematical form:

To translate the problem, we express the quantities mathematically we *want* (the objective function) and *know* (in this case, a constraint equation). Because we *want* the cost to be a minimum, the objective function will give the total cost of building the pen with the partition.

Although we are not explicitly given a constraint equation we *know*, we do *know* the total area of the pen is 2560 square feet. We will be able to use this known value to create a constraint equation.

Because we have a geometric optimization problem, we will draw a picture, define variables, and label the picture to help us create the objective function and the constraint equation. We will define x to be the length, in feet, of the partition and the sides parallel to the partition. We will let y be the length, in feet, of the remaining sides. See **Figure 3.5.5**.

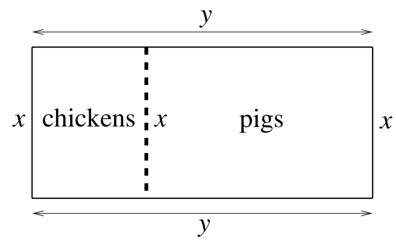


Figure 3.5.5: Rectangular pen with a partition and variables x and y labeled

The objective function will represent the cost of all the fencing needed to enclose the pen with the partition because we *want* to minimize cost. We must express the cost of each side of the pen and the partition using our variables and the cost per foot, and then we add the expressions to get the total cost. We will call the objective function C because it represents cost. Adding the cost of each side and then the partition gives

$$\begin{aligned} \textit{want:}\ \ C &= 8x + 8x + 8y + 8y + 24x \implies \\ C &= 40x + 16y \end{aligned}$$

We can also create the constraint equation using our picture. We know the total area of the pen is 2560 square feet, so we can create an equation we *know* representing the area. The area of the pen is equal to its width times its length:

$$\textit{know:}\ \ xy = 2560$$

2. State the objective function in terms of one variable:

Because the objective function has two variables, x and y, we must solve the constraint equation for one variable and substitute the result into the objective function. We can choose either x or y to solve for in the constraint equation, but we will solve for y:

$$\begin{aligned} xy &= 2560 \\ y &= \frac{2560}{x} \end{aligned}$$

Substituting y into the objective function we get

$$\begin{aligned} C &= 40x + 16y \\ &= 40x + 16\left(\frac{2560}{x}\right) \\ &= 40x + \frac{40,960}{x} \end{aligned}$$

Thus, the objective function in terms of one variable, x, is

$$C(x) = 40x + \frac{40,960}{x}$$

3. Find the interval on which the objective function must be optimized:

To find the interval, we must consider the context of the problem and determine a realistic domain for the objective function. Again, the dimensions of the pen must be positive. In this example, the dimensions are given by x and y, so we have

$$x > 0 \ \text{ and } \ y > 0$$

However, because the objective function is in terms of x, we need to substitute for y in the second inequality to ensure our restrictions lead us to an interval consisting of x-values. Recall $y = \dfrac{2560}{x}$ from step 2. Substituting for y in the second inequality gives

$$\begin{aligned} y > 0 &\implies \\ \frac{2560}{x} &> 0 \end{aligned}$$

Like the previous example, we cannot solve this inequality algebraically for x (if we multiply both sides by x, we lose the variable!). Thus, we have to simply observe this quotient and note that the numerator is positive and the denominator will be positive as long as $x > 0$. Therefore, the quotient itself will be positive when $x > 0$.

So the only we restriction we have is that $x > 0$.

Thus, we must maximize the objective function, $C(x) = 40x + \dfrac{40,960}{x}$, on the interval $(0, \infty)$. Note that the interval is open.

4. Use calculus to find the optimal solution:

To find the optimal solution, we need to find the critical values of the objective function, C, we found in step 2 and use an appropriate test to find the absolute maximum.

The critical values of C are the x-values where $C'(x) = 0$ or $C'(x)$ does not exist, and they are in the domain of C (i.e., they are in the interval we found in step 3). To find the critical values of C, we must find $C'(x)$ first. Before doing so, we will rewrite the function C in order to avoid using the Quotient Rule:

$$\begin{aligned} C(x) &= 40x + \frac{40,960}{x} \\ &= 40x + 40,960x^{-1} \end{aligned}$$

Finding the derivative and simplifying gives

$$\begin{aligned} C'(x) &= 40 - 40,960x^{-2} \\ &= 40 - \frac{40,960}{x^2} \end{aligned}$$

To find the critical values of C, we find the x-values where $C'(x) = 0$ or $C'(x)$ does not exist. First, let's find the x-values where $C'(x) = 0$:

$$\begin{aligned} C'(x) = 0 &\implies \\ 40 - \frac{40,960}{x^2} &= 0 \\ 40 &= \frac{40,960}{x^2} \\ \left(x^2\right)(40) &= \frac{40,960}{x^2}\left(x^2\right) \\ 40x^2 &= 40,960 \\ x^2 &= 1024 \\ \implies x = -32 &\text{ and } x = 32 \end{aligned}$$

Because the dimensions of the pen must be positive, we can disregard the negative x-value. Hence, $C'(x) = 0$ when $x = 32$.

Now, we find the x-values where $C'(x)$ does not exist. $C'(x) = 40 - \dfrac{40,960}{x^2}$ does not exist when $x^2 = 0$ (remember we are looking at the domain of C' when determining where it does not exist). Solving the equation $x^2 = 0$ for x gives $x = 0$. Hence, $C'(x)$ does not exist when $x = 0$.

Recall that, technically, $C'(x)$ also does not exist when $x < 0$ because we restricted the domain (i.e., interval) of C to be $(0, \infty)$. However, as discussed previously, we would not consider these to be "important" x-values when finding where $C'(x)$ does not exist. Furthermore, we are looking for the critical values of C, and these x-values would not be critical values because they are not in the restricted domain (i.e., interval) of C. Therefore, the only "important" x-value where $C'(x)$ does not exist is $x = 0$.

Of the x-values $x = 32$ and $x = 0$, only $x = 32$ is in the interval $(0, \infty)$. Thus, it is the only critical value of C.

Because the interval $(0, \infty)$ is open, we cannot use the Closed Interval Method to find the absolute minimum. However, because the objective function only has one critical value in the interval and is twice differentiable on the interval, we can again use either the First Derivative Test or the Second Derivative Test (because the critical value occurs when $C'(x) = 0$) to find the absolute extremum. We will use the First Derivative Test in this example.

Because we already know the critical value, $x = 32$, we can go straight to creating the sign chart of $C'(x)$ to determine the behavior of the objective function at the critical value.

We will place the critical value (also a partition number) on a number line, and indicate that it is in the interval with a solid dot. We will also place a dotted line at $x = 0$ on the number line so we remember not to select x-values to test that are outside of the interval $(0, \infty)$.

Now, we need to determine the sign of $C'(x)$ on the intervals $(0, 32)$ and $(32, \infty)$. We will choose the x-values $x = 20$ and $x = 40$ to test:

$$\begin{aligned} C'(x) &= 40 - \frac{40,960}{x^2} \implies \\ C'(20) &= 40 - \frac{40,960}{(20)^2} = -62.4 < 0 \\ C'(40) &= 40 - \frac{40,960}{(40)^2} = 14.4 > 0 \end{aligned}$$

Using this information, we can fill in the sign chart of $C'(x)$. Because we are also interested in the information this yields for C, we include that information below the number line. See **Figure 3.5.6**.

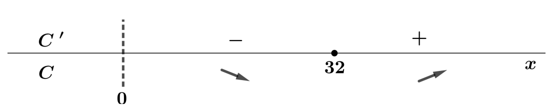


Figure 3.5.6: Sign chart of $C'(x)$ with the corresponding information for $C(x) = 40x + \dfrac{40,960}{x}$

Using **Theorem 3.2**, we see that a local minimum occurs at $x = 32$ (note that we also double check that C is defined at $x = 32$, which it is). Because $x = 32$ is the only critical value in the interval $(0, \infty)$ and C is continuous on the interval, we know this local minimum is also the absolute minimum. Now, we double check that we are in fact looking for an absolute minimum, which we are!

5. Reread the problem to be sure you answer the original question:

Looking back at the problem, we see that we need to find the length of the partition that will minimize Farmer Ben's cost. In other words, we only need the value of x. We found that the absolute minimum occurs at $x = 32$ feet, so the partition needs to be 32 feet long to minimize cost.

■

■ **Example 7** Zachary needs to construct a box whose length is twice its width and has a volume of 50 cubic feet. The material used to build the top and bottom of the box costs \$10 per square foot, and the material used to build the sides costs \$6 per square foot. Find Zachary's minimum cost of constructing the box.

Solution:
This problem asks us to *minimize* the cost of constructing the box, so we proceed with the optimization process:

1. Translate the word problem into mathematical form:

To translate the problem, we express the quantities mathematically we *want* (the objective function) and *know* (in this case, a constraint equation). Because we *want* the cost to be a minimum, the objective function will give the cost of constructing the box.

Although we are not explicitly given a constraint equation we *know*, we do *know* the volume of the the box is 50 cubic feet. We will be able to use this known value to create a constraint equation.

Because we have a geometric optimization problem, we will draw a picture, define variables, and label the picture to help us create the objective function and constraint equation. We will define x to be the width, in feet, of the box. This means the length, in feet, of the box will be $2x$ because the length is twice the width. We will let h be the height, in feet, of the box. See **Figure 3.5.7**.

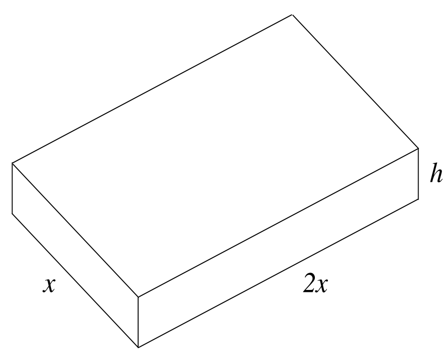


Figure 3.5.7: Box whose length is twice its width

The objective function must represent the cost of all the materials needed to construct the box because we *want* to minimize cost. We must express the cost of each side of the box using our variables and then add the expressions to get the total cost. Note that the cost of each side is given by its area, in square feet, multiplied by its respective cost per square foot.

We will call the objective function C because it represents cost. Adding the cost of materials to make each side of the box (starting with the top and bottom which cost \$10 per square foot), we get

$$\begin{aligned} \textit{want:}\quad & C = 10(x\cdot 2x)+10(x\cdot 2x)+6(x\cdot h)+6(2x\cdot h)+6(x\cdot h)+6(2x\cdot h) \implies \\ & C = 20x^2+20x^2+6xh+12xh+6xh+12xh \implies \\ & C = 40x^2+36xh \end{aligned}$$

We can also create the constraint equation using our picture. We know the volume of the box is 50 cubic feet, so we can create an equation we *know* representing the volume. The volume of the box is equal to its length times its width times its height:

$$\textit{know:}\ 2x \cdot x \cdot h = 50 \implies$$
$$2x^2 h = 50$$

2. State the objective function in terms of one variable:

Because the objective function has two variables, x and h, we must solve the constraint equation for one variable and substitute the result into the objective function. We can choose either x or h to solve for in the constraint equation, but we will solve for h because it is easier:

$$2x^2 h = 50 \implies$$
$$h = \frac{50}{2x^2} = \frac{25}{x^2}$$

Substituting h into the objective function gives

$$\begin{aligned} C &= 40x^2 + 36xh \\ &= 40x^2 + 36x\left(\frac{25}{x^2}\right) \\ &= 40x^2 + \frac{900}{x} \end{aligned}$$

Thus, the objective function in terms of one variable, x, is

$$C(x) = 40x^2 + \frac{900}{x}$$

3. Find the interval on which the objective function must be optimized:

To find the interval, we must consider the context of the problem and determine a realistic domain for the objective function. The dimensions of the box must be positive, or there will be no box! In this example, the dimensions are given by x, $2x$, and h, so we have

$$x > 0, \ \ 2x > 0, \ \text{ and } \ h > 0$$

Notice that solving the second inequality, $2x > 0$, for x gives $x > 0$, which we already have from the first inequality. However, because the objective function is in terms of x, we need to substitute for h in the third inequality to ensure our restrictions lead us to an interval consisting of x-values. Recall $h = \frac{25}{x^2}$ from step 2. Substituting for h in the second inequality gives

$$h > 0 \implies$$
$$\frac{25}{x^2} > 0$$

Like the previous examples, we cannot solve this inequality algebraically for x (if we multiply both sides by x^2, we lose the variable!). Thus, we have to simply observe this quotient and note that the numerator is positive and the denominator will always be positive as long as $x \neq 0$ because x is squared. Hence, this quotient will always be positive provided $x \neq 0$.

Thus, we have $x > 0$ and $x \neq 0$, which means the only restriction is $x > 0$.

Therefore, we must maximize the objective function, $C(x) = 40x^2 + \dfrac{900}{x}$, on the interval $(0, \infty)$. Note that the interval is open.

4. Use calculus to find the optimal solution:

To find the optimal solution, we need to find the critical values of the objective function, C, we found in step 2 and use an appropriate test to find the absolute maximum.

The critical values of C are the x-values where $C'(x) = 0$ or $C'(x)$ does not exist, and they are in the domain of C (i.e., they are in the interval we found in step 3). To find the critical values of C, we must find $C'(x)$ first. Before doing so, we will rewrite the function C in order to avoid using the Quotient Rule:

$$\begin{aligned} C(x) &= 40x^2 + \frac{900}{x} \\ &= 40x^2 + 1800x^{-1} \end{aligned}$$

Finding the derivative and simplifying gives

$$\begin{aligned} C'(x) &= 80x - 900x^{-2} \\ &= 80x - \frac{900}{x^2} \end{aligned}$$

To find the critical values of C, we find the x-values where $C'(x) = 0$ or $C'(x)$ does not exist. First, let's find the x-values where $C'(x) = 0$:

$$\begin{aligned} C'(x) &= 0 \implies \\ 80x - \frac{900}{x^2} &= 0 \\ 80x &= \frac{900}{x^2} \\ \left(x^2\right)80x &= \frac{900}{x^2}\left(x^2\right) \\ 80x^3 &= 900 \\ x^3 &= \frac{45}{4} \\ \implies x &= \sqrt[3]{\frac{45}{4}} \approx 2.2407 \end{aligned}$$

Now, we find the x-values where $C'(x)$ does not exist. $C'(x) = 80x - \dfrac{900}{x^2}$ does not exist when $x^2 = 0$ (remember we are looking at the domain of C' when determining where it does not exist). Solving the equation $x^2 = 0$ for x gives $x = 0$. Hence, $C'(x)$ does not exist when $x = 0$.

N *Recall that, technically, $C'(x)$ also does not exist when $x < 0$ because we restricted the domain (i.e., interval) of C to be $(0, \infty)$. However, as discussed previously, we would not consider these to be "important" x-values when finding where $C'(x)$ does not exist. Furthermore, we are looking for the critical values of C, and these x-values would not be critical values because they are not in the restricted domain (i.e., interval) of C. Therefore, the only "important" x-value where $C'(x)$ does not exist is $x = 0$.*

Of the x-values $x = \sqrt[3]{\frac{45}{4}}$ and $x = 0$, only $x = \sqrt[3]{\frac{45}{4}}$ is in the interval $(0, \infty)$. Thus, it is the only critical value of C.

Because the interval $(0, \infty)$ is open, we cannot use the Closed Interval Method to find the absolute minimum. However, because the objective function only has one critical value in the interval and is twice differentiable on the interval, we can again use either the First Derivative Test or the Second Derivative Test (because the critical value occurs when $C'(x) = 0$) to find the absolute extremum. We will use the Second Derivative Test in this example.

Because we already know the critical value, $x = \sqrt[3]{\frac{45}{4}}$, we can go straight to finding $C''(x)$ and evaluating it at the critical value. To avoid using the Quotient Rule, we will use the original form of $C'(x)$ to find $C''(x)$:

$$C'(x) = 80x - 900x^{-2} \implies$$
$$C''(x) = 80 + 1800x^{-3}$$

Because we only have to find the value of $C''(x)$ when $x = \sqrt[3]{\frac{45}{4}}$ to determine if there is an absolute minimum, there is no need to continue simplifying the second derivative. We now find $C''\left(\sqrt[3]{\frac{45}{4}}\right)$:

$$C''\left(\sqrt[3]{\frac{45}{4}}\right) = 80 + 1800\left(\sqrt[3]{\frac{45}{4}}\right)^{-3} = 240 > 0$$

Using **Theorem 3.4**, we see that C is concave up at $x = \sqrt[3]{\frac{45}{4}}$, so there is a local minimum at $x = \sqrt[3]{\frac{45}{4}}$. Because $x = \sqrt[3]{\frac{45}{4}}$ is the only critical value in the interval $(0, \infty)$ and C is continuous on the interval, this local minimum is also the absolute minimum of the objective function on the interval (which is good because our goal was to minimize the cost!).

5. Reread the problem to be sure you answer the original question:

Looking back at the problem, we see that we need to find the minimum cost of constructing the box. We know the absolute minimum occurs at $x = \sqrt[3]{\frac{45}{4}}$ feet, so we need to evaluate the cost (objective) function we found in step 2 at this x-value:

$$C\left(\sqrt[3]{\frac{45}{4}}\right) = 40\left(\sqrt[3]{\frac{45}{4}}\right)^2 + \frac{900}{\sqrt[3]{\frac{45}{4}}} \approx \$602.49$$

Thus, Zachary's minimum cost of constructing the box is \$602.49.

■

Try It # 4:
A manufacturer is going to make rectangular storage containers with open tops. The volume of each container must be 10 cubic meters, and the length of each container will be twice its width. The cost of the material for the four sides of the container is \$4 per square meter. The cost of the material for the bottom of the container is \$8 per square meter. Find the manufacturer's minimum cost of making each container.

Try It Answers

1. $x = 50$; $y = 50$
2. \$28.50
3. 24 inches by 24 inches by 12 inches; 6912 cubic inches
4. \$115.86

EXERCISES

Basic Skills Practice

1. Find two nonnegative numbers whose sum is 40 and their product is a maximum.

2. Find two numbers whose difference is 16 and their product is a minimum.

3. Find two positive numbers whose product is 30 and their sum is a minimum.

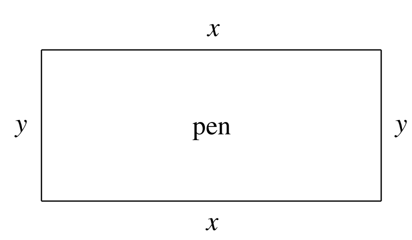
4. The revenue function for Star Bright Sneakers is given by $R(x) = 208x - 0.2x^2$ dollars, where x is the number of pairs of sneakers sold. How many pairs of sneakers must the company sell to maximize its revenue?

5. Reid's Pizzeria has a daily profit function given by $P(x) = -60 + 12x - \frac{1}{2}x^2$ dollars, where x is the number of pizzas sold. How many pizzas must her pizzeria sell to maximize profit?

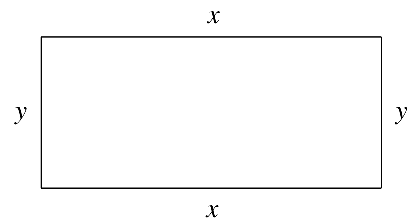
6. Cut That Grass! sells a popular lawn mower and has determined its revenue function is given by $R(x) = -.04x^2 + 350x$ dollars, where x is the number of lawn mowers sold.
   (a) How many lawn mowers must the company sell to maximize revenue?
   (b) What is the maximum revenue?

7. Shake It Up, an appliance company that sells high-end blenders, has a monthly profit function given by $P(x) = -0.3x^2 + 450x - 10,500$ dollars, where x is the number of blenders sold.
   (a) How many blenders must the company sell to maximize profit?
   (b) What is the maximum profit?

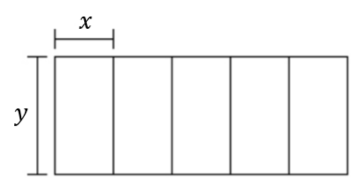
8. You have 400 ft of fencing to construct a rectangular pen for cattle. What are the dimensions of the pen that maximize its area?



9. You need to construct a fence around a rectangular area of 1600 square feet. What are the dimensions of the pen that will minimize the amount of material needed to construct the fence?

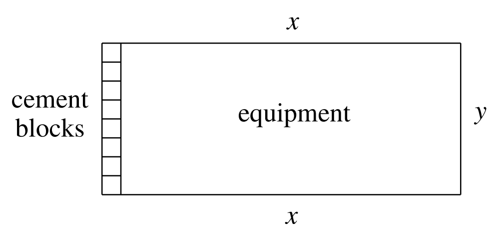


10. Roy Lee, a famed ostrich farmer, wants to enclose a rectangular area and then divide it into five pens with fencing parallel to one side of the rectangle as shown below. He will use 420 feet of fencing to complete the job.

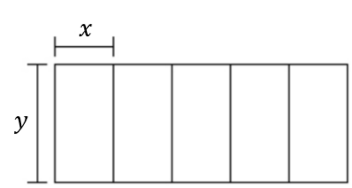


 (a) Find the dimensions of each pen that will maximize the total area of all five pens.
 (b) What is the largest possible total area of all five pens?

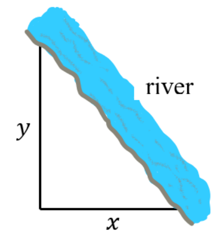
11. The manager of a garden store wants to build a 600 square foot rectangular enclosure on the store's parking lot in order to display some equipment. Three sides of the enclosure will be built of redwood fencing, at a cost of \$7 per foot. The fourth side will be built of cement blocks, at a cost of \$14 per foot. Find the dimensions of the enclosure that will minimize cost.



12. You are building five identical pens adjacent to each other such that they have a total area of 1000 m^2 as shown in the figure below. Find the dimensions of each pen that will minimize the amount of fencing needed. Round your answers to three decimal places, if necessary.



13. Bob needs to fence in a right-angled triangular region that will border a (mostly straight) river as shown below. The fencing for the left border costs \$9 per foot, and the lower border costs \$3 per foot. Bob doesn't need any fencing along the side of the river. He has \$630 to spend, and he wants as much area as possible.



    (a) Find the dimensions of the region that will provide the largest possible area. *Hint: The area of a triangle is* $\frac{1}{2} \cdot base \cdot height$.

    (b) What is the largest possible area of the region?

Intermediate Skills Practice

14. Find two nonnegative numbers x and y whose product is 250 and $2x + 5y$ is minimized.

15. Find two positive numbers x and y such that $x + y = 10$ and x^2y^2 is maximized.

16. Find two numbers x and y such that $x + y = 15$ and $x^2 - y$ is minimized.

17. Super Rad Bikez specializes in selling glow in the dark bikes for kids. It has determined its price-demand function to be $p(x) = 200 - 0.8x$, where x is the number of bikes sold at a price of \$$p$ each. Find the number of bikes the company must sell to maximize its revenue.

18. A curling iron manufacturer has revenue and cost functions, both in dollars, given by $R(x) = 35x - 0.1x^2$ and $C(x) = 4x + 2000$, respectively, where x is the number of curling irons made and sold.

    (a) How many curling irons must the company sell to maximize its profit?

    (b) What is the company's maximum profit?

19. The price-demand and cost functions, both in dollars, for a company that makes and sells cordless drills are given by $p(x) = 143 - 0.04x$ and $C(x) = 75,000 + 75x$, respectively, where x is the number of cordless drills made and sold.

    (a) Find the price per drill that will maximize profit.

    (b) Find the cost when profit is maximized.

20. When Mrs. Stewart, a theater owner, charges \$5 dollars per ticket, she sells 250 tickets. Through market analysis, the owner learns that for every one dollar she raises the price, she will lose 10 customers. What price should she charge for each ticket to maximize revenue?

21. The total cost, in dollars, for Marsha to make x oven mitts is given by $C(x) = 64 + 1.5x + 0.01x^2$.
    (a) Find a function that gives the average cost per oven mitt, $\overline{C}(x)$.
    (b) How many oven mitts should Marsha make to minimize her average cost?

22. Mrs. Barker has 400 feet of fencing to enclose a rectangular vegetable garden. What should the dimensions of her garden be in order to enclose the largest area?

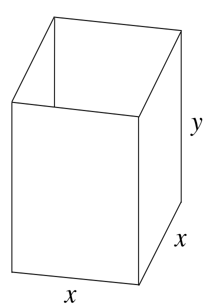
23. Ainzley has 600 feet of fencing available to construct a rectangular pen with a fence divider down the middle of the pen. What are the dimensions of the pen that will enclose the largest total area?

24. Brian has 800 ft of fencing to build a pen for his hogs. If he has a river on his property that he can use for one side of the pen, what is the maximum area of the pen?

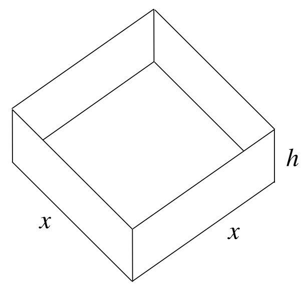
25. Mrs. Savage wants to enclose a rectangular garden. The area of the garden is 800 square feet. The fence along three sides costs \$6 per foot, and the fence along the fourth side, which needs to be reinforced to keep out her cows, costs \$18 per foot. Find the length of the reinforced fence that will minimize Mrs. Savage's cost of enclosing the garden.

26. Angus has \$240 dollars to buy fencing material to enclose a rectangular plot of land. The fencing for the north and south sides costs \$6 per foot, and the fencing for the east and west sides costs \$3 per foot.
    (a) Find the dimensions of the plot with the largest area.
    (b) For this largest plot, how much money will be used to build the east and west sides?

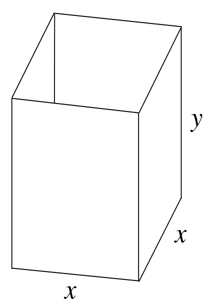
27. A box with a square base and an open top must have a volume of 216 in^3. Find the dimensions of the box so that the least amount of material is used to make the box. Round your answers to three decimal places, if necessary.



28. Coker is constructing a box for his cat to sleep in. The plush material for the square bottom of the box costs $5/ft^2$, and the material for the sides costs $2/ft^2$. Coker needs a box with a volume of 4 ft^3. Find the dimensions of the box that will minimize cost. Round your answers to three decimal places, if necessary.



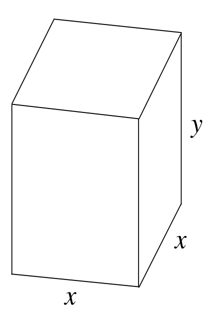
29. Kirby has 1875 cm^2 of material, and he needs to make a box with a square base and an open top that has the largest volume possible.



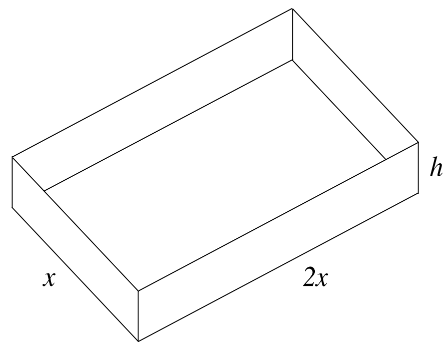
(a) Find the dimensions of his box.

(b) What is the volume of his box?

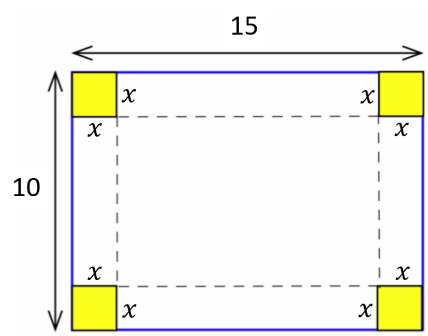
30. Fred is going to make rectangular storage containers. Each container will have a square base and a volume of 128 cubic feet. The cost of material for the top and four sides is $2 per square foot, and the cost of material for the bottom is $6 per square foot. Find the minimum cost for which Fred can make each storage container.



31. Sarah Jo needs to construct a box without a top whose length is twice its width and whose volume is 72 ft^3. The material used to build the bottom of the box costs \$8/ft^2, and the material used to build the sides costs \$4/ft^2. Find the dimensions of the box that will minimize Sarah Jo's cost of constructing the box.



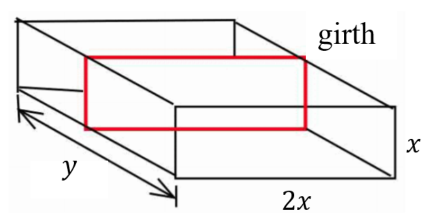
32. You need to form a 10-inch by 15-inch piece of tin into a box (without a top) by cutting a square from each corner of the tin and folding up the sides. Round your answers below to three decimal places, if necessary.



(a) Find the dimensions of the box with the largest volume.

(b) What is the largest possible volume?

33. U.S. postal regulations state that the sum of the length and girth (distance around) of a package must be no more than 108 inches. Find the dimensions of an acceptable box that has the largest volume possible if its end is a rectangle whose length is twice its width.



Mastery Practice

34. You must construct a box with a square base and an open top that will hold 100 cubic inches of water. Round your answers below to three decimal places, if necessary.
    (a) Find the dimensions of the box that will require using the least amount of material.
    (b) What is the least amount of material needed to make the box?

35. Suppose the box in the previous exercise (Exercise #35) is made of different materials for the bottom and the sides. If the bottom material costs \$0.05 per square inch and the side material costs \$0.03 per square inch, what is the cost of the least expensive box that will hold 100 cubic inches of water?

36. Find two nonnegative numbers whose sum is 36 and the sum of their squares is a minimum.

37. Suppose you decide to create a rectangular garden in the corner of your yard and enclose it with fencing. Two sides of the garden will be bounded by your existing fence, so you only need to enclose the other two sides. You have 80 feet of fencing. Find the maximum area of your new garden.

38. A rectangular box with a square bottom and closed top is to be made from two materials. The material for the sides costs \$1.50 per square foot, and the material for the top and bottom costs \$3.00 per square foot. If you are willing to spend \$15 on the box, what is the largest volume it can contain? Round your answer to three decimal places, if necessary.

39. A rectangle is inscribed with its base on the x-axis and its upper corners on the parabola $y = 3 - x^2$. What are the dimensions of such a rectangle with the greatest possible area?

40. Party Tune, a company that makes sing-a-long microphones for kids, has found that it can sell 500 microphones each month when the price of a microphone is \$86. For every one dollar decrease in price, Party Tune can sell 5 additional microphones each month.
    (a) Find the price Party Tune must charge per microphone to maximize its revenue.
    (b) What is the maximum revenue?

41. A rectangular storage container with an open top is to have a volume of 28 cubic meters. The length of its base is twice the width. Material for the base costs 10 dollars per square meter. Material for the sides costs 7 dollars per square meter. Find the cost of materials for the cheapest such container.

42. An open box is to be made from a 10-inch by 16-inch piece of cardboard by cutting out squares of equal size from the four corners and folding up the sides.
    (a) Find the length and width of each square so that the volume of the box is maximized.
    (b) What is the largest volume of the box?

43. Find two positive numbers x and y such that $x + y = 10$ and $y - \frac{1}{x}$ is a maximum.

44. The price-demand function for a certain flashlight is given by $p(x) = 15 - 0.001x$, where x is the number of flashlights sold at a price of \$$p$ each. The flashlight manufacturer has fixed costs of \$10,000 per month, and it costs \$3.00 to make each flashlight.

    (a) How many flashlights must be sold to maximize the manufacturer's profit?

    (b) What is the maximum profit?

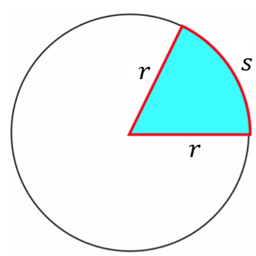
45. The top and bottom margins of a poster are 2 cm long, and the side margins are each 4 cm long. If the area of printed material on the poster is fixed at 384 cm^2, find the dimensions of the poster with the smallest area. Round your answers to three decimal places, if necessary.

46. A farmer wants to start raising cows, horses, goats, and sheep, and desires to have a rectangular pasture for the animals to graze in. However, no two different kinds of animals can graze together. In order to minimize the amount of fencing she will need, she has decided to enclose a large rectangular area and then divide it into four equally-sized pens by adding three segments of fencing inside the large rectangle that are parallel to two existing sides. She has decided to purchase 7500 ft of fencing to use. What is the maximum possible area that each of the four pens will enclose?

47. Vicki makes and sells backpack danglies. The cost, in dollars, for Vicki to make x danglies is given by $C(x) = 75 + 2x + 0.015x^2$. Find Vicki's minimum average cost for making danglies.

48. Heather needs to construct a box whose length is three times its width and whose volume is 50 cubic feet. The material used to build the top and bottom of the box costs \$10 per square foot, and the material used to build the sides costs \$6 per square foot. What is the minimum cost of the box?

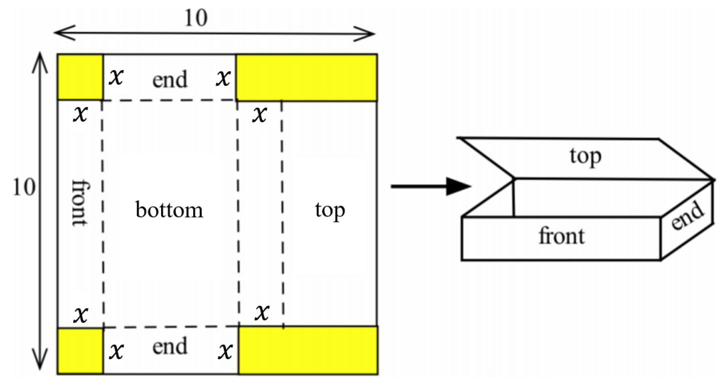
49. You have 100 feet of fencing to build a pen in the shape of a circular sector (see the "pie slice" shown below). The area of such a sector is given by $A = \frac{rs}{2}$, where r is the radius of the circle and s is the arc length of the sector. What value of r maximizes the enclosed area?



50. Find the positive integer that minimizes the sum of the number and its reciprocal.

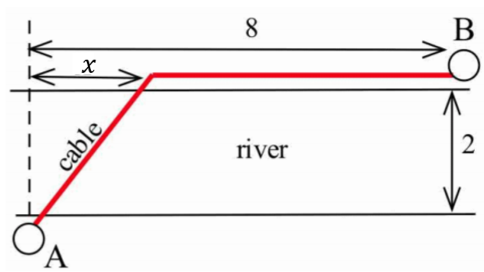
51. A small business that sells wind chimes can sell 100 wind chimes each month when the price per wind chime is \$25. For every dollar the price is raised, 10 fewer wind chimes will be sold. If the business has a cost function given by $C(x) = 9x + 1000$ dollars, where x is the number of wind chimes produced, find the revenue of the business when its profit is maximized.

52. Find the length of the cut, x, to make a box with the largest volume formed from a 10-inch by 10-inch piece of cardboard cut and folded as shown below.



53. You own a small airplane that holds a maximum of 20 passengers. It costs you \$100 per flight from St. Thomas to St. Croix for gas and wages plus an additional \$6 per passenger for the extra gas required by the extra weight. The charge per passenger is \$30 each if 10 people charter your plane (10 is the minimum number you will fly), and this charge is reduced by \$1 per passenger for each passenger over 10 who travels. What number of passengers on a flight will maximize your profit?

54. You have been asked to determine the least expensive route for a telephone cable that connects Allenville with Boontown. The towns are separated by a river that is 2 km wide, and the horizontal (land) distance between them is 8 km (see the figure below). If it costs \$5000 per mile to lay the cable on land and \$8000 per mile to lay the cable across the river (with the cost of the cable included), find the cost of the least expensive route. Round your answer to the nearest dollar.



55. Dr. Whitfield claims the "productivity levels" of people in various fields can be described as a function of their "career age" t by $p(t) = e^{-at} - e^{-bt}$, where a and b are constants depending on the field, and career age is approximately 20 less than the actual age of the individual. Based on this model, at what career ages do (a) mathematicians ($a = 0.03$, $b = 0.05$), (b) geologists ($a = 0.02$, $b = 0.04$), and (c) historians ($a = 0.02$, $b = 0.03$) reach their maximum productivity? Round your answers to the nearest year.

56. You have T dollars to buy fencing material to enclose a rectangular plot of land. The fencing for the north and south sides costs \$$A$ per foot, and the fencing for the east and west sides costs \$$B$ per foot. Find the dimensions of the plot with the largest area.

Communication Practice

57. Briefly describe the five steps of the optimization process.

58. When using the optimization process to solve an optimization problem involving geometric regions, you should translate the problem by first doing what?

59. Under what circumstance is a constraint equation required to solve an optimization problem?

60. After finding the interval on which to maximize or minimize an objective function in step 2 of the optimization process, describe the calculus techniques that can be used to solve the problem in step 3 as well as the condition(s) for using each.

61. If a continuous function has only one critical value, $x = 4$, and a local maximum at $x = 4$, explain why the function also has an absolute maximum at $x = 4$.

IV

Chapter 4

4. Antidifferentiation

A pool cover company makes custom fitted pool covers. If a pool is rectangular, the company can easily determine how much material is needed to make the cover by finding the area of the rectangle. This is also true for other pools with nice geometric shapes. However, what if a pool is shaped like a heart as shown in **Figure 4.0.1**?

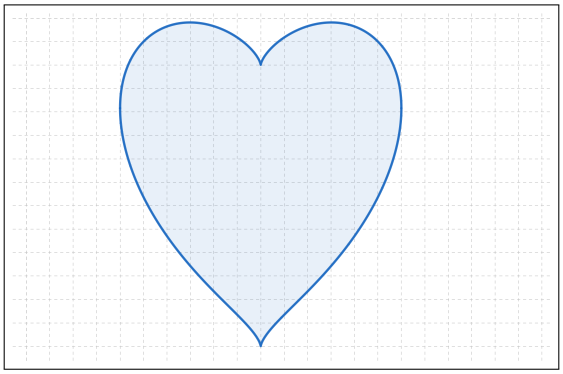


Figure 4.0.1: Heart-shaped pool

We cannot find the area of the shape above using formulas from previous geometry classes. We could estimate the area using the grid shown, but with the tools of calculus, we can find the *exact* area by describing it as the area between the graphs of the functions $f(x) = x^{2/3} + \sqrt{9 - x^2}$ and $g(x) = x^{2/3} - \sqrt{9 - x^2}$ on the interval $[-3, 3]$ as shown in **Figure 4.0.2**:

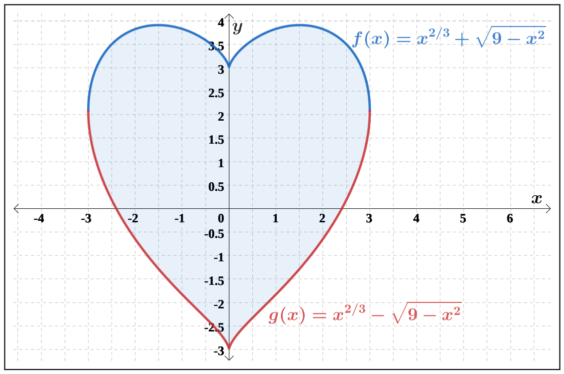


Figure 4.0.2: Heart-shaped region whose area is bounded by the graphs of f and g on the interval $[-3, 3]$

In this chapter, we will investigate two main concepts. First, we will explore **antidifferentiation**, the process of finding a function whose derivative we are given. Second, we will study definite integrals, which are used to find the areas of shapes like the heart shown above, the change in a quantity when its rate of change is known, the average value of a function, and many more real-world quantities. Finally, we will discuss the relationship between these two seemingly differing topics: **The Fundamental Theorem of Calculus**.

4.1 Antiderivatives: Introductory Rules

When a process can be done, mathematicians are generally interested in whether the process can be undone. For instance, we know that given a function, we can find its derivative. Now we ask the question, given a derivative, can we find the function it is the derivative of?

The function we take the derivative of and get the desired derivative is called an **antiderivative**. Let's consider the derivative $f'(x) = x^2$. Suppose we want to find the antiderivative of this function (i.e., we want to find the function we can take the derivative of and get x^2). We know that the process of taking the derivative decreases powers, so we might try $y = x^3$ as the antiderivative because it seems it would reverse the differentiation process by increasing the power of x.

Let's take the derivative of $y = x^3$ to see if it equals $f'(x)$:

$$y' = 3x^2 \neq f'(x)$$

Thus, $y = x^3$ is not an antiderivative of $f'(x) = x^2$ because its derivative, $3x^2$, does not equal x^2. It is close, but off by a multiple of 3. Because we know the property $\dfrac{d}{dx}(c \cdot f(x)) = c \cdot \dfrac{d}{dx}(f(x))$, we will now try $y = \frac{1}{3}x^3$:

$$y' = \frac{1}{3} \cdot 3x^2 = x^2 = f'(x)$$

The function $y = \frac{1}{3}x^3$ does have the derivative we are looking for! Now, consider the function $y = \frac{1}{3}x^3 + 5$. Taking this derivative we get

$$y' = \frac{1}{3} \cdot 3x^2 + 0 = x^2 = f'(x)$$

Both of these functions are antiderivatives of f'! It turns out there are infinitely many antiderivatives of f' because we could add any constant to $\frac{1}{3}x^3$ and still get x^2 when taking the derivative (remember the derivative of a constant is zero). Thus, we can write the antiderivative in the form $\frac{1}{3}x^3 + C$, where C can be any real number.

N *This number C represents shifting a function up or down by a certain number of units, and while this will change the tangent lines at points, it will not change their slopes!*

Learning Objectives:

In this section, you will learn how to use introductory antiderivative rules to calculate antiderivatives of functions and solve problems involving real-world applications. Upon completion you will be able to:

- Calculate an antiderivative of a constant function.
- Calculate an antiderivative of a power function.
- Calculate an antiderivative of a power function of the form $f(x) = x^{-1} = \dfrac{1}{x}$.
- Calculate an antiderivative of an exponential function (base e and base b).
- Calculate an antiderivative of a function involving sums, differences, and constant multiples of constant, power, and/or exponential functions.
- Calculate the most general antiderivative of a function using the Introductory Antiderivative Rules.
- Understand the mathematical notation used to represent antiderivatives as well as the relevant terminology.
- Calculate an indefinite integral using the Introductory Antiderivative Rules.
- Solve for the constant of integration of a specific antiderivative found using the Introductory Antiderivative Rules.

- Calculate a specific antiderivative using the Introductory Antiderivative Rules.
- Calculate a specific antiderivative using the Introductory Antiderivative Rules, and evaluate it at a particular x-value.
- Calculate a specific antiderivative of a rate of change function involving a real-world scenario, including marginal revenue, marginal cost, and marginal profit, using the Introductory Antiderivative Rules.
- Calculate a specific antiderivative of a rate of change function involving a real-world scenario, including marginal revenue, marginal cost, and marginal profit, using the Introductory Antiderivative Rules, and evaluate it at a particular x-value.

Before we continue, we will formally state the definition of an antiderivative:

Definition
F is an **antiderivative** of f if $F'(x) = f(x)$. ▪

Let's see this definition in action with an example.

▪ **Example 1** Verify that $F(x) = xe^x - e^x + 12$ is an antiderivative of $f(x) = xe^x$.

Solution:
To verify that F is an antiderivative of f, we will find its derivative and see if it equals $f(x)$:

$$\begin{aligned}\frac{d}{dx}(xe^x - e^x + 12) &= \frac{d}{dx}(xe^x) - e^x + 0 \\ &= x\left(\frac{d}{dx}(e^x)\right) + e^x\left(\frac{d}{dx}(x)\right) - e^x \\ &= xe^x + e^x(1) - e^x \\ &= xe^x\end{aligned}$$

Because $F'(x) = f(x)$, we know F is an antiderivative of f.

▪

General Antiderivatives

As discussed in the introduction, a function will have infinitely many antiderivatives. We add the constant C when finding an antiderivative to represent the family of functions which are all antiderivatives of the function. An antiderivative of this form is called the **most general antiderivative** because it represents all possible antiderivatives.

The mathematical notation used to represent antiderivatives as well as the relevant terminology are summarized below:

Definition

Given a function f, the **indefinite integral** of f, denoted

$$\int f(x)\,dx,$$

represents *all* antiderivatives of f. If F is an antiderivative of f, then

$$\int f(x)\,dx = F(x) + C,$$

where C is an arbitrary constant and $F(x)+C$ is the **most general antiderivative** of f.

The process of finding the most general antiderivative of a function is referred to as **integration**, and the symbol $\int$ is called an **integral sign**. The function f is called the **integrand**, and the expression dx indicates that we are integrating with respect to x. The constant C is called the **constant of integration**.

Using this integral notation, we can represent finding the antiderivative of x^2 in the introduction by writing

$$\int x^2\,dx = \frac{1}{3}x^3 + C,$$

where C is the constant of integration.

To find the most general antiderivative of a function, we need rules that will reverse the differentiation process. The following table shows the Introductory Antiderivative Rules we need for the scope of this textbook, along with the corresponding derivative rules for comparison:

Theorem 4.1 Introductory Antiderivative Rules

Constant: $\int k\,dx = kx + C$, where k is any real number	$\frac{d}{dx}(kx) = k$
Power: $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$, where n is any real number with $n \neq -1$	$\frac{d}{dx}(x^n) = nx^{n-1}$
Special Case: $\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$	$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$
Exponential: $\int b^x\,dx = \frac{1}{\ln(b)}b^x + C$, where b is any positive real number	$\frac{d}{dx}(b^x) = b^x\ln(b)$
Special Case: $\int e^x\,dx = e^x + C$	$\frac{d}{dx}(e^x) = e^x$
Sum/Difference: $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$	$\frac{d}{dx}(f(x) \pm g(x)) = \frac{d}{dx}(f(x)) \pm \frac{d}{dx}(g(x))$
Constant Multiple: $\int kf(x)\,dx = k\int f(x)\,dx$, where k is any real number	$\frac{d}{dx}(k \cdot f(x)) = k\left(\frac{d}{dx}(f(x))\right)$

This table is similar to the table in **Section 2.3**, but it gives rules for finding antiderivatives instead of derivatives. You may notice a small difference between the two tables: The antiderivative of $y = \frac{1}{x}$ is the natural logarithm of **the absolute value** of x (plus the constant of integration). This may seem like a problem at first, but it is actually a good thing: It ensures the function and its antiderivative have the same domain. Let's verify this antiderivative rule:

Example 2 Show that $f(x) = \ln\lvert x\rvert$ is an antiderivative of $f'(x) = \frac{1}{x}$.

Solution:
We start by rewriting the function f as a piecewise-defined function:

$$f(x) = \begin{cases} \ln(-x) & x < 0 \\ \ln(x) & x > 0 \end{cases}$$

To find $f'(x)$, we have to find $\frac{d}{dx}(\ln(-x))$ and $\frac{d}{dx}(\ln(x))$ for where they are defined. The latter is one of the Introductory Derivative Rules, so we know

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$$

Now, we just have to find $\frac{d}{dx}(\ln(-x))$. Using the Chain Rule, we have

$$\begin{aligned}\frac{d}{dx}(\ln(-x)) &= \frac{\frac{d}{dx}(-x)}{-x} \\ &= \frac{-1}{-x} \\ &= \frac{1}{x}\end{aligned}$$

Thus, we see the derivative of $\ln(-x)$ is also $\frac{1}{x}$.

Therefore, we can conclude that the derivative of $\ln|x|$ is in fact $\frac{1}{x}$, and this verifies

$$\int \frac{1}{x}\,dx = \ln|x| + C$$

■

N *Although the notation in our definition of antiderivative states that F is an antiderivative of f if $F'(x) = f(x)$, we can translate this definition using different notation. For instance, in the previous example, we were asked to verify f was an antiderivative of f'. This notation may actually be more intuitive because if we start with the function f', its antiderivative will be f.*

Now that we trust all of our antiderivative rules, let's get some practice using them!

■ **Example 3** Find the most general antiderivative of each of the following functions.

a. $f'(x) = x^{14}$

b. $g'(x) = 4x^{-1.8}$

c. $h'(x) = \frac{2}{x}$

d. $K'(x) = \frac{6}{x^4}$

e. $J'(x) = 3^x + 12$

Solution:

a. $f'(x) = x^{14}$ is a power function, and the power is *not* equal to -1. Thus, we can use the Power Rule for antiderivatives:

$$\begin{aligned}\int f'(x)\,dx &= \int x^{14}\,dx \\ &= \frac{1}{14+1}\cdot x^{14+1} + C \\ &= \frac{1}{15}x^{15} + C\end{aligned}$$

This tells us the most general antiderivative of $f'(x) = x^{14}$, which represents all its antiderivatives, is $f(x) = \frac{1}{15}x^{15} + C$.

Note that we can check our answer when finding an antiderivative of a function by finding the derivative (of our answer) and checking that the two functions are equivalent. For instance, to verify that $f(x) = \frac{1}{15}x^{15} + C$ is the most general antiderivative of $f'(x) = x^{14}$, we find its derivative:

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{15}x^{15} + C\right) &= 15 \cdot \frac{1}{15}x^{15-1} + 0 \\ &= x^{14}\end{aligned}$$

Thus, we know $f(x) = \frac{1}{15}x^{15} + C$ is the most general antiderivative of $f'(x) = x^{14}$.

When finding all antiderivatives, it is a common mistake to forget the constant of integration! Without it, you have only found ***one*** *antiderivative, not* ***all*** *antiderivatives!*

b. $g'(x) = 4x^{-1.8}$ is a power function, and the power is *not* equal to -1. So we can use the Power Rule for antiderivatives. First, we start with the Constant Multiple Rule:

$$\begin{aligned}\int g'(x)\,dx &= \int 4x^{-1.8}\,dx \\ &= 4\int x^{-1.8}\,dx \\ &= 4\left(\frac{1}{-1.8+1} \cdot x^{-1.8+1} + C\right) \\ &= \frac{4}{-0.8}x^{-0.8} + C \\ &= -5x^{-0.8} + C\end{aligned}$$

This tells us the most general antiderivative of $g'(x) = 4x^{-1.8}$, which represents all its antiderivatives, is $g(x) = -5x^{-0.8} + C$. Remember, you can always check your answer by finding its derivative!

You might be wondering why we do not write $4C$ for the constant of integration. C stands for an arbitrary constant. Multiplying an arbitrary constant by 4 leaves us with another arbitrary constant. It is not wrong to write $4C$ in this problem, it is just unnecessary.

c. Recall $h'(x) = \frac{2}{x}$. Typically, if we have a constant divided by a power function, we rewrite the function by bringing the power function to the numerator (we also used this algebraic technique when finding derivatives). However, if we rewrite the function, we get $y = \frac{2}{x} = 2x^{-1}$. Notice the power equals -1, so we cannot use the general Power Rule for antiderivatives. Why? If we add 1 to the power -1, we get 0!

Thus, this is the one situation where we *do not* want to rewrite the function, but leave the x in the denominator. Using the other Introductory Antiderivative Rules yields

$$\begin{aligned}\int h'(x)\,dx &= \int \frac{2}{x}\,dx \\ &= 2\int \frac{1}{x}\,dx \\ &= 2\ln|x| + C\end{aligned}$$

This tells us the most general antiderivative of $h'(x) = \frac{2}{x}$, which represents all its antiderivatives, is $h(x) = 2\ln|x| + C$. Remember, you can always check your answer by finding its derivative!

d. Recall $K'(x) = \frac{6}{x^4}$. Because this function consists of a constant divided by a power function and the power is equal to 4 (and not 1), we can rewrite the function by bringing x^4 to the numerator. Remember, the only time we do not want to bring the function to the numerator is if it will result in a power of -1 in the numerator. Bringing x^4 to the numerator allows us to use the Power Rule for antiderivatives. We must also incorporate the Constant Multiple Rule:

$$\begin{aligned}
\int K'(x)\,dx &= \int \frac{6}{x^4}\,dx \\
&= \int 6x^{-4}\,dx \\
&= 6\int x^{-4}\,dx \\
&= 6\left(\frac{1}{-4+1}\cdot x^{-4+1} + C\right) \\
&= 6\left(\frac{1}{-3}x^{-4+1} + C\right) \\
&= \frac{6}{-3}x^{-3} + C \\
&= -2x^{-3} + C
\end{aligned}$$

This tells us the most general antiderivative of $K'(x) = \frac{6}{x^4}$, which represents all its antiderivatives, is $K(x) = -2x^{-3} + C$. Remember, you can always check your answer by taking its derivative!

e. Recall $J'(x) = 3^x + 12$. We start by using the Sum Rule for antiderivatives:

$$\begin{aligned}
\int J'(x)\,dx &= \int (3^x + 12)\,dx \\
&= \int 3^x\,dx + \int 12\,dx \\
&= \frac{1}{\ln(3)}3^x + 12x + C
\end{aligned}$$

This tells us the most general antiderivative of $J'(x) = 3^x + 12$, which represents all its antiderivatives, is $J(x) = \frac{1}{\ln(3)}3^x + 12x + C$. Remember, you can always check your answer by finding its derivative!

*You may again be wondering why we do not have two constants of integration: one from the first integral and another from the second. Similar to part **b** where we did not need to write 4C, adding two arbitrary constants will result in another arbitrary constant. Again, it is not wrong to write two different constants; it is just unnecessary.*

■

Try It # 1:
Find the most general antiderivative of each of the following functions.

a. $f'(x) = \dfrac{3}{x^6}$

b. $g'(x) = -4 + 9^x$

c. $H'(x) = \dfrac{7}{x} + \dfrac{2}{5}$

▪ **Example 4** Find each of the following indefinite integrals.

a. $\displaystyle\int \left(3x^7 - 15\sqrt{x} + \frac{14}{x}\right) dx$

b. $\displaystyle\int \frac{x^2e^x + 12x^3 - 16}{x^2}\, dx$

c. $\displaystyle\int \left(8x^3 - \sqrt[5]{x}\right)\left(4 - x^9\right) dx$

Solution:

a. We start by rewriting each of the terms of the indefinite integral as power functions, except for the last term because we do not want to have a power equal to -1 in the numerator. Then, we use the Sum/Difference Rule and the Constant Multiple Rule:

$$\begin{aligned}\int \left(3x^7 - 15\sqrt{x} + \frac{14}{x}\right) dx &= \int \left(3x^7 - 15x^{\frac{1}{2}} + \frac{14}{x}\right) dx \\ &= \int 3x^7\, dx - \int 15x^{\frac{1}{2}}\, dx + \int \frac{14}{x}\, dx \\ &= 3\int x^7\, dx - 15\int x^{\frac{1}{2}}\, dx + 14\int \frac{1}{x}\, dx\end{aligned}$$

Now, we can use the Introductory Antiderivative Rules:

$$\begin{aligned}3\int x^7\, dx - 15\int x^{\frac{1}{2}}\, dx + 14\int \frac{1}{x}\, dx &= 3\left(\frac{1}{7+1}\cdot x^{7+1}\right) - 15\left(\frac{1}{\frac{1}{2}+1}\cdot x^{\frac{1}{2}+1}\right) + 14\ln|x| + C \\ &= 3\left(\frac{1}{8}x^8\right) - 15\left(\frac{1}{\frac{3}{2}}x^{\frac{3}{2}}\right) + 14\ln|x| + C \\ &= \frac{3}{8}x^8 - \frac{15}{\frac{3}{2}}x^{\frac{3}{2}} + 14\ln|x| + C \\ &= \frac{3}{8}x^8 - 15\left(\frac{2}{3}\right)x^{\frac{3}{2}} + 14\ln|x| + C \\ &= \frac{3}{8}x^8 - 10x^{\frac{3}{2}} + 14\ln|x| + C\end{aligned}$$

N *When using the Power Rule for antiderivatives, it may be easier to think of dividing by the new exponent as multiplying by its reciprocal, especially when the new exponenent is a fraction. For instance, in the solution of part* **a** *above, instead of dividing 15 by* $\frac{3}{2}$ *and writing* $\dfrac{15}{\frac{3}{2}}$*, we can think of this instead as multiplying by the reciprocal of the new exponent and simply write* $15\left(\dfrac{2}{3}\right)$ *when we first find the antiderivative.*

b. Recall we must find $\int \frac{x^2e^x + 12x^3 - 16}{x^2}\,dx$. Unlike derivatives, **there is no Product or Quotient Rule for antiderivatives**. If the integrand consists of a product or quotient of functions, we must apply other antidifferentiation techniques. Here, we will algebraically manipulate the function so we can use the Introductory Antiderivative Rules. However, there is another integration technique, called substitution, that we will learn in **Section 4.2** that may be used in some situations. There are other techniques as well, but they are beyond the scope of this textbook.

Because the function in the integrand has a single term in its denominator, x^2, we will divide each term in the numerator by x^2. Then, we can use laws of exponents to ensure each term is in the form required to find its antiderivative. Rewriting the integral in this way gives

$$\begin{aligned}\int \frac{x^2e^x + 12x^3 - 16}{x^2}\,dx &= \int \left(\frac{x^2e^x}{x^2} + \frac{12x^3}{x^2} - \frac{16}{x^2}\right)dx \\ &= \int \left(e^x + 12x - 16x^{-2}\right)dx\end{aligned}$$

Now, we will use the Sum/Difference Rule and the Constant Multiple Rule in one step. After that, we will apply the other Introductory Antiderivative Rules:

$$\begin{aligned}\int \left(e^x + 12x - 16x^{-2}\right)dx &= \int e^x\,dx + 12\int x\,dx - 16\int x^{-2}\,dx \\ &= e^x + 12\left(\frac{1}{1+1}\cdot x^{1+1}\right) - 16\left(\frac{1}{-2+1}\cdot x^{-2+1}\right) + C \\ &= e^x + 12\left(\frac{1}{2}x^2\right) - 16\left(\frac{1}{-1}x^{-1}\right) + C \\ &= e^x + 6x^2 + 16x^{-1} + C\end{aligned}$$

*We must be cautious here for two reasons. First, we are adding 1 to a negative number. In short, $-2+1=-1$, **not** -3. Second, one of the terms in the antiderivative has a power of -1. This occurs after finding the antiderivative. This is perfectly fine! We only have to be careful if we have a term with a power of -1 **in the integrand**.*

c. Recall we must find $\int \left(8x^3 - \sqrt[5]{x}\right)\left(4 - x^9\right)dx$. Due to the fact that there is no Product Rule for antiderivatives, we must use algebraic manipulation. We start by multiplying the integrand and then use laws of exponents. Rewriting the integral in this way gives

$$\begin{aligned}\int \left(8x^3 - \sqrt[5]{x}\right)\left(4 - x^9\right)dx &= \int \left(32x^3 - 8x^{12} - 4\sqrt[5]{x} + x^9\sqrt[5]{x}\right)dx \\ &= \int \left(32x^3 - 8x^{12} - 4x^{\frac{1}{5}} + x^9x^{\frac{1}{5}}\right)dx \\ &= \int \left(32x^3 - 8x^{12} - 4x^{\frac{1}{5}} + x^{9+\frac{1}{5}}\right)dx \\ &= \int \left(32x^3 - 8x^{12} - 4x^{\frac{1}{5}} + x^{\frac{46}{5}}\right)dx\end{aligned}$$

Next, we will use the Sum/Difference Rule, Constant Multiple Rule, and then Power Rule for indefinite integrals (i.e., antiderivatives) to find the answer:

$$\begin{aligned}
\int\left(32x^3-8x^{12}-4x^{\frac{1}{5}}+x^{\frac{46}{5}}\right)dx &= 32\int x^3\,dx-8\int x^{12}\,dx-4\int x^{\frac{1}{5}}\,dx+\int x^{\frac{46}{5}}\,dx\\
&= 32\left(\frac{1}{3+1}\cdot x^{3+1}\right)-8\left(\frac{1}{12+1}\cdot x^{12+1}\right)-4\left(\frac{1}{\frac{1}{5}+1}\cdot x^{\frac{1}{5}+1}\right)+\frac{1}{\frac{46}{5}+1}\cdot x^{\frac{46}{5}+1}+C\\
&= 32\left(\frac{1}{4}x^4\right)-8\left(\frac{1}{13}x^{13}\right)-4\left(\frac{1}{\frac{6}{5}}x^{\frac{6}{5}}\right)+\frac{1}{\frac{51}{5}}x^{\frac{51}{5}}+C\\
&= \frac{32}{4}x^4-\frac{8}{13}x^{13}-\frac{4}{\frac{6}{5}}x^{\frac{6}{5}}+\frac{5}{51}x^{\frac{51}{5}}+C\\
&= 8x^4-\frac{8}{13}x^{13}-4\left(\frac{5}{6}\right)x^{\frac{6}{5}}+\frac{5}{51}x^{\frac{51}{5}}+C\\
&= 8x^4-\frac{8}{13}x^{13}-\frac{10}{3}x^{\frac{6}{5}}+\frac{5}{51}x^{\frac{51}{5}}+C
\end{aligned}$$

▪

N *As we get more practice using the Introductory Antiderivative Rules and become more comfortable with them, we will condense some of the steps in our calculations.*

Try It # 2:
Find each of the following indefinite integrals.

a. $\int 8x^{32}\,dx$

b. $\int\left(e^x-13x^{-2.4}\right)dx$

c. $\int \frac{7^x\sqrt[3]{x}+x^{-2/3}}{x^{1/3}}\,dx$

d. $\int\left(x^{32}+14\right)\left(x^{34}-30\right)dx$

Specific Antiderivatives

While it is helpful to be able to find *all* antiderivatives of a function, which is given by the most general antiderivative, there are times when we need to find a **specific antiderivative**.

For instance, if we know the marginal cost function for a company, we may want to use it to find the cost function. However, finding the most general antiderivative gives infinitely many functions. We will need more information other than just the derivative to get one, specific antiderivative!

This information will come in the form of a point that the graph of the antiderivative must pass through. Knowing a specific point that the graph of the antiderivative passes through will allow us to solve for the constant of integration, C, and find a unique, or specific, antiderivative.

We can use the following steps to find a specific antiderivative:

Finding a Specific Antiderivative

If we are given the derivative of a function and a point the function passes through, we can find the corresponding specific antiderivative by doing the following:

1. Find the most general antiderivative using previous techniques.
2. Solve for the constant of integration, C, using the point we are given.
3. Substitute this value of C into the most general antiderivative.

▪ **Example 5** Find the constant of integration of $f(x)$ if $f'(x) = e^x + x^2 - 68$ and $f(0) = 22$.

Solution:
First, we find the most general antiderivative of f':

$$\begin{aligned} f(x) &= \int f'(x)\,dx \\ &= \int \left(e^x + x^2 - 68\right)\,dx \\ &= \int e^x\,dx + \int x^2\,dx \int -68\,dx \\ &= e^x + \frac{1}{3}x^3 - 68x + C \end{aligned}$$

Thus, $f(x) = e^x + \frac{1}{3}x^3 - 68x + C$ for some particular value of C. Because we know that $f(0) = 22$, we can substitute $x = 0$ into $f(x)$ and set the function equal to 22. This will allow us to solve for C:

$$\begin{aligned} f(x) = e^x + \frac{1}{3}x^3 - 68x + C &\implies \\ f(0) = e^0 + \frac{1}{3}(0)^3 - 68(0) + C = 22 &\implies \\ 1 + 0 - 0 + C &= 22 \\ C &= 21 \end{aligned}$$

Thus, the specific value of C for $f(x)$ is 21.

Note that if we were asked to find the specific antiderivative, we would simply substitute this value of C into the most general antiderivative. This would give us $f(x) = e^x + \frac{1}{3}x^3 - 68x + 21$.

▪

▪ **Example 6** Find $f(x)$ if $f'(x) = \dfrac{4x^3 + 19}{7x}$ and $f(6) = 11$.

Solution:
Again, we find the most general antiderivative of f' and then solve for C. First, we must find the most general antiderivative of f':

$$\begin{aligned} f(x) &= \int f'(x)\,dx \\ &= \int \frac{4x^3 + 19}{7x}\,dx \end{aligned}$$

Be careful! Before we can find the antiderivative, we need to algebraically manipulate the integrand. Remember, there is no Quotient Rule for antiderivatives!

Because there is only one term in the denominator of the integrand, $7x$, we will divide each term in the numerator by $7x$. Then, we can use laws of exponents to ensure the first term of the resulting sum is in the form required to find its antiderivative. We will leave the x in the denominator of the second term because we have a special Introductory Antiderivative Rule we use when finding the antiderivative of $\frac{1}{x}$. Therefore,

$$\begin{aligned} f(x) &= \int \frac{4x^3+19}{7x}\,dx \\ &= \int \left(\frac{4x^3}{7x}+\frac{19}{7x}\right)dx \\ &= \int \left(\frac{4}{7}x^2+\frac{19}{7}\left(\frac{1}{x}\right)\right)dx \end{aligned}$$

Next, we find the most general antiderivative:

$$\begin{aligned} \int \left(\frac{4}{7}x^2+\frac{19}{7}\left(\frac{1}{x}\right)\right)dx &= \frac{4}{7}\int x^2\,dx+\frac{19}{7}\int \frac{1}{x}\,dx \\ &= \frac{4}{7}\cdot\frac{1}{3}x^3+\frac{19}{7}\ln|x|+C \\ &= \frac{4}{21}x^3+\frac{19}{7}\ln|x|+C \end{aligned}$$

Now, we can solve for the constant of integration, C, using the point we are given, $f(6)=11$:

$$\begin{aligned} f(x) &= \frac{4}{21}x^3+\frac{19}{7}\ln|x|+C \implies \\ f(6) &= \frac{4}{21}\cdot 6^3+\frac{19}{7}\ln|6|+C = 11 \implies \\ & \frac{288}{7}+\frac{19}{7}\ln(6)+C = 11 \\ & C = -\frac{211}{7}-\frac{19}{7}\ln(6) \end{aligned}$$

Substituting this value for C into the most general antiderivative gives us the specific antiderivative:

$$f(x) = \frac{4}{21}x^3+\frac{19}{7}\ln|x|-\frac{211}{7}-\frac{19}{7}\ln(6)$$

We can check our work by substituting $x=6$ into $f(x)$ and verifying $f(6)=11$ and by taking the derivative of f to verify $f'(x)=\frac{4x^3+19}{7x}$.

■

■ **Example 7** Given that $f''(x)=2e^x-\frac{8}{x^3}$, $f'(2)=2e^2+3$, and $f(4)=2e^4+10$, find $f(x)$.

Solution:
We will start by finding $f'(x)$. We have its derivative, $f''(x)$, and a point its graph passes through, $\left(2,\ 2e^2+3\right)$. Therefore, to find $f'(x)$, we will first find the most general antiderivatives of f'':

$$\begin{aligned}
f'(x) &= \int f''(x)\,dx \\
&= \int \left(2e^x - \frac{8}{x^3}\right)\,dx \\
&= \int \left(2e^x - 8x^{-3}\right)\,dx \\
&= 2\int e^x\,dx - 8\int x^{-3}\,dx \\
&= 2e^x - 8\cdot\frac{1}{-2}x^{-2} + C \\
&= 2e^x + 4x^{-2} + C
\end{aligned}$$

Using the point $\left(2,\ 2e^2+3\right)$, we can find the specific form of $f'(x)$:

$$\begin{aligned}
f'(x) = 2e^x + 4x^{-2} + C &\implies \\
f'(2) = 2e^2 + 4(2)^{-2} + C = 2e^2 + 3 &\implies \\
2e^2 + 1 + C = 2e^2 + 3& \\
C = 2&
\end{aligned}$$

This tells us $f'(x) = 2e^x + 4x^{-2} + 2$.

Now that we have the function f' and we know that the graph of f passes through the point $\left(4,\ 2e^4+10\right)$, we can repeat this process to find $f(x)$. First, we find the most general antiderivative of f':

$$\begin{aligned}
f(x) &= \int f'(x)\,dx \\
&= \int \left(2e^x + 4x^{-2} + 2\right)\,dx \\
&= 2\int e^x\,dx + 4\int x^{-2}\,dx + \int 2\,dx \\
&= 2e^x + 4\cdot\frac{1}{-1}x^{-1} + 2x + C \\
&= 2e^x - 4x^{-1} + 2x + C
\end{aligned}$$

Next, we use the point $\left(4,\ 2e^4+10\right)$ to determine the value of C:

$$\begin{aligned}
f(x) = 2e^x - 4x^{-1} + 2x + C &\implies \\
f(4) = 2e^4 - 4(4)^{-1} + 2(4) + C = 2e^4 + 10 &\implies \\
2e^4 - 1 + 8 + C = 2e^4 + 10& \\
2e^4 + 7 + C = 2e^4 + 10& \\
C = 3&
\end{aligned}$$

Thus, $f(x) = 2e^x - 4x^{-1} + 2x + 3$.

■

Try It # 3:
Find $f(x)$ if

a. $f'(x) = 8x^3 - 12x^2 + 13$ and $f(-1) = 18$.

b. $f'(x) = 19e^x - 11x^5 + x^2$ and $f(0) = -30$.

c. $f''(x) = 19e^x - 11x^5 + x^2$, $f'(0) = -30$, and $f(0) = 30$.

Applications

Recall that a marginal function, such as marginal cost, marginal revenue, or marginal profit, is the derivative of a function. If we are given a marginal cost, marginal revenue, or marginal profit function, for example, as well as an amount for either producing or selling a particular number of items, we can find the original cost, revenue, or profit function.

Example 8 The marginal profit function for the sports themed breakfast restaurant Muffin But Net is given by $P'(x) = -0.08x + 6$ dollars per muffin, where x is number of muffins sold. If the restaurant's profit from selling 60 muffins is \$120, find the restaurant's profit function.

Solution:
To find the profit function, P, we start by finding the most general antiderivative of P'. Then, because we know $P(60) = 120$, we can find the specific profit function.

$$\begin{aligned} P(x) &= \int P'(x)\,dx \\ &= \int (-0.08x + 6)\;dx \\ &= -0.08\int x\,dx + \int 6\,dx \\ &= -0.08\cdot\frac{1}{2}x^2 + 6x + C \\ &= -0.04x^2 + 6x + C \end{aligned}$$

Now, we can use the point $(60, 120)$ to solve for C and determine the specific antiderivative:

$$\begin{aligned} P(x) = -0.04x^2 + 6x + C &\implies \\ P(60) = -0.04(60)^2 + 6(60) + C = 120 &\implies \\ 216 + C = 120 & \\ C = -96 & \end{aligned}$$

Thus, the profit function for Muffin But Net is $P(x) = -0.04x^2 + 6x - 96$ dollars when x muffins are sold.

∎

Example 9 The pasta food truck Al Dente on the Western Front has found its marginal revenue function to be $r(x) = 15(0.975)^x$ dollars per pasta bowl when x pasta bowls are sold.

a. Find the revenue function for Al Dente on the Western Front.
b. Determine the revenue earned when 50 pasta bowls are sold.

Solution:

a. At first glance, it may appear as though we are not given enough information to find the specific revenue function, which we will call R. We can find the most general antiderivative of r, but we are not explicitly given a point that the revenue function passes through. We need to remember, however, that if the food truck does not sell any pasta bowls, its revenue will be \$0. Thus, we know $R(0) = 0$ dollars.

To find $R(x)$, we will start by finding the most general antiderivative of r:

$$\begin{aligned} R(x) &= \int r(x)\,dx \\ &= \int 15(0.975)^x\,dx \\ &= 15\cdot\frac{1}{\ln(0.975)}(0.975)^x + C \\ &= 15\cdot\frac{(0.975)^x}{\ln(0.975)} + C \end{aligned}$$

Now, using the point $(0,0)$, we can solve for C:

$$\begin{aligned} R(x) &= 15\cdot\frac{(0.975)^x}{\ln(0.975)} + C \implies \\ R(0) &= 15\cdot\frac{(0.975)^0}{\ln(0.975)} + C = 0 \implies \\ 15\cdot\frac{1}{\ln(0.975)} &+ C = 0 \\ C &= -\frac{15}{\ln(0.975)} \end{aligned}$$

Thus, the revenue function for Al Dente on the Western Front is $R(x) = 15\cdot\frac{(0.975)^x}{\ln(0.975)} - \frac{15}{\ln(0.975)}$ dollars, where x is the number of pasta bowls sold.

b. To determine the revenue when 50 pasta bowls are sold, we calculate $R(50)$:

$$\begin{aligned} R(x) &= 15\cdot\frac{(0.975)^x}{\ln(0.975)} - \frac{15}{\ln(0.975)} \implies \\ R(50) &= 15\cdot\frac{(0.975)^{50}}{\ln(0.975)} - \frac{15}{\ln(0.975)} \\ &\approx \$425.40 \end{aligned}$$

Thus, the revenue from selling 50 pasta bowls is \$425.40.

■

■ **Example 10** The high end bakery Loaf of my Life specializes in producing large quantities of artisanal freshly baked bread loaves. The bakery's marginal revenue function is $R'(x) = 0.07x + 3.6$ and its marginal cost function is $C'(x) = 16.65 - 0.02x$, both in dollars per loaf, where x is number of bread loaves baked and sold. Also, the bakery has determined that baking and selling 300 bread loaves will result in a loss in profit of \$459. How much profit will the bakery earn if it bakes and sells 360 bread loaves?

Solution:
To find the profit the bakery earns from selling 360 loaves, we first need to find the profit function, P. Then, we can use the function to determine the profit from selling 360 loaves.

The profit function is equal to the revenue function minus the cost function or, in symbols, $P = R - C$. Thus, $P' = R' - C'$. Because we have the functions R' and C', we can subtract the two to obtain the marginal profit function, P'. Then, we can find the specific antiderivative of P', which will be the profit function, P.

We start by finding the marginal profit function, P':

$$\begin{aligned}P'(x) &= R'(x) - C'(x)\\ &= (0.07x + 3.6) - (16.65 - 0.02x)\\ &= 0.07x + 3.6 - 16.65 + 0.02x\\ &= 0.09x - 13.05\end{aligned}$$

Now that we have the function P', we can find the profit function, P. First, we find the most general antiderivative of P':

$$\begin{aligned}P(x) &= \int P'(x)\,dx\\ &= \int (0.09x - 13.05)\,dx\\ &= 0.09\int x\,dx - \int 13.05\,dx\\ &= 0.09\cdot\frac{1}{2}x^2 - 13.05x + C\\ &= 0.045x^2 - 13.05x + C\end{aligned}$$

Now, we solve for C using the point $(300, -459)$ to find the specific antiderivative:

$$\begin{aligned}P(x) = 0.045x^2 - 13.05x + C &\implies\\ P(300) = 0.045(300)^2 - 13.05(300) + C = -459 &\implies\\ 135 + C = -459&\\ C = -594&\end{aligned}$$

Therefore, the bakery's profit function is given by $P(x) = 0.045x^2 - 13.05x - 594$ dollars when x loaves are baked and sold.

Now, we need to find the bakery's profit when it bakes and sells 360 loaves. In other words, we need to calculate $P(360)$:

$$\begin{aligned}P(x) &= 0.045x^2 - 13.05x - 594 \implies\\ P(360) &= 0.045(360)^2 - 13.05(360) - 594\\ &= \$540\end{aligned}$$

Thus, the bakery's profit from selling 360 loaves is \$540.

▪

Try It # 4:
The pastry shop Beignet and the Jets has found its daily marginal cost function to be $C'(x) = 3 - 0.06x$ dollars per beignet when x beignets are made each day. If it costs the pastry shop \$65 each day to make 20 beignets, how much does it cost the shop to make 40 beignets each day?

Example 11 A car is traveling at a rate of 88 feet per second (or 60 miles per hour) when the brakes are applied. This causes the car to decelerate at a constant rate of 16 feet per second per second. How many seconds elapse before the car stops?

Solution:
When the car comes to a stop, its velocity will be 0 feet per second. Thus, we need to find the velocity function of the car, set it equal to 0 feet per second, and then solve for the time, t. This will tell us how many seconds elapse before the car stops.

To find the velocity function, v, we will first create an acceleration function, a, based on the information in the problem. Because an acceleration function is the derivative of the velocity function, we can find the antiderivative of a to get the velocity function, v. Also, because we know the car is initially traveling at a rate of 88 feet per second, we will be able to find the specific antiderivative of a.

If we let t be the number of seconds since the brakes are applied, we can write the acceleration function as $a(t) = -16$ feet per second per second because we are told the deceleration is constant. Now, to find the velocity function v, we start by finding the most general antiderivative of a:

$$\begin{aligned} v(t) &= \int a(t)\,dt \\ &= \int -16\,dt \\ &= -16t + C \end{aligned}$$

Now, we can find the specific antiderivative because we know the car is initially traveling at a rate of 88 feet per second. In other words, we know $v(0) = 88$ feet per second. This information allows us to solve for the constant of integration, C:

$$\begin{aligned} v(t) &= -16t + C \implies \\ v(0) &= -16(0) + C = 88 \implies \\ C &= 88 \end{aligned}$$

Thus, $v(t) = -16t + 88$ feet per second, where t is time, in seconds, since the brakes are applied.

Finally, we can find the time when the car stops by setting the velocity function equal to 0 and solving for t:

$$\begin{aligned} v(t) &= 0 \implies \\ -16t + 88 &= 0 \\ -16t &= -88 \\ t &= 5.5 \end{aligned}$$

The car will come to a complete stop 5.5 seconds after the brakes are applied.

Try It # 5:
Suppose the car in the previous example has a deceleration given by $a(t) = -12t$, where t is in seconds and $a(t)$ is in feet per second per second. How long will it take the car to come to a complete stop? Round your answer to two decimal places, if necessary.

Try It Answers

1. **a.** $f(x) = -\frac{3}{5}x^{-5} + C$

   b. $g(x) = -4x + \frac{1}{\ln(9)}9^x + C$

   c. $H(x) = 7\ln|x| + \frac{2}{5}x + C$

2. **a.** $\frac{8}{33}x^{33} + C$

   b. $e^x + \frac{13}{1.4}x^{-1.4} + C$

   c. $\frac{1}{\ln(7)}7^x + \ln|x| + C$

   d. $\frac{1}{67}x^{67} + \frac{2}{5}x^{35} - \frac{10}{11}x^{33} - 420x + C$

3. **a.** $f(x) = 2x^4 - 4x^3 + 13x + 25$

   b. $f(x) = 19e^x - \frac{11}{6}x^6 + \frac{1}{3}x^3 - 49$

   c. $f(x) = 19e^x - \frac{11}{42}x^7 + \frac{1}{12}x^4 - 49x + 11$

4. \$89

5. 3.83 seconds

EXERCISES

Basic Skills Practice

For Exercises 1 - 3, verify the indefinite integral by finding the derivative of the right-hand side of the equation. Remember that C is a constant.

1. $\int 19^x \, dx = \frac{1}{\ln(19)} 19^x + C$

2. $\int xe^x \, dx = e^x(x-1) + C$

3. $\int \frac{\ln(x)}{x^2} \, dx = -\frac{\ln(x)+1}{x} + C$

For Exercises 4 and 5, find the most general antiderivative of the function.

4. $f'(x) = x^7$

5. $g'(x) = e^x + x - 2$

For Exercises 6 - 8, find the indefinite integral.

6. $\int \left(\frac{1}{x} + \frac{1}{x^2}\right) dx$

7. $\int \left(x^{2/3} - 3x^4 + e^x\right) dx$

8. $\int \left(x^4 - \frac{3}{x} + x^{7/6} + 13\right) dx$

For Exercises 9 - 11, find the constant of integration, C, of $f(x)$ using the given information.

9. $f'(x) = 9x^3 + 3x^9$ and $f(0) = -2$

10. $f'(x) = 18t^{-1} + 4t + 24t^{-2}$ and $f(1) = 0$

11. $f'(x) = 37e^x + 17x^{-0.8}$ and $f(0) = -76$

For Exercises 12 - 14, find the specific antiderivative of the rate of change function using the given information.

12. $h'(x) = 3x^{0.2} + 3x^{-0.2}$ and $h(1) = 6$

13. $f'(t) = 2t^{-1} - \frac{1}{t^4}$ and $f(-1) = 0$

14. $g'(x) = x^3 + x^2 + x + 1$ and $g(0) = 1$

15. The weekly marginal cost function for Ash's poke bowl restaurant is given by $C'(x) = 90 - 0.1x$ dollars per poke bowl when x poke bowls are sold each week. The restaurant has fixed costs of \$450 each week. Find a function representing the restaurant's cost, in dollars, when x poke bowls are sold each week. *Hint: If the restaurant does not sell any poke bowls, its cost will equal its fixed costs.*

16. The marginal revenue function for a particular brand of graphing calculator is given by $R'(x) = 185 - 0.4x$ dollars per calculator when x calculators are sold. Find the revenue when 100 calculators are sold. *Hint: If the company does not sell any calculators, its revenue is* \$0.

17. The marginal profit function for a designer shoe label is given by $P'(x) = -0.6x + 475$ dollars per pair of shoes when x pairs of shoes are sold. If the profit from selling 800 pairs of shoes is \$177,500, find the profit function, in dollars, when x pairs of shoes are sold.

18. An object moves at a rate of $f'(t) = 0.09t^2 + 0.18t$ meters per second after t seconds have passed. The object is 59 meters to the right of its initial position after 10 seconds. Find a function that gives the position of the object, in meters, after t seconds.

Intermediate Skills Practice

For Exercises 19 - 21, find the indefinite integral.

19. $\int \left(18x^{-17/14} + x^{-4} - 10^x\right)\,dx$

20. $\int \left(-19e^x - 7x^{5/3} + 22\right)\,dx$

21. $\int \left(8x^{11} - 12x^{-1} - 5x\right)\,dx$

For Exercises 22 - 26, find the most general antiderivative of the function.

22. $f'(x) = x^2(x-1)$

23. $j'(x) = \dfrac{2x^2 + x - 9}{x}$

24. $g'(t) = (t-8)(t+12)$

25. $h'(x) = \dfrac{16x^7 - 20x^{18}}{x^6}$

26. $k'(t) = \dfrac{3(t+13)(t-16)}{t-16}$

For Exercises 27 - 29, find the constant of integration of $f(x)$ using the given information.

27. $f'(x) = 18x^8 - 20e^x - 3$ and $f(-1) = 1$

28. $f'(x) = -8\left(x^2 - 16\right)(x+1)$ and $f(3) = 1526$

29. $f'(x) = \dfrac{-3x^6 + 19x^{14} + 5x^{12}}{x^7}$ and $f(1) = \dfrac{101}{24}$

For Exercises 30 - 32, find the specific antiderivative of the rate of change function using the given information.

30. $f'(x) = x\left(x^2 - 1\right)(2x - 10)$ and $f(-1) = \dfrac{1}{3}$

31. $g'(u) = e^u\left(1 - u^3 e^{-u}\right)$ and $g(2) = e^2$

32. $R'(t) = t^{-5}\left(0.07t^3 + 5.93t^4 - 1.28t^9\right)$ and $R(1) = 0$

33. The marginal revenue function for a company that sells lawnmowers is given by $R'(x) = -.08x + 350$ dollars per lawnmower when x lawnmowers are sold. Find the company's revenue function, in dollars, when x lawnmowers are sold.

34. The marginal cost function for Mark Hall's Bulk Greeting Cards is given by $C'(x) = 3x^{-1/2}$ dollars per box when x boxes of greeting cards are sold. If the cost of making and shipping 25 boxes of cards is \$35, find the company's cost function, in dollars, when x boxes of greeting cards are sold.

35. Bike Tykes is a company that makes bikes for children. The company's weekly marginal profit function is given by $P'(x) = 30x - 0.3x^2 - 250$ dollars per bike when x bikes are sold each week. If the company's profit from selling 9 bikes each week is \$0 (i.e., the break-even quantity is 9 bikes), find the company's profit when 80 bikes are sold each week.

36. The velocity of a hummingbird flying in a straight line is given by $v(t) = 9t^2 - 9$ meters per second after t seconds have passed. If the hummingbird is 6 meters to the right of its initial position after 2 seconds have passed, find a function that gives the position of the bird, in meters, after t seconds.

Mastery Practice

37. Verify that $\displaystyle\int \left(\log_7(x) + 10x^4\right)\, dx = \frac{x\ln(x) - x}{(\ln 7)} + 2x^5 + C.$

For Exercises 38 - 41, find the most general antiderivative of the function.

38. $f'(u) = \left(\frac{9}{4}\right)^u - 12u^{-3/4}$

39. $H'(x) = \left(3x^{-2} + 6x^5\right)\left(-x^{-3} + 11x\right)$

40. $g'(t) = 7t^{-1} + \dfrac{2}{3}t^5 - t^{-10/3} + 2^5$

41. $F'(x) = \dfrac{5x^2(7^x) - x^5 + 26x}{x^2}$

For Exercises 42 - 45, find the indefinite integral.

42. $\displaystyle\int \frac{9 + 7e^{-t}}{e^{-t}}\, dt$

43. $\displaystyle\int \left(-39x^{20} + 49x^{1/3} - 19\right) dx$

44. $\displaystyle\int \frac{-8u^{-10} + 18u^{17}}{u^5}\, du$

45. $\displaystyle\int \left(9^x(8 \cdot 9^{-x} - 43) + \frac{8}{5}x^3\right) dx$

46. Find the constant of integration of $f(x)$ if $f'(x) = x^2\left(6 + 2x^{-2}e^x\right)$ and $f(1) = 2 + 2e$.

47. Find the constant of integration of y if $\dfrac{dy}{dx} = -x^2 + e^x + \dfrac{1}{x}$ and $y(1) = 0$.

For Exercises 48 - 50, find the specific antiderivative of the rate of change function using the given information.

48. $f'(x) = x^{1.8} + x^{-1.8}$ and $f(1) = 0$

49. $g'(t) = \left(9 - \sqrt{t}\right)\left(6 + t^{5/2}\right)$ and $g(0) = -18$

50. $R'(x) = \dfrac{20 + 3 \cdot 5^{-x}}{5^{-x}}$ and $R(0) = \dfrac{20}{\ln(5)}$

51. Find $f(x)$ if $f''(x) = 6x + 12$, $f'(-2) = 0$, and $f(1) = 20$.

52. A hair dryer manufacturer has marginal revenue and marginal cost functions, both in dollars per hair dryer, given by $R'(x) = 38 - 0.2x$ and $C'(x) = 5$, respectively, where x is the number of hair dryers made and sold. If the fixed costs for the manufacturer are \$2000, find

    (a) the cost function.

    (b) the revenue function.

    (c) the profit function.

53. A toddler is squeezing a tube of toothpaste. The rate at which toothpaste leaves the tube is given by $r(t) = -0.8t^3$ ounces per second after t seconds. If the tube originally contains 8 ounces of toothpaste, how long will it take for the tube to be empty? Round your answer to three decimal places, if necessary.

54. A company has a marginal profit function given by $P'(x) = -\frac{3\sqrt{x}}{2} + 23$ dollars per item when x items are sold. If the company breaks even when 100 items are sold, find the company's profit when 145 items are sold.

55. The rate of change of a company's total sales, in millions of dollars per month, after t months is given by $s(t) = \frac{1}{3\sqrt[3]{t}} + \frac{0.1}{\sqrt{t}} + 6$. After 8 months, the company earned 10 million dollars in sales. Find a function representing the company's total sales after t months.

56. The velocity of an object moving along a horizontal line is given by $v(t) = 3t^2 - 6t - 5$ feet per second, where t is time in seconds. Initially, the object is 6 feet to the right of its point of origin. Find the position of the object after 2 seconds.

Communication Practice

57. If F and G are antiderivatives of the same function, does $F(x) = G(x)$? Explain.

58. How can we check our answer after finding an antiderivative?

59. Given $\int f(x)\,dx = F(x) + C$ explain the terminology used for each of the following:

    (a) $\int$

    (b) $f(x)$

    (c) $F(x)$

    (d) C

60. Does $\int \left(4x^3 - 3x^2 + 5\right)\,dx = x^4 - 6x + 5x + C$? Explain.

61. Under what circumstance(s) are we able to find a specific antiderivative?

62. Explain why $\int \frac{6x^3 + 4x^6 - 5x^2}{3x^4}\,dx \neq \frac{2}{x} + \frac{4x^2}{3} - \frac{5}{3x^2}$.

63. Does $\int \left(x^2 + 5x^{-3}\right)\left(7x^{0.4} - 5\right)\,dx = \left(\frac{1}{3}x^3 - \frac{5}{2}x^{-2}\right)\left(5x^{1.4} - 5x\right)$? Explain.

4.2 Antiderivatives: Substitution

In the last section, we learned introductory techniques for finding antiderivatives. In this section, we will learn a more advanced technique to find the antiderivative of a function that is a composition of functions.

To develop the technique, let's consider the indefinite integral $\int 200x\left(x^2+9\right)^{99}\,dx$. So far, the only technique we have to find the antiderivative of the integrand is multiplying x^2+9 by itself 99 times and then multiplying by $200x$. After that, we would be able to use the Introductory Antiderivative Rules from **Section 4.1** to find the antiderivative. This process would take an incredibly long time!

We claim that $F(x) = \left(x^2+9\right)^{100}$ is actually an antiderivative of $f(x) = 200x(x^2+9)^{99}$. Let's check:

$$\begin{aligned}\frac{d}{dx}\left(\left(x^2+9\right)^{100}\right) &= 100\left(x^2+9\right)^{99}\cdot\frac{d}{dx}\left(x^2+9\right)\\ &= 100\left(x^2+9\right)^{99}\cdot(2x)\\ &= 200x\left(x^2+9\right)^{99}\end{aligned}$$

This works! Thus, we know the most general antiderivative of f is given by $\int 200x\left(x^2+9\right)^{99}\,dx = \left(x^2+9\right)^{100}+C$. This result suggests the existence of a "reverse Chain Rule" for finding antiderivatives. The process of reversing the Chain Rule to find an antiderivative is called **substitution** or ***u*-substitution**.

N *The term u-substitution comes from the substitution variable traditionally being called u. There is no specific need to call the variable u, so if you prefer to call it something else, feel free to do so. In this textbook, we will use the historic variable.*

Learning Objectives:

In this section, you will learn how to use the method of substitution to calculate antiderivatives of functions and solve problems involving real-world applications. Upon completion you will be able to:

- Determine the appropriate u-substitution that should be applied when using the method of substitution.
- Find the new indefinite integral in terms of u and du that results after applying the appropriate u-substitution when using the method of substitution.
- Calculate the most general antiderivative of a function using the method of substitution.
- Calculate an indefinite integral using the method of substitution.
- Solve for the constant of integration of a specific antiderivative found using the method of substitution.
- Calculate a specific antiderivative using the method of substitution.
- Calculate a specific antiderivative using the method of substitution, and evaluate it at a particular x-value.
- Calculate a specific antiderivative of a rate of change function involving a real-world scenario, including marginal revenue, marginal cost, and marginal profit, using the method of substitution.
- Calculate a specific antiderivative of a rate of change function involving a real-world scenario, including marginal revenue, marginal cost, and marginal profit, using the method of substitution, and evaluate it at a particular x-value.

The Substitution Method

As discussed previously, the method of substitution works by reversing the Chain Rule. Recall that the Chain Rule enables us to find the derivative of a function that is a composition of functions:

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

Reversing the Chain Rule means

$$\int f'(g(x)) \cdot g'(x)\,dx = f(g(x)) + C$$

The idea behind substitution is that if part of the integrand contains a composite function that was found as a result of using the Chain Rule, we identify the *inside* function of the composite function and its corresponding derivative and perform a substitution so we can obtain a new integral in which we can use the Introductory Antiderivative Rules to find the antiderivative. In other words, we identify what we referred to as $g(x)$ and $g'(x)$ when learning the Chain Rule so we can obtain an integral of the form

$$\int f'(u)\,du$$

where $u = g(x)$ and $du = g'(x)\,dx$.

> **N** *When we write $du = g'(x)\,dx$, we are treating du and dx as the change in u and x, respectively. du and dx are called* **differentials***. Any further discussion about the differentials du and dx is beyond the scope of this textbook, so when finding the derivative of $u = g(x)$, we will simply write $du = g'(x)\,dx$.*

Ideally, $\int f'(u)\,du$ will be an indefinite integral we can solve using the Introductory Antiderivative Rules in **Section 4.1**. The result will be an antiderivative function with the variable u, so we will have to substitute back for u to get the antiderivative in terms of the original variable. We now summarize this process formally:

The Substitution Method

To find the antiderivative of a function using the method of substitution, follow these five steps:

1. Look carefully at the integrand, and select $u = g(x)$ so that $g'(x)$ is also part of the integrand (or a multiple of it). Remember when selecting $u = g(x)$ that $g(x)$ is the *inside* function of the composite function in the integrand.
2. Take the derivative of $u = g(x)$ to obtain $du = g'(x)\,dx$.
3. Rewrite the integral in terms of u and du only.

   > **N** *If you cannot fully rewrite the integral without the original variable, then, in most cases, the choice of u will not work! In this case, you must return to step 1 and try selecting a different u.*

4. Find $\int f'(u)\,du$ using the Introductory Antiderivative Rules.

   > **N** *We may need to assess our choice of u at this step: If we cannot find an antiderivative of $f'(u)$ using the Introductory Antiderivative Rules, then our choice of u will not work! We must return to step 1 and choose a different part of the integrand for u.*

5. Substitute $u = g(x)$ so the antiderivative is in terms of the original variable, x.

It is very important to note that there may be more than one choice for u that will allow us to find the antiderivative of a function using the method of substitution. In other words, we may select different quantities for u and still be able to rewrite the integrand in terms of u and du for both. Although the new integrands may not look the same after substituting u and du, their antiderivatives will be mathematically equivalent (assuming antiderivatives can be found).

How do we know if we selected the correct u? As stated previously, if we can rewrite the given integral entirely in terms of u and du and find the resulting antiderivative using the Introductory Antiderivative Rules, then we have chosen a substitution that will work. If not, we must go back and select a different u.

Let's get some practice using the method of substitution!

■ **Example 1** Find each of the following indefinite integrals using the method of substitution.

a. $\displaystyle\int 200x(x^2+9)^{99}\,dx$

b. $\displaystyle\int x^2e^{4x^3+5}\,dx$

c. $\displaystyle\int \frac{7x^3}{6x^4-15}\,dx$

Solution:

a. As discussed previously, to find $\int 200x(x^2+9)^{99}\,dx$, we could algebraically manipulate the function and then use the Introductory Antiderivative Rules. However, doing so would be computationally intense and take a long time.

Looking at the integrand carefully, we see that part of it can be written as a composite function, say $y=\left(x^2+9\right)^{99}$, where the *inside* function is $g(x)=x^2+9$ and the *outside* function can be written as $y=x^{99}$. Note that you can compose the two function to check the composition. Thus, we will attempt to use the method of substitution to find the antiderivative:

1. As discussed previously, the integrand contains a composite function, so we consider letting u equal the *inside* function $g(x)$, where $g(x)=x^2+9$.

   Remember, for the method of substitution to work, we also need the derivative of the *inside* function, $g'(x)$, to appear in the integrand (or a multiple of it). If we let $g(x)=x^2+9$, then $g'(x)=2x$. Although $g'(x)=2x$ is not explicitly given in the integrand, there is a multiple of it: $200x$. As long as the degree of the variable is correct, we can use algebraic manipulation to adjust for the constant (we will do that in step 3).

   Thus, it seems we should let $u=x^2+9$.

2. Taking the derivative of $u=x^2+9$ gives $du=2x\,dx$, which is part of the integrand; it only differs by a constant.

3. We need to rewrite the indefinite integral in terms of u and du. To do this, we substitute x^2+9 in the integrand with u and use the fact that $du=2x\,dx$ to help us rewrite the rest of the integrand. Looking at the integrand again, we can see fairly easily where we will substitute u. To help us see the terms that will remain after we substitute x^2+9 with u, let's rearrange the terms in the integrand, but leave the constant in front:

$$\int 200x\left(x^2+9\right)^{99}\,dx = \int 200\cdot\left(x^2+9\right)^{99}\cdot x\,dx$$

Thus, in order to write the integral completely in terms of u and du, we also need to substitute for $x\,dx$ (the constant 200 is okay; we just need the integral to be completely in terms of u and du).

Next, we need to find an expression to substitute for $x\,dx$ in the integrand. Remember, the constant 200 is okay, and $\left(x^2+9\right)^{99}$ will become u^{99}. Given $du = 2x\,dx$, we can divide both sides of this equation by 2, or multiply both sides by $\frac{1}{2}$, and then we will have an expression that is equivalent to $x\,dx$ that we can substitute:

$$\begin{aligned} du &= 2x\,dx \implies \\ \frac{1}{2}\,du &= \frac{1}{2}\cdot 2x\,dx \\ \frac{1}{2}\,du &= x\,dx, \text{ or } x\,dx = \frac{1}{2}\,du \end{aligned}$$

Now, we are ready to formally rewrite the integral. As discussed previously, we leave the constant in front, substitute x^2+9 with u, and substitute $x\,dx$ with $\frac{1}{2}\,du$:

$$\int 200\cdot\left(x^2+9\right)^{99}\cdot x\,dx = \int 200\cdot u^{99}\cdot\frac{1}{2}\,du$$

Moving the constant $\frac{1}{2}$ to the front and simplifying so the function is "ready" to integrate in step 4 gives

$$\begin{aligned} \int 200\cdot u^{99}\cdot\frac{1}{2}du &= \int 200\cdot\frac{1}{2}u^{99}\,du \\ &= \int 100u^{99}\,du \end{aligned}$$

So far, it looks as though we chose a correct substitution (i.e., u) because the integral is written completely in terms of u and du!

4. Next, we compute the indefinite integral we found at the end of step 3 using the Introductory Antiderivative Rules from **Section 4.1**:

$$\begin{aligned} \int 100u^{99}\,du &= 100\cdot\frac{1}{100}u^{100}+C \\ &= u^{100}+C \end{aligned}$$

   Because we were able to compute this integral, we know $u = x^2+9$ is a correct substitution!

5. Finally, we rewrite the most general antiderivative we found at the end of step 4 in terms of x by substituting $u = x^2+9$:

$$u^{100}+C = \left(x^2+9\right)^{100}+C$$

Therefore,

$$\int 200x(x^2+9)^{99}\,dx = \left(x^2+9\right)^{99}+C$$

Remember, you can always check that your antiderivative is correct by finding its derivative!

b. Recall that we must find $\int x^2 e^{4x^3+5}\,dx$. Unlike the integrand in part **a**, there is no possible way to algebraically manipulate this integrand in order to use the Introductory Antiderivative Rules.

Looking at the integrand carefully, we see that part of it can be written as a composite function, say $y = e^{4x^3+5}$, where the *inside* function is $g(x) = 4x^3 + 5$ and the *outside* function can be written as $y = e^x$. Note that you can compose the two function to check the composition. Thus, we will attempt to use the method of substitution to find the antiderivative:

1. As discussed previously, the integrand contains a composite function, so we consider letting u equal the *inside* function $g(x)$, where $g(x) = 4x^3 + 5$. Now, we need to see if $g'(x) = 12x^2$ is in the integrand. Again, although this derivative is not explicitly given in the integrand, there is a multiple of it: $1x^2$.

   Thus, it seems we should let $u = 4x^3 + 5$.

2. Taking the derivative of $u = 4x^3 + 5$ gives $du = 12x^2\,dx$, which is part of the integrand; it only differs by a constant.

3. We need to rewrite the indefinite integral in terms of u and du. To do this, we substitute $4x^3 + 5$ in the integrand with u and use the fact that $du = 12x^2\,dx$ to help us rewrite the rest of the integrand. Looking at the integrand again, we can see fairly easily where we will substitute u. To help us see the terms that will remain after we substitute $4x^3 + 5$ with u, let's rearrange the terms in the integrand:

$$\int x^2 e^{4x^3+5}\,dx = \int e^{4x^3+5} \cdot x^2\,dx$$

   Thus, in order to write the integral completely in terms of u and du, we also need to substitute for $x^2\,dx$.

   Next, we need to find an expression to substitute for $x^2\,dx$ in the integrand. Remember, e^{4x^3+5} will become e^u. Given $du = 12x^2\,dx$, we can divide both sides of this equation by 12, or multiply both sides by $\frac{1}{12}$, and then we will have quantity that is equivalent to $x^2\,dx$ that we can substitute:

$$\begin{aligned} du &= 12x^2\,dx \implies \\ \frac{1}{12}\,du &= \frac{1}{12}\cdot 12x^2\,dx \\ \frac{1}{12}\,du &= x^2\,dx, \text{ or } x^2\,dx = \frac{1}{12}\,du \end{aligned}$$

   Now, we are ready to formally rewrite the integral. We substitute $4x^3 + 5$ with u, and we substitute $x^2\,dx$ with $\frac{1}{12}\,du$:

$$\int e^{4x^3+5} \cdot x^2\,dx = \int e^u \cdot \frac{1}{12}\,du$$

   Moving the constant $\frac{1}{12}$ to the front so the function is "ready" to integrate in step 4 gives

$$\int e^u \cdot \frac{1}{12} du = \int \frac{1}{12} e^u\,du$$

   So far, it looks as though we chose a correct substitution (i.e., u) because the integral is written completely in terms of u and du!

4. Next, we compute the indefinite integral we found at the end of step 3 using the Introductory Antiderivative Rules from **Section 4.1**:

$$\int \frac{1}{12}e^u\, du = \frac{1}{12}e^u + C$$

   Because we were able to compute this integral, we know $u = 4x^3 + 5$ is a correct substitution!

5. Finally, we rewrite the most general antiderivative we found at the end of step 4 in terms of x by substituting $u = 4x^3 + 5$:

$$\frac{1}{12}e^u + C = \frac{1}{12}e^{4x^3+5} + C$$

Therefore,

$$\int x^2 e^{4x^3+5}\, dx = \frac{1}{12}e^{4x^3+5} + C$$

Remember, you can always check that your antiderivative is correct by finding its derivative!

N *We could have chosen a different substitution for this integrand: $u = e^{4x^3+5}$. The solution using this function for u is similar to that above. We stated previously that there may be more than one choice for u in which the method of substitution will work. However, it is important to realize that not every integrand will have a choice for the substitution; it just depends on the particular function in the integrand.*

c. Recall that we must find $\displaystyle\int \frac{7x^3}{6x^4-15}\, dx$. Because the denominator of the integrand, $6x^4 - 15$, consists of more than one term, we cannot algebraically manipulate the function (i.e., divide every term in the numerator by the denominator) in order to use the Introductory Antiderivative Rules.

Rewriting the integrand in the form $\displaystyle\int 7x^3 \cdot \frac{1}{6x^4-15}\, dx$, we see that part of it can be written as a composite function, say $y = \dfrac{1}{6x^4-15}$, where the *inside* function is $g(x) = 6x^4 - 15$ and the *outside* function can be written as $y = \dfrac{1}{x}$. Note that you can compose the two function to check the composition. Thus, we will attempt to use substitution to find the antiderivative:

1. As discussed previously, the integrand contains a composite function, so we consider letting u equal the *inside* function $g(x)$, where $g(x) = 6x^4 - 15$. Now, we need to see if $g'(x) = 24x^3$ is in the integrand. Again, although this derivative is not explicitly given in the integrand, there is a multiple of it: $7x^3$.

   Thus, it seems we should let $u = 6x^4 - 15$.

2. Taking the derivative of $u = 6x^4 - 15$ gives $du = 24x^3\, dx$, which is part of the integrand; it only differs by a constant.

3. We need to rewrite the indefinite integral in terms of u and du. To do this, we substitute $6x^4 - 15$ in the integrand with u and use the fact that $du = 24x^3\, dx$ to help us rewrite the rest of the integrand. Looking at the integrand again, we can see fairly easily where we will substitute u. To help us see the terms that will remain after we substitute $6x^4 - 15$ with u, let's rearrange the terms in the integrand, but leave the constant in front:

$$\int 7x^3 \cdot \frac{1}{6x^4-15}\, dx = \int 7 \cdot \frac{1}{6x^4-15} \cdot x^3\, dx$$

Thus, in order to write the integral completely in terms of u and du, we also need to substitute for $x^3\ dx$.

Next, we need to find an expression to substitute for $x^3\ dx$ in the integrand. Remember, $\dfrac{1}{6x^4-15}$ will become $\dfrac{1}{u}$. Given $du = 24x^3\ dx$, we can divide both sides of this equation by 24, or multiply both sides by $\frac{1}{24}$, and then we will have an expression that is equivalent to $x^3\ dx$ that we can substitute:

$$\begin{aligned} du &= 24x^3\ dx \implies \\ \frac{1}{24}\ du &= \frac{1}{24}\cdot 24x^3\ dx \\ \frac{1}{24}\ du &= x^3\ dx, \text{ or } x^3\ dx = \frac{1}{24}\ du \end{aligned}$$

Now, we are ready to formally rewrite the integral. We leave the constant in front, substitute $6x^4-15$ with u, and substitute $x^3\ dx$ with $\frac{1}{24}\ du$:

$$\int 7\cdot\frac{1}{6x^4-15}\cdot x^3\ dx = \int 7\cdot\frac{1}{u}\cdot\frac{1}{24}\ du$$

Moving the constant $\frac{1}{24}$ to the front and simplifying so the function is "ready" to integrate in step 4 gives

$$\begin{aligned} \int 7\cdot\frac{1}{u}\cdot\frac{1}{24}du &= \int 7\cdot\frac{1}{24}\cdot\frac{1}{u}\ du \\ &= \int \frac{7}{24}\frac{1}{u}\ du \end{aligned}$$

So far, it looks as though we chose a correct substitution (i.e., u) because the integral is written completely in terms of u and du!

4. Next, we compute the indefinite integral we found at the end of step 3 using the Introductory Antiderivative Rules from **Section 4.1**:

$$\int \frac{7}{24}\frac{1}{u}\ du = \frac{7}{24}\ln|u| + C$$

Because we were able to compute this integral, we know $u = 6x^4-15$ is a correct substitution!

5. Finally, we rewrite the most general antiderivative we found at the end of step 4 in terms of x by substituting $u = 6x^4-15$:

$$\frac{7}{24}\ln|u| + C = \frac{7}{24}\ln\left|6x^4-15\right| + C$$

Therefore,

$$\int \frac{7x^3}{6x^4-15}\ dx = \frac{7}{24}\ln\left|6x^4-15\right| + C$$

Remember, you can always check that your antiderivative is correct by finding its derivative!

■

The integrands in the previous example were selected for a specific purpose: to demonstrate the possible cases we may encounter, in this textbook, when using the method of substitution to find an antiderivative. In part **a**, we found the antiderivative of a composite function in which the *inside* function was raised to a power. In part **b**, we found the antiderivative of a composite function in which the *inside* function was the power of an exponential function. Finally, in part **c**, we found the antiderivative of a composite function in which the *inside* function was the denominator. We now summarize these three cases:

Possible Substitution Cases

When using the method of substitution to find an antiderivative, looking for one of the following formats in the integrand may help you determine the correct substitution $u = g(x)$, where $g(x)$ is the *inside* function:

1. $\int (g(x))^n \cdot g'(x)\, dx$, where n is any real number with $n \neq -1$
2. $\int b^{(g(x))} \cdot g'(x)\, dx$, where b is any positive real number
3. $\int \frac{1}{g(x)} \cdot g'(x)\, dx$

From this point forward, we will use these three cases to help us determine u when using the method of substitution, or at least help us make a good first choice! Remember, there may be more than one choice for u in which the method of substitution will enable us to find the antiderivative.

Try It # 1:
Find each of the following indefinite integrals using the method of substitution.

a. $\int 3x^2\left(x^3-3\right)^2\, dx$

b. $\int 4xe^{-5x^2+1}\, dx$

c. $\int \frac{-8x^4}{7x^5-10}\, dx$

Example 2 Find each of the following indefinite integrals.

a. $\int \left(8x^3-5\right)e^{8x^4-20x}\, dx$

b. $\int \frac{(\ln(x))^{20}}{x}\, dx$

c. $\int \frac{-3}{\sqrt{x}-1}\, dx$

Solution:

a. To find $\int \left(8x^3-5\right)e^{8x^4-20x}\, dx$, we must use the method of substitution. There is no way to algebraically manipulate the integrand in order to use the Introductory Antiderivative Rules.

Let's consider the exponential function as a composite function where the *inside* function is given by $g(x) = 8x^4 - 20x$. Note that this is the second case in our list of possible substitution cases.

1. We let u equal the *inside* function $g(x)$, where $g(x) = 8x^4 - 20x$. Thus, we let $u = 8x^4 - 20x$.
2. Taking the derivative of $u = 8x^4 - 20x$ gives $du = \left(32x^3 - 20\right)\, dx$.
3. We need to rewrite the indefinite integral in terms of u and du. To do this, we substitute $8x^4 - 20x$ in the integrand with u and use the fact that $du = \left(32x^3 - 20\right)\, dx$ to help us rewrite the rest of the integrand.

Looking at the integrand again, we can see fairly easily where we will substitute u. To help us see the terms that will remain after we substitute $8x^4 - 20x$ with u, let's rearrange the terms in the integrand:

$$\int \left(8x^3 - 5\right) e^{8x^4 - 20x}\, dx = \int e^{8x^4 - 20x} \left(8x^3 - 5\right)\, dx$$

Thus, in order to write the integral completely in terms of u and du, we also need to substitute for $\left(8x^3 - 5\right)\, dx$.

Next, we need to find an expression to substitute for $\left(8x^3 - 5\right)\, dx$ in the integrand. Remember, $e^{8x^4 - 20x}$ will become e^u.

We know $du = \left(32x^3 - 20\right)\, dx$, and although the right-hand side of this equation is not exactly the expression we need to substitute for, $\left(8x^3 - 5\right)\, dx$, it is a multiple of it! In other words, if we divide both sides of the equation $du = \left(32x^3 - 20\right)\, dx$ by 4, or multiply both sides by $\frac{1}{4}$, we will have an expression that is equivalent to $\left(8x^3 - 5\right)\, dx$ that we can substitute:

$$\begin{aligned} du &= \left(32x^3 - 20\right)\, dx \implies \\ \frac{1}{4}\, du &= \frac{1}{4}\left(32x^3 - 20\right)\, dx \\ \frac{1}{4}\, du &= \left(8x^3 - 5\right)\, dx, \text{ or } \left(8x^3 - 5\right)\, dx = \frac{1}{4}\, du \end{aligned}$$

Now, we are ready to formally rewrite the integral. We substitute $8x^4 - 20x$ with u, and we substitute $\left(8x^3 - 5\right)\, dx$ with $\frac{1}{4}\, du$:

$$\int e^{8x^4 - 20x} \left(8x^3 - 5\right)\, dx = \int e^u \cdot \frac{1}{4}\, du$$

Moving the constant $\frac{1}{4}$ to the front so the function is "ready" to integrate in step 4 gives

$$\int e^u \cdot \frac{1}{4}\, du = \int \frac{1}{4} e^u\, du$$

So far, it looks as though we chose a correct substitution (i.e., u) because the integral is written completely in terms of u and du!

4. Next, we compute the indefinite integral we found at the end of step 3 using the Introductory Antiderivative Rules from **Section 4.1**:

$$\int \frac{1}{4} e^u\, du = \frac{1}{4} e^u + C$$

Because we were able to compute this integral, we know $u = 8x^4 - 20x$ is a correct substitution!

5. Finally, we rewrite the most antiderivative we found at the end of step 4 in terms of x by substituting $u = 8x^4 - 20x$:

$$\frac{1}{4} e^u + C = \frac{1}{4} e^{8x^4 - 20x} + C$$

Therefore,

$$\int \left(8x^3 - 5\right) e^{8x^4 - 20x}\, dx = \frac{1}{4} e^{8x^4 - 20x} + C$$

Remember, you can always check that your antiderivative is correct by finding its derivative!

b. To find $\int \frac{(\ln(x))^{20}}{x}\, dx$, we must use the method of substitution. There is no way to algebraically manipulate the integrand in order to use the Introductory Antiderivative Rules.

With this function, the substitution may be tricky to see. The most complicated part of the integrand is the numerator, so let's consider the numerator to be a composite function in which the *inside* function, $g(x) = \ln(x)$, is raised to a power, 20. Note that an *inside* function raised to a power is the first case in our list of possible substitution cases.

1. As discussed previously, we will let u equal the *inside* function $g(x)$, where $g(x) = \ln(x)$. Thus, we let $u = \ln(x)$.

2. Taking the derivative of $u = \ln(x)$ gives $du = \frac{1}{x}\, dx$.

3. We need to rewrite the indefinite integral in terms of u and du. To do this, we substitute $\ln(x)$ in the integrand with u and use the fact that $du = \frac{1}{x}\, dx$ to help us rewrite the rest of the integrand. Looking at the integrand again, we can see fairly easily where we will substitute u. To help us see the terms that will remain after we substitute $\ln(x)$ with u, let's rearrange the terms in the integrand:

$$\int \frac{(\ln(x))^{20}}{x}\, dx = \int (\ln(x))^{20} \cdot \frac{1}{x}\, dx$$

Thus, in order to write the integral completely in terms of u and du, we also need to substitute for $\frac{1}{x}\, dx$.

Next, we need to find an expression to substitute for $\frac{1}{x}\, dx$ in the integrand. Remember, $(\ln(x))^{20}$ will become u^{20}. Because $du = \frac{1}{x}\, dx$, we already have the expression we need! In other words, we know $\frac{1}{x}\, dx = du$.

Thus, we are ready to formally rewrite the integral. We substitute $\ln(x)$ with u, and we substitute $\frac{1}{x}\, dx$ with du:

$$\int (\ln(x))^{20} \cdot \frac{1}{x}\, dx = \int u^{20}\, du$$

The function is now in a form we can integrate. Also, it looks as though we chose a correct substitution (i.e., u) because the integral is written completely in terms of u and du!

4. Next, we compute the indefinite integral we found at the end of step 3 using the Introductory Antiderivative Rules from **Section 4.1**:

$$\int u^{20}\, du = \frac{1}{21}u^{21} + C$$

Because we were able to compute this integral, we know $u = \ln(x)$ is a correct substitution!

5. Finally, we rewrite the most general antiderivative we found at the end of step 4 in terms of x by substituting $u = \ln(x)$:

$$\frac{1}{21}u^{21} + C = \frac{1}{21}(\ln(x))^{21} + C$$

Therefore,

$$\int \frac{(\ln(x))^{20}}{x}\,dx = \frac{1}{21}(\ln(x))^{21} + C$$

Remember, you can always check that your antiderivative is correct by finding its derivative!

c. Again, we cannot use introductory techniques to find $\int \frac{-3}{\sqrt{x-1}}\,dx$.

Looking at the integrand, it appears the most complicated part of the function is the denominator. So let's try using the third case in our list of possible substitution cases and assume the denominator, $\sqrt{x-1}$, is the *inside* function, $g(x)$.

1. As discussed previously, we will let u equal the *inside* function $g(x)$, where $g(x) = \sqrt{x-1}$. Thus, we let $u = \sqrt{x-1}$.

2. Using the Chain Rule to take the derivative of $u = \sqrt{x-1}$ gives

$$\begin{aligned} du &= \frac{1}{2}(x-1)^{-\frac{1}{2}}\,dx \\ &= \frac{1}{2}\cdot\frac{1}{(x-1)^{\frac{1}{2}}}\,dx \\ &= \frac{1}{2}\cdot\frac{1}{\sqrt{x-1}}\,dx \end{aligned}$$

3. We need to rewrite the indefinite integral in terms of u and du. To do this, we substitute $\sqrt{x-1}$ in the integrand with u and try to use the fact that $du = \frac{1}{2}\cdot\frac{1}{\sqrt{x-1}}\,dx$ to help us rewrite the rest of the integrand.

Notice that if we substitute $\sqrt{x-1}$ with u in the integrand, we will have -3 in the numerator, u in the denominator, and only dx left to replace in order to write the integral in terms of the variable u. This presents a potential problem! If we "solve" the equation $du = \frac{1}{2}\cdot\frac{1}{\sqrt{x-1}}\,dx$ for dx to find an equivalent expression to substitute for dx, the expression will contain the variable x. Thus, it appears as though we are not able to substitute for dx so that the integral is completely in terms of u and du.

However, because finding du involved using the Chain Rule, we can still rewrite the integrand in terms of u and du by substituting in a slightly different way. We will still let $u = \sqrt{x-1}$, and therefore, we still have $du = \frac{1}{2\sqrt{x-1}}\,dx$. The difference in the way we will approach the substitution now is instead of substituting u in the denominator of the integrand like we tried previously, we will view the denominator of the integrand, $\sqrt{x-1}$, as part of du. We will first rearrange the terms in the original integrand to help us understand the idea behind substituting in this way:

$$\int \frac{-3}{\sqrt{x-1}}\,dx = \int -3\cdot\frac{1}{\sqrt{x-1}}\,dx$$

Now, recall that $du = \frac{1}{2}\cdot\frac{1}{\sqrt{x-1}}\,dx$. Multiplying both sides of this equation by 2 will allow us to replace the expression $\frac{1}{\sqrt{x-1}}\,dx$ in the orginal integrand with $2\,du$. Starting with du and multiplying both sides of the equation by 2 gives

$$du = \frac{1}{2} \cdot \frac{1}{\sqrt{x-1}}\ dx \implies$$
$$2\ du = 2 \cdot \frac{1}{2} \frac{1}{\sqrt{x-1}}\ dx$$
$$2\ du = \frac{1}{\sqrt{x-1}}\ dx, \text{ or } \frac{1}{\sqrt{x-1}}\ dx = 2\ du$$

Thus, we can substitute $2\ du$ in the integrand for $\frac{1}{\sqrt{x-1}}\ dx$ and formally rewrite the integral in terms of u and du:

$$\int -3 \cdot \frac{1}{\sqrt{x-1}}\ dx = \int -3 \cdot 2\ du$$

Multiplying the constants so the function is "ready" to integrate in step 4 gives

$$\int -3 \cdot 2\ du = \int -6\ du$$

So far, it looks as though this substitution, where $u = \sqrt{x-1}$, will work because the integral is written entirely in terms of u and du!

4. Next, we compute the indefinite integral we found at the end of step 3 using the Introductory Antiderivative Rules from **Section 4.1**:

$$\int -6\ du = -6u + C$$

Because we were able to compute this integral, we know $u = \sqrt{x-1}$ is a correct substitution!

5. Finally, we rewrite the most general antiderivative we found at the end of step 4 in terms of x by substituting $u = \sqrt{x-1}$:

$$-6u + C = -6\sqrt{x-1} + C$$

Therefore,

$$\int \frac{-3}{\sqrt{x-1}}\ dx = -6\sqrt{x-1} + C$$

Remember, you can always check that your antiderivative is correct by finding its derivative!

You may be wondering why for the previous indefinite integral, $\int \frac{-3}{\sqrt{x-1}}\ dx$, we did not use the first case in our list of possible substitution cases: an *inside* function raised to a power. If we write the denominator in exponential form and then bring it to the numerator, we can rewrite the indefinite integral as

$$\int \frac{-3}{\sqrt{x-1}}\ dx = \int \frac{-3}{(x-1)^{\frac{1}{2}}}\ dx$$
$$= \int -3(x-1)^{-\frac{1}{2}}\ dx$$

Given this form of the integral, it appears we can also let $u = x-1$. Let's go ahead and rework this problem using the substitution $u = x-1$, and see if it will work!

1. As stated previously, we will assume the integral is of the form $\int -3(x-1)^{-\frac{1}{2}}\,dx$, and we will let $u = x-1$.

2. Taking the derivative of $u = x-1$ gives $du = 1dx$, or $du = dx$.

3. We need to rewrite the indefinite integral in terms of u and du. To do this, we substitute $x-1$ in the integrand with u and use the fact that $du = dx$ to help us rewrite the rest of the integrand. Notice that after substituting $x-1$ with u, the only term left in the integrand with an x that we need to substitute for is dx. However, because $du = dx$, we know $dx = du$! Thus, we can proceed with rewriting the integral at this point. We will substitute $x-1$ with u and dx with du:

$$\int -3(x-1)^{-\frac{1}{2}}\,dx = \int -3u^{-\frac{1}{2}}\,du$$

   The function is in a form we can integrate. Also, so far it looks as though we chose a correct substitution (i.e., u) because the integral is written completely in terms of u and du!

4. Next, we compute the indefinite integral we found at the end of step 3 using the Introductory Antiderivative Rules from **Section 4.1**:

$$\begin{aligned}\int -3u^{-\frac{1}{2}}\,du &= -3\cdot\frac{2}{1}u^{\frac{1}{2}}+C\\ &= -6u^{\frac{1}{2}}+C\end{aligned}$$

   Because we were able to compute this integral, we know $u = x-1$ is also a correct substitution!

5. Finally, we rewrite the most general antiderivative we found at the end of step 4 in terms of x by substituting $u = x-1$:

$$\begin{aligned}-6u^{\frac{1}{2}}+C &= -6(x-1)^{\frac{1}{2}}+C\\ &= -6\sqrt{x-1}+C\end{aligned}$$

Therefore,

$$\int \frac{-3}{\sqrt{x-1}}\,dx = -6\sqrt{x-1}+C$$

This is the same general antiderivative we found using the substitution $u = \sqrt{x-1}$ previously! Hence, there may be more than one correct choice for u when using the method of substitution.

▪

Try It # 2:
Find each of the following indefinite integrals.

a. $\int \left(9x^2-8x\right)e^{6x^3-8x^2}\,dx$

b. $\int \frac{\ln(x)}{x}\,dx$

c. $\int \frac{2x^4}{\sqrt[3]{8x^5-9}}\,dx$

In the previous example, we found the most general antiderivative of a function using two different substitutions (one with $u = \sqrt{x-1}$ and another with $u = x-1$). Thus, we have established that more than one substitution may work, but what will it look like if the substitution we select does not work?

There are two ways a substitution will not work: Either the integrand cannot be written entirely in terms of u and du, or even if the integrand can be written in terms of u and du, the function cannot be integrated using the Introductory Antiderivative Rules. We will see one of these possibilities in the next example.

■ **Example 3** Find $\displaystyle\int \frac{e^{2x}}{e^{2x}+4}\,dx$.

Solution:
Because the denominator of the integrand, $e^{2x}+4$, consists of more than one term, we cannot algebraically manipulate the function in order to use the Introductory Antiderivative Rules. Thus, we will use the method of substitution to find the antiderivative:

1. Looking at the integrand, we may notice the terms e^{2x} immediately. So let's try using the second case in our list of possible substitution cases and assume the exponent, $2x$, is the *inside* function $g(x)$.

   Thus, we let $u = 2x$.

2. Taking the derivative of $u = 2x$ gives $du = 2\,dx$.

3. We need to rewrite the indefinite integral in terms of u and du. To do this, we substitute $2x$ in the integrand with u and use the fact that $du = 2\,dx$ to help us rewrite the rest of the integrand. Looking at the integrand again, we can see fairly easily where we will substitute u. The only term containing the variable x that will remain after substituting u in the integrand is dx.

   Thus, in order to write the integral completely in terms of u and du, we also need to substitute for dx. Given $du = 2\,dx$, we can divide both sides of this equation by 2, or multiply both sides by $\frac{1}{2}$, and then we will have an expression that is equivalent to dx that we can substitute:

$$\begin{aligned} du &= 2\,dx \implies \\ \frac{1}{2}\,du &= \frac{1}{2}\cdot 2\,dx \\ \frac{1}{2}\,du &= dx, \text{ or } dx = \frac{1}{2}\,du \end{aligned}$$

   Now, we are ready to formally rewrite the integral. We substitute $2x$ with u and dx with $\frac{1}{2}\,du$:

$$\int \frac{e^{2x}}{e^{2x}+4}\,dx = \int \frac{e^{u}}{e^{u}+4}\cdot\frac{1}{2}\,du$$

   Moving the constant $\frac{1}{2}$ to the front so the function is "ready" to integrate in step 4 gives

$$\int \frac{e^{u}}{e^{u}+4}\cdot\frac{1}{2}\,du = \int \frac{1}{2}\frac{e^{u}}{e^{u}+4}\,du$$

   It looks as though we chose a correct substitution (i.e., u) because the integral is written completely in terms of u and du. However, the Introductory Antiderivative Rules do not provide a way to find the antiderivative of this integrand. Also, we cannot algebraically manipulate the function because the denominator, $e^{u}+4$, has two terms. Thus, we should go back to step 1 and select a new u with which to attempt the method of substitution.

! *Even though we were able to rewrite the integrand in terms of u and du only in this example, we were not able to find the antiderivative of the resulting integrand.*

Rather than working through another substitution attempt with a different u, we will let you try it:

■

Try It # 3:
Find $\displaystyle\int \frac{e^{2x}}{e^{2x}+4}\,dx$.

We have been emphasizing that for the method of substitution to work, we must be able to rewrite the indefinite integral completely in terms of u and du. While this is absolutely true, there may be a situation where it seems as though a particular choice for substitution will not work because the original variable, say x, still appears in the integrand after substituting.

However, there may be an algebraic way to substitute for the remaining variable so that the integrand can still be rewritten entirely in terms of u and du and the substitution process may continue. We will learn how to incorporate this additional integration technique in the next example.

■ **Example 4** Find $\displaystyle\int \frac{9x}{x-10}\,dx$.

Solution:
Because the denominator of the integrand, $x-10$, consists of more than one term, we cannot algebraically manipulate the function in order to use the Introductory Antiderivative Rules. Thus, we will use the method of substitution to find the antiderivative:

1. Looking at the integrand, it seems the most complicated part of the function is the denominator. So let's try using the third case in our list of possible substitution cases and assume the denominator, $x-10$, is the *inside* function $g(x)$.

   Thus, we let $u = x-10$.

2. Taking the derivative of $u = x-10$ gives $du = dx$.

3. We need to rewrite the indefinite integral in terms of u and du. To do this, we substitute $x-10$ in the integrand with u and use the fact that $du = dx$ to help us rewrite the rest of the integrand. Looking at the integrand again, we can see fairly easily where we will substitute u. To help us see the terms that will remain after we substitute $x-10$ with u, let's rearrange the terms in the integrand, but leave the constant in front:

$$\int \frac{9x}{x-10}\,dx = \int 9\cdot\frac{1}{x-10}\cdot x\,dx$$

   Thus, in order to write the integral completely in terms of u and du, we also need to substitute for $x\,dx$. Remember, $\dfrac{1}{x-10}$ will become $\dfrac{1}{u}$. We have a big problem! Because $du = dx$, it appears there is no way to write the integrand entirely in terms of u. We could substitute u for $x-10$ and du for dx, but we would still have the x remaining in the integrand.

   While it may appear as though our choice for u will not work and that we need to go back to step 1 and select a different u, we can actually still proceed if we recall that $u = x-10$ and solve this equation for x. Then, we can substitute the resulting expression for x in the integrand, and our integral will be written entirely in terms of u and du. We begin by solving $u = x-10$ for x:

$$u = x-10 \implies u+10 = x$$

   This tells us $x = u+10$. Substituting this expression for x in the integrand will result in the following integral in terms of u and du:

$$\int 9 \cdot \frac{1}{u} \cdot (u+10) \cdot du$$

Now, we need to simplify the integrand so it is "ready" to integrate in step 4. Remember, there is no Product Rule for antiderivatives. Distributing $\frac{1}{u}$ and simplifying gives

$$\begin{aligned}\int 9 \cdot \frac{1}{u} \cdot (u+10) \cdot du &= \int 9\left(\frac{u}{u} + \frac{10}{u}\right) du \\ &= \int 9\left(1 + \frac{10}{u}\right) du \\ &= \int \left(9 + \frac{90}{u}\right) du\end{aligned}$$

The function is ready to integrate in step 4! Notice we left the u in the denominator of the second term. Remember, we only move the denominator to the numerator if the term has a power *other than* 1. Also, it looks so far as though we chose a correct substitution (i.e., u) because the integral is written completely in terms of u and du!

4. Next, we compute the indefinite integral we found at the end of step 3 using the Introductory Antiderivative Rules from **Section 4.1**:

$$\int \left(9 + \frac{90}{u}\right) du = 9u + 90\ln|u| + C$$

Because we were able to compute this integral, we know $u = x - 10$ is a correct substitution!

5. Finally, we rewrite the most general antiderivative we found at the end of step 4 in terms of x by substituting $u = x - 10$:

$$9u + 90\ln|u| + C = 9(x-10) + 90\ln|x-10| + C$$

Therefore,

$$\int \frac{9x}{x-10}\, dx = 9(x-10) + 90\ln|x-10| + C$$

Remember, you can always check that your antiderivative is correct by finding its derivative!

▪

Try It # 4:
Find $\int 2x(6x-5)^3\, dx$.

Specific Antiderivatives

At the heart of it, finding specific antiderivatives with the method of substitution is the same as finding them without the method of substitution, which we learned in **Section 4.1**: Find the most general antiderivative and then use the point given to find the value of the constant of integration, C.

Example 5 Find $f(x)$ for each of the following using the given information.

a. $f'(x) = e^{2-3x}$ and $f(0) = -3$

b. $f'(x) = (20x-50)\left(x^2-5x+6\right)^{12} + 5$ and $f(3) = 18$

Solution:

a. Because f is the antiderivative of f', we know

$$\begin{aligned} f(x) &= \int f'(x)\,dx \\ &= \int e^{2-3x}\,dx \end{aligned}$$

To find the antiderivative, we need to use the method of substitution because we cannot use the Introductory Antiderivative Rules (nor can we algebraically manipulate the function in order to use them). We proceed with the method of substitution:

1. We let $u = 2-3x$. Note that this is the second case in our list of possible substitution cases.

2. Taking the derivative of u gives $du = -3\,dx$.

3. We need to rewrite the indefinite integral in terms of u and du. To do this, we substitute $2-3x$ in the integrand with u and use the fact that $du = -3\,dx$ to help us rewrite the rest of the integrand. Looking at the integrand again, we can see fairly easily where we will substitute u. The only term containing the variable x that will remain after substituting u in the integrand is dx.

   Thus, in order to write the integral completely in terms of u and du, we also need to substitute for dx. Given $du = -3\,dx$, we can divide both sides of this equation by -3, or multiply both sides by $-\frac{1}{3}$, and then we will have an expression that is equivalent to dx that we can substitute:

$$\begin{aligned} du &= -3\,dx \implies \\ -\frac{1}{3}\,du &= -\frac{1}{3}\cdot(-3)\,dx \\ -\frac{1}{3}\,du &= dx, \text{ or } dx = -\frac{1}{3}\,du \end{aligned}$$

   Now, we can formally rewrite the integral by substituting u for $2-3x$ and $-\dfrac{1}{3}\,du$ for dx:

$$\begin{aligned} \int e^{2-3x}\,dx &= \int e^u \cdot\left(-\frac{1}{3}\right)du \\ &= \int -\frac{1}{3}e^u\,du \end{aligned}$$

4. Next, we compute the indefinite integral:

$$\int -\frac{1}{3}e^u\,du = -\frac{1}{3}e^u + C$$

5. Finally, we rewrite the most general antiderivative in terms of x by substituting $u = 3-2x$:

$$-\frac{1}{3}e^u + C = -\frac{1}{3}e^{2-3x} + C$$

Therefore, $f(x) = -\frac{1}{3}e^{2-3x} + C$. This is the most general antiderivative, and we can find the specific antiderivative using the fact that $f(0) = -3$. Recall that this means we know that the graph of f passes through the point $(0, -3)$.

We substitute $x = 0$ into $f(x)$, and then set $f(x)$ equal to -3. This will allow us to solve for C:

$$\begin{aligned} f(x) &= -\frac{1}{3}e^{2-3x} + C \implies \\ f(0) &= -\frac{1}{3}e^{2-3(0)} + C = -3 \implies \\ -\frac{1}{3}e^{2} + C &= -3 \\ C &= -3 + \frac{1}{3}e^{2} \end{aligned}$$

Substituting this value of C into the most general antiderivative gives us the specific antiderivative:

$$f(x) = -\frac{1}{3}e^{2-3x} - 3 + \frac{1}{3}e^{2}$$

Therefore, the specific antiderivative of f' is given by $f(x) = -\dfrac{1}{3}e^{2-3x} - 3 + \dfrac{1}{3}e^{2}$.

b. Recall $f'(x) = (20x-50)\left(x^2-5x+6\right)^{12} + 5$ and $f(3) = 18$. Again, because f is the antiderivative of f', we know

$$\begin{aligned} f(x) &= \int f'(x)\,dx \\ &= \int \left((20x-50)\left(x^2-5x+6\right)^{12} + 5\right) dx \end{aligned}$$

Notice the integrand consists of the sum of two functions. One function is a composite function which will require the method of substitution, and the other is a constant function which we can integrate using the Introductory Antiderivative Rules. Thus, to find the antiderivative of the integrand, we need to use the Sum Rule for antiderivatives first:

$$\int \left((20x-50)\left(x^2-5x+6\right)^{12} + 5\right) dx = \int (20x-50)\left(x^2-5x+6\right)^{12}\,dx + \int 5\,dx$$

We will use the method of substitution to evaluate the first integral, and we know $\displaystyle\int 5\,dx = 5x + C$.

Now, we will find the first integral using the method of substitution:

1. We let $u = x^2 - 5x + 6$. Note that this is the first case in our list of possible substitution cases.

2. Taking the derivative of u gives $du = (2x-5)\,dx$.

3. We need to rewrite the indefinite integral in terms of u and du. To do this, we substitute x^2-5x+6 in the integrand with u and use the fact that $du = (2x-5)\,dx$ to help us rewrite the rest of the integrand. Looking at the integrand again, we can see fairly easily where we will substitute u. To help us see the terms that will remain after we substitute x^2-5x+6 with u, let's rearrange the terms in the integrand:

$$\int (20x-50)\left(x^2-5x+6\right)^{12}\,dx = \int \left(x^2-5x+6\right)^{12}(20x-50)\,dx$$

Thus, in order to write the integral completely in terms of u and du, we also need to substitute for $(20x-50)\ dx$ in the integrand. Remember, $\left(x^2-5x+6\right)^{12}$ will become u^{12}.

We know $du=(2x-5)\ dx$, and although the right-hand side of this equation is not exactly the expression we need to substitute for, $(20x-50)\ dx$, it is a multiple of it! In other words, if we multiply both sides of the equation $du=(2x-5)\ dx$ by 10, we will have an expression that we can substitute:

$$\begin{aligned} du &= (2x-5)\ dx \implies \\ 10\ du &= 10(2x-5)\ dx \\ 10\ du &= (20x-50)\ dx, \text{ or } (20x-50)\ dx = 10\ du \end{aligned}$$

Now, we formally rewrite the integral. We substitute x^2-5x+6 with u and $(20x-50)\ dx$ with $10\ du$:

$$\begin{aligned} \int \left(x^2-5x+6\right)^{12}(20x-50)\ dx &= \int u^{12}\cdot 10\ du \\ &= \int 10u^{12}\ du \end{aligned}$$

4. Next, we compute the indefinite integral:

$$\begin{aligned} \int 10u^{12}\ du &= 10\cdot\frac{1}{13}u^{13}+C \\ &= \frac{10}{13}u^{13}+C \end{aligned}$$

5. Finally, we rewrite the most general antiderivative we found at the end of step 4 in terms of x by substituting $u=x^2-5x+6$:

$$\frac{10}{13}u^{13}+C=\frac{10}{13}\left(x^2-5x+6\right)^{13}+C$$

Be careful! This is only part of the most general antiderivative of f'. Recall that we also need to add the antiderivative of the second function in the original integrand. Doing so will give us the most general antiderivative:

$$\begin{aligned} f(x) &= \int f'(x)\ dx \\ &= \int \left((20x-50)(x^2-5x+6)^{12}+5\right)\ dx \\ &= \int (20x-50)(x^2-5x+6)^{12}\ dx + \int 5\ dx \\ &= \frac{10}{13}\left(x^2-5x+6\right)^{13}+5x+C \end{aligned}$$

Therefore, $f(x)=\frac{10}{13}\left(x^2-5x+6\right)^{13}+5x+C$. This is the most general antiderivative, and we can find the specific antiderivative using the point $(3,18)$. Substituting $x=3$ into $f(x)$ and setting $f(x)$ equal to 18 allows us to solve for C:

$$\begin{aligned} f(x) &= \frac{10}{13}\left(x^2-5x+6\right)^{13}+5x+C \implies \\ f(3) &= \frac{10}{13}\left((3)^2-5(3)+6\right)^{13}+5(3)+C=18 \implies \\ &\frac{10}{13}(0)^{13}+15+C=18 \\ &C=3 \end{aligned}$$

Substituting this value of C into the most general antiderivative gives us the specific antiderivative:

$$f(x) = \frac{10}{13}\left(x^2 - 5x + 6\right)^{13} + 5x + 3$$

Therefore, the specific antiderivative of f' is given by $f(x) = \frac{10}{13}\left(x^2 - 5x + 6\right)^{13} + 5x + 3$.

■

Try It # 5:
Find $f(x)$ for each of the following using the given information.

a. $f'(x) = 4x^4 \sqrt[3]{2x^5 + 64}$ and $f(0) = -36$

b. $f'(x) = \frac{\ln(x)}{x} + 4x^3 - 12x$ and $f(e) = e^4 - 6e^2 + 82$

Applications

We can use the method of substitution to solve problems involving the same types of applications we encountered using introductory antidifferentiation techniques, only now we can apply the method of substitution to more complicated functions.

■ **Example 6** The marginal cost function for Donut Let The Good Life Pass You By bakery is given by $C'(x) = 0.2x(0.913)^{(0.1x^2)}$ dollars per dozen donuts when x dozen donuts are made. What is the bakery's cost of making 100 dozen donuts if

a. it costs the bakery \$50 to make 5 dozen donuts?
b. it costs the bakery \$80 to make 36 donuts?
c. it costs the bakery \$120 if no donuts are made?

Solution:
To find the cost of making 100 dozen donuts in any of the above scenarios in parts **a**, **b**, or **c**, we need to find the bakery's cost function first. We can find the cost function, C, by finding the antiderivative of the marginal cost function, C'. After finding the most general antiderivative, we can find the specific antiderivative associated with parts **a**, **b**, and **c** based on the information given in each part. After obtaining the specific antiderivative, or cost function C, for each part, we can use the function to determine the bakery's cost of making 100 dozen donuts.

Thus, before moving to parts **a**, **b**, and **c**, we must find the most general antiderivative of C':

$$\begin{aligned} C(x) &= \int C'(x)\,dx \\ &= \int 0.2x(0.913)^{(0.1x^2)}\,dx \end{aligned}$$

We will use the method of substitution to find the antiderivative because there is a composite function in the integrand:

1. We let $u = 0.1x^2$. Note that this is the second case in our list of possible substitution cases.

2. Taking the derivative of u gives $du = 0.2x\,dx$.

3. We must rewrite the integral in terms of u and du. To do this, we substitute $0.1x^2$ in the integrand with u and use the fact that $du = 0.2x\,dx$ to help us rewrite the rest of the integrand. Looking at the integrand again, we can see fairly easily where we will substitute u. To help us see the terms that will remain after we substitute $0.1x^2$ with u, let's rearrange the terms in the integrand, but leave the constant in front:

$$\int 0.2x(0.913)^{(0.1x^2)}\,dx = \int 0.2\cdot(0.913)^{(0.1x^2)}\cdot x\,dx$$

Thus, in order to write the integral completely in terms of u and du, we also need to substitute for $x\,dx$ (the constant 0.2 is okay; we just need the integral to be completely in terms of u and du). Because $du = 0.2x\,dx$, we can divide both sides of this equation by 0.2, or multiply both sides by $\frac{1}{0.2}$, and then we will have an expression that is equivalent to $x\,dx$ that we can substitute:

$$\begin{aligned} du &= 0.2x\,dx \implies \\ \frac{1}{0.2}\,du &= \frac{1}{0.2}\cdot 0.2x\,dx \\ 5\,du &= x\,dx, \text{ or } x\,dx = 5\,du \end{aligned}$$

Now, we rewrite the integral. We substitute $0.1x^2$ with u and $x\,dx$ with $5\,du$:

$$\begin{aligned} \int 0.2\cdot(0.913)^{(0.1x^2)}\cdot x\,dx &= \int 0.2\cdot(0.913)^u\cdot 5\,du \\ &= \int 0.2\cdot 5\cdot(0.913)^u\,du \\ &= \int (0.913)^u\,du \end{aligned}$$

4. Next, we compute the indefinite integral:

$$\int (0.913)^u\,du = \frac{1}{\ln(0.913)}(0.913)^u + C$$

5. Finally, we rewrite the most general antiderivative in terms of x by substituting $u = 0.1x^2$:

$$\frac{1}{\ln(0.913)}(0.913)^u + C = \frac{1}{\ln(0.913)}(0.913)^{(0.1x^2)} + C$$

Thus, the most general antiderivative representing the cost function is given by $C(x) = \frac{1}{\ln(0.913)}(0.913)^{(0.1x^2)} + C$ dollars. Now, we move to part **a** and use the information given to find a specific antiderivative (i.e., a specific cost function).

a. We are told it costs the bakery \$50 to make 5 dozen donuts. Thus, we know $C(5) = 50$ dollars. Substituting $x = 5$ into $C(x)$ and setting $C(x)$ equal to 50 allows us to solve for the constant of integration:

$$\begin{aligned} C(x) &= \frac{1}{\ln(0.913)}(0.913)^{(0.1x^2)} + C \implies \\ C(5) &= \frac{1}{\ln(0.913)}(0.913)^{(0.1(5)^2)} + C = 50 \implies \\ &\frac{(0.913)^{2.5}}{\ln(0.913)} + C = 50 \\ &C = 50 - \frac{(0.913)^{2.5}}{\ln(0.913)} \end{aligned}$$

Substituting this value for C gives us the specific antiderivative, or specific cost function:

$$C(x) = \frac{1}{\ln(0.913)}(0.913)^{(0.1x^2)} + 50 - \frac{(0.913)^{2.5}}{\ln(0.913)}$$

Finally, we calculate $C(100)$ to determine the bakery's cost when it makes 100 dozen donuts:

$$\begin{aligned} C(x) &= \frac{1}{\ln(0.913)}(0.913)^{\left(0.1x^2\right)} + 50 - \frac{(0.913)^{2.5}}{\ln(0.913)} \implies \\ C(100) &= \frac{1}{\ln(0.913)}(0.913)^{\left(0.1(100)^2\right)} + 50 - \frac{(0.913)^{2.5}}{\ln(0.913)} \\ &\approx \$58.75 \end{aligned}$$

Thus, the bakery's cost for making 100 dozen donuts is \$58.75 (assuming it costs \$50 to make 5 dozen donuts).

b. Now, let's assume it costs the bakery \$80 to make 36 donuts. We will find a different specific antiderivative corresponding to this information (i.e., a different specific cost function), and then evaluate the function to find the cost of making 100 dozen donuts!

The variable x is in dozen donuts, not donuts! 36 donuts is equivalent to 3 dozen donuts because 12 donuts make one dozen. Thus, we know it costs the bakery \$80 to make 3 dozen donuts. In other words, $C(3) = 80$. Substituting $x = 3$ into $C(x)$ and setting $C(x)$ equal to 80 allows us to solve for the constant of integration:

$$\begin{aligned} C(x) &= \frac{1}{\ln(0.913)}(0.913)^{\left(0.1x^2\right)} + C \implies \\ C(3) &= \frac{1}{\ln(0.913)}(0.913)^{\left(0.1(3)^2\right)} + C = 80 \implies \\ &\frac{(0.913)^{0.9}}{\ln(0.913)} + C = 80 \\ &C = 80 - \frac{(0.913)^{0.9}}{\ln(0.913)} \end{aligned}$$

Substituting this value for C gives us the specific antiderivative, or specific cost function:

$$C(x) = \frac{1}{\ln(0.913)}(0.913)^{\left(0.1x^2\right)} + 80 - \frac{(0.913)^{0.9}}{\ln(0.913)}$$

Finally, we calculate $C(100)$ to determine the bakery's cost when it makes 100 dozen donuts:

$$\begin{aligned} C(x) &= \frac{1}{\ln(0.913)}(0.913)^{\left(0.1x^2\right)} + 80 - \frac{(0.913)^{0.9}}{\ln(0.913)} \implies \\ C(100) &= \frac{1}{\ln(0.913)}(0.913)^{\left(0.1(100)^2\right)} + 80 - \frac{(0.913)^{0.9}}{\ln(0.913)} \\ &\approx \$90.12 \end{aligned}$$

Thus, the bakery's cost for making 100 dozen donuts is \$90.12 (assuming it costs \$80 to make 3 dozen donuts).

c. Finally, we will find the cost function given $C(0) = 120$. We use this information to find the value of the constant of integration:

$$\begin{aligned} C(x) &= \frac{1}{\ln(0.913)}(0.913)^{(0.1x^2)} + C \implies \\ C(0) &= \frac{1}{\ln(0.913)}(0.913)^{(0.1(0)^2)} + C = 120 \implies \\ &\frac{1}{\ln(0.913)}(0.913)^{0} + C = 120 \\ &\frac{1}{\ln(0.913)} \cdot 1 + C = 120 \\ &C = 120 - \frac{1}{\ln(0.913)} \end{aligned}$$

Substituting this value for C gives us the specific antiderivative, or specific cost function:

$$C(x) = \frac{1}{\ln(0.913)}(0.913)^{(0.1x^2)} + 120 - \frac{1}{\ln(0.913)}$$

Finally, we calculate $C(100)$ to determine the bakery's cost when it makes 100 dozen donuts:

$$\begin{aligned} C(x) &= \frac{1}{\ln(0.913)}(0.913)^{(0.1x^2)} + 120 - \frac{1}{\ln(0.913)} \implies \\ C(100) &= \frac{1}{\ln(0.913)}(0.913)^{(0.1(100)^2)} + 120 - \frac{1}{\ln(0.913)} \\ &\approx \$130.99 \end{aligned}$$

Thus, the bakery's cost for making 100 dozen donuts is \$130.99 (assuming it costs \$120 when the bakery does not make any donuts).

■

■ **Example 7** When x day passes are sold, the marginal revenue function for the musical aquarium Tuna Piano is given by $R'(x) = \left(\frac{7}{171}x^2 + 28\right)\left(\frac{1}{513}x^3 + 4x\right)^{-\frac{1}{3}}$ dollars per day pass.

a. Find the aquarium's revenue function.
b. Determine the amount of revenue the aquarium earns when 90 day passes are sold.

Solution:

a. The revenue function, R, is the antiderivative of the marginal revenue function, R'. Thus, we know

$$\begin{aligned} R(x) &= \int R'(x)\,dx \\ &= \int \left(\frac{7}{171}x^2 + 28\right)\left(\frac{1}{513}x^3 + 4x\right)^{-\frac{1}{3}}\,dx \end{aligned}$$

We will use the method of substitution to find the antiderivative because there is a composite function in the integrand:

1. We let $u = \frac{1}{513}x^3 + 4x$. Note that this is the first case in our list of possible substitution cases.

2. Taking the derivative of u gives $du = \left(\frac{3}{513}x^2 + 4\right)\,dx = \left(\frac{1}{171}x^2 + 4\right)\,dx$.

3. We must rewrite the integral in terms of u and du. To do this, we substitute $\frac{1}{513}x^3+4x$ in the integrand with u and use the fact that $du=\left(\frac{1}{171}x^2+4\right)\,dx$ to help us rewrite the rest of the integrand. Looking at the integrand again, we can see fairly easily where we will substitute u. To help us see the terms that will remain after we substitute $\frac{1}{513}x^3+4x$ with u, let's rearrange the terms in the integrand:

$$\int\left(\frac{7}{171}x^2+28\right)\left(\frac{1}{513}x^3+4x\right)^{-\frac{1}{3}}\,dx=\int\left(\frac{1}{513}x^3+4x\right)^{-\frac{1}{3}}\cdot\left(\frac{7}{171}x^2+28\right)\,dx$$

Thus, in order to write the integral completely in terms of u and du, we also need to substitute for $\left(\frac{7}{171}x^2+28\right)\,dx$. Because $du=\left(\frac{1}{171}x^2+4\right)\,dx$, we can multiply both sides of this equation by 7, and then we will have an expression that is equivalent to $\left(\frac{7}{171}x^2+28\right)\,dx$ that we can substitute:

$$\begin{aligned}du&=\left(\frac{1}{171}x^2+4\right)\,dx\implies\\7\,du&=7\left(\frac{1}{171}x^2+4\right)\,dx\\7\,du&=\left(\frac{7}{171}x^2+28\right)\,dx,\text{ or }\left(\frac{7}{171}x^2+28\right)\,dx=7\,du\end{aligned}$$

Now, we rewrite the integral. We substitute $\frac{1}{513}x^3+4x$ with u and $\left(\frac{7}{171}x^2+28\right)\,dx$ with $7\,du$:

$$\begin{aligned}\int\left(\frac{1}{513}x^3+4x\right)^{-\frac{1}{3}}\cdot\left(\frac{7}{171}x^2+28\right)\,dx&=\int u^{-\frac{1}{3}}\cdot 7\,du\\&=\int 7u^{-\frac{1}{3}}\,du\end{aligned}$$

4. Next, we compute the indefinite integral:

$$\begin{aligned}\int 7u^{-\frac{1}{3}}\,du&=7\cdot\frac{3}{2}u^{\frac{2}{3}}+C\\&=\frac{21}{2}u^{\frac{2}{3}}+C\end{aligned}$$

5. Finally, we rewrite the most general antiderivative in terms of x by substituting $u=\frac{1}{513}x^3+4x$:

$$\frac{21}{2}u^{\frac{2}{3}}+C=\frac{21}{2}\left(\frac{1}{513}x^3+4x\right)^{\frac{2}{3}}+C$$

Thus, the most general antiderivative representing the aquarium's revenue function is given by $R(x)=\frac{21}{2}\left(\frac{1}{513}x^3+4x\right)^{\frac{2}{3}}+C$ dollars when x day passes are sold. It may appear as though this is the "best" revenue function we can find because there is no explicit function value given in the problem that would allow us to solve for C and find a specific antiderivative.

However, if no day passes are sold, then the revenue is \$0.00. Thus, we actually do have a function value: $R(0)=0$! Substituting $x=0$ into $R(x)$ and setting $R(x)$ equal to 0 allows us to solve for C:

$$R(x) = \frac{21}{2}\left(\frac{1}{513}x^3 + 4x\right)^{\frac{2}{3}} + C \implies$$
$$R(0) = \frac{21}{2}\left(\frac{1}{513}(0)^3 + 4(0)\right)^{\frac{2}{3}} + C = 0 \implies$$
$$\frac{21}{2}(0)^{\frac{2}{3}} + C = 0$$
$$C = 0$$

Thus, the specific antiderivative, or the revenue function, is given by $R(x) = \frac{21}{2}\left(\frac{x^3}{513} + 4x\right)^{\frac{2}{3}}$ dollars, where x is the number of day passes sold.

b. To find the aquarium's revenue from selling 90 day passes, we use the specific antiderivative we found in part **a** and calculate $R(90)$:

$$R(x) = \frac{21}{2}\left(\frac{x^3}{513} + 4x\right)^{\frac{2}{3}} \implies$$
$$R(90) = \frac{21}{2}\left(\frac{(90)^3}{513} + 4(90)\right)^{\frac{2}{3}}$$
$$\approx \$1542.79$$

Selling 90 day passes will result in a revenue of \$1542.79 for the aquarium.

▪

Try It # 6:
Dice Dice Baby has found that its marginal profit function for making and selling polyhedral dice sets is given by $P'(x) = \frac{6x}{\sqrt{7x^2+20}}$ dollars per dice set when x dice sets are made and sold. If the company's profit from making and selling 500 dice sets is \$240, what is the company's profit from making and selling 600 dice sets?

Try It Answers

1. **a.** $\frac{1}{3}\left(x^3 - 3\right)^3 + C$

   b. $-\frac{2}{5}e^{-5x^2+1} + C$

   c. $\frac{8}{35}\ln\left|7x^5 - 10\right| + C$

2. **a.** $\frac{1}{2}e^{6x^3-8x^2} + C$

   b. $\frac{1}{2}(\ln(x))^2 + C$

   c. $\frac{3}{40}\left(8x^5 - 9\right)^{2/3} + C$

3. $\frac{1}{2}\ln\left|e^{2x}+4\right|+C$

4. $\frac{1}{90}(6x-5)^5+\frac{5}{72}(6x-5)^4+C$

5. **a.** $f(x)=\frac{3}{10}\left(2x^5+64\right)^{4/3}-\frac{564}{5}$

   b. $f(x)=\frac{1}{2}(\ln(x))^2+x^4-6x^2+81.5$

6. \$466.78

EXERCISES

Basic Skills Practice

For Exercises 1 - 3, find an appropriate substitution for the indefinite integral (i.e., find u).

1. $\displaystyle\int 2x\left(x^2+4\right)^{2/3}\,dx$

2. $\displaystyle\int \frac{(\ln(x))^7}{x}\,dx$

3. $\displaystyle\int \frac{24x^2}{8x^3-7}\,dx$

For Exercises 4 - 6, find an appropriate substitution for the indefinite integral, and rewrite the integral in terms of the new variable (i.e., rewrite the integral in terms of u and du).

4. $\displaystyle\int 9x^8\left(x^9-13\right)^{0.7}\,dx$

5. $\displaystyle\int 8e^{4x}\,dx$

6. $\displaystyle\int 3x^2\sqrt{x^3+4}\,dx$

For Exercises 7 - 12, find the most general antiderivative of the function using the method of substitution.

7. $f'(x) = 100(5x-13)^{-0.4}$

8. $g'(x) = 30\dfrac{(\ln(x))^9}{x}$

9. $h'(t) = 32t^3e^{t^4}$

10. $j'(x) = \dfrac{12}{(5-x)^2}$

11. $m'(x) = \dfrac{72x^3}{9x^4-15}$

12. $k'(t) = 33t^2\sqrt{t^3-9}$

For Exercises 13 - 15, find the constant of integration, C, of $f(x)$ using the method of substitution and the given information.

13. $f'(x) = 36x^3\left(4-2x^4\right)^{7/2}$ and $f(0) = -500$

14. $f'(x) = \dfrac{24(\ln(x))^{11}}{x}$ and $f(e) = 4$

15. $f'(x) = 28e^{7x}$ and $f(1) = e^7$

For Exercises 16 - 18, find the specific antiderivative of the rate of change function using the given information.

16. $f'(x) = 70(1-7x)^{2/3}$ and $f(-1) = 200$

17. $g'(t) = 8te^{-t^2}$ and $g(0) = 10$

18. $h'(x) = \dfrac{2x}{x^2+16}$ and $h(2) = \ln(20)$

19. A company that makes high-end toaster ovens has a marginal cost function given by $C'(x) = 7\sqrt{14x+169}$ dollars per toaster oven, where x is the number of toaster ovens produced. If the company has fixed costs of \$100, find the cost of producing 200 toaster ovens. *Hint: If the company does not produce any toaster ovens, its cost will equal its fixed costs.*

20. The food truck Crepes by Monica has a marginal profit function given by $P'(x) = \dfrac{20x}{3(x^2-1000)^{2/3}}$ dollars per crepe when x crepes are sold. Monica has determined she loses \$100 if she does not sell any crepes. Find Monica's profit function, in dollars, when x crepes are sold.

21. A company has a weekly marginal revenue function given by $R'(x) = \dfrac{500}{x+2}$ dollars per item when x items are sold each week. Find the company's revenue function, in dollars, when x items are sold each week. *Hint: If the company does not sell any items each week, its revenue is* \$0.

22. A company that makes collector, limited edition action figures has a marginal cost function given by $C'(x) = 2.5\sqrt{5x+16}$ dollars per action figure, where x is the number of action figures made. If the cost of making 20 action figures is \$500, find the cost of making 100 action figures.

23. Leroy Jenkins is running in a straight line. His velocity after t seconds is $v(t) = \dfrac{90t}{t^2+4}$ feet per second. If his starting point is position 0 feet, find a function that gives his position, in feet, after t seconds.

Intermediate Skills Practice

For Exercises 24 - 26, find an appropriate substitution for the indefinite integral, and rewrite the integral in terms of the new variable.

24. $\displaystyle\int (8x+4)\left(x^2+x+18\right)^{9/4}\,dx$

25. $\displaystyle\int \frac{x^2+4}{\left(x^3+12x-2\right)^5}\,dx$

26. $\displaystyle\int \frac{4}{x(\ln(x))^2}\,dx$

For Exercises 27 - 32, find the most general antiderivative of the function.

27. $f'(x) = 4x^{-2}e^{1/x}$

28. $g'(x) = \dfrac{24-12x^3}{\left(112+8x-x^4\right)^2}$

29. $h'(t) = 10t\left(15t^2-240\right)^{2/3}$

30. $f'(x) = 7x^{-1}(\ln(x))^4$

31. $m'(x) = \dfrac{12x^2-2}{8x^3-4x+5}$

32. $R'(t) = \left(6t^2-18t+24\right)\sqrt{2t^3-9t^2+24t+22}$

For Exercises 33 - 35, find the constant of integration of $f(x)$ using the given information.

33. $f'(x) = \dfrac{10(\ln(x))^9}{x}$ and $f(e) = 20$

34. $f'(x) = \dfrac{9x^2-24x}{3x^3-12x^2+18}$ and $f(1) = \ln(9)$

35. $f'(x) = \left(x^2+1\right)e^{2x^3+6x-8}$ and $f(1) = 1$

For Exercises 36 - 38, find the specific antiderivative of the rate of change function using the given information.

36. $h'(x) = \dfrac{8x}{\left(5-x^2\right)^{5/3}}$ and $h(2) = -10$

37. $f'(t) = (t+1)^7$ and $f(0) = 1$

38. $g'(x) = (x+2)e^{-x^2-4x-3}$ and $g(-3) = \dfrac{3}{2}$

39. Generic Corp, a manufacturer of doodads, has a daily marginal cost function given by $C'(x) = 0.77(0.04x + 0.18)\left(0.02x^2 + 0.18x + 30.28\right)^{-2/9}$ dollars per doodad when x doodads are made each day. The fixed costs for Generic Corp are \$15 per day. How much does it cost the company to produce 300 doodads each day?

40. A manufacturer has determined that an employee with d days of production experience will be able to produce items at a rate of $r(d) = 6e^{-0.2d}\left(1 - e^{-0.2d}\right)^2$ items per day. If an employee with no experience can produce 3 items each day, how many items can an employee with 20 days of experience produce each day? Round your answer to the nearest item, if necessary.

41. The marginal profit function for selling x boxes of Thick Mint Scout cookies is given by $P'(x) = \dfrac{4x}{\sqrt{4x^2 + 29}}$ dollars per box. If the profit from selling 100 boxes is \$0 (i.e., the break-even quantity is 100 boxes), find the profit from selling 325 boxes.

42. The velocity of a particle moving in a straight line is given by $v(t) = \dfrac{6t}{(t^2 + 1)^4}$ meters per second after t seconds. If the particle is initially 2 meters to the right of its point of origin, find the position function, in meters, of the particle after t seconds.

Mastery Practice

For Exercises 43 and 44, find an appropriate substitution for the indefinite integral.

43. $\displaystyle\int \frac{18(\ln(x))^7}{5x}\, dx$

44. $\displaystyle\int (6x - 12)\left(2x^2 - 8x + 14\right)^{13}\, dx$

For Exercises 45 - 47, find an appropriate substitution for the indefinite integral, and rewrite the integral in terms of the new variable.

45. $\displaystyle\int \frac{5e^{5x}}{e^{5x} + 9}\, dx$

46. $\displaystyle\int \left(15x^3 + 10x - 5\right)e^{3x^4 + 4x^2 - 4x - 6}\, dx$

47. $\displaystyle\int 3x(4 - x)^{-2/5}\, dx$

For Exercises 48 - 52, find the most general antiderivative of the function.

48. $h'(x) = 16x^2 + 2e^{8x}$

49. $P'(t) = \dfrac{9t}{2 - 3t}$

50. $g'(x) = \dfrac{e^{2x} + e^{-2x}}{e^{2x} - e^{-2x}}$

51. $f'(x) = \dfrac{1}{x\ln\left(\sqrt{x}\right)}$

52. $j'(t) = 7^{2t}$

53. Find the constant of integration of $f(x)$ if $f(0) = -27$ and $f'(x) = \dfrac{x^2}{5x^3 + 1}$.

54. Find the constant of integration of y if $\dfrac{dy}{dx} = \dfrac{(\ln(x))^2}{8x}$ and $y(5) = 0$.

For Exercises 55 - 58, find the specific antiderivative of the rate of change function using the given information.

55. $f'(x) = e^{-x}\left(2e^{-x} - 3\right)^9$ and $f(0) = \dfrac{1}{20}$

56. $g'(t) = \dfrac{4\sqrt[3]{t^{-1} + 20}}{t^2}$ and $g(-1) = 61$

57. $h'(x) = x\sqrt{x - 7}$ and $h(8) = 5$

58. $j'(t) = \dfrac{e^{9t}}{e^{9t} + 6}$ and $j(0) = \dfrac{\ln(7)}{9}$

59. Pack Men, a moving company specializing in safely transporting arcade game cabinets, has a yearly marginal profit function given by $P'(x) = \dfrac{2.57x^{4/3}}{\sqrt{x^{7/3} + 90}} - 3.4$ dollars per arcade cabinet, where x is the number of arcade cabinets transported each year. The break-even quantity for Pack Men is 129 arcade cabinets moved each year. Find the company's profit when it moves 500 arcade cabinets each year.

60. The weekly marginal revenue function for Shoe Fly, a company that makes insect themed footwear, is given by $R'(x) = 300 - 260e^{-0.01x}$ dollars per pair of shoes when x pairs of shoes are sold each week. Find Shoe Fly's weekly revenue function when x pairs of shoes are sold each week.

61. The rate at which the amount of money in a bank account is changing is given by $a(t) = 4000e^{0.08t}$ dollars per year, where t is time in years since the account was opened. If there is \$111,277.05 in the account after 10 years, how much money will be in the account after 20 years?

62. The marginal cost function for a company is given by $\dfrac{d}{dx}C(x) = 200x + \dfrac{15}{2\sqrt{15x + 169}}$ dollars per item, where x is the number of items produced. If the company has fixed costs of \$2000, find the company's cost function when x items are produced.

63. Felicia walks in a straight line at a rate of $v(t) = \dfrac{10\ln(t+1)}{t+1}$ feet per second after t seconds have passed. If her starting point is position 0 feet, find Felicia's position, in feet, after she has walked for 30 seconds. Round your answer to three decimal places, if necessary.

Communication Practice

64. How do we know if the substitution we picked will work when using the method of substitution?

65. What do we do if the substitution we picked does not work when using the method of substitution?

66. If we let $u = 4 - x^3$, does $\int x^2 e^{4-x^3}\,dx = \int e^u\,du$? Explain.

67. Is $u = 4x^2 - x + 5$ an appropriate substitution to solve $\int (8x-1)\left(4x^2 - x + 5\right)^{3/2}\,dx$? Explain.

68. Is $u = 12x^2 - 22$ an appropriate substitution to solve $\int \left(12x^2 - 22\right)\left(4x^3 - 22x + 18\right)^{2/7}\,dx$? Explain.

69. If, after performing a substitution, there is still an x term in the integrand, explain how we can attempt to continue writing the integrand completely in terms of u and du.

4.3 The Definite Integral

Tetra, Patrick Orchard's greyhound, is pictured below in **Figure 4.3.1**. She can reach almost 45 miles per hour, or 20 meters per second, at maximum velocity. If she travels at this velocity for 2 seconds, she will have traveled (20 m/s) · (2 s) = 40 meters.

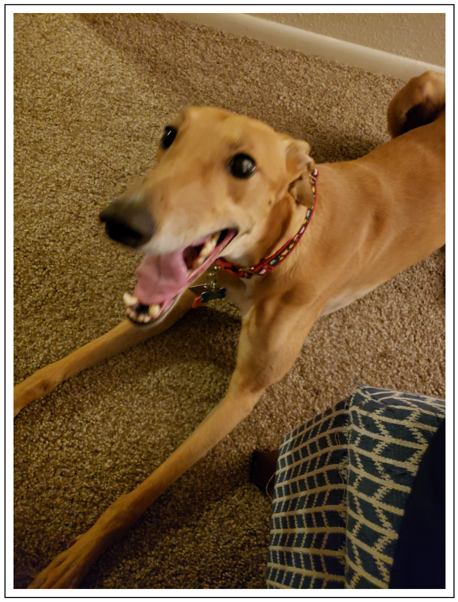


Figure 4.3.1: Tetra, Patrick Orchard's greyhound

In this scenario, we are assuming Tetra's velocity remains constant at 20 meters per second for 2 seconds. However, if her velocity varies during the 2 second time interval, we will not be able to (easily) determine the distance she has traveled. If we are unable to calculate the exact distance traveled by an object because its velocity is not constant, we can *estimate* the distance traveled using a technique involving **Riemann sums**.

Also, if we think of the distance traveled as the *change* in the relevant position function (assuming the velocity function is nonnegative, which it will be in this textbook), we can extend this idea of estimating distance (or change in the position function) to estimating the *change* (or *net change*) in any quantity, such as profit, revenue, or cost, given the corresponding rate of change function, such as the marginal profit, marginal revenue, or marginal cost function. Thus, we can also use Riemann sums to estimate the change in a quantity given its rate of change function if we are unable to calculate the change in the quantity exactly.

Learning Objectives:

In this section, you will learn how to calculate a Riemann sum to estimate the area/net area between a curve and the x-axis, calculate the exact value of a definite integral geometrically, estimate a definite integral using a Riemann sum, and estimate the change (or net change) in a quantity using a Riemann sum. You will also learn properties of definite integrals as well as how to use the properties to find the values of definite integrals. Upon completion you will be able to:

- Write the mathematical formula for a Riemann sum using correct notation.
- Calculate a left-hand Riemann sum.
- Calculate a right-hand Riemann sum.
- Calculate a midpoint Riemann sum.

- Calculate a Riemann sum given the graph of a function.
- Calculate a Riemann sum given a table of function values.
- Calculate a Riemann sum given the rule of a function.
- Determine whether a Riemann sum represents an overestimate or underestimate, if possible.
- Estimate the area/net area between a curve and the x-axis by calculating a Riemann Sum.
- Estimate the change/net change in a real-world quantity, including revenue, cost, and profit, by calculating a Riemann sum.
- Estimate the distance traveled by an object, or the change in its position (assuming the velocity function is nonnegative), by calculating a Riemann sum.
- Express the definite integral as the limit of a Riemann sum.
- Calculate the value of a definite integral geometrically by finding the exact net area between the curve and the x-axis.
- Calculate the value of a definite integral to find the exact net change in a real-word quantity, including revenue, cost, and profit, by calculating the exact net area between the graph of the corresponding rate of change function and the x-axis.
- Calculate the value of a definite integral to find the distance traveled by an object, or the change in its position (assuming the velocity function is nonnegative), by calculating the exact area under its velocity curve.
- Estimate the value of a definite integral by calculating a Riemann sum.
- Calculate the value of a definite integral using the Properties of the Definite Integral.
- Combine several definite integrals into a single definite integral using the Properties of the Definite Integral.
- Calculate the value of a definite integral using given values of other related definite integrals and the Properties of the Definite Integral.
- Calculate the smallest and largest possible values of a definite integral using the Comparison Properties of the Definite Integral.

Exact Area Under a Curve

Let's look at the scenario mentioned in the introduction graphically. We assumed Tetra's velocity was constant at a rate of 20 meters per second for 2 seconds. The graph of her (constant) velocity function, $f(t) = 20$ meters per second, is shown in **Figure 4.3.2**, and the region under the velocity curve between $t = 0$ and $t = 2$ seconds is shaded:

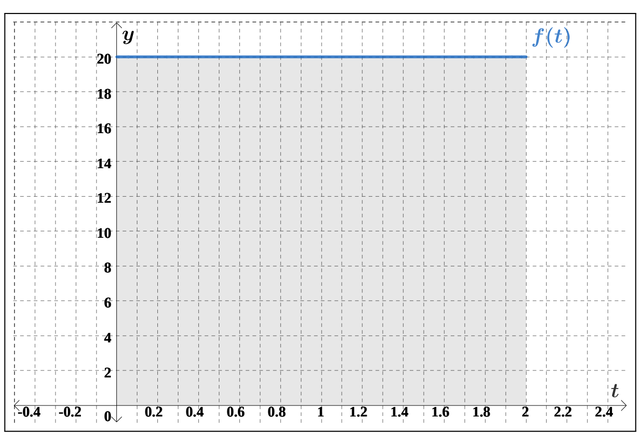


Figure 4.3.2: Shaded region under Tetra's velocity curve on the interval $[0,2]$

Recall that we calculated the distance Tetra traveled during the 2 second time interval to be (20 m/s) · (2 s) = 40 meters. Notice the area of the shaded region, which is a rectangle, is also 40. This is not a coincidence! Given a nonnegative function for velocity, the area between the *t*-axis, the velocity function, and the *t*-coordinates will give the distance traveled. Because we assumed Tetra's velocity remained constant for the entire 2 seconds, we were able to calculate the distance she traveled *exactly*.

Calculating distance traveled is just one application of finding the area under a curve. Before we investigate other applications, let's practice finding the exact area under a curve.

! *The area under an object's velocity curve only represents the distance traveled by the object if the velocity curve is nonnegative. As stated previously, the velocity functions in this textbook are nonnegative.*

Example 1 The graph of f is shown in **Figure 4.3.3**. Find the exact area between the curve and the x-axis on each of the following intervals.

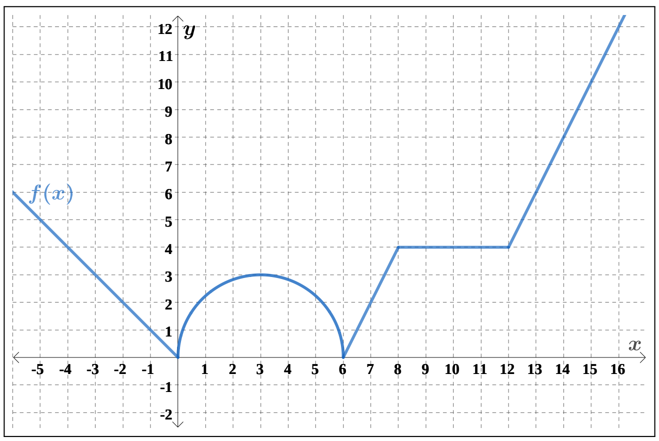


Figure 4.3.3: Graph of a function f

a. $[0,6]$
b. $[-4,3]$
c. $[10,14]$

Solution:

a. The area we are trying to find is that of a semicircle, which is shaded in **Figure 4.3.4**:

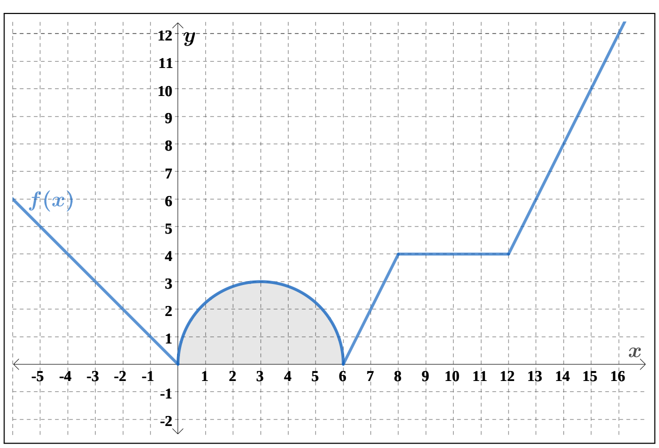


Figure 4.3.4: Graph of a function, f, in which the region of a semicircle is shaded

The area of a semicircle, which is half of a circle, is given by $A(r) = \dfrac{\pi r^2}{2}$, where r is the length of its radius. The radius here is 3, so the area is $\dfrac{\pi(3)^2}{2} = \dfrac{9\pi}{2}$.

b. To find the area corresponding to the interval $[-4, 3]$, we will break the region into two separate regions: a triangle on the interval $[-4, 0]$ and a quarter circle on the interval $[0, 3]$. See **Figure 4.3.5**.

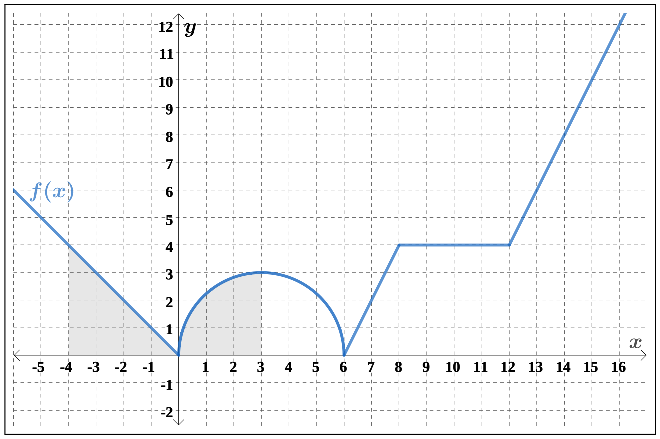


Figure 4.3.5: Graph of a function, f, in which the regions of a triangle and quarter circle are shaded

We can find the area of each region and then add them together. The formula for the area of a triangle is given by $A_{\text{triangle}} = \dfrac{bh}{2}$, where b is the length of its base and h is the length of its height. The length of the base of this triangle is 4, and its height is also 4. So the area of this region is $\dfrac{4 \cdot 4}{2} = 8$. The quarter circle is the area of a circle divided by 4, so the area of this portion is $\dfrac{\pi(3)^2}{4} = \dfrac{9\pi}{4}$. Thus, the total area is exactly $8 + \dfrac{9\pi}{4}$.

c. To find the area corresponding to the interval $[10, 14]$, we must find the area of the region that is shaded in **Figure 4.3.6**:

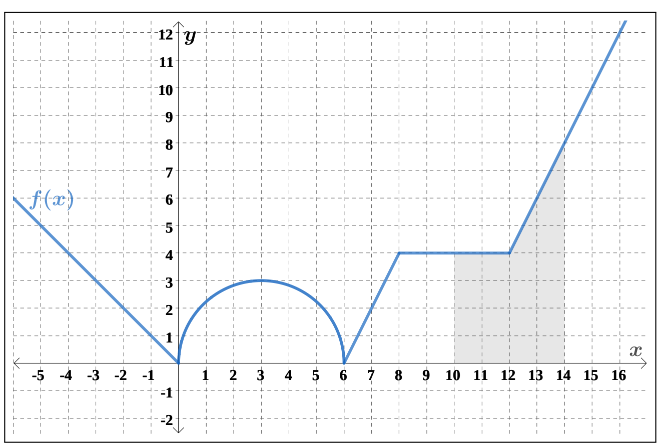


Figure 4.3.6: Graph of a function, f, in which a region is shaded on the interval $[10, 14]$

This is not a shape we are familiar with, so we need to divide the region into geometric shapes of which we know how to find the areas. Let's create a triangle on top of a square as shown in **Figure 4.3.7**:

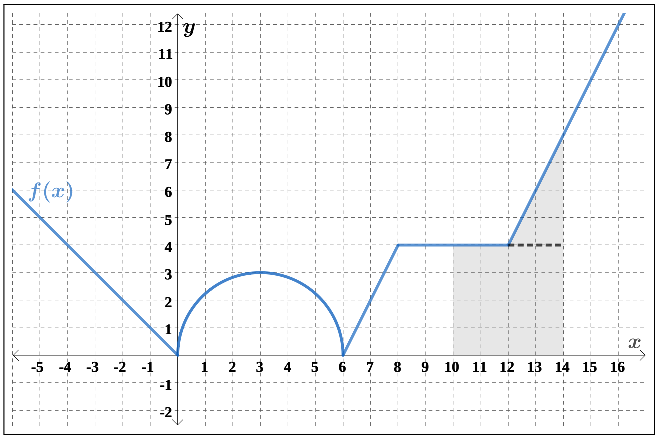


Figure 4.3.7: Graph of a function, f, in which the regions of a square and triangle are shaded

Now, we have a square with a length and width of 4 and a triangle with a base of 2 and a height of 4. Therefore, the exact area of the shaded region is $4^2 + \dfrac{2 \cdot 4}{2} = 20$.

This is not the only way to view this region! We could view it as a rectangle (with its base on the interval $[10, 12]$*) and a trapezoid. No matter how we divide the region, we should still arrive at the same answer.*

▪

Try It # 1:
Using **Figure 4.3.3**, find the exact area between the graph of f and the x-axis on the interval $[-4, 12]$.

▪ **Example 2** Find the exact area under the curve $y = -\frac{1}{2}x + 1$ and above the x-axis on the interval $[-2, 2]$.

Solution:
We begin by graphing the function as shown in **Figure 4.3.8**. The shaded region represents the area we are trying to find:

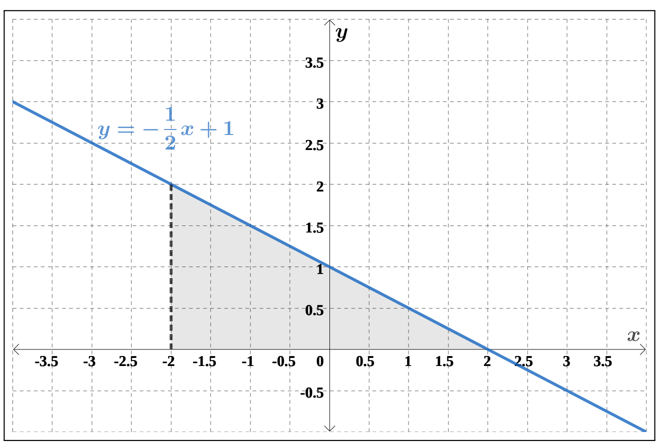


Figure 4.3.8: Graph of the function $y = -\frac{1}{2}x + 1$

The shaded region is a triangle! The height of the triangle is 2, and its base is 4. Thus, the area is $\dfrac{2 \cdot 4}{2} = 4$.

▪

Try It # 2:
Find the exact area under the curve $y = 2x - 4$ on the interval $[2, 10]$.

Riemann Sums

In very few cases is the graph of a function defined by nice geometric regions like we have seen in the previous examples. In fact, returning to the discussion in the introduction, it is unlikely Tetra's velocity remains constant at 20 meters per second during the 2 second interval (or any object's velocity).

For instance, on the greyhound racetracks, the dogs start at a velocity of 0 meters per second. Tetra can accelerate to her maximum velocity within 6 seconds. Let's assume that Tetra starts from rest and accelerates at an increasing rate until she reaches her top speed. Assuming her velocity is $f(t)$ meters per second after t seconds, the distance she travels is represented by the area of the region shaded in **Figure 4.3.9** (because the velocity curve is nonnegative). In other words, her distance is given by the area below the velocity curve, above the t-axis, and between the t-values 0 and 6 seconds:

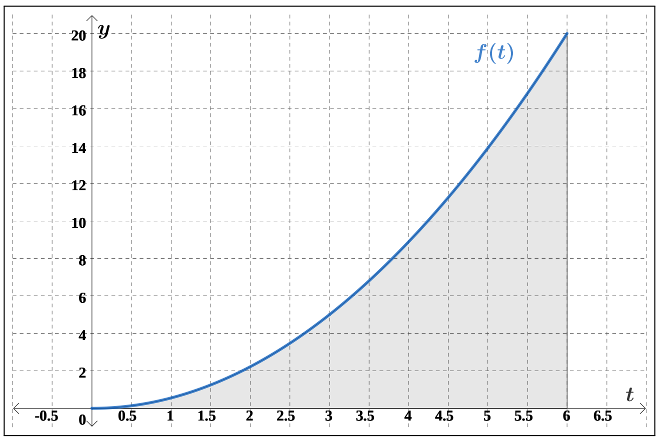


Figure 4.3.9: Shaded region under Tetra's velocity curve on the interval $[0, 6]$

Unfortunately, we do not know how to calculate the area of the shape represented by this shaded region. However, we can estimate the area by dividing the region into subintervals and creating rectangles of which we can calculate the areas. For example, if we divide the region into 6 subintervals and, as a result, create 6 rectangles, each with a base of width 1 and a height equal to the y-value of a point on the function, we obtain the graph and rectangles shown in **Figure 4.3.10**:

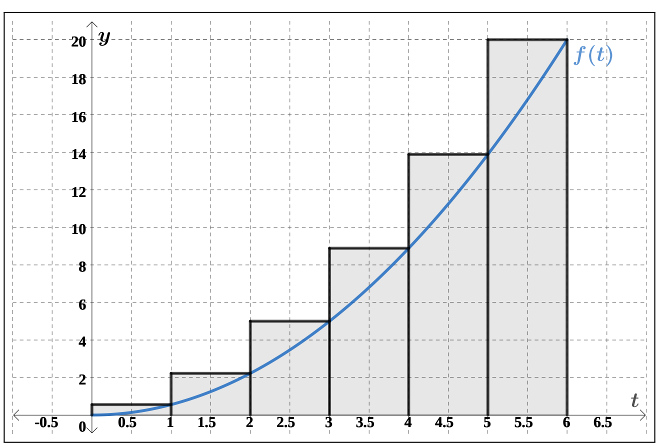


Figure 4.3.10: Six rectangles whose areas provide an estimate for the area under the velocity curve on the interval $[0, 6]$

The areas of all 6 rectangles added together will give us an estimate for the exact area under the curve between $t = 0$ and $t = 6$ seconds. Notice in this particular case, adding the rectangles together results in an overestimate of the actual area under the curve (the shaded portion of each rectangle extends above the velocity function).

Because a large portion of the time we cannot use geometric formulas to find the exact area of a region under a curve, we will use this technique of dividing the region into subintervals to create rectangles, which we can calculate the areas of, to obtain an estimate of the total area.

Before we can fully develop the ideas for estimating the area under a curve by dividing the region into subintervals, we need to discuss the mathematical notation that represents adding several quantities together, what mathematicians call **summation notation** or **Sigma notation**.

Summation Notation

As mentioned previously, we can estimate the area under a curve by dividing the region into subintervals to create rectangles and then add the areas of the rectangles together. When we use this approach with several subintervals (i.e., rectangles), it is helpful to use the mathematical notation for adding several values together: **Sigma** (Σ) notation.

For example, say we want to calculate the following sum:

$$\sum_{k=1}^{5} k^2$$

Before calculating this sum, we will first familiarize ourselves with some terminology. The variable (typically i, j, k, m, or n) used in the summation is called the **counter** or **index** variable. The function to the right of the sigma is called the **summand**, while the numbers below and above the sigma are called the **lower** and **upper limits** of the summation.

To begin calculating this sum, we start with the index variable at the lower limit of the summation: $k = 1$. Substituting $k = 1$ into the summand, we get $(1)^2$. Next, we add this value to the value of the summand when substituting the next (integer) value of the index variable: $k = 2$. Substituting $k = 2$ gives $(2)^2$. So thus far, we have $(1)^2 + (2)^2$. We continue in this manner until we get to the upper limit of the summation, $k = 5$:

$$\begin{aligned}\sum_{k=1}^{5} k^2 &= (1)^2 + (2)^2 + (3)^2 + (4)^2 + (5)^2 \\ &= 1 + 4 + 9 + 16 + 25 \\ &= 55\end{aligned}$$

■ **Example 3** Find the value of $\displaystyle\sum_{i=2}^{6} \frac{i+3}{4}$.

Solution:

Starting with the index variable at $i = 2$, we have

$$\begin{aligned}\sum_{i=2}^{6} \frac{i+3}{4} &= \frac{2+3}{4} + \frac{3+3}{4} + \frac{4+3}{4} + \frac{5+3}{4} + \frac{6+3}{4} \\ &= \frac{5}{4} + \frac{6}{4} + \frac{7}{4} + \frac{8}{4} + \frac{9}{4} \\ &= \frac{35}{4}\end{aligned}$$

■

Try It # 3:

Find the value of $\sum_{j=0}^{4} 2^j$.

Riemann Sum Notation

To learn how to calculate a sum in which we are adding the areas of rectangles to estimate the area under a curve as well as how to apply the summation notation we just discussed, let's return to our motivating scenario regarding the greyhound Tetra.

The graph of Tetra's (non-constant) velocity function shown in **Figures 4.3.9** and **4.3.10** is given by $f(t) = \frac{5}{9}t^2$ meters per second after t seconds. To estimate the distance she traveled between $t = 0$ and $t = 6$ seconds, we need to estimate the area under the curve (because the velocity function is nonnegative), which we can do by dividing the region into subintervals of equal width and creating rectangles of which we can calculate the areas.

Again, let's assume we are dividing the region into 6 subintervals of equal width, so the width of each subinterval (i.e., rectangle) is 1 second. We will let the height of each rectangle be given by the y-value of the function corresponding to the x-value that is the right endpoint of each subinterval (although we could select any x-value in each subinterval to determine the y-value, or height, of each rectangle). See **Figure 4.3.11**.

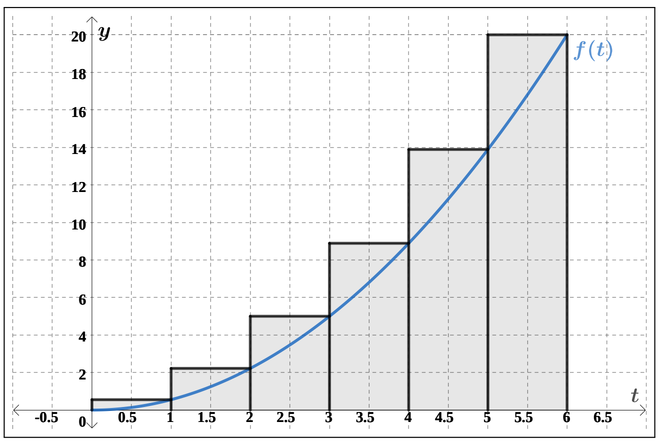


Figure 4.3.11: Six rectangles whose areas provide an estimate for the area under the velocity curve on the interval $[0,6]$

Thus, the heights of the rectangles are given by the function values $f(1)$, $f(2)$, $f(3)$, $f(4)$, $f(5)$, and $f(6)$. Remembering that $f(t) = \frac{5}{9}t^2$ meters per second and that the width of each rectangle is 1 second, we multiply the height and the width of each rectangle and then add the areas of all 6 rectangles:

$$\begin{aligned}
\text{area} &\approx f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1 + f(5)\cdot 1 + f(6)\cdot 1 \\
&= \frac{5}{9}(1)^2\cdot 1 + \frac{5}{9}(2)^2\cdot 1 + \frac{5}{9}(3)^2\cdot 1 + \frac{5}{9}(4)^2\cdot 1 + \frac{5}{9}(5)^2\cdot 1 + \frac{5}{9}(6)^2\cdot 1 \\
&= \frac{5}{9}\cdot 1 + \frac{20}{9}\cdot 1 + 5\cdot 1 + \frac{80}{9}\cdot 1 + \frac{125}{9}\cdot 1 + 20\cdot 1 \\
&= \frac{455}{9} \approx 51 \text{ meters}
\end{aligned}$$

Therefore, Tetra traveled approximately 51 meters during the 6 second interval. Remember, as noted previously, this is an overestimate of the area under the velocity curve, so this is an overestimate of the distance she traveled.

Note that to create the six rectangles we used to estimate Tetra's distance, we first divided the interval $[0,6]$ into 6 subintervals of equal width (each of width $\frac{6-0}{6} = 1$). Then, we selected the right endpoint of each subinterval to determine the height of each rectangle (given by the y-value of the function). Calculating the heights of the rectangles using the *right* endpoint of each subinterval means we are calculating a **right-hand sum**, and it is denoted R_n, where n is the number of subintervals (i.e., rectangles).

In this particular example, selecting the right endpoints to obtain the height of each rectangle gives more area than is actually under the curve (i.e., it gives an overestimate). Therefore, we know Tetra traveled approximately 51 meters, but definitely not more than that. This will not always be true! There may be cases where we use the right endpoints and obtain an underestimate of the area under a curve.

In general, if we want to estimate the area under a curve on the interval $[a,b]$ (i.e., $a \leq x \leq b$), where a and b are real numbers, we begin by dividing the interval into n subintervals of equal width given by Δx:

$$\Delta x = \frac{b-a}{n}$$

We will denote the x-coordinates of the subintervals on a number line as shown in **Figure 4.3.12**:

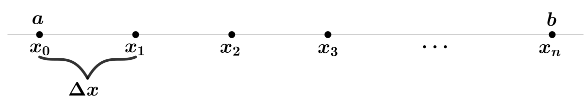


Figure 4.3.12: Number line showing the notation for the x-coordinates of n subintervals on the interval $[a,b]$

Definition
The x-value we choose in each subinterval to determine the height of a particular rectangle is called a **sample point**. ▪

Using this notation, to estimate the area under a curve taking the sample points to be the right endpoints of each subinterval, we calculate

$$\sum_{i=1}^{n} f(x_i) \cdot \Delta x = f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + f(x_3) \cdot \Delta x + \cdots f(x_n) \cdot \Delta x$$

If we let $n = 6$ and $\Delta x = 1$, then the right-hand side of this equation is exactly what we obtained when we estimated Tetra's distance using the right-hand sum previously:

$$\begin{aligned}\sum_{i=1}^{6} f(x_i) \cdot \Delta x &= f(x_1) \cdot 1 + f(x_2) \cdot 1 + f(x_3) \cdot 1 + f(x_4) \cdot 1 + f(x_5) \cdot 1 + f(x_6) \cdot 1 \\ &= f(1) \cdot 1 + f(2) \cdot 1 + f(3) \cdot 1 + f(4) \cdot 1 + f(5) \cdot 1 + f(6) \cdot 1\end{aligned}$$

Recalling that Tetra's velocity function is given by $f(t) = \frac{5}{9}t^2$ meters per second after t seconds and continuing to calculate this sum gives the same distance we calculated previously:

$$\begin{aligned}f(1) \cdot 1 + f(2) \cdot 1 + f(3) \cdot 1 + f(4) \cdot 1 + f(5) \cdot 1 + f(6) \cdot 1 &= \frac{5}{9} \cdot 1 + \frac{20}{9} \cdot 1 + 5 \cdot 1 + \frac{80}{9} \cdot 1 + \frac{125}{9} \cdot 1 + 20 \cdot 1 \\ &= \frac{455}{9} \approx 51 \text{ meters}\end{aligned}$$

In general, we can choose any x-value within each subinterval to determine the height of each rectangle. For instance, if we calculate the heights of the rectangles using the left endpoint of each subinterval, then we are calculating a **left-hand sum**. A left-hand sum is denoted L_n, where n is the number of subintervals.

Referring back to the motivating example with Tetra, we could have used the left endpoint of each subinterval to determine the heights of the rectangles. The graph of Tetra's velocity function, $f(t) = \frac{5}{9}t^2$ meters per second, is shown in **Figure 4.3.13** with rectangles of width 1 second and heights given by the left endpoint of each subinterval:

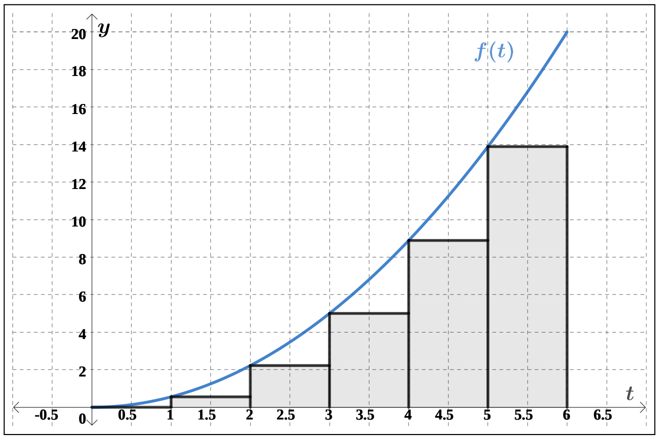


Figure 4.3.13: Six rectangles whose areas provide an estimate for the area under the velocity curve on the interval $[0,6]$

Using the parlance of the number line in **Figure 4.3.12** with $n = 6$, the left-hand sum L_6 is given by

$$\begin{aligned}\sum_{i=0}^{5} f(x_i)\cdot\Delta x &= f(x_0)\cdot 1 + f(x_1)\cdot 1 + f(x_2)\cdot 1 + f(x_3)\cdot 1 + f(x_4)\cdot 1 + f(x_5)\cdot 1\\ &= f(0)\cdot 1 + f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1 + f(5)\cdot 1\end{aligned}$$

Recalling that Tetra's velocity function is given by $f(t) = \frac{5}{9}t^2$ meters per second after t seconds, we have

$$\begin{aligned}f(0)\cdot 1 + f(1)\cdot 1 + f(2)\cdot 1 + f(3)\cdot 1 + f(4)\cdot 1 + f(5)\cdot 1 &= \frac{5}{9}(0)^2\cdot 1 + \frac{5}{9}(1)^2\cdot 1 + \frac{5}{9}(2)^2\cdot 1 + \frac{5}{9}(3)^2\cdot 1 + \frac{5}{9}(4)^2\cdot 1 + \frac{5}{9}(5)^2\cdot 1\\ &= 0\cdot 1 + \frac{5}{9}\cdot 1 + \frac{20}{9}\cdot 1 + 5\cdot 1 + \frac{80}{9}\cdot 1 + \frac{125}{9}\cdot 1\\ &= \frac{275}{9} \approx 31 \text{ meters}\end{aligned}$$

In general, a left-hand sum is given by

$$\sum_{i=0}^{n-1} f(x_i)\cdot\Delta x = f(x_0)\cdot\Delta x + f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + \cdots f(x_{n-1})\cdot\Delta x$$

The difference between the formula for a left-hand sum and the formula for a right-hand sum is subtle, but important. The index for the left-hand sum starts at 0 and ends at $n-1$, whereas for the right-hand sum, the index starts at 1 and ends at n.

Notice that for the left-hand sum we calculated, the area of all the rectangles is *less* than the total area under the velocity curve. This means we calculated an underestimate. Thus, we know Tetra traveled somewhere between approximately 31 and 51 meters during this 6 second interval.

Be careful again not to assume that a left-hand sum will always give an underestimate. If the function is *increasing* (as it is here), then a left-hand sum will give an underestimate and a right-hand sum will give an overestimate. If the function is *decreasing*, the opposite is true: The left-hand sum will give an overestimate, and the right-hand sum will give an underestimate. However, if the function is not always increasing or decreasing, then there is no guarantee that these sums will be an over or underestimate!

As stated previously, we can choose any x-value within each subinterval to determine the heights of the rectangles. We need not restrict ourselves to the endpoints. For the purposes of this textbook, we will use the endpoints or the midpoint of each subinterval as our sample points.

The sums we have developed thus far to estimate the area under a curve have a special name: They are called **Riemann sums** (named for the famous and prolific German mathematician Bernhard Riemann). We will now define Riemann sums formally:

Definition

Given a function f defined on an interval $[a,b]$ and divided into n subintervals, let $\Delta x = \dfrac{b-a}{n}$ be the width of each subinterval and x_i^* be any x-value in the i^{th} subinterval. A **Riemann sum** defined for f is

$$\sum_{i=1}^{n} f\left(x_i^*\right)\cdot\Delta x = f\left(x_1^*\right)\cdot\Delta x + f\left(x_2^*\right)\cdot\Delta x + f\left(x_3^*\right)\cdot\Delta x + \cdots + f\left(x_n^*\right)\cdot\Delta x$$

The x_i^*-value we choose in each subinterval to determine the height of a particular rectangle is called a **sample point**.

■

N *It is possible to precisely define Riemann sums for intervals of differing sizes, but that is beyond the scope of this textbook.*

■ **Example 4** Estimate the area under the graph of $f(t) = \dfrac{5}{9}t^2$ on the interval $[0,6]$ using 6 subintervals of equal width and a midpoint Riemann sum.

Solution:

The width of each subinterval is given by $\Delta x = \dfrac{b-a}{n} = \dfrac{6-0}{6} = 1$. This means the subintervals are partitioned by the x-values 0, 1, 2, 3, 4, 5, and 6. See **Figure 4.3.14**.

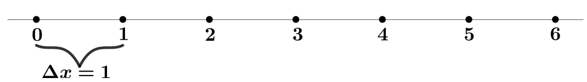


Figure 4.3.14: Number line showing the x-values partitioning six subintervals on the interval $[0,6]$

The midpoints (i.e., sample points) of these subintervals are

$$x_1^* = 0.5,\ x_2^* = 1.5,\ x_3^* = 2.5,\ x_4^* = 3.5,\ x_5^* = 4.5, \text{ and } x_6^* = 5.5$$

Using this information, we will graph the velocity function and the rectangles corresponding to the midpoint Riemann sum, M_6. See **Figure 4.3.15**.

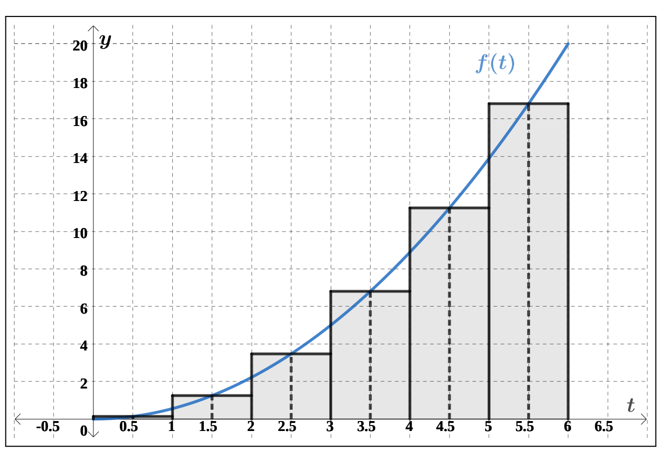


Figure 4.3.15: Six rectangles whose areas provide an estimate for the area under the velocity curve on the interval $[0,6]$

Using the summation notation representing the fact that the sample points can be *any* x-value in each subinterval, we have

$$\begin{aligned}
\sum_{i=1}^{6} f\left(x_i^*\right)\cdot\Delta x &= f\left(x_1^*\right)\cdot\Delta x + f\left(x_2^*\right)\cdot\Delta x + f\left(x_3^*\right)\cdot\Delta x + f\left(x_4^*\right)\cdot\Delta x + f\left(x_5^*\right)\cdot\Delta x + f\left(x_6^*\right)\cdot\Delta x \\
&= f(0.5)\cdot 1 + f(1.5)\cdot 1 + f(2.5)\cdot 1 + f(3.5)\cdot 1 + f(4.5)\cdot 1 + f(5.5)\cdot 1 + f(6.5)\cdot 1 \\
&= \frac{5}{9}(0.5)^2\cdot 1 + \frac{5}{9}(1.5)^2\cdot 1 + \frac{5}{9}(2.5)^2\cdot 1 + \frac{5}{9}(3.5)^2\cdot 1 + \frac{5}{9}(4.5)^2\cdot 1 + \frac{5}{9}(5.5)^2\cdot 1 \\
&= \frac{5}{36}\cdot 1 + \frac{5}{4}\cdot 1 + \frac{125}{36}\cdot 1 + \frac{245}{36}\cdot 1 + \frac{45}{4}\cdot 1 + \frac{605}{36}\cdot 1 \\
&= \frac{715}{18} \approx 40 \text{ meters}
\end{aligned}$$

Unlike the left-hand and right-hand sums we computed earlier for this function, we cannot determine if this midpoint Riemann sum is an over or an underestimate of the area! It is an approximation (and much better than the other two), but there is no guarantee that it is larger or smaller than the exact area under the curve.

■

■ **Example 5** Estimate the area under the curve of $f(x) = (x-1)^4$ on the interval $[0,2]$ using

a. a left-hand Riemann sum with 2 subintervals of equal width.
b. a right-hand Riemann sum with 4 subintervals of equal width.
c. a midpoint Riemann sum with 6 subintervals of equal width.

Solution:

a. Dividing the interval $[0,2]$ into 2 subintervals of equal width means that each subinterval will have width $\Delta x = \frac{b-a}{n} = \frac{2-0}{2} = 1$. Thus, the subintervals are $[0,1]$ and $[1,2]$. The left endpoints of each subinterval are 0 and 1, respectively. Thus, the sample points of each subinterval are

$$x_0 = 0 \text{ and } x_1 = 1$$

Hence, recalling that $f(x) = (x-1)^4$, the left-hand Riemann sum is

$$\begin{aligned}
\sum_{i=0}^{1} f(x_i)\cdot\Delta x &= f(x_0)\cdot\Delta x + f(x_1)\cdot\Delta x \\
&= f(0)\cdot 1 + f(1)\cdot 1 \\
&= (0-1)^4\cdot 1 + (1-1)^4\cdot 1 \\
&= 1 + 0 \\
&= 1
\end{aligned}$$

b. Recall that we must estimate the area under the curve of $f(x) = (x-1)^4$ on the interval $[0,2]$ using a right-hand Riemann sum with 4 subintervals of equal width.

Dividing the interval into 4 subintervals of equal width means that each subinterval will have width $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$. We can sketch a number line to help us determine the sample points x_1, x_2, x_3, and x_4. See **Figure 4.3.16**.

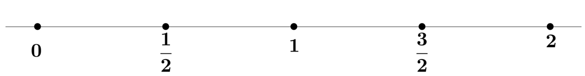


Figure 4.3.16: Number line showing the x-values partitioning four subintervals on the interval $[0,2]$

The sample points of each subinterval are

$$x_1 = \frac{1}{2},\ x_2 = 1,\ x_3 = \frac{3}{2}, \text{ and } x_4 = 2$$

Recalling that $f(x) = (x-1)^4$, we now calculate the right-hand Riemann sum to estimate the area:

$$\begin{aligned}\sum_{i=1}^{4} f(x_i)\cdot\Delta x &= f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + f(x_3)\cdot\Delta x + f(x_4)\cdot\Delta x \\ &= f\left(\frac{1}{2}\right)\cdot\frac{1}{2} + f(1)\cdot\frac{1}{2} + f\left(\frac{3}{2}\right)\cdot\frac{1}{2} + f(2)\cdot\frac{1}{2} \\ &= \left(\frac{1}{2}-1\right)^4\cdot\frac{1}{2} + (1-1)^4\cdot\frac{1}{2} + \left(\frac{3}{2}-1\right)^4\cdot\frac{1}{2} + (2-1)^4\cdot\frac{1}{2} \\ &= \frac{9}{16}\end{aligned}$$

c. Recall that we must estimate the area under the curve of $f(x) = (x-1)^4$ on the interval $[0,2]$ using a midpoint Riemann sum with 6 subintervals of equal width.

Dividing the interval $[0,2]$ further into 6 subintervals means that each subinterval will have width $\Delta x = \frac{b-a}{n} = \frac{2-0}{6} = \frac{1}{3}$. The number line shown in **Figure 4.3.17** will help us find the sample points (i.e., midpoints):

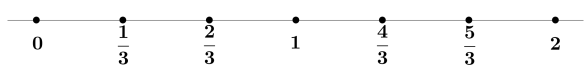


Figure 4.3.17: Number line showing the x-values partitioning six subintervals on the interval $[0,2]$

The sample points of each subinterval are

$$x_1^* = \frac{1}{6},\ x_2^* = \frac{1}{2},\ x_3^* = \frac{5}{6},\ x_4^* = \frac{7}{6},\ x_5^* = \frac{3}{2}, \text{ and } x_6^* = \frac{11}{6}$$

Therefore, recalling that $f(x) = (x-1)^4$, the midpoint Riemann sum is

$$\begin{aligned}\sum_{i=1}^{6} f\left(x_i^*\right)\cdot\Delta x &= f\left(x_1^*\right)\cdot\Delta x + f\left(x_2^*\right)\cdot\Delta x + f\left(x_3^*\right)\cdot\Delta x + f\left(x_4^*\right)\cdot\Delta x + f\left(x_5^*\right)\cdot\Delta x + f\left(x_6^*\right)\cdot\Delta x \\ &= f\left(\frac{1}{6}\right)\cdot\frac{1}{3} + f\left(\frac{1}{2}\right)\cdot\frac{1}{3} + f\left(\frac{5}{6}\right)\cdot\frac{1}{3} + f\left(\frac{7}{6}\right)\cdot\frac{1}{3} + f\left(\frac{3}{2}\right)\cdot\frac{1}{3} + f\left(\frac{11}{6}\right)\cdot\frac{1}{3} \\ &= \left(\frac{1}{6}-1\right)^4\cdot\frac{1}{3} + \left(\frac{1}{2}-1\right)^4\cdot\frac{1}{3} + \left(\frac{5}{6}-1\right)^4\cdot\frac{1}{3} + \left(\frac{7}{6}-1\right)^4\cdot\frac{1}{3} + \left(\frac{3}{2}-1\right)^4\cdot\frac{1}{3} + \left(\frac{11}{6}-1\right)^4\cdot\frac{1}{3} \\ &= \frac{707}{1944}\end{aligned}$$

■

Try It # 4:
Estimate the area under the curve of $f(x) = -x^2 + 12x - 20$ on the interval $[2, 10]$ using 4 subintervals of equal width and a

a. right-hand Riemann sum.
b. left-hand Riemann sum.
c. midpoint Riemann sum.

Example 6 The graph of a function f is shown in **Figure 4.3.18**.

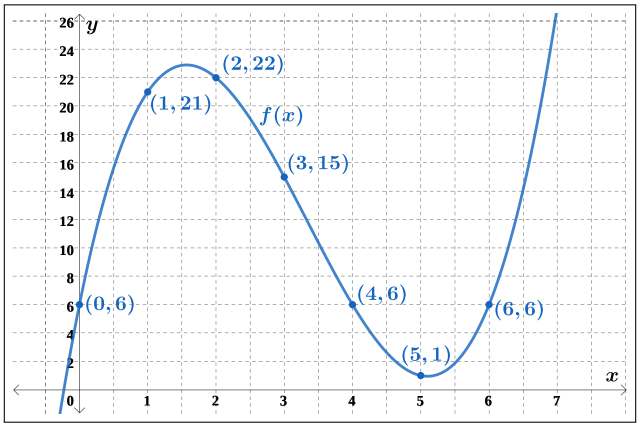


Figure 4.3.18: Graph of a function f

Estimate the area under the curve on the interval $[0, 6]$ using 3 subintervals of equal width and a

a. midpoint Riemann sum.
b. left-hand Riemann sum.

Solution:

a. Because $\Delta x = \dfrac{b-a}{n} = \dfrac{6-0}{3} = 2$, we have the following 3 subintervals for the interval $[0, 6]$ shown in **Figure 4.3.19**:

Figure 4.3.19: Number line showing the x-values partitioning three subintervals on the interval $[0, 6]$

The sample points (i.e., midpoints) of each subinterval are

$$x_1^* = 1, \ x_2^* = 3, \text{ and } x_3^* = 5$$

The rectangles we will use to estimate the area under the curve are shown in **Figure 4.3.20**:

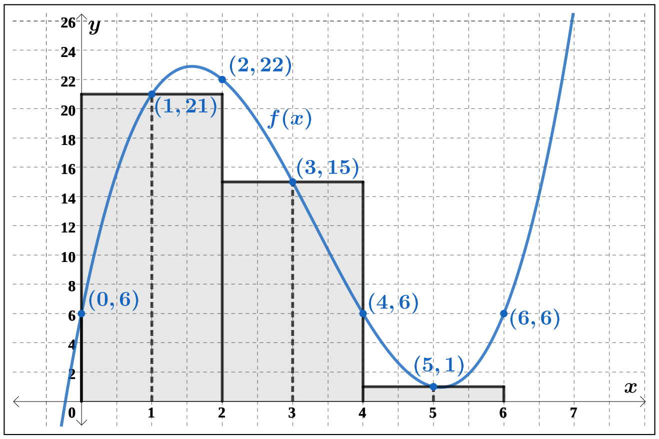


Figure 4.3.20: Three rectangles whose areas provide an estimate for the area under the curve on the interval $[0,6]$

From the graph, we can identify the heights (and widths) of the rectangles to calculate the midpoint Riemann sum:

$$\begin{aligned}\sum_{i=1}^{3} f\left(x_i^*\right)\cdot\Delta x &= f\left(x_1^*\right)\cdot\Delta x + f\left(x_2^*\right)\cdot\Delta x + f\left(x_3^*\right)\cdot\Delta x\\ &= f(1)\cdot 2 + f(3)\cdot 2 + f(5)\cdot 2+\\ &= 21\cdot 2 + 15\cdot 2 + 1\cdot 2\\ &= 74\end{aligned}$$

b. Recall that we must estimate the area under the curve on the interval $[0,6]$ using 3 subintervals of equal width and a left-hand Riemann sum.

We can look at the number line showing the subintervals again to help us determine the sample points (see **Figure 4.3.19**). Thus, for a left-hand Riemann sum, the sample points are

$$x_0 = 0,\ x_1 = 2, \text{ and } x_2 = 4$$

The rectangles we will use to estimate the area under the curve are shown in **Figure 4.3.21**:

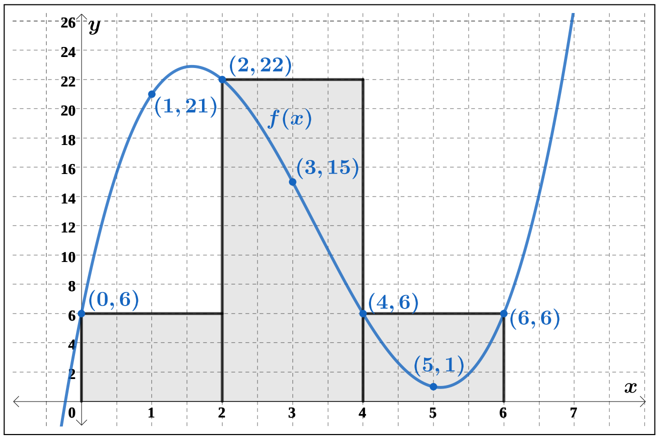


Figure 4.3.21: Three rectangles whose areas provide an estimate for the area under the curve on the interval $[0,6]$

The left-hand Riemann sum is

$$\begin{aligned}\sum_{i=0}^{2} f(x_i)\cdot\Delta x &= f(x_0)\cdot\Delta x + f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x\\ &= f(0)\cdot 2 + f(2)\cdot 2 + f(4)\cdot 2\\ &= 6\cdot 2 + 22\cdot 2 + 6\cdot 2\\ &= 68\end{aligned}$$

▪

Try It # 5:
Estimate the area under the curve of f shown in **Figure 4.3.18** on the interval $[0,6]$ using 6 subintervals of equal width and a right-hand Riemann sum.

Without more information, it is impossible to determine if the sums in the previous example and try it are over or underestimates of the exact area. We can only determine this information for left- and right-hand Riemann sums if the function is either strictly increasing or strictly decreasing.

▪ **Example 7** Particular values of a positive function f are given in **Table 4.1**. Use the table to estimate the area under the graph of f on the interval $[-3,-0.5]$ using 5 subintervals of equal width and a right-hand Riemann sum.

x	-3	-2.5	-2	-1.5	-1	-0.5
$f(x)$	2.3	7.1	9	1.4	1	2

Table 4.1: Particular values of $f(x)$

Solution:
Each subinterval has a width of $\Delta x = \dfrac{b-a}{n} = \dfrac{-0.5-(-3)}{5} = \dfrac{2.5}{5} = 0.5$, and the endpoints of the subintervals are given in the table. Thus, for a right-hand Riemann sum, we have

$$\begin{aligned}\sum_{i=1}^{5} f(x_i)\cdot\Delta x &= f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + f(x_3)\cdot\Delta x + f(x_4)\cdot\Delta x + f(x_5)\cdot\Delta x\\ &= f(-2.5)\cdot 0.5 + f(-2)\cdot 0.5 + f(-1.5)\cdot 0.5 + f(-1)\cdot 0.5 + f(-0.5)\cdot 0.5\\ &= 7.1\cdot 0.5 + 9\cdot 0.5 + 1.4\cdot 0.5 + 1\cdot 0.5 + 2\cdot 0.5\\ &= 10.25\end{aligned}$$

▪

Try It # 6:
Using **Table 4.1**, estimate the area under the graph of f on the interval $[-3,-0.5]$ using 5 subintervals of equal width and a left-hand Riemann sum.

Applications

Recall that when given a nonnegative velocity function (measured in units of distance per unit of time) of an object, the area under the velocity curve between two particular values of time, $t = a$ and $t = b$, gives the distance the object traveled during that time period. In other words, the area under the velocity curve gives the object's *change* in position during that interval of time.

This is true more generally! The area under the graph of a rate of change function on an interval $[a,b]$ represents the overall change in the relevant quantity on the interval.

For example, suppose f' represents the rate of traffic over a bridge, where $f'(x)$ is measured in cars per minute and x is measured in minutes. If we calculate the area below the graph of f', above the x-axis, and between the x-values 1 and 2, then we are calculating the number of cars that passed over the bridge during that minute. In other words, we are calculating the *change* in the number of cars passing over the bridge from time $x = 1$ to $x = 2$ minutes.

If we are given a rate of change function whose graph can be defined by geometric shapes we can calculate the areas of, then we can calculate the change in the relevant quantity (i.e., the area under the graph of the rate of change function) *exactly*. However, if the graph of the rate of change function cannot be defined in terms of nice geometric shapes, then we can use Riemann sums to estimate the change in the quantity (i.e., the area under the graph of the rate of change function).

Before we get into calculations, let's learn how to find the units of this overall change. Previously, we saw that if $f(t)$ is measured in meters per second and t is measured in seconds, then the units of the area under the graph of f are meters. This is true in general: The units of the area under the curve of f are the units of $f(x)$ multiplied by the units of x. Usually, the units of $f(x)$ are measured as "per" something because the function represents a rate of change (meters per second, miles per hour, chickens per roost, etc.), and x is measured as the unit after the "per" (seconds, hours, roosts, etc.). This means that the units of the area will be the units that are stated first: meters, miles, chickens, etc. This is not always the case, but it is true often enough to be mentioned here.

▪ **Example 8** **Table 4.2** gives the units for x and $f(x)$, where f is a positive function. Determine the appropriate units for the area under the graph of f on the interval $[a,b]$.

Units for x	Units for $f(x)$	Units for Area
people	automobiles per person	**a.**
seconds	miles per second per second	**b.**
hours	kilowatts	**c.**
pancakes	dollars per pancake	**d.**

Table 4.2: Units for x and $f(x)$

Solution:
The units for the area under the graph of f are the units for $f(x)$ multiplied by the units for x and then divided where possible. Remember that the "per" can be thought of as "dividing by" the unit.

a. If we multiply the units of $f(x)$, automobiles per person, by the units of x, people, we can divide the "people" part of the units. Thus, the units for the area under the graph of f in this case are automobiles.

b. If we multiply miles per second per second and seconds, we can divide the "seconds", but we still have another "seconds" unit! This means the area under the graph of f will have units in miles per second.

c. We multiply kilowatts and hours to obtain the unit kilowatt-hours.

N *Electric companies often measure kilowatt-hours to determine how much to bill a customer.*

d. Dollars per pancake times pancakes will leave dollars for the units.

▪

Try It # 7:
Table 4.3 gives the units for x and $f(x)$, where f is a positive function. Determine the appropriate units for the area under the graph of f on the interval $[a,b]$.

Units for x	Units for $f(x)$	Units for Area
pancakes	dollars per pancake	**a.**
feet	pounds	**b.**
fortnights	cubits per fortnight per fortnight	**c.**

Table 4.3: Units for x and $f(x)$

Now that we understand how to find the units for the area under a curve, let's apply the concept of finding the area under a curve to calculate the *change* in a quantity given its corresponding rate of change function (i.e., derivative).

Recall that if the graph of a rate of change function can be defined by geometric shapes that we can calculate the areas of, then we can find the change in the relevant quantity (i.e., the area under the graph of the rate of change function) *exactly*. However, if the graph of the rate of change function cannot be defined by geometric shapes, then we can use Riemann sums to estimate the change in the quantity (i.e., the area under the graph of the rate of change function).

■ **Example 9** The velocity of a car after t seconds is given by $v(t)$ meters per second, and the graph of v is shown in **Figure 4.3.22**.

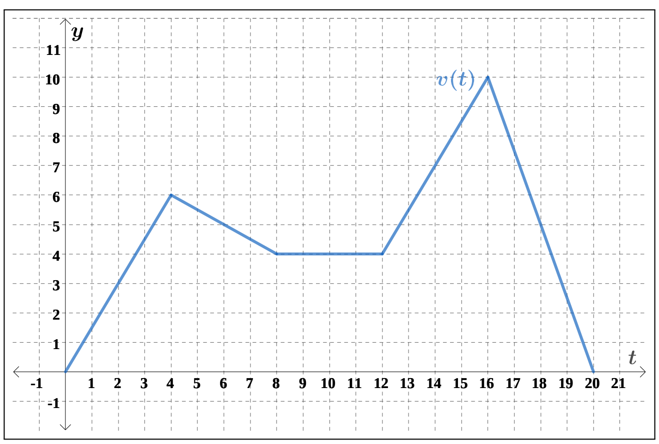


Figure 4.3.22: Graph of a velocity function v

Use the graph to find the distance the car traveled

a. during the first 8 seconds.
b. during the first 12 seconds.
c. between the 12^{th} and 20^{th} seconds.

Solution:
Because the velocity function of the car is nonnegative and it is the rate of change function of the car's position function, we can calculate the distance the car traveled by finding the *change* in the position of the car (i.e., by finding the area under the car's velocity curve). Note again that finding the change in the car's position gives us the distance it traveled because its velocity function is nonnegative. Remember, the velocity functions discussed in this textbook are nonnegative.

Before we begin finding the distance the car traveled during each interval, note that we can calculate the distances *exactly* because the graph of v is defined by geometric shapes of which we can find the areas. If it were not, we would need to use Riemann sums to estimate the distances (areas).

a. The distance traveled by the car during the first 8 seconds is the *change* in its position between $t = 0$ and $t = 8$ seconds because the velocity function is nonnegative. The change in the car's position is given by the area below the graph of v on the interval $[0,8]$. The region whose area we must calculate is shaded in **Figure 4.3.23**:

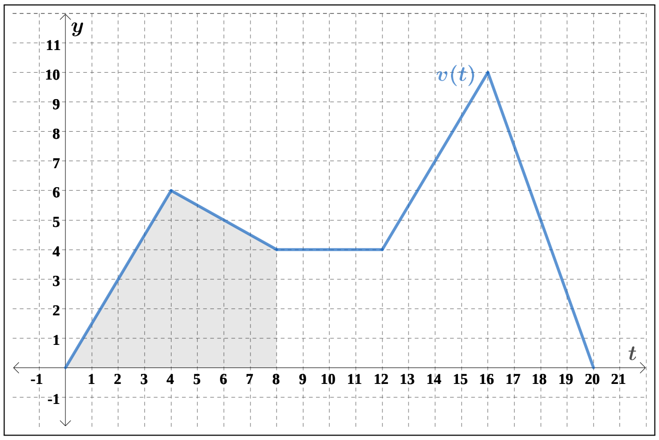


Figure 4.3.23: Shaded region below the velocity curve on the interval $[0,8]$

This is not an elementary shape that we can find the area of directly. However, we can divide the region into smaller shapes and add the areas of the shapes together to obtain an *exact* answer (similar to the procedure used in Example 1). One way to divide the region into shapes that we can find the areas of is shown in **Figure 4.3.24**:

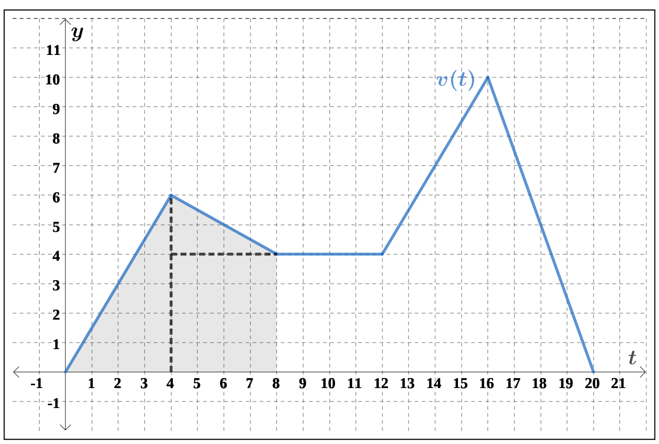


Figure 4.3.24: Graph of the velocity curve in which the regions of two triangles and a square are shaded

The region consists of a triangle and a triangle above a square. The first triangle has a base of 4 and a height of 6. This means its area is $\dfrac{4\cdot 6}{2} = 12$. The square has a length and width of 4, so its area is 16. The triangle above the square has a base of 4 and a height of 2 (from $y = 4$ to $y = 6$). Thus, its area is $\dfrac{4\cdot 2}{2} = 4$. Adding all of these areas together gives the exact area under the velocity curve on the interval $[0,8]$: $12+16+4 = 32$.

Next, we must determine the correct units. $v(t)$ is measured in meters per second, and t is measured in seconds. Multiplying these units together yields meters as the correct units. Hence, during the first 8 seconds, the car travels exactly 32 meters.

b. The distance traveled by the car during the first 12 seconds is the *change* in its position between $t = 0$ and $t = 12$ seconds because the velocity function is nonnegative. The change in the car's position is given by the area below the graph of v on the interval $[0, 12]$. The region whose area we must calculate is shaded in **Figure 4.3.25**:

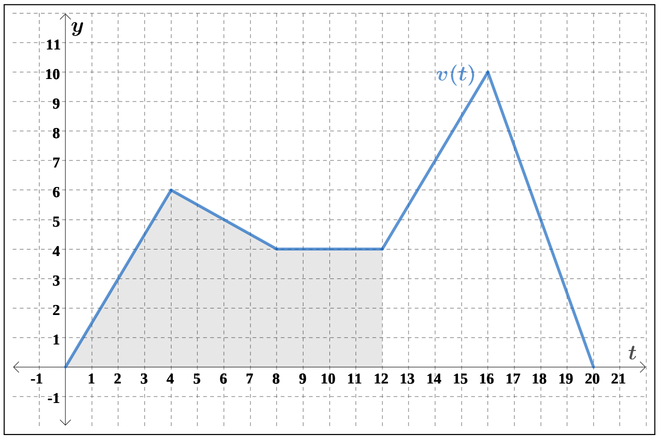


Figure 4.3.25: Shaded region below the velocity curve on the interval $[0, 12]$

To find the distance traveled on the interval $[0, 12]$, we can take our answer from part **a** (32 meters) and add the distance traveled from $t = 8$ to $t = 12$ seconds. The car is traveling at a constant rate of 4 meters per second during these 4 seconds, meaning it travels another 16 meters. Thus, the total distance traveled during the first 12 seconds is $32 + 16 = 48$ meters.

c. The distance traveled by the car between the 12^{th} and 20^{th} seconds is the *change* in its position during this time period because the velocity function is nonnegative. The change in the car's position is given by the area below the graph of v on the interval $[12, 20]$. The region whose area we must calculate is shaded in **Figure 4.3.26**:

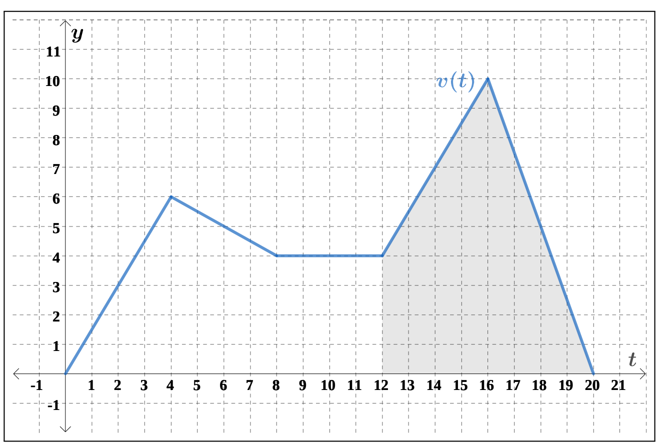


Figure 4.3.26: Shaded region below the velocity curve on the interval $[12, 20]$

This region can be divided into several smaller regions consisting of a square and two triangles (see **Figure 4.3.27**), but there are many other ways it can be divided into smaller regions.

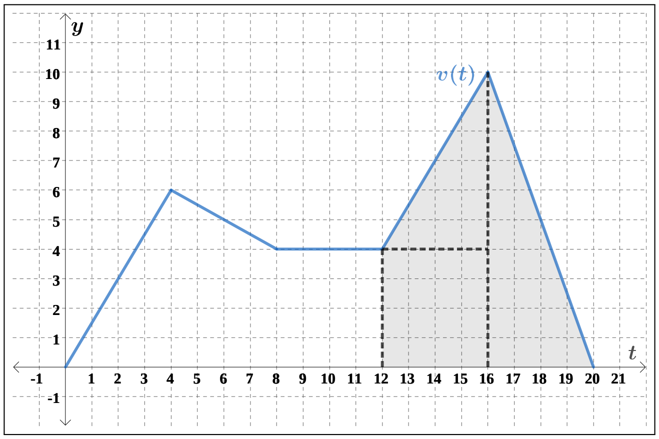


Figure 4.3.27: Graph of the velocity curve in which the regions of two triangles and a square are shaded

The square has a length of 4 (from $x = 12$ to $x = 16$) and a width of 4, so its area is 16. The triangle above it has the same length for its base, 4, and its height is 6 (from $y = 4$ to $y = 10$). So its area is $\frac{4 \cdot 6}{2} = 12$. The last triangle also has a base of 4 (from $x = 16$ to $x = 20$), and it has a height of 10. So its area is 20. Adding these areas together shows the car traveled $16 + 12 + 20 = 48$ meters between the 12^{th} and 20^{th} seconds.

■

Try It # 8:
Find the distance traveled by the car in the previous example between the 4^{th} and 14^{th} seconds.

Remember, if we are unable to calculate the area under a curve exactly, we can estimate the area using Riemann sums. We will see this in the next example while continuing to find the change in a quantity given a rate of change function.

■ **Example 10** Blowing Baubles, a company that makes hand blown glass items, has a marginal revenue function given by $R'(x)$ dollars per bauble when x thousand baubles are sold. The graph of R' is shown in **Figure 4.3.28**. Use the graph of R' and the points indicated to estimate the change in revenue from selling the first 16,000 baubles using 4 subintervals of equal width and a midpoint Riemann sum.

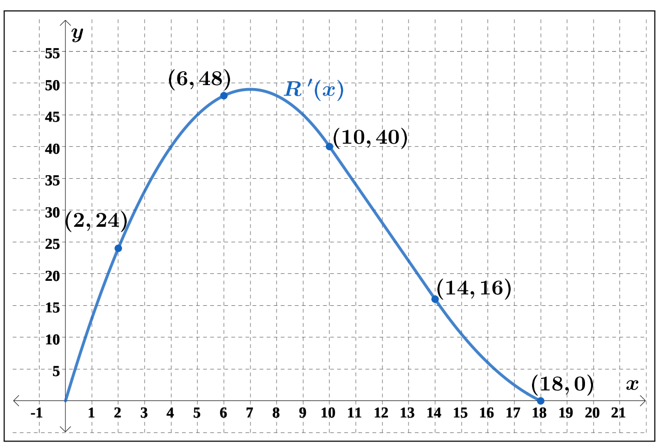


Figure 4.3.28: Graph of the marginal revenue function, R'

Solution:
The change in revenue from selling the first 16,000 baubles corresponds to the change in revenue when the number of baubles sold increases from 0 to 16,000. In other words, it is the change in the revenue function, R, from $x = 0$ to

$x = 16$ baubles. This change is represented by the area under the graph of the rate of change function, R', on the interval $[0, 16]$. The region corresponding to this area is shaded in **Figure 4.3.29**:

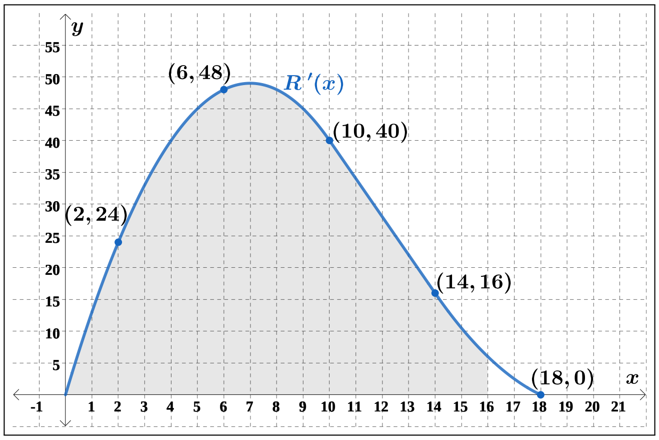


Figure 4.3.29: Shaded region under the graph of R' on the interval $[0, 16]$

Because the graph of R' cannot be defined in terms of geometric shapes that we can calculate the areas of, we must use the indicated Riemann sum to *estimate* the area under the curve on the interval $[0, 16]$.

For this midpoint Riemann sum, the width of each subinterval is $\Delta x = \dfrac{16-0}{4} = 4$. Using the graph in **Figure 4.3.29**, we can determine the sample points (i.e., midpoints) of each subinterval:

$$x_1^* = 2,\ x_2^* = 6,\ x_3^* = 10, \text{ and } x_4^* = 14$$

Thus, the midpoint Riemann sum is

$$\begin{aligned}\sum_{i=1}^{4} R'\left(x_i^*\right)\cdot\Delta x &= R'\left(x_1^*\right)\cdot\Delta x + R'\left(x_2^*\right)\cdot\Delta x + R'\left(x_3^*\right)\cdot\Delta x + R'\left(x_4^*\right)\\ &= R'(2)\cdot 4 + R'(6)\cdot 4 + R'(10)\cdot 4 + R'(14)\cdot 4\\ &= 24\cdot 4 + 48\cdot 4 + 40\cdot 4 + 16\cdot 4\\ &= 512\end{aligned}$$

Now, we will determine the units. $R'(x)$ is measured in dollars per bauble and x is measured in thousands of baubles. When we multiply these units, we are left with thousand dollars after dividing the baubles unit.

Therefore, the company's revenue increases by approximately 512 thousand dollars, or \$512,000, from selling the first 16,000 baubles.

The graph of R' and corresponding estimating rectangles are shown on the same axes in **Figure 4.3.30**:

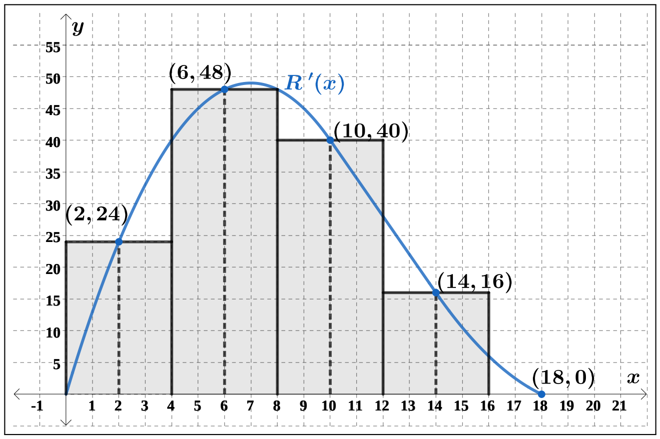


Figure 4.3.30: Four rectangles whose areas provide an estimate for the change in revenue on the interval $[0, 16]$

The area under a rate of change function gives the ***change*** *in the related quantity, not (necessarily) a total amount. However, in this example, the change in revenue when the number of baubles sold increases from* 0 *to* $16,000$ *is actually equal to the total revenue from selling the first* $16,000$ *baubles. Why? The total revenue from selling the first* $16,000$ *baubles can be thought of as the "initial" revenue, or the revenue when* $x = 0$ *baubles are sold, plus the change in revenue when the number of baubles sold increases from 0 to 16,000. However, if the company does not sell any baubles, it will not have any revenue. In other words, if* $x = 0$ *items, then the revenue is* \$0. *Thus, the revenue from selling the first* $16,000$ *baubles is equivalent to the change in revenue from selling the first* $16,000$ *baubles.*

▪

Try It # 9:
Estimate the revenue Blowing Baubles earns when the number of baubles sold increases from 2000 to 18,000 using

a. a right-hand Riemann sum with 4 subintervals of equal width.
b. a left-hand Riemann sum with 4 subintervals of equal width.

▪ **Example 11** Rats Off to You is an exterminating company with a marginal cost function given by $C'(x) = -0.06x^2 + 3.6x + 8$ dollars per home when x homes are treated for pests. If the company has fixed costs of \$1000, estimate the company's cost of treating the first 30 homes using 6 subintervals of equal width and a left-hand Riemann sum.

Solution:
As stated previously, the cost of treating the first 30 homes is not necessarily equal to the *change* in cost when the first 30 homes are treated. To find the cost of treating the first 30 homes, we need to find the change in cost when the number of homes treated increases from $x = 0$ to $x = 30$ homes and then add the company's cost when $x = 0$ homes are treated (i.e., the company's fixed costs).

We will start by finding the change in the cost function, C, from $x = 0$ to $x = 30$ homes, which means we need to find the area under the graph of C' on this interval.

Because the graph of C' is a parabola, it cannot be defined in terms of geometric shapes that we can calculate the areas of to get an exact answer. Thus, we will have to use the indicated Riemann sum to *estimate* the area under the curve on the interval $[0, 30]$. Also, the units for our final answer should be "dollars per home" times "homes", which is just "dollars".

To find the approximation, we begin by dividing the interval $[0,30]$ into 6 subintervals of equal width $\Delta x = \dfrac{30-0}{6} = 5$. See **Figure 4.3.31**.

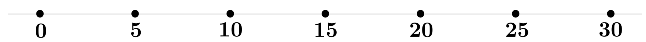


Figure 4.3.31: Number line showing the x-values partitioning six subintervals on the interval $[0,30]$

Using the left endpoint of each subinterval to determine the heights of the rectangles and recalling that $C'(x) = -0.06x^2 + 3.6x + 8$, we have the following left-hand Riemann sum:

$$\begin{aligned}\sum_{i=0}^{5} C'(x_i)\cdot\Delta x &= C'(x_0)\cdot\Delta x + C'(x_1)\cdot\Delta x + C'(x_2)\cdot\Delta x + C'(x_3)\cdot\Delta x + C'(x_4)\cdot\Delta x + C'(x_5)\cdot\Delta x\\ &= C'(0)\cdot 5 + C'(5)\cdot 5 + C'(10)\cdot 5 + C'(15)\cdot 5 + C'(20)\cdot 5 + C'(25)\cdot 5\\ &= 8\cdot 5 + 24.5\cdot 5 + 38\cdot 5 + 48.5\cdot 5 + 56\cdot 5 + 60.5\cdot 5\\ &= \$1177.50\end{aligned}$$

Therefore, the change in cost from treating the first 30 homes is approximately \$1177.50.

Now, we must add the company's cost when $x = 0$ homes are treated. In other words, we must add the company's fixed costs, \$1000:

$$\$1177.50 + \$1000 = \$2177.50$$

Thus, the first 30 homes cost Rats Off to You approximately \$2177.50 to treat.

N *If, for some reason, the company did not have any fixed costs (i.e., when $x = 0$ homes are treated the cost is \$0), then the cost of treating the first 30 homes would equal the change in cost from treating the first 30 homes.*

■

Try It # 10:
Estimate the cost of Rats Off to You treating the 15^{th} through the 40^{th} home using 5 subintervals of equal width and a midpoint Riemann sum.

■ **Example 12** **Table 4.4** shows the velocity of a runner during the first twelve seconds of her race.

t (seconds)	0	2	4	6	8	10	12
$v(t)$ (feet per second)	0	6.7	9.9	10.1	10.2	9.4	9.7

Table 4.4: Table showing the velocities of a runner

Using the table, estimate the distance traveled by the runner

a. from the 6^{th} to the 12^{th} second using 3 subintervals of equal width and a right-hand Riemann sum.

b. during the first 12 seconds using 3 subintervals of equal width and a left-hand Riemann sum.

Solution:

Because we are assuming the velocity functions in this textbook are nonnegative, the distance the runner travels during the time intervals in both parts **a** and **b** is given by the *change* in her position on the intervals. Recall that the change in her position is determined by the area of the region under the velocity curve.

We do not know what the graph of the velocity function looks like because we are only given certain function values. Thus, our only option is to estimate the area of the region under the graph of v using the indicated Riemann sum.

Before we begin estimating the distance traveled on each interval, we will determine the units for the answers. To determine the units, we multiply "feet per second" and "seconds" and then divide the unit "seconds". Thus, the estimations for the distances traveled will be in feet.

a. We need to estimate the area of the region under the graph of v on the interval $[6,12]$ using 3 subintervals of equal width and a right-hand Riemann sum. The width of each subinterval is $\Delta t = \dfrac{12-6}{3} = 2$, so the sample points (i.e., right endpoints) of the subintervals are $t_1 = 8$, $t_2 = 10$, and $t_3 = 12$. Using the function values in **Table 4.4**, the right-hand Riemann sum is

$$\begin{aligned}\sum_{i=1}^{3} v(t_i)\cdot\Delta t &= v(t_1)\cdot\Delta t + v(t_2)\cdot\Delta t + v(t_3)\cdot\Delta t\\ &= v(8)\cdot 2 + v(10)\cdot 2 + v(12)\cdot 2\\ &= 10.2\cdot 2 + 9.4\cdot 2 + 9.7\cdot 2\\ &= 58.6 \text{ feet}\end{aligned}$$

Thus, the runner ran approximately 58.6 feet between the 6^{th} and 12^{th} seconds.

b. We need to estimate the area of the region under the graph of v on the interval $[0,12]$ using 3 subintervals of equal width and a left-hand Riemann sum. The width of each subinterval is $\Delta x = \dfrac{12-0}{3} = 4$, which gives the sample points $t_0 = 0,\ t_1 = 4$, and $t_2 = 8$ using the left endpoint of each subinterval. Thus, the left-hand Riemann sum is

$$\begin{aligned}\sum_{i=0}^{2} v(t_i)\cdot\Delta t &= v(t_0)\cdot\Delta t + v(t_1)\cdot\Delta t + v(t_2)\cdot\Delta t\\ &= v(0)\cdot 4 + v(4)\cdot 4 + v(8)\cdot 4\\ &= 0\cdot 4 + 9.9\cdot 4 + 10.2\cdot 4\\ &= 80.4 \text{ feet}\end{aligned}$$

Thus, the runner ran approximately 80.4 feet during the first 12 seconds of her race.

■

Try It # 11:
Estimate the distance traveled by the runner in the previous example

a. during the first 6 seconds using 3 subintervals of equal width and a left-hand Riemann sum.
b. during the first 12 seconds using 6 subintervals of equal width and a right-hand Riemann sum.

Net Area

We have seen many times throughout this textbook that functions with negative values are incredibly useful. For instance, a profit function might be negative indicating a loss in profit, or a marginal cost function may be negative to indicate the cost per item is decreasing.

The functions we discussed in the first part of this section were nonnegative (i.e., on or above the x-axis). Now, we will extend the idea of calculating the area between a function and the x-axis on some interval to more general functions that may have negative values. Let's investigate what happens when we use a Riemann sum to estimate the area between a function and the x-axis on an interval in which the function is negative!

▪ **Example 13** Estimate the area between $f(x) = 2 - \sqrt{x+7}$ and the x-axis on the interval $[-2,2]$ using 8 subintervals of equal width and a right-hand Riemann sum. Round your answer to two decimal places, if necessary.

Solution:

The width of each subinterval is $\Delta x = \dfrac{2-(-2)}{8} = 0.5$. **Figure 4.3.32** shows the graph of the function f and the estimating rectangles:

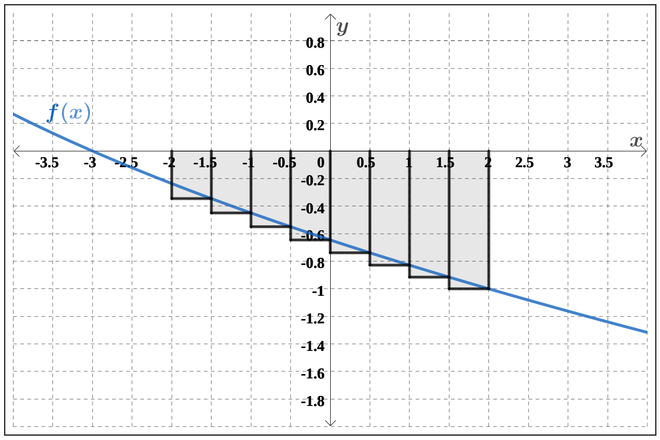


Figure 4.3.32: Eight rectangles whose areas provide an estimate for the area between the curve and the x-axis on the interval $[-2,2]$

Recalling that $f(x) = 2 - \sqrt{x+7}$ and using the right endpoints of the subintervals to calculate the heights of the rectangles, we obtain the right-hand Riemann sum

$$\begin{aligned}\sum_{i=1}^{8} f(x_i)\cdot\Delta x &= f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + f(x_3)\cdot\Delta x + f(x_4)\cdot\Delta x + f(x_5)\cdot\Delta x + f(x_6)\cdot\Delta x + f(x_7)\cdot\Delta x + f(x_8)\cdot\Delta x \\ &= f(-1.5)\cdot 0.5 + f(-1)\cdot 0.5 + f(-0.5)\cdot 0.5 + f(0)\cdot 0.5 + f(0.5)\cdot 0.5 + f(1)\cdot 0.5 + f(1.5)\cdot 0.5 + f(2)\cdot 0.5 \\ &= \left(2-\sqrt{5.5}\right)\cdot 0.5 + \left(2-\sqrt{6}\right)\cdot 0.5 + \left(2-\sqrt{6.5}\right)\cdot 0.5 + \left(2-\sqrt{7}\right)\cdot 0.5 + \left(2-\sqrt{7.5}\right)\cdot 0.5 + \left(2-\sqrt{8}\right)\cdot 0.5 \\ &\quad + \left(2-\sqrt{8.5}\right)\cdot 0.5 + \left(2-\sqrt{9}\right)\cdot 0.5 \\ &\approx -2.74\end{aligned}$$

We may look at this answer and think that it is incorrect because the physical area of any region cannot be negative! That is indeed true, which leads us to a very important result: Using a Riemann sum to estimate the area between a curve and the x-axis on an interval in which the function is negative will give a negative answer. In other words, it will *count* the physical area as negative. This is due to the fact that the function values (i.e., y-values) determining the heights of each rectangle are negative.

So for this question, we must take the absolute value of the sum to obtain the area of the region:

$$|-2.74| = 2.74$$

▪

The previous example leads us to the concept of **net area**. In short, when calculating Riemann sums, the areas of the regions between the curve and the x-axis where the function is negative will count as *negative* values, and the areas of the regions between the curve and the x-axis where the function is positive will count as *positive values*, as we have seen previously in this section.

Thus, Riemann sums can be used to estimate both area and net area. Furthermore, if we *increase* the number of subintervals (i.e., rectangles) used in a Riemann sum, the estimation will approach the *exact* answer.

For example, consider estimating the area between the graph of $f(x) = x^3 - 6x^2 + 8x + 5$ and the x-axis on the interval $[0,4]$ using a right-hand Riemann sum with $n = 4$ subintervals (i.e., rectangles). The length of each subinterval would equal $\dfrac{4-0}{4} = 1$ (see **Figure 4.3.33**). If we increase the number of subintervals to say $n = 8$, then each subinterval would have length $\dfrac{4-0}{8} = \dfrac{1}{2}$. Looking at **Figure 4.3.34**, we see that using $n = 8$ subintervals instead of $n = 4$ subintervals provides a better estimate for the area under the curve:

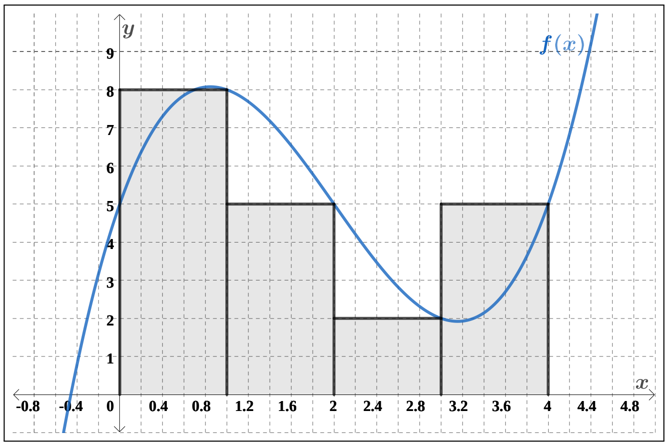


Figure 4.3.33: Rectangles representing a right-hand Riemann on the interval $[0,4]$, where $f(x) = x^3 - 6x^2 + 8x + 5$ and $n = 4$

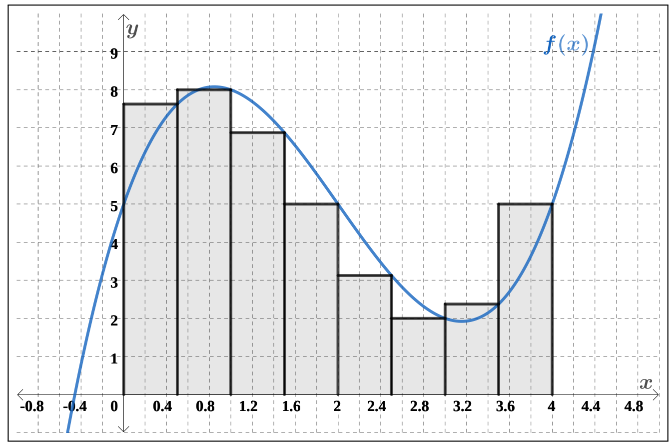


Figure 4.3.34: Rectangles representing a right-hand Riemann on the interval $[0,4]$, where $f(x) = x^3 - 6x^2 + 8x + 5$ and $n = 8$

If we continue increasing the number of subintervals, our approximations will become even more accurate. Using $n = 16$ subintervals for the Riemann sum is shown in **Figure 4.3.35**, and letting the number of subintervals, n, go to infinity is shown in **Figure 4.3.36**. Notice that when we let n go to infinity (i.e., we let $n \to \infty$), we arrive at the exact area under the curve:

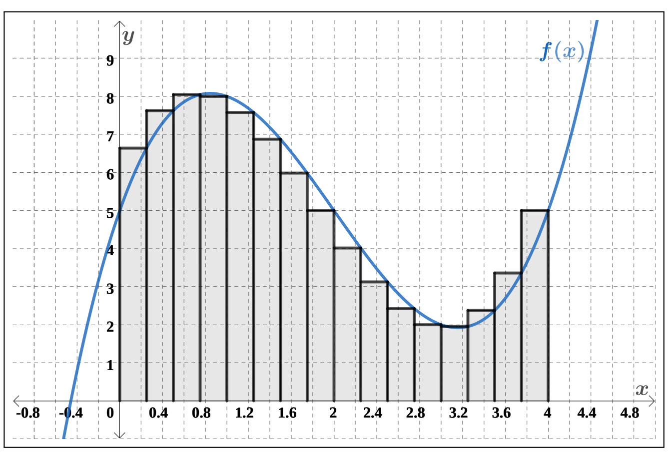


Figure 4.3.35: Rectangles representing a right-hand Riemann on the interval $[0,4]$, where $f(x) = x^3 - 6x^2 + 8x + 5$ and $n = 16$

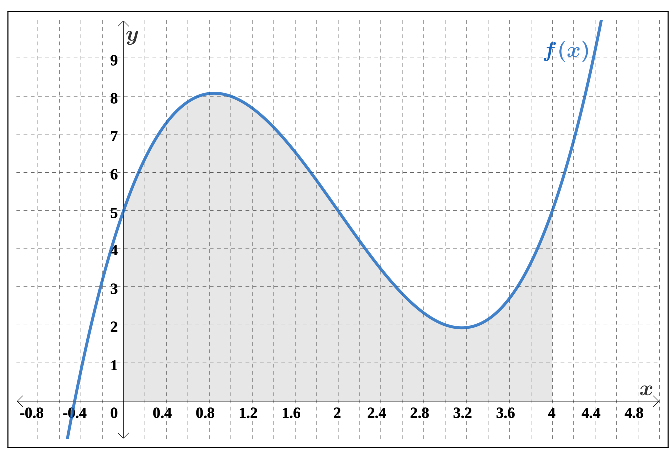


Figure 4.3.36: Rectangles representing a right-hand Riemann on the interval $[0,4]$, where $f(x) = x^3 - 6x^2 + 8x + 5$ and $n \to \infty$

This leads us to the following definition:

Definition

If f is a continuous function, the **definite integral** of f from a to b is the net area of the regions between the graph of f and the x-axis on the interval $[a,b]$. In symbols, we write

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right)\cdot \Delta x$$

where a is the **lower limit of integration** and b is the **upper limit of integration**. ▪

*This limit may not always exist or may depend on the choice of x_i. If either of these is the case, we say that the function is **not integrable** on that interval. We will not be considering such functions in this textbook.*

There is a very good reason that the definite integral contains the same symbol as the indefinite integral from **Sections 4.1** and **4.2**. We will see why in the next section! We will also use the same terminology we learned in **Section 4.1**: $f(x)$ is called the integrand, and the symbol $\int$ is the integral sign.

Notice that because the definite integral is defined as the limit of a Riemann sum, and Riemann sums *count* areas below the x-axis negatively, the definite integral will also *count* areas below the x-axis negatively.

More specifically, $\displaystyle\int_a^b f(x)\,dx$ will *count* the areas of regions above the x-axis positively and the areas of regions below the x-axis negatively. Therefore, the definite integral gives the exact net area of a region.

▪ **Example 14** The graph of f is shown in **Figure 4.3.37**. Use the graph to find each of the following.

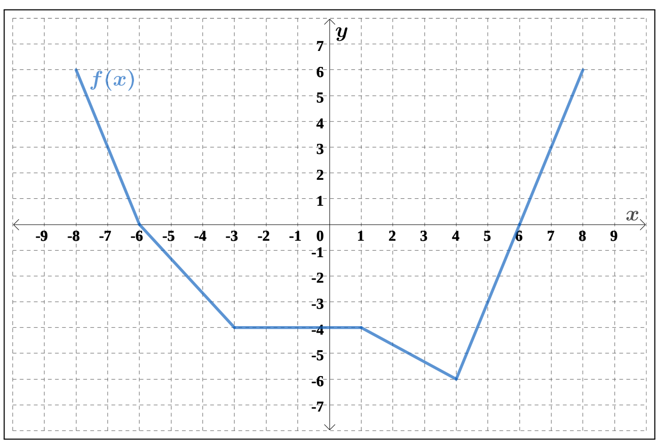


Figure 4.3.37: Graph of a function f

a. $\displaystyle\int_{-3}^{0} f(x)\,dx$

b. $\displaystyle\int_{4}^{8} f(x)\,dx$

c. $\displaystyle\int_{-8}^{8} f(x)\,dx$

Solution:

a. Recall that $\int_{-3}^{0} f(x)\,dx$ represents the exact net area of the regions bounded between the graph of f and the x-axis on the interval $[-3,0]$. The areas of the regions above the x-axis are counted positively, and the areas of the regions below the x-axis are counted negatively.

Because the graph of f is below the x-axis on this entire interval, we must find the area of the region and *count* it as negative. The region whose area we must calculate is shaded in **Figure 4.3.38**:

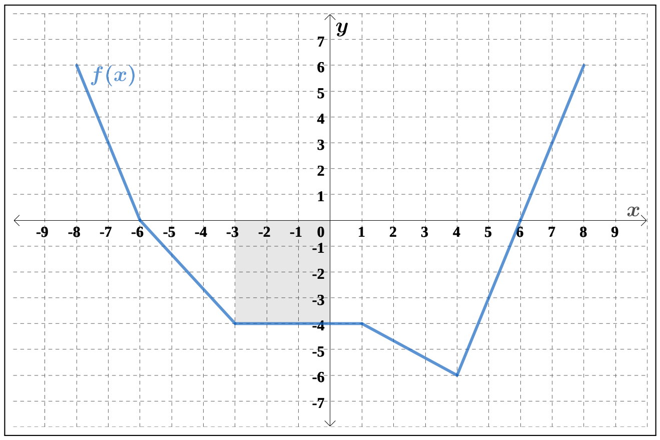


Figure 4.3.38: Shaded region between the curve and the x-axis on the interval $[-3,0]$

The shaded region is a square of length 3 and width 4, so the area of the square is 12. Because this region is *below* the x-axis, the value of the definite integral will be negative: $\int_{-3}^{0} f(x)\,dx = -12$.

b. To calculate $\int_{4}^{8} f(x)\,dx$, we must find the net area of the regions shaded in **Figure 4.3.39**:

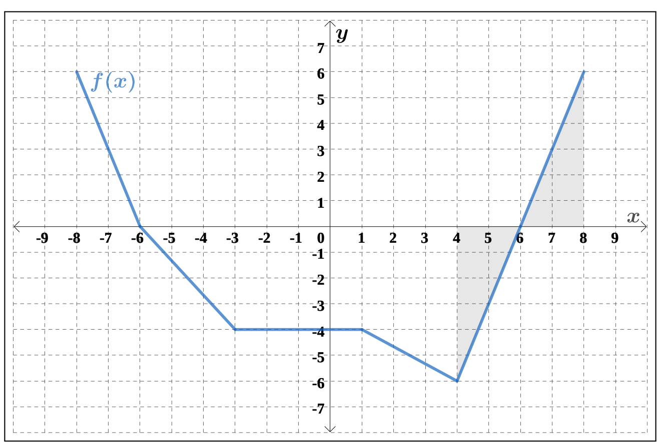


Figure 4.3.39: Shaded regions between the curve and the x-axis on the interval $[4,8]$

There are two regions: a triangle below the x-axis whose area we will *count* as negative and a triangle above the x-axis whose area we will *count* as positive. The triangle below the x-axis has a base of 2 and a height of

6. So the area of the triangle is $\frac{2\cdot 6}{2} = 6$. The triangle above the x-axis also has a base of 2 and a height of 6, so its area is $\frac{2\cdot 6}{2} = 6$. Thus, $\int_4^8 f(x)\,dx = -6+6 = 0$.

c. To calculate $\int_{-8}^{8} f(x)\,dx$, we must find the net area of the regions shaded in **Figure 4.3.40**:

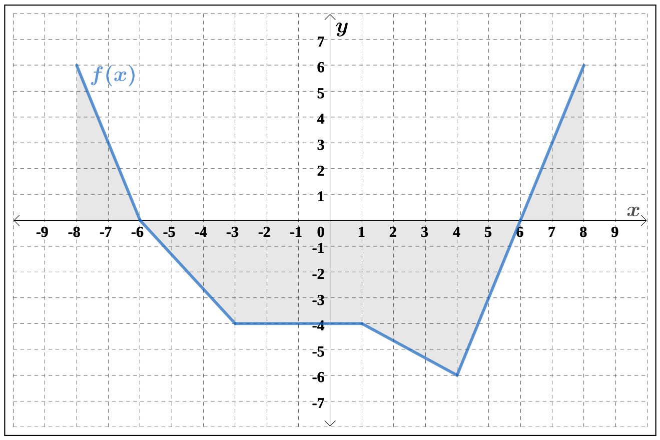


Figure 4.3.40: Shaded regions between the curve and the x-axis on the interval $[-8, 8]$

We will divide these regions into several smaller regions of which we can calculate the areas. One way to divide the regions is shown in **Figure 4.3.41**:

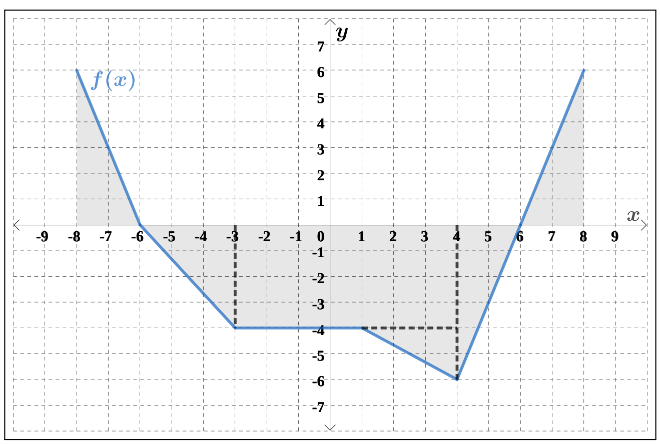


Figure 4.3.41: Graph of a function, f, in which the regions of five triangles and one rectangle are shaded

We will start by finding the areas above the x-axis, and then we will find the areas below the x-axis. Above the x-axis, there are two triangles. The first triangle has a base of length 2 (from $x = -8$ to $x = -6$) and a height of length 6. The second triangle also has a base of length 2 and a height of length 6. Therefore, each of these triangles has an area of 6, so the total area of the regions above the x-axis is 12. Note that this area will be counted positively in our final calculation.

The regions below the x-axis consist of three triangles and a rectangle. The rectangle has length 7 (from $x = -3$ to $x = 4$) and a width of 4, so its area is 28. The farthest left triangle has a base of 3 (from $x = -6$ to $x = -3$) and a height of 4, so its area is 6. The triangle on the far right has an area of 6 (as discussed in part **b**). The last triangle has a base of 3 (from $x = 1$ to $x = 4$) and a height of 2 (from $y = -4$ to $y = -6$). Thus, its area is 3. So the total area of the regions below the x-axis is $28+6+6+3=43$. Note that this area will be counted negatively in our final calculation.

Combining this information, we can calculate the value of the definite integral:

$$\int_{-8}^{8} f(x)\,dx = 12-43=-31$$

■

$\int_a^b f(x)\,dx =$ *(Area above x-axis) − (Area below x-axis). When we say "area" here, we mean the positive concept of area as we have learned it previously. The definite integral counts area above the x-axis as positive and area below the x-axis as negative.*

Try It # 12:

Using the graph of f shown in **Figure 4.3.37**, find $\int_{-8}^{-3} f(x)\,dx$.

As seen previously, not every graph will consist of nice geometric shapes we can calculate the areas of exactly to find the value of a definite integral. In these cases, all we can do is estimate the definite integral using a Riemann sum. Recall from our previous discussion and example that a Riemann sum provides an estimate of the net area if the function is negative on part of the interval.

■ **Example 15** The graph of f is shown in **Figure 4.3.42**.

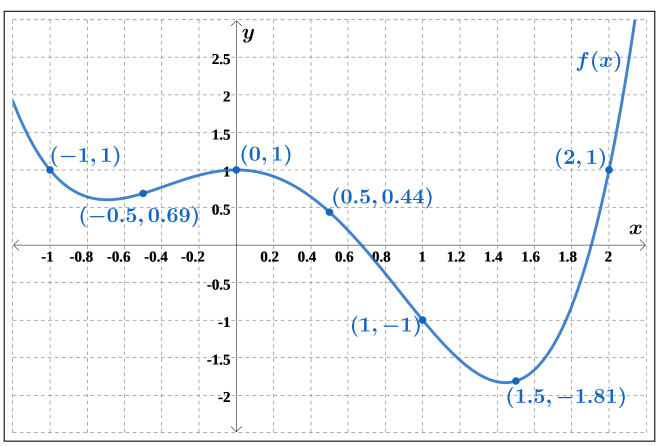


Figure 4.3.42: Graph of a function, f, in which certain points are indicated

Use the graph and the points indicated to estimate

a. $\int_{-1}^{1} f(x)\,dx$ using a right-hand Riemann sum with 4 subintervals of equal width.

b. $\int_{-1}^{2} f(x)\,dx$ using a midpoint Riemann sum with 3 subintervals of equal width.

Solution:

a. The interval for this definite integral is $[-1,1]$, so the width of each subinterval is $\Delta x = \frac{b-a}{n} = \frac{1-(-1)}{4} = 0.5$. Thus, the sample points (i.e., right endpoints) of the subintervals are $x_1=-0.5$, $x_2=0$, $x_3=0.5$, and $x_4=1$. **Figure 4.3.43** shows the graph of f and the estimating rectangles on the same axes:

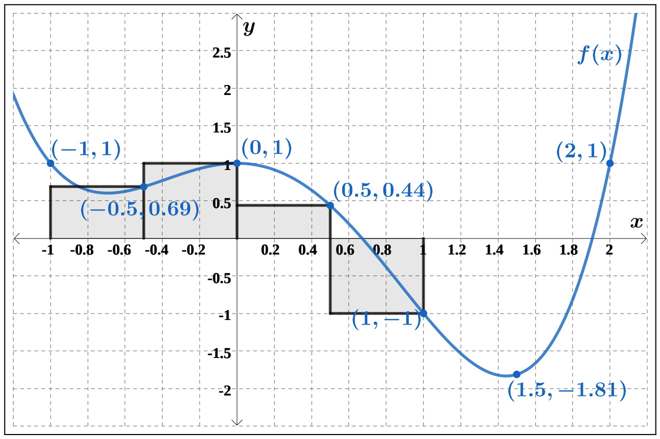


Figure 4.3.43: Four estimating rectangles for the net area on the interval $[-1,1]$

Using the indicated points on the graph for the heights of the rectangles, we obtain the following right-hand Riemann sum:

$$\begin{aligned}\sum_{i=1}^{4} f(x_i)\cdot\Delta x &= f(x_1)\cdot\Delta x + f(x_2)\cdot\Delta x + f(x_3)\cdot\Delta x + f(x_4)\cdot\Delta x\\ &= f(-0.5)\cdot 0.5 + f(0)\cdot 0.5 + f(0.5)\cdot 0.5 + f(1)\cdot 0.5\\ &= 0.69\cdot 0.5 + 1\cdot 0.5 + 0.44\cdot 0.5 + (-1)\cdot 0.5\\ &= 0.565\end{aligned}$$

Thus,

$$\int_{-1}^{1} f(x)\,dx \approx 0.565$$

b. Recall that we must estimate $\int_{-1}^{2} f(x)\,dx$ using a midpoint Riemann sum with 3 subintervals of equal width.

Similar to part **a**, we first find the width of each subinterval: $\Delta x = \dfrac{2-(-1)}{3} = 1$. Therefore, the sample points (i.e., midpoints) of the subintervals are $x_1^* = -0.5$, $x_2^* = 0.5$, and $x_3^* = 1.5$. **Figure 4.3.44** shows the graph of f and the estimating rectangles on the same axes:

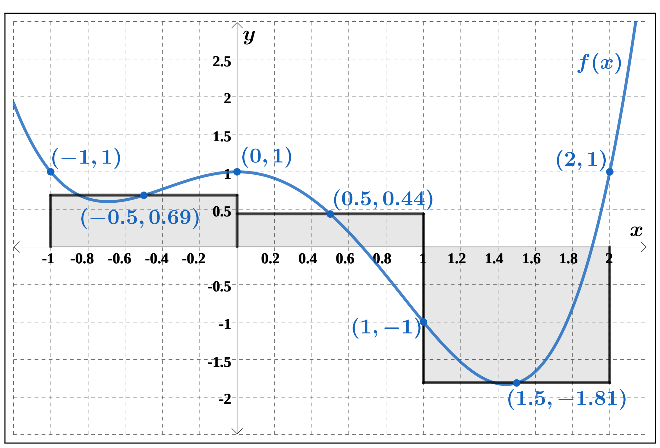


Figure 4.3.44: Three estimating rectangles for the net area on the interval $[-1,2]$

Using the indicated points on the graph for the heights of the rectangles, we obtain the following midpoint Riemann sum:

$$\begin{aligned}\sum_{i=1}^{3} f\left(x_i^*\right)\cdot\Delta x &= f\left(x_1^*\right)\cdot\Delta x + f\left(x_2^*\right)\cdot\Delta x + f\left(x_3^*\right)\cdot\Delta x\\ &= f(-0.5)\cdot 1 + f(0.5)\cdot 1 + f(1.5)\cdot 1\\ &= 0.69\cdot 1 + 0.44\cdot 1 + (-1.81)\cdot 1\\ &= -0.68\end{aligned}$$

Thus,

$$\int_{-1}^{2} f(x)\,dx \approx -0.68$$

■

Try It # 13:
Particular values of a function f are shown in **Table 4.5**.

x	6	6.2	6.4	6.6	6.8	7	7.2	7.4	7.6	7.8	8
$f(x)$	−2	−7	−4	0	1	3	0	−1	2	3	8

Table 4.5: Particular values of $f(x)$

Use the information in the table to estimate $\int_{6}^{8} f(x)\,dx$ using

a. 5 subintervals of equal width and a right-hand Riemann sum.
b. 10 subintervals of equal width and a left-hand Riemann sum.

Applications

The applications of the definite integral are the same as the area under a curve we discussed previously (namely, finding the change in a quantity on an interval given its rate of change function). Only now, because the definite integral *counts* the areas of regions below the x-axis negatively and the areas of regions above the x-axis positively, the net area of the regions will give us the *net change* in a quantity from one x-value to another.

In other words, $\int_{a}^{b} f'(x)\,dx$ gives the net change in f on the interval $[a,b]$.

The definite integral counts negative units as a loss and positive units as a gain. For instance, if the change in revenue is (or is estimated to be) −\$4000 when the number of items a company sells increases from 100 to 200 items, this means that overall, the company's revenue decreased by \$4000. If the change in revenue is (or is estimated to be) \$6000 when the number of items a company sells increases from 100 to 200 items, this means that overall, the company's revenue increased by \$6000.

Be careful when interpreting the change (or net change) in a quantity. This amount represents an overall increase or decrease in the quantity (unless the value is zero), not necessarily a total amount.

We will now continue working examples in which we find the change (or net change) in a quantity on an interval given its rate of change function. Remember, this change is represented by the definite integral, which we can calculate exactly if the graph of the rate of change function can be defined in terms of geometric shapes of which we can calculate the areas. If not, then we can estimate the definite integral using a Riemann sum.

■ **Example 16** The marginal profit function for the bakery Rye of the Ancient Mariner is given by $P'(x) = 0.8x + \dfrac{610}{x+1} - 110$ dollars per loaf of bread when x loaves of anchovy rye bread are sold.

a. Estimate the change in profit when the number of loaves sold increases from 100 to 200 using 4 subintervals of equal width and a right-hand Riemann sum.

b. If the bakery has a loss in profit of $2000 when it does not sell any loaves, estimate the bakery's profit from selling the first 150 loaves using 5 subintervals of equal width and a left-hand Riemann sum.

Solution:

a. The change (or net change) in profit when the number of loaves sold increases from 100 to 200 is given by the definite integral of the marginal profit function: $\int_{100}^{200} P'(x)\,dx$. Because the graph of P' cannot be defined in terms of geometric shapes we can calculate the areas of, we must use the indicated Riemann sum to estimate the value of the definite integral. This will give us an estimate for the change in profit.

The width of each subinterval is $\Delta x = \dfrac{200-100}{4} = 25$. So the four subintervals are $[100,125]$, $[125,150]$, $[150,175]$, and $[175,200]$, and the sample points (i.e., right endpoints) of the subintervals are $x_1 = 125$, $x_2 = 150$, $x_3 = 175$, and $x_4 = 200$. Thus, recalling that $P'(x) = 0.8x + \dfrac{610}{x+1} - 110$, the right-hand Riemann sum is

$$\begin{aligned}\sum_{i=1}^{4} P'(x_i)\cdot\Delta x &= P'(x_1)\cdot\Delta x + P'(x_2)\cdot\Delta x + P'(x_3)\cdot\Delta x + P'(x_4)\cdot\Delta x \\ &= P'(125)\cdot 25 + P'(150)\cdot 25 + P'(175)\cdot 25 + P'(200)\cdot 25 \\ &\approx \$2384.54\end{aligned}$$

Hence, Rye of the Ancient Mariner's profit increases by $2384.54 when the number of loaves sold increases from 100 to 200.

> ! *Be careful! Remember that the change in a quantity is not necessarily the value of the quantity (it is a change!). We calculated the change in profit in part* **a** *to be $ 2,384.54. This does not mean the profit from selling 200 loaves is $ 2,384.54!*

b. Recall that the bakery's profit from selling the first 150 loaves is not necessarily equal to the change (or net change) in profit when the number of loaves sold increases from 0 to 150. In addition to calculating this change in profit, we must also add the bakery's profit when it does not sell any loaves (i.e., when $x = 0$ loaves).

We will start by calculating the change in profit from selling the first 150 loaves, which is given by the definite integral of the marginal profit function: $\int_{0}^{150} P'(x)\,dx$. Because the graph of P' cannot be defined in terms of geometric shapes we can calculate the areas of, we must use the indicated Riemann sum to estimate the value of the definite integral.

The width of each subinterval is $\Delta x = \dfrac{150-0}{5} = 30$. So the five subintervals are $[0,30]$, $[30,60]$, $[60,90]$, $[90,120]$, and $[120,150]$, and the sample points (i.e., left endpoints) of the subintervals are $x_1 = 0$, $x_2 = 30$, $x_3 = 60$, $x_4 = 90$, and $x_5 = 120$.

Thus, recalling that $P'(x) = 0.8x + \dfrac{610}{x+1} - 110$, the left-hand Riemann sum is

$$\begin{aligned}\sum_{i=0}^{4} P'(x_i)\cdot\Delta x &= P'(x_0)\cdot\Delta x + P'(x_1)\cdot\Delta x + P'(x_2)\cdot\Delta x + P'(x_3)\cdot\Delta x + P'(x_4)\cdot\Delta x \\ &= P'(0)\cdot 30 + P'(30)\cdot 30 + P'(60)\cdot 30 + P'(90)\cdot 30 + P'(120)\cdot 30 \\ &\approx \$10,242.66\end{aligned}$$

Therefore, the change in profit from selling the first 150 loaves is approximately \$10,242.66.

Now, we must add the profit when $x = 0$ loaves are sold. We are told that the bakery loses \$2000 in profit if it does not sell any loaves, so we must "add" −\$2000 to our approximation for the change in profit:

$$\$10,242.66 - \$2000 = \$8242.66$$

Hence, Rye of the Ancient Mariner's profit from selling the first 150 loaves is approximately \$8242.66.

■

■ **Example 17** Particular values of the marginal profit function for Short Stop in the Name of Glove, a company that makes and sells designer baseball gloves, are shown in **Table 4.6**, where x is the number of baseball gloves sold and $P'(x)$ is measured in dollars per glove.

x	80	90	100	110	120	130	140	150	160
$P'(x)$	−38.38	−31.22	−23.90	−16.45	−8.92	−1.31	6.36	14.07	21.81

Table 4.6: Particular values of $P'(x)$

Use the information in the table to estimate

a. the change in profit when the number of gloves made and sold increases from 80 to 110 using 3 subintervals of equal width and a left-hand Riemann sum.

b. the change in profit when the number of gloves made and sold increases from 80 to 160 using 8 subintervals of equal width and a right-hand Riemann sum.

Solution:

a. In short, we need to estimate $\int_{80}^{110} P'(x)\,dx$. The width of each subinterval is given by $\Delta x = \dfrac{110-80}{3} = 10$, so the sample points (i.e., left endpoints) of the subintervals are $x_0 = 80$, $x_1 = 90$, and $x_2 = 100$. Thus, the left-hand Riemann sum is

$$\begin{aligned}\sum_{i=0}^{2} P'(x_i)\cdot\Delta x &= P'(x_0)\cdot\Delta x + P'(x_1)\cdot\Delta x + P'(x_2)\cdot\Delta x \\ &= P'(80)\cdot 10 + P'(90)\cdot 10 + P'(100)\cdot 10 \\ &= (-38.38)\cdot 10 + (-31.22)\cdot 10 + (-23.90)\cdot 10 \\ &= -\$935\end{aligned}$$

Therefore, $\int_{80}^{110} P'(x)\,dx \approx -\$935$, so the profit decreases by approximately \$935 when the number of gloves made and sold increases from 80 to 110.

b. Similarly, we need to estimate $\int_{80}^{160} P'(x)\,dx$. The width of each subinterval is given by $\Delta x = \dfrac{160-80}{8} = 10$, so the sample points (i.e., right endpoints) of the subintervals are $x_1 = 90$, $x_2 = 100$, $x_3 = 110$, $x_4 = 120$, $x_5 = 130$, $x_6 = 140$, $x_7 = 150$, and $x_8 = 160$. Thus, we have the following right-hand Riemann sum:

$$\begin{aligned}\sum_{i=1}^{8} P'(x_i)\cdot\Delta x &= P'(x_1)\cdot\Delta x + P'(x_2)\cdot\Delta x + P'(x_3)\cdot\Delta x + P'(x_4)\cdot\Delta x + P'(x_5)\cdot\Delta x\\ &\quad + P'(x_6)\cdot\Delta x + P'(x_7)\cdot\Delta x + P'(x_8)\cdot\Delta x\\ &= P'(90)\cdot 10 + P'(100)\cdot 10 + P'(110)\cdot 10 + P'(120)\cdot 10 + P'(130)\cdot 10\\ &\quad + P'(140)\cdot 10 + P'(150)\cdot 10 + P'(160)\cdot 10\\ &= (-31.22)\cdot 10 + (-23.90)\cdot 10 + (-16.45)\cdot 10 + (-8.92)\cdot 10 + (-1.31)\cdot 10\\ &\quad + 6.36\cdot 10 + 14.07\cdot 10 + 21.81\cdot 10\\ &= -\$395.60\end{aligned}$$

Therefore, $\int_{80}^{160} P'(x)\,dx \approx -\$395.60$, so the profit decreases by approximately \$395.60 when the number of gloves made and sold increases from 80 to 160. Notice that by selling more gloves (compared to part **a**) that Short Stop in the Name of Glove will lose less in profit.

▪

Try It # 14:
Butterfly Construction Equipment has a marginal cost function given by $C'(x)$ dollars per dump truck when x dump trucks are produced. The graph of C' is shown in **Figure 4.3.45**. If the company has fixed costs of \$100,000, use the graph and the points indicated to estimate the cost of producing the first 1600 dump trucks using a left-hand Riemann sum with 4 subintervals of equal width.

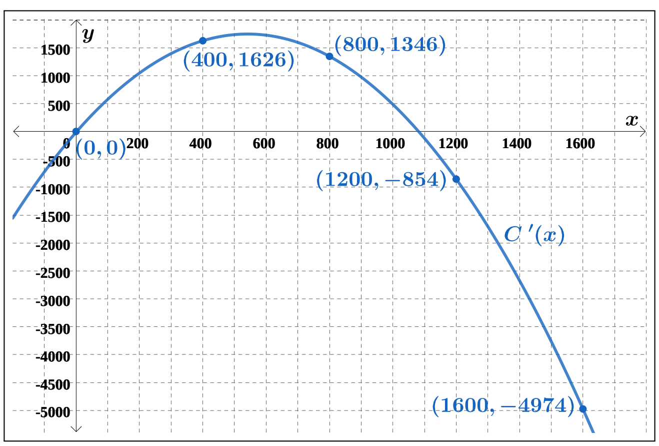


Figure 4.3.45: Graph of the marginal cost function C'

Properties of the Definite Integral

Now that we have seen some uses of the definite integral, we will discuss more general properties of definite integrals that hold whether we are working with a purely mathematical object (i.e., a definite integral with no context) or a real-world scenario, such as revenue or cost.

Properties of the Definite Integral

Let f and g be continuous functions on the interval $[a,b]$.

1. $\int_c^c f(x)\,dx = 0$

   If the limits of integration are the same, then there is no area.

2. $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

   Reversing the order of the limits of integration will change the sign of the definite integral.

3. $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

   The definite integral of a sum (or difference) is the sum (or difference) of the definite integrals.

4. If k is a constant, then $\int_a^b k \cdot f(x)\,dx = k \cdot \int_a^b f(x)\,dx$

   If we multiply a function by a constant and then find the definite integral, it is equivalent to finding the definite integral of the function and then multiplying by the constant.

5. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

   A region can be divided into two separate regions, and the area will remain the same whether we calculate it all as one region or as two separate regions.

N *Although our justification for the last property works best for c in the interval $[a,b]$, the property will hold for c not in that interval!*

We will now see these properties in practice!

Example 18 If $\int_0^8 f(x)\,dx = 10$ and $\int_0^5 f(x)\,dx = 15$, find $\int_5^8 f(x)\,dx$.

Solution:

Using Property #5, we will rewrite the first definite integral in terms of the definite integral we know and the definite integral we are solving for, and then we can substitute the known values:

$$\int_0^8 f(x)\,dx = \int_0^5 f(x)\,dx + \int_5^8 f(x)\,dx \implies$$
$$10 = 15 + \int_5^8 f(x)\,dx$$
$$-5 = \int_5^8 f(x)\,dx$$

Thus, $\int_5^8 f(x)\,dx = -5$.

Try It # 15:
If $\int_{-1}^{5} f(x)\,dx = -3$ and $\int_{2}^{5} f(x)\,dx = 4$, find $\int_{-1}^{2} f(x)\,dx$.

▪ **Example 19** If $\int_{-3}^{2} f(x)\,dx = 3$ and $\int_{-3}^{2} g(x)\,dx = 12$, find each of the following.

a. $\int_{-3}^{2} [g(x) - f(x)]\,dx$

b. $\int_{-3}^{2} 2f(x)\,dx$

c. $\int_{2}^{-3} [-7f(x) + 3g(x)]\,dx + \int_{1}^{1} 8f(x)\,dx$

Solution:

a. We start by separating $\int_{-3}^{2} [g(x) - f(x)]\,dx$ into two definite integrals using Property #3, and then we substitute the known values:

$$\begin{aligned}\int_{-3}^{2} [g(x) - f(x)]\,dx &= \int_{-3}^{2} g(x)\,dx - \int_{-3}^{2} f(x)\,dx \\ &= 12 - 3 \\ &= 9\end{aligned}$$

b. To find $\int_{-3}^{2} 2f(x)\,dx$, we will use Property #4:

$$\begin{aligned}\int_{-3}^{2} 2f(x)\,dx &= 2\int_{-3}^{2} f(x)\,dx \\ &= 2 \cdot 3 \\ &= 6\end{aligned}$$

c. Recall that we must find $\int_{2}^{-3} [-7f(x) + 3g(x)]\,dx + \int_{1}^{1} 8f(x)\,dx$.

First, notice that by Property #1, $\int_{1}^{1} 8f(x)\,dx = 0$. Using this fact and Property #3 yields

$$\begin{aligned}\int_{2}^{-3} [-7f(x) + 3g(x)]\,dx + \int_{1}^{1} 8f(x)\,dx &= \int_{2}^{-3} [-7f(x) + 3g(x)]\,dx + 0 \\ &= \int_{2}^{-3} -7f(x)\,dx + \int_{2}^{-3} 3g(x)\,dx\end{aligned}$$

Next, we use Property #2 and Property #4 for both definite integrals, and then we substitute the known values:

$$\begin{aligned}&= -7\int_{2}^{-3} f(x)\,dx + 3\int_{2}^{-3} g(x)\,dx \\ &= 7\int_{-3}^{2} f(x)\,dx - 3\int_{-3}^{2} g(x)\,dx \\ &= 7 \cdot 3 - 3 \cdot 12 \\ &= -15\end{aligned}$$

▪

Try It # 16:
If $\int_{9}^{27} g(t)\,dt = -4$ and $\int_{9}^{27} f(t)\,dt = 13$, find $\int_{27}^{9} [f(t) - 2g(t)]\,dt$.

▪ **Example 20** The graphs of the functions f and g are shown in **Figures 4.3.46 and 4.3.47**, respectively. Use the graphs to find each of the following.

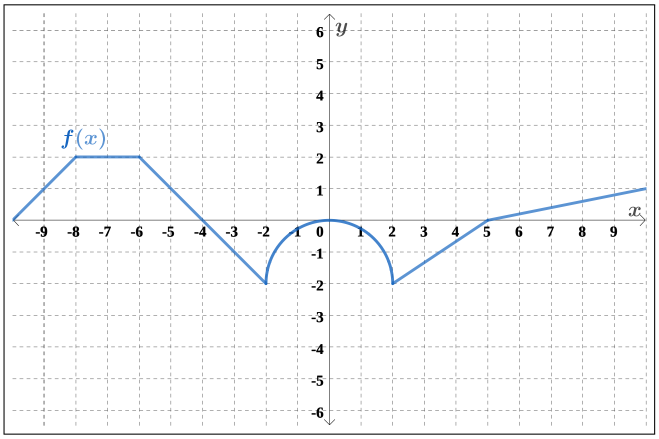


Figure 4.3.46: Graph of a function f

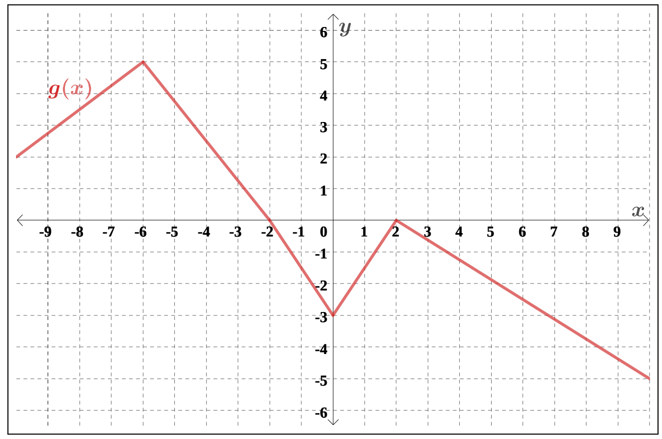


Figure 4.3.47: Graph of a function g

a. $\int_{-6}^{2} [f(x) + g(x)]\,dx$

b. $\int_{10}^{0} \left[3f(x) - \frac{g(x)}{2}\right] dx$

Solution:

a. Using Property #3, we can restate the problem:

$$\int_{-6}^{2} [f(x) + g(x)]\,dx = \int_{-6}^{2} f(x)\,dx + \int_{-6}^{2} g(x)\,dx$$

Thus, we will find $\int_{-6}^{2} f(x)\,dx$ and $\int_{-6}^{2} g(x)\,dx$ separately, and then we will add their values together.

To find $\int_{-6}^{2} f(x)\,dx$ graphically, we need to find the net area of the regions shaded in **Figure 4.3.48**:

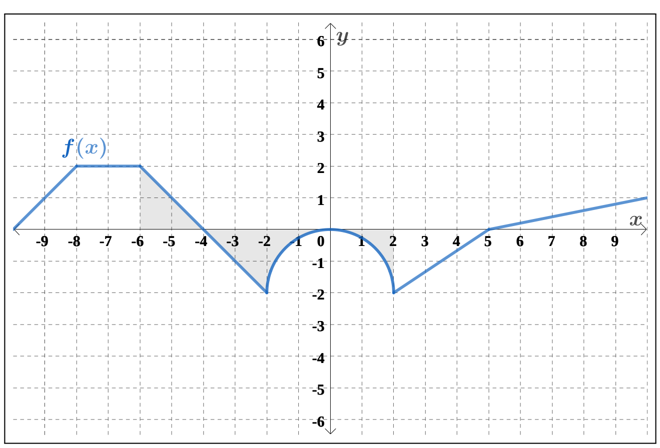


Figure 4.3.48: Shaded regions between the graph of f and the x-axis on the interval $[-6, 2]$

These shaded regions can be divided in the manner shown in **Figure 4.3.49** so that we can calculate the net area of the regions:

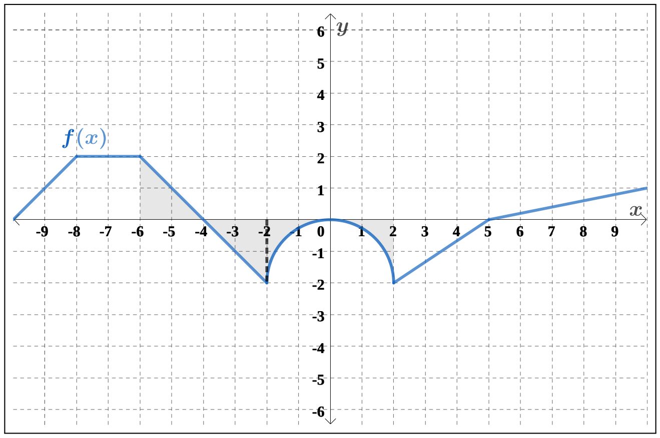


Figure 4.3.49: Shaded regions between the graph of f and the x-axis on the interval $[-6, 2]$

The first region is a triangle above the x-axis. It has a base of 2 (from $x = -6$ to $x = -4$) and a height of 2. Thus, its area is $\dfrac{2 \cdot 2}{2} = 2$.

The second region is also a triangle. It has a base of 2 (from $x = -4$ to $x = -2$) and a height of 2, so its area is $\dfrac{2 \cdot 2}{2} = 2$. This region is below the x-axis, so its contribution to the definite integral will be negative.

The third shaded region is a bit harder to describe. First, consider a rectangle whose length is 4 (from $x = -2$ to $x = 2$) and width is 2 (from $y = 0$ to $y = -2$). The area of this rectangle, then, is 8. However, this rectangle covers more area than we need for our problem. To only include the region we need, we can subtract the area of the semicircle that is inscribed in the rectangle from the area of the rectangle!

Again, the area of the rectangle is 8. The semicircle has a radius of 2, so its area is $\dfrac{\pi \cdot 2^2}{2} = 2\pi$. Thus, the area of this region is $8 - 2\pi$. However, because this region is below the x-axis, its area will be counted negatively.

Combining all of this information gives

$$\begin{aligned} \int_{-6}^{2} f(x)\,dx &= 2 + (-2) - (8 - 2\pi) \\ &= 0 - 8 + 2\pi \\ &= 2\pi - 8 \end{aligned}$$

Next, we need to find $\int_{-6}^{2} g(x)\,dx$, which is the net area of the regions shaded in **Figure 4.3.50**:

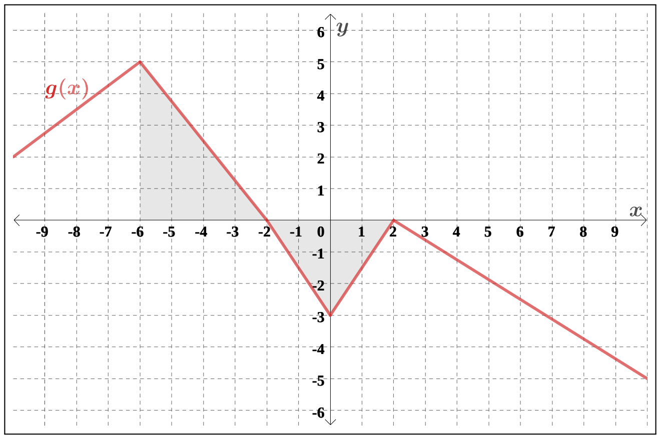


Figure 4.3.50: Shaded regions between the graph of g and the x-axis on the interval $[-6, 2]$

There are two triangles; one above the x-axis and one below. The triangle above the x-axis has a base of 4 (from $x = -6$ to $x = -2$) and a height of 5. Thus, its area is $\dfrac{4 \cdot 5}{2} = 10$.

The triangle below the x-axis has a base of 4 (from $x = -2$ to $x = 2$) and a height of 3. Thus, its area is $\dfrac{4 \cdot 3}{2} = 6$. Because this region is below the x-axis, its area will be counted negatively.

Combining all of this information gives

$$\begin{aligned}\int_{-6}^{2} g(x)\,dx &= 10 - 6 \\ &= 4\end{aligned}$$

Finally, adding the values of both definite integrals gives

$$\begin{aligned}\int_{-6}^{2} [f(x) + g(x)]\,dx &= \int_{-6}^{2} f(x)\,dx + \int_{-6}^{2} g(x)\,dx \\ &= (2\pi - 8) + 4 \\ &= 2\pi - 4\end{aligned}$$

b. We will start by using Property #3 and Property #4 to separate the definite integral and move the constants to the front, respectively:

$$\begin{aligned}\int_{10}^{0} \left[3f(x) - \frac{g(x)}{2}\right] dx &= \int_{10}^{0} 3f(x)\,dx - \int_{10}^{0} \frac{g(x)}{2}\,dx \\ &= 3\int_{10}^{0} f(x)\,dx - \frac{1}{2}\int_{10}^{0} g(x)\,dx\end{aligned}$$

At this point, you may be tempted to look at the graphs of f and g and calculate the net areas of the region between the graphs and the x-axis on the interval $[0, 10]$. However, the technique we have been using previously to find the value of a definite integral by calculating the net area only works when *the lower limit of integration is smaller than the upper limit of integration*.

Thus, before we can look at the graphs of f and g and calculate the net areas, we must switch the limits of integration for both definite integrals using Property #2:

$$3\int_{10}^{0} f(x)\,dx - \frac{1}{2}\int_{10}^{0} g(x)\,dx = -3\int_{0}^{10} f(x)\,dx + \frac{1}{2}\int_{0}^{10} g(x)\,dx$$

Now that the limits of integration are in numerical order, we can proceed by finding $\int_0^{10} f(x)\,dx$ and $\int_0^{10} g(x)\,dx$ separately and then substituting their values.

To find $\int_0^{10} f(x)\,dx$ graphically, we need to find the net area of the regions shaded in **Figure 4.3.51**:

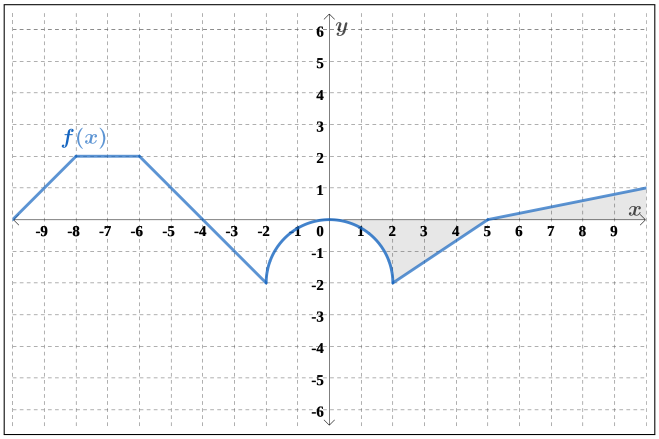


Figure 4.3.51: Shaded regions between the graph of f and the x-axis on the interval $[0, 10]$

We will divide these regions into three smaller regions as shown in **Figure 4.3.52**:

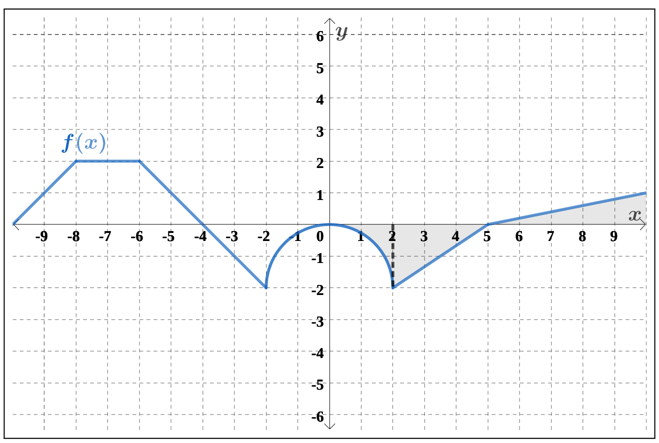


Figure 4.3.52: Three shaded regions between the graph of f and the x-axis on the interval $[0, 10]$

For the first region, which is from $x = 0$ to $x = 2$, we will use the technique we used previously: We will calculate the area of the square from $x = 0$ to $x = 2$ and then subtract the area of the quarter circle inscribed in the square. The square has a length of 2 and a width of 2. Thus, its area is 4. The quarter circle has a radius of 2, so its area is $\frac{\pi \cdot 2^2}{4} = \pi$. Therefore, the area of this shaded region is $4 - \frac{\pi \cdot 2^2}{4} = 4 - \pi$. Because this region is below the x-axis, we will count its area as negative when calculating the definite integral.

The second region is a triangle with a base of 3 (from $x = 2$ to $x = 5$) and a height of 2. Thus, its area is $\frac{3 \cdot 2}{2} = 3$. Again, because this region is below the x-axis, we will count its area as negative.

The last region is a triangle above the x-axis. It has a base of 5 (from $x = 5$ to $x = 10$) and a height of 1. So its area is $\frac{5 \cdot 1}{2} = \frac{5}{2}$. This region is above the x-axis, so we will count its area as positive.

Now, we calculate the definite integral:

$$\begin{aligned}\int_0^{10} f(x)\,dx &= -(4-\pi) - 3 + \frac{5}{2} \\ &= \pi - \frac{9}{2}\end{aligned}$$

Next, we need to find $\int_0^{10} g(x)\,dx$, which is the net area of the regions shaded in **Figure 4.3.53**:

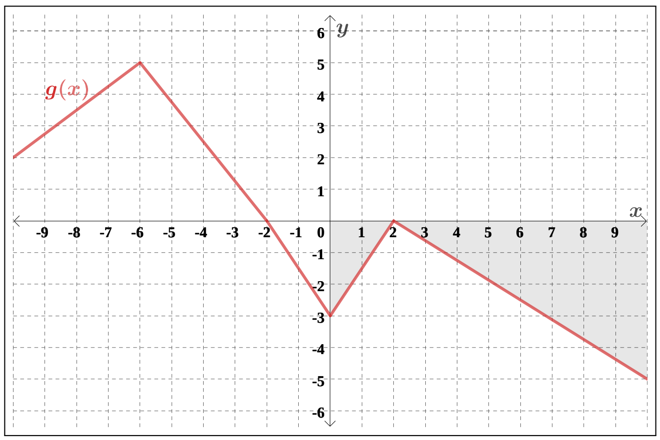


Figure 4.3.53: Shaded regions between the graph of g and the x-axis on the interval $[0, 10]$

There are two triangles, both of which are below the x-axis. The first triangle has a base of 2 (from $x = 0$ to $x = 2$) and a height of 3, so its area is 3. The second triangle has a base of 8 (from $x = 2$ to $x = 10$) and a height of 5, so its area is 20. Both of these triangles are below the x-axis, so their areas will both be counted negatively when calculating the definite integral.

Thus, we have

$$\begin{aligned}\int_0^{10} g(x)\,dx &= (-3) + (-20) \\ &= -23\end{aligned}$$

Now, combining all of this information, we can calculate the final answer:

$$\begin{aligned}\int_{10}^{0} \left[3f(x) - \frac{g(x)}{2}\right] dx &= -3\int_0^{10} f(x)\,dx + \frac{1}{2}\int_0^{10} g(x)\,dx \\ &= -3\left(\pi - \frac{9}{2}\right) + \frac{1}{2}(-23) \\ &= -3\pi + \frac{27}{2} - \frac{23}{2} \\ &= 2 - 3\pi\end{aligned}$$

■

Try It # 17:
Using **Figures 4.3.46** and **4.3.47**, find

a. $\int_{10}^{2} 3f(x)\,dx.$

b. $\int_{-10}^{2} [2g(x) - f(x)]\,dx.$

Comparison Properties of the Definite Integral

A picture can sometimes tell us more about a function than the results of computations. Comparing functions by their graphs as well as by their algebraic expressions can often give new insight into the process of integration.

Intuitively, we might say that if $f(x)$ is greater than $g(x)$, then the same should hold for their definite integrals: If both functions are positive and $f(x)$ is greater than $g(x)$, then the area between the graph of f and the x-axis will contain the area between the graph of g and the x-axis, and perhaps more. This is true even if the functions are not strictly positive!

This leads us to three comparison properties:

Comparison Properties of the Definite Integral

Let f and g be continuous functions.

1. If $f(x) \geq 0$ for $a \leq x \leq b$, then

$$\int_{a}^{b} f(x)\,dx \geq 0$$

2. If $f(x) \geq g(x)$ for $a \leq x \leq b$, then

$$\int_{a}^{b} f(x)\,dx \geq \int_{a}^{b} g(x)\,dx$$

3. If L and M are constants such that $L \leq f(x) \leq M$ for $a \leq x \leq b$, then

$$L(b-a) \leq \int_{a}^{b} f(x)\,dx \leq M(b-a)$$

Let's examine Comparison Property #3 more closely with the graph of a function such as the one shown in **Figure 4.3.54**:

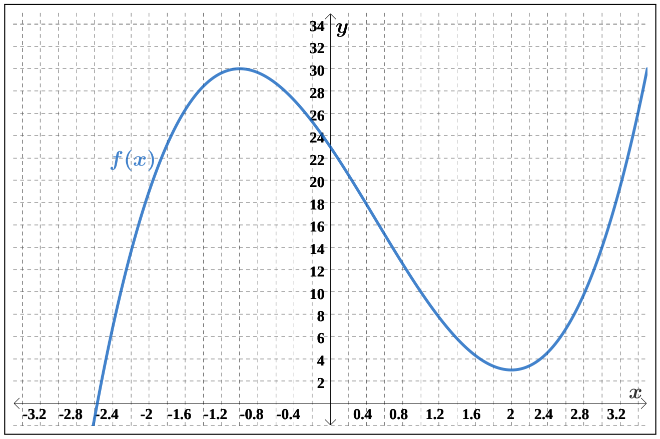


Figure 4.3.54: Graph of a function f

On the interval $[-2,3]$, the absolute minimum of f is $y = 3$, and it occurs at $x = 2$. Now, consider the rectangle on this interval with width 3. This rectangle is graphed and shaded on the same axes as f in **Figure 4.3.55**:

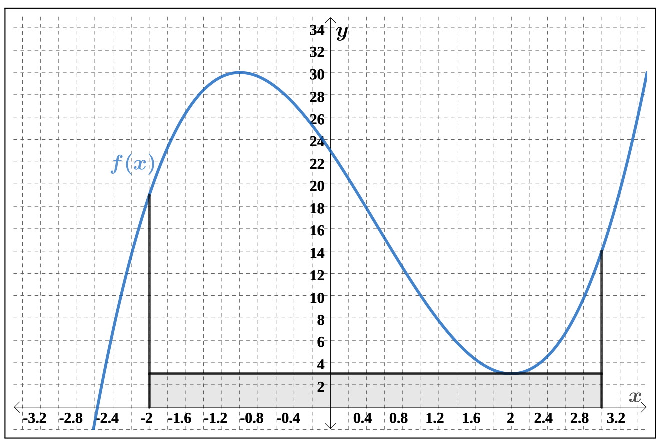


Figure 4.3.55: Graph of f and a shaded rectangle on the interval $[-2,3]$ with width 3

The entire area of this rectangle is contained within the area between the graph of f and the x-axis! This means that the area of this rectangle, which has a width of 3 and length given by $3-(-2)$, is less than the value of the definite integral of f on the interval. In symbols,

$$3(3-(-2) \leq \int_{-2}^{3} f(x)\,dx \implies$$
$$15 \leq \int_{-2}^{3} f(x)\,dx$$

Similarly, we see the absolute maximum of f on the interval $[-2,3]$ is $y = 30$, and it occurs at $x = -1$. Again, we will consider a rectangle on the interval, but this time with width 30. This rectangle is graphed and shaded on the same axes as f in **Figure 4.3.56**:

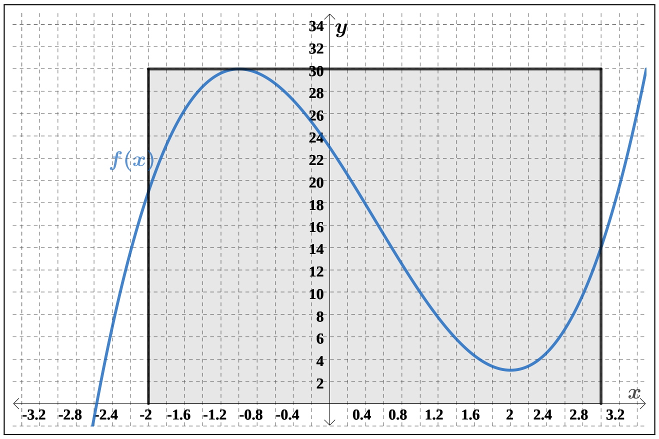


Figure 4.3.56: Graph of f and a shaded rectangle on the interval $[-2,3]$ with width 30

The area of this rectangle completely contains the area between the graph of f and the x-axis. This rectangle has a width of 30 and a length given by $3-(-2)$. In symbols,

$$\int_{-2}^{3} f(x)\,dx \le 30(3-(-2)) \implies$$
$$\int_{-2}^{3} f(x)\,dx \le 150$$

Thus, by Comparison Property #3, we have

$$15 \le \int_{-2}^{3} f(x)\,dx \le 150$$

Hence, we know the smallest and largest possible values of $\int_{-2}^{3} f(x)\,dx$ are 15 and 150, respectively.

Even though our justification for this property was with a function that was positive on an interval, the inequalities are true even if $f(x)$ is negative on part or all of the interval $[a,b]$.

■ **Example 21** If f is a continuous function such that $-2 \le f(x) \le 7$ on the interval $[-4,3]$, determine the smallest and largest possible values of $\int_{-4}^{3} f(x)\,dx$.

Solution:
Recall that Comparison Property #3 states that

$$L(b-a) \le \int_{a}^{b} f(x)\,dx \le M(b-a)$$

if L and M are constants such that $L \le f(x) \le M$ for $a \le x \le b$.

Because $-2 \le f(x) \le 7$ and L is the absolute minimum of f on the interval, we know $L = -2$. By Comparison Property #3, we have $-2(3-(-4)) \le \int_{-4}^{3} f(x)\,dx$. In other words, $-14 \le \int_{-4}^{3} f(x)\,dx$. Thus, the smallest possible value of this definite integral is -14.

Likewise, the absolute maximum of f on the interval is $M = 7$, so Comparison Property #3 says $\int_{-4}^{3} f(x)\,dx \le 7(3-(-4))$. In other words, $\int_{-4}^{3} f(x)\,dx \le 49$. Thus, the largest possible value of this definite integral is 49.

In symbols,

$$-14 \leq \int_{-4}^{3} f(x)\,dx \leq 49$$

■

Try It # 18:
If g is a continuous function such that $-14 \leq g(x) \leq 12$ on the interval $[0,5]$, determine the

a. smallest possible value of $\int_{0}^{5} g(x)\,dx$.

b. largest possible value of $\int_{0}^{5} g(x)\,dx$.

■ **Example 22** Determine the smallest and largest possible values of $\int_{2}^{10} -x\ln(x)\,dx$. Round your answers to two decimal places, if necessary.

Solution:
We must use Comparison Property #3 to find the smallest and largest possible values of the definite integral. Therefore, we must find the absolute maximum and minimum values of the function $y = -x\ln(x)$ on the interval $[2,10]$.

If we graph this function, we see that it is decreasing on the interval $[2,10]$. We could also find the derivative to determine that it is decreasing on the interval. In short, this means the absolute maximum occurs at $x = 2$ and is $-2\ln(2)$, and the absolute minimum occurs at $x = 10$ and is $-10\ln(10)$.

Thus, the smallest value that $\int_{2}^{10} -x\ln(x)\,dx$ could be is $-10\ln(10)\cdot(10-2) \approx -184.21$, and the largest it could be is $-2\ln(2)\cdot(10-2) \approx -11.09$.

■

Try It # 19:

a. Determine the smallest possible value of $\int_{-4}^{2} \left(2x^2 - \frac{x^4}{4}\right) dx$.

b. Determine the largest possible value of $\int_{-4}^{2} \left(2x^2 - \frac{x^4}{4}\right) dx$.

■ **Example 23** Compare $\int_{0}^{1} \sqrt{1+x^2}\,dx$ and $\int_{0}^{1} \sqrt{1+x}\,dx$. Determine the smallest and largest possible values of each of these definite integrals.

Solution:
The graphs of $f(x) = \sqrt{1+x^2}$ and $g(x) = \sqrt{1+x}$ are shown on the same axes in **Figure 4.3.57**:

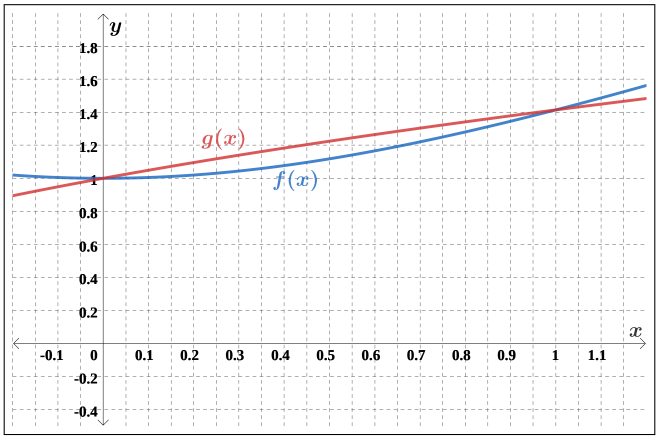


Figure 4.3.57: Graphs of the functions f and g

From the graphs, we see $f(x) \leq g(x)$ on the interval $[0,1]$. Using Comparison Property #2, we have

$$\int_0^1 \sqrt{1+x^2}\,dx \leq \int_0^1 \sqrt{1+x}\,dx$$

Furthermore, for both of these functions, we see that the absolute minimum value on the interval $[0,1]$ is $y=1$, which occurs at $x=0$. Thus, both definite integrals must be greater than or equal to $1(1-0)=1$. The absolute maximum value for both of these functions on the interval is $y=\sqrt{2}$, which occurs at $x=1$. So they must both be less than or equal to $\sqrt{2}(1-0)=\sqrt{2}$. Combining all of these inequalities shows the following:

$$1 \leq \int_0^1 \sqrt{1+x^2}\,dx \leq \int_0^1 \sqrt{1+x}\,dx \leq \sqrt{2}$$

While it is true in this example that the inequalities are strict (meaning we could use $<$ instead of $\leq$), it is not the case in general with the Comparison Properties of the Definite Integral!

■

Try It # 20:

Compare $\displaystyle\int_{-1}^{3}\left(10-x^2\right)\,dx$ and $\displaystyle\int_{-1}^{3}(-x)\,dx$. Determine the smallest and largest possible values of each of these definite integrals.

Try It Answers

1. $28+\dfrac{9\pi}{2}$
2. 64
3. 31
4. **a.** 80
   b. 80
   c. 88

5. 71

6. 10.4

7. **a.** dollars

   b. feet pounds

   c. cubits per fortnight

8. 47 meters

9. **a.** \$416,000

   b. \$512,000

10. \$1465.63

11. **a.** 33.2 feet

    b. 112 feet

12. 0

13. **a.** 2.8

    b. -1

14. $\$947,200$

15. -7

16. -21

17. **a.** 0.5

    b. $30 - 2\pi$

18. **a.** -70

    b. 60

19. **a.** -192

    b. 24

20. $-12 \le \int_{-1}^{3} (-x)\, dx \le \int_{-1}^{3} \left(10 - x^2\right) dx \le 40$

EXERCISES

Basic Skills Practice

For Exercises 1 - 3, compute the sum.

1. $\sum_{i=1}^{3} i^3$

2. $\sum_{i=1}^{3} \frac{i(i+1)}{2}$

3. $\sum_{k=1}^{3} 3\left(2^k\right)$

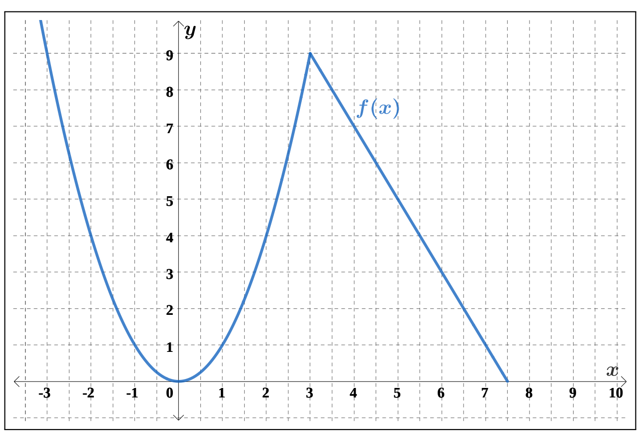
For exercises 4 - 6, write the expression using summation notation (i.e., Σ notation).

4. $\frac{0^3}{4}+\frac{1^3}{4}+\frac{2^3}{4}+\frac{3^3}{4}+\frac{4^3}{4}+\frac{5^3}{4}+\frac{6^3}{4}$

5. $\left(5(1)^2+2(1)\right)+\left(5(2)^2+2(2)\right)+\left(5(3)^2+2(3)\right)+\left(5(4)^2+2(4)\right)$

6. $\left(0^4+4\right)+\left(1^4+4\right)+\left(2^4+4\right)+\left(3^4+4\right)+\left(4^4+4\right)+\left(5^4+4\right)+\left(6^4+4\right)$

7. Use the graph of f shown below to calculate each of the following.



(a) A right-hand Riemann sum on $[0,6]$ using 6 subintervals of equal width

(b) A left-hand Riemann sum on $[0,6]$ using 6 subintervals of equal width

(c) A right-hand Riemann sum on $[-2,2]$ using 4 subintervals of equal width

(d) A midpoint Riemann sum on $[-3,7]$ using 5 subintervals of equal width

(e) A left-hand Riemann sum on $[1,5]$ using 2 subintervals of equal width

8. Particular values of $f(x)$ are shown in the table below. Use the information to calculate each of the following.

x	3	4	5	6	7	8	9	10
$f(x)$	9	8	5	0	7	16	27	40

(a) A left-hand Riemann sum on $[3,7]$ using 4 subintervals of equal width

(b) A right-hand Riemann sum on $[3,7]$ using 4 subintervals of equal width

(c) A left-hand Riemann sum on $[3,10]$ using 7 subintervals of equal width

(d) A midpoint Riemann sum on $[3,9]$ using 3 subintervals of equal width

(e) A right-hand Riemann sum on $[3,9]$ using 6 subintervals of equal width

9. Estimate the area under the graph of $f(x) = 2^x + 9x$ on the interval $[0,3]$ using a right-hand Riemann sum with 3 subintervals of equal width.

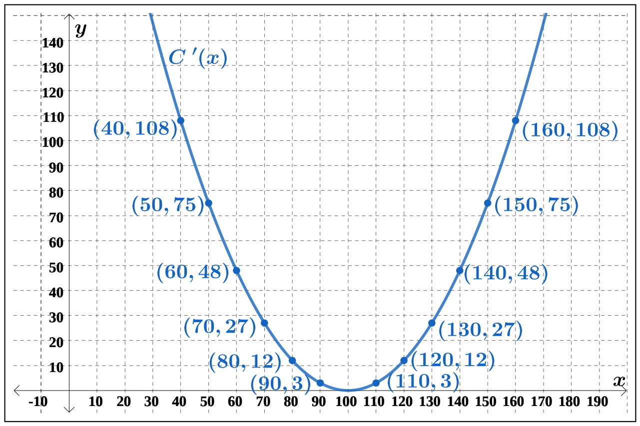
10. Estimate the area under the graph of $f(x) = x^3 - 6x^2 + 80x$ on the interval $[6,12]$ using a left-hand Riemann sum with 6 subintervals of equal width.

11. Estimate the area under the graph of $f(x) = -x^2 + 6x + 7$ on the interval $[-1,7]$ using a midpoint Riemann sum with 4 subintervals of equal width.

12. The table below gives the units for x and $f(x)$. Determine the appropriate units for the area between the graph of f and the x-axis on an interval $[a,b]$.

Units for x	Units for $f(x)$	Units for Area
gallons	miles per gallon	**a.**
minutes	kerfuffles per minute	**b.**
seconds	charges per second	**c.**
ears of corn	kernels per ear of corn	**d.**

13. The graph of a company's marginal cost function, C', is shown below, where x is the number of items produced and $C'(x)$ is measured in dollars per item. Selected points are also shown on the graph.



(a) Estimate the change in cost when the number of items produced increases from 60 to 120 using a left-hand Riemann sum with 3 subintervals of equal width.

(b) Estimate the change in cost when the number of items produced increases from 60 to 120 using a right-hand Riemann sum with 3 subintervals of equal width.

(c) Estimate the change in cost when the number of items produced increases from 60 to 120 using a midpoint Riemann sum with 3 subintervals of equal width.

14. Particular values of the marginal profit function of a company that sells packs of water balloons are shown in the table below, where x is the number of packs of water balloons sold and $P'(x)$ is measured in dollars per pack of water balloons.

x	50	60	70	80	90	100	110	120	130
$P'(x)$	−20	−12	−8	3	10	17	24	30	31

(a) Estimate the change in profit when the number of packs of water balloons sold increases from 50 to 130 using a right-hand Riemann sum with 8 subintervals of equal width.

(b) Estimate the change in profit when the number of packs of water balloons sold increases from 50 to 130 using a left-hand Riemann sum with 8 subintervals of equal width.

15. Rock 'n' Stroll makes musical baby strollers. The company's weekly marginal profit function is given by $P'(x) = 30x - 0.3x^2 - 250$ dollars per stroller when x strollers are sold.

(a) Estimate the change in profit when the number of strollers sold increases from 20 to 25 using a left-hand Riemann sum with 5 subintervals of equal width.

(b) Estimate the change in profit when the number of strollers sold increases from 20 to 25 using a right-hand Riemann sum with 5 subintervals of equal width.

16. The marginal revenue function for a company that sells lawn chairs for kids is given by $R'(x) = 25 - 1.5\sqrt{x}$ dollars per lawn chair when x lawn chairs are sold.

    (a) Estimate the change in revenue when the number of lawn chairs sold increases from 60 to 120 using a left-hand Riemann sum with 4 subintervals of equal width.

    (b) Estimate the change in revenue when the number of lawn chairs sold increases from 400 to 450 using a midpoint Riemann sum with 5 subintervals of equal width.

17. The velocity of an object after t seconds is $v(t)$ feet per second, and the graph of v is shown below.

    (a) Estimate the distance traveled by the object after six seconds using a right-hand Riemann sum with three subintervals of equal width.

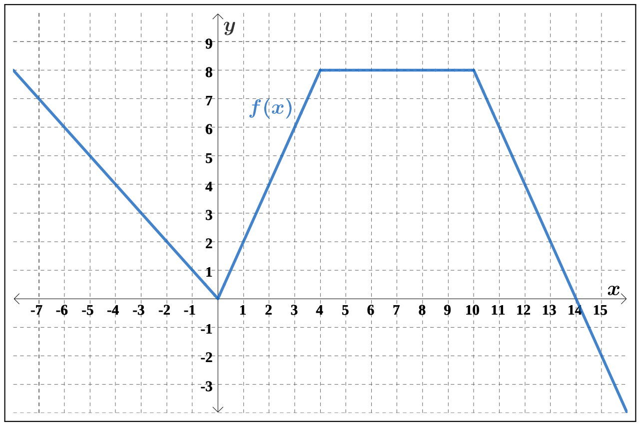
    (b) Is the value of the sum you calculated in part (a) an overestimate or an underestimate of the distance traveled?

18. The table below gives the velocity of Jack, in feet per second, every 2 seconds for the first 14 seconds of his race. Use the table to estimate how far Jack ran during the first 14 seconds using a left-hand Riemann sum with 7 subintervals of equal width.

t (s)	0	2	4	6	8	10	12	14
v (ft/s)	0	5.2	6.4	7.2	9.3	9.5	8.7	8.5

19. An object moves at a rate of $f'(t) = 0.09t^2 + 0.18t$ meters per second after t seconds have passed. Estimate the distance traveled by this object during the first five seconds using a right-hand Riemann sum with five subintervals of equal width. Round your answer to two decimal places, if necessary.

20. The graph of f is shown below. Use the graph to find each of the following.



(a) $\displaystyle\int_0^4 f(x)\,dx$

(b) $\displaystyle\int_{-4}^0 f(x)\,dx$

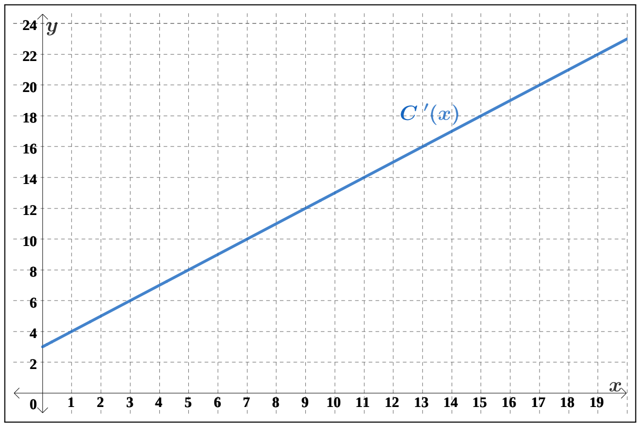
(c) $\displaystyle\int_4^{10} f(x)\,dx$

For exercises 21 and 22, use geometric regions to find the value of the definite integral exactly.

21. $\displaystyle\int_2^7 4\,dx$

22. $\displaystyle\int_{-3}^2 x\,dx$

23. The graph of a company's marginal cost function, C', is shown below, where x is the number of items produced and $C'(x)$ is measured in dollars per item.

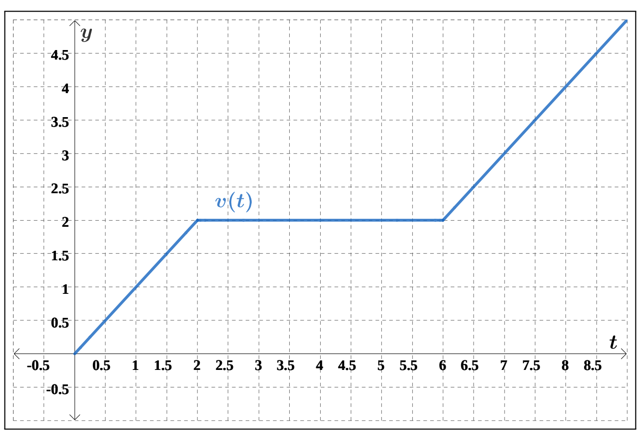


(a) Find the exact change in cost when production increases from 3 items to 9 items.

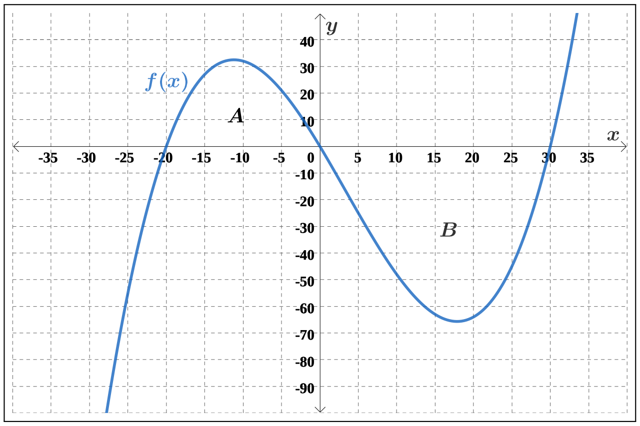
(b) Find the exact change in cost when production increases from 1 item to 15 items.

(c) Find the exact change in cost when production increases from 5 items to 19 items.

24. The velocity of an object after t seconds is $v(t)$ feet per second, and the graph of v is shown below. Find the exact distance the object traveled during the first four seconds.



25. The graph of f is shown below. Use the indicated areas of the regions between the graph of f and the x-axis to find each of the following.



Area of A: 427
Area of B: 1260

(a) $\displaystyle\int_{-20}^{0} 4f(x)\,dx$

(b) $\displaystyle\int_{-20}^{30} -\frac{1}{2}f(x)\,dx$

26. If $\displaystyle\int_{-5}^{4} f(x)\,dx = 6$, $\displaystyle\int_{4}^{9} f(x)\,dx = 2$, and $\displaystyle\int_{-5}^{9} g(x)\,dx = -3$, find each of the following.

(a) $\displaystyle\int_{-5}^{9} \frac{1}{2}g(x)\,dx$

(b) $\displaystyle\int_{-5}^{4} -7f(x)\,dx$

(c) $\displaystyle\int_{-5}^{9} f(x)\,dx$

(d) $\displaystyle\int_{9}^{-5} g(x)\,dx$

(e) $\displaystyle\int_{-5}^{9} [g(x) - f(x)]\,dx$

(f) $\displaystyle\int_{-5}^{9} \left[8f(x) - \frac{g(x)}{3}\right]\,dx$

27. If f is a continuous function such that $3 \le f(x) \le 8$ on the interval $[2,9]$, use Comparison Property #3 to determine the (a) largest and (b) smallest possible values of $\displaystyle\int_{2}^{9} f(x)\,dx$.

28. If f is a continuous function such that $-5 \leq f(x) \leq 1$ on the interval $[-10,-6]$, use Comparison Property #3 to determine the (a) largest and (b) smallest possible values of $\int_{-10}^{-6} f(x)\,dx$.

29. If f is a continuous function such that $-12 \leq f(x) \leq -4$ on the interval $[-7,13]$, use Comparison Property #3 to determine the (a) largest and (b) smallest possible values of $\int_{-7}^{13} f(x)\,dx$.

For Exercises 30 - 32, use Comparison Property #3 to find the (a) largest and (b) smallest possible values of the definite integral.

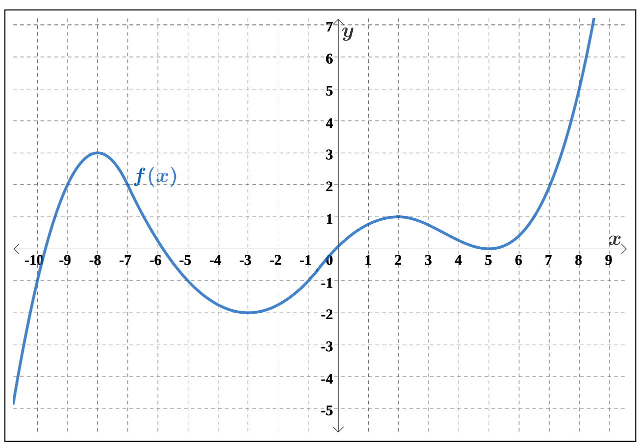
30. $\int_{0}^{8} (2x+5)\,dx$

31. $\int_{-2}^{3} \left(x^2+1\right)\,dx$

32. $\int_{-1}^{1} -3x^3\,dx$

Intermediate Skills Practice

33. Use the graph of f shown below to calculate each of the following.



(a) A left-hand Riemann sum on $[-10,-7]$ using 3 subintervals of equal width

(b) A right-hand Riemann sum on $[-10,-7]$ using 3 subintervals of equal width

(c) A midpoint Riemann sum on $[-8,0]$ using 4 subintervals of equal width

(d) A right-hand Riemann sum on $[-1,8]$ using 3 subintervals of equal width

(e) A left-hand Riemann sum on $[2,8]$ using 2 subintervals of equal width

34. Particular values of $f(x)$ are shown in the table below. Use the information to calculate each of the following.

x	-1	-0.5	0	0.5	1	1.5	2	2.5	3
$f(x)$	-28	-11.38	0	6.88	10	10.13	8	4.38	0

(a) A left-hand Riemann sum on $[-1,3]$ using 8 subintervals of equal width

(b) A right-hand Riemann sum on $[-1,3]$ using 8 subintervals of equal width

(c) A midpoint Riemann sum on $[-1,3]$ using 4 subintervals of equal width

(d) A left-hand Riemann sum on $[-1,1]$ using 2 subintervals of equal width

(e) A right-hand Riemann sum on $[-0.5,2]$ using 5 subintervals of equal width

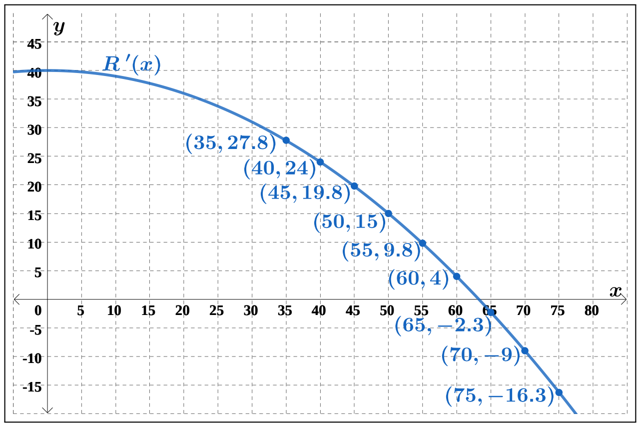
For Exercises 35 - 37, estimate the definite integral using the indicated Riemann sum. Round your answer to three decimal places, if necessary.

35. $\int_{-1}^{2}\left(x^3-x+3\right)\,dx$; right-hand Riemann sum; 3 subintervals of equal width

36. $\int_{1}^{5}\left(9-x^2\right)\,dx$; left-hand Riemann sum; 4 subintervals of equal width

37. $\int_{4}^{10}\left(4+\sqrt{x}-0.5x\right)\,dx$; midpoint Riemann sum; 3 subintervals of equal width

38. Flashy Fashion Dolls has a marginal revenue function given by $R'(x)$ dollars per doll, where x is the number of dolls sold. The graph of R' is shown below as well as selected points.



(a) Estimate the change in revenue when the number of dolls sold increases from 40 to 70 using a left-hand Riemann sum with 3 subintervals of equal width.

(b) Is the value of the sum you calculated in part (a) an overestimate or an underestimate of the change in revenue?

(c) Estimate the change in revenue when the number of dolls sold increases from 40 to 70 using a right-hand Riemann sum with 3 subintervals of equal width.

(d) Is the value of the sum you calculated in part (c) an overestimate or an underestimate of the change in revenue?

(e) Estimate the change in revenue when the number of dolls sold increases from 40 to 70 using a midpoint Riemann sum with 3 subintervals of equal width.

39. Particular values of the marginal cost function for Kuddly Kites, a company that sells kites shaped like different animals, are shown in the table below, where x is the number of kites produced and $C'(x)$ is measured in dollars per kite.

x	200	208	216	224	232	240	248	256	264	272	280
$C'(x)$	2.00	3.75	4.75	5.25	5.60	5.40	5.10	4.50	3.70	2.50	1.25

(a) Estimate the change in cost when the number of kites produced increases from 208 to 272 using a right-hand Riemann sum with 4 subintervals of equal width.

(b) Estimate the change in cost when the number of kites produced increases from 208 to 272 using a left-hand Riemann sum with 4 subintervals of equal width.

40. The marginal profit function for a company is given by $P'(x) = \dfrac{400}{\sqrt{x+1}} - 12$ dollars per item when x items are sold.

    (a) Estimate the change in profit when the number of items sold increases from 1108 to 1123 using a left-hand Riemann sum with 6 subintervals of equal width.

    (b) Estimate the change in profit when the number of items sold increases from 1108 to 1123 using a right-hand Riemann sum with 6 subintervals of equal width.

    (c) Estimate the change in profit from selling the first 1000 items using a left-hand Riemann sum with 4 subintervals of equal width.

41. A company that makes camping tents has a marginal cost function given by $C'(x) = 15 + \dfrac{300}{0.2x+4}$ dollars per tent, where x is the number of tents produced.

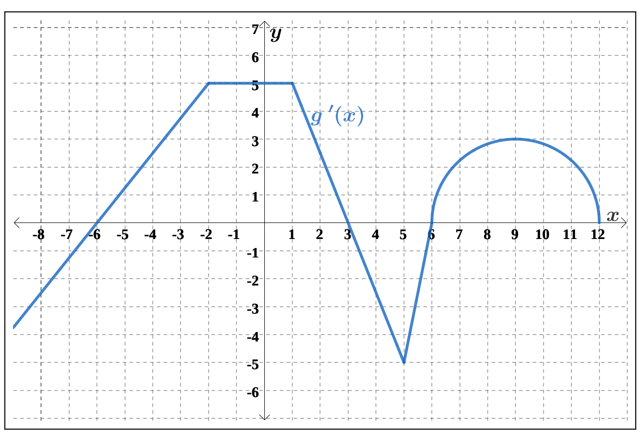
    (a) Estimate the change in cost when the number of tents produced increases from 140 to 206 using a left-hand Riemann sum with 3 subintervals of equal width.

    (b) Estimate the change in cost when the number of tents produced increases from 50 to 66 using a midpoint Riemann sum with 8 subintervals of equal width.

42. The table below gives the velocity of an object, in meters per second, every 5 seconds for 1 minute. Estimate the distance traveled by the object during the last 30 seconds using a left-hand Riemann sum with 3 subintervals of equal width.

t (s)	0	5	10	15	20	25	30	35	40	45	50	55	60
v (m/s)	0	0.6	1.5	2.4	3.0	2.3	2.0	1.8	0.4	1.3	2.1	2.7	3.5

43. The graph of g' is shown below. Use the graph to find each of the following.



    (a) $\displaystyle\int_0^3 g'(x)\,dx$

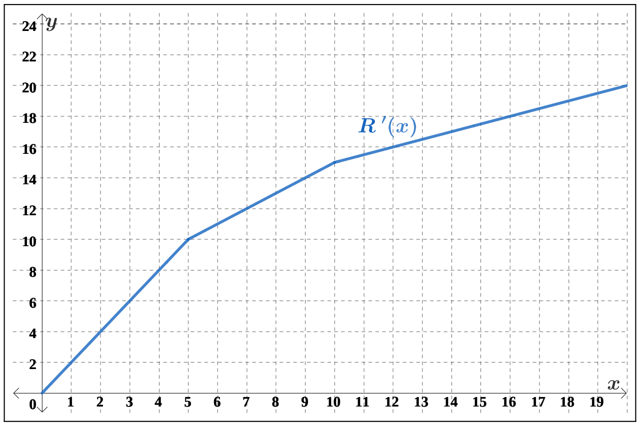
    (b) $\displaystyle\int_0^{12} g'(x)\,dx$

For exercises 44 and 45, use geometric regions to find the value of the definite integral.

44. $\displaystyle\int_{-4}^{-1} -3\,dx$

45. $\displaystyle\int_{0}^{6} (2x+5)\,dx$

46. The graph of a company's marginal revenue function, R', is shown below, where x is the number of items sold and $R'(x)$ is measured in dollars per item.



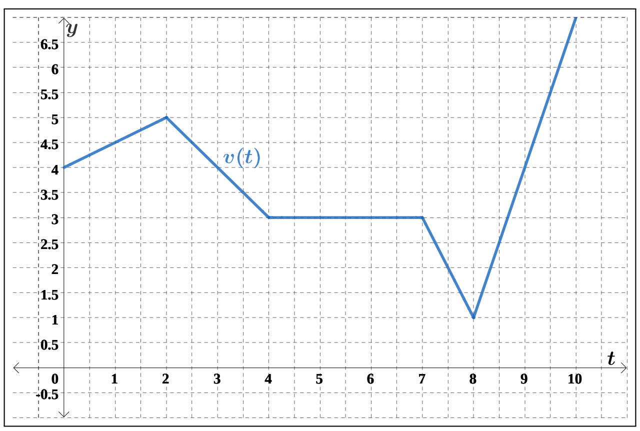
(a) Find the change in revenue when the first 5 items are sold.

(b) Find the change in revenue when the number of items sold increases from 3 items to 10 items.

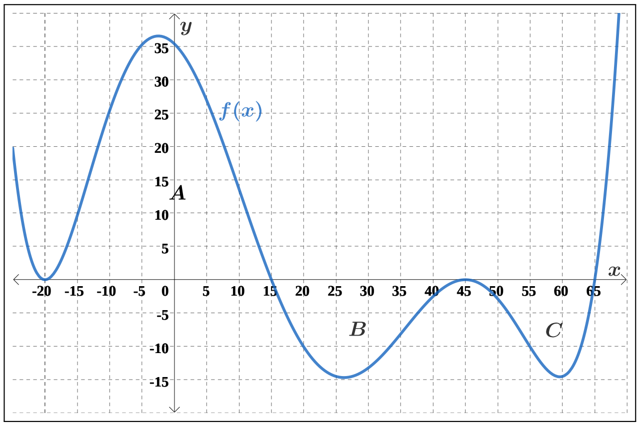
(c) Find the change in revenue when the number of items sold increases from 2 items to 16 items.

(d) Find the revenue from selling the first 7 items. *Hint: If the company does not sell any items, it will not have any revenue.*

47. The velocity of an object after t seconds is $v(t)$ feet per second, and the graph of v is shown below. Find the distance the object traveled during the first eight seconds.



48. The graph of f is shown below. Use the indicated areas of the regions between the graph of f and the x-axis to find each of the following.



Area of A: 736
Area of B: 248
Area of C: 151

(a) $\displaystyle\int_{15}^{45} -5f(x)\,dx + \int_{15}^{-20} \frac{f(x)}{4}\,dx$

(b) $\displaystyle\int_{-20}^{45} 2f(x)\,dx - \int_{65}^{15} 0.3f(x)\,dx$

49. If $\displaystyle\int_{-6}^{-2} f(x)\,dx = 17$, $\displaystyle\int_{3}^{-2} f(x)\,dx = -5$, $\displaystyle\int_{-6}^{3} g(x)\,dx = 8$, and $\displaystyle\int_{-6}^{3} h(x)\,dx = -12$, find each of the following.

(a) $\displaystyle\int_{-6}^{3} [4f(x) - 9g(x)]\,dx$

(b) $\displaystyle\int_{-6}^{3} -\frac{5}{2}g(x)\,dx + \int_{3}^{-6} h(x)\,dx$

(c) $\displaystyle\int_{-6}^{3} \left[-\frac{h(x)}{4} + 0.5g(x)\right]dx + \int_{3}^{3} \sqrt{2}f(x)\,dx$

For Exercises 50 - 52, find the (a) largest and (b) smallest possible values of the definite integral.

50. $\displaystyle\int_{-8}^{4} \left(x^5 + 5x^4 - 35x^3\right)dx$

51. $\displaystyle\int_{5}^{9} \frac{-x^2}{3-x}\,dx$

52. $\displaystyle\int_{-2}^{0} (x-4)e^x\,dx$

Mastery Practice

53. Calculate $\sum_{i=0}^{5} \frac{1}{i+1}$.

54. Write the following expression using summation notation:

$$\frac{1}{2}f(x_0)+\frac{1}{2}f(x_1)+\frac{1}{2}f(x_2)+\frac{1}{2}f(x_3)+\frac{1}{2}f(x_4)+\frac{1}{2}f(x_5)$$

For Exercises 55 - 57, estimate the definite integral using the indicated Riemann sum. Round your answer to three decimal places, if necessary.

55. $\int_{-2}^{3}(8x^2-2x^4)\,dx$; right-hand Riemann sum; 5 subintervals of equal width

56. $\int_{-3}^{0} e^{3x}\,dx$; left-hand Riemann sum; 5 subintervals of equal width

57. $\int_{-4}^{4}(x^3-16x)\,dx$; midpoint Riemann sum; 4 subintervals of equal width

58. Particular values of $f(x)$ are shown in the table below. Use the information to calculate each of the following.

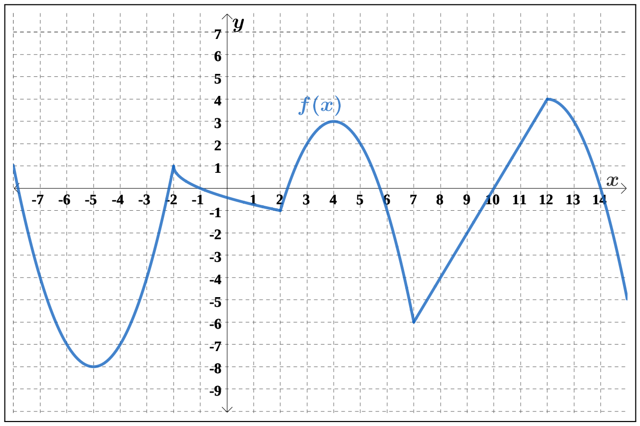
x	0	0.5	1	1.5	2	2.5	3
$f(x)$	3	1.8	1	0.43	0	−0.33	−0.6

(a) A midpoint Riemann sum on $[0,3]$ using 3 subintervals of equal width

(b) A right-hand Riemann sum on $[0,3]$ using 6 subintervals of equal width

(c) A left-hand Riemann sum on $[0,3]$ using 3 subintervals of equal width

59. Use the graph of f shown below to calculate each of the following.



(a) A left-hand Riemann sum on $[2,10]$ using 4 subintervals of equal width

(b) A right-hand Riemann sum on $[2,10]$ using 4 subintervals of equal width

(c) A midpoint Riemann sum on $[-3,9]$ using 3 subintervals of equal width

(d) A right-hand Riemann sum on $[1,6]$ using 5 subintervals of equal width

60. The table below gives the units for x and $f(x)$. Determine the appropriate units for the area between the graph of f and the x-axis on an interval $[a,b]$.

Units for x	Units for $f(x)$	Units for Area
mice	cats per mouse	**a.**
seconds	hit points per second	**b.**
inches	dollars per inch	**c.**

61. Russ and Jay are driving back from track practice, when suddenly a squirrel runs onto the road. Russ slams on the brakes, and his velocity, in feet per second, every second until his car stops is shown in the table below. Find the best possible (a) upper and (b) lower estimates for the distance the car traveled from the time Russ hit the brakes until the car came to a stop. *(Note: The squirrel was okay!)*

t (s)	0	1	2	3	4
v (ft/s)	50	42	27	12	0

62. Reid makes and sells custom bracelets using colorful rubber bands. Her marginal profit function is given by $\dfrac{dP}{dx} = \dfrac{8}{\sqrt{0.01x^2+1}} - 4$ dollars per bracelet when x bracelets are made and sold.

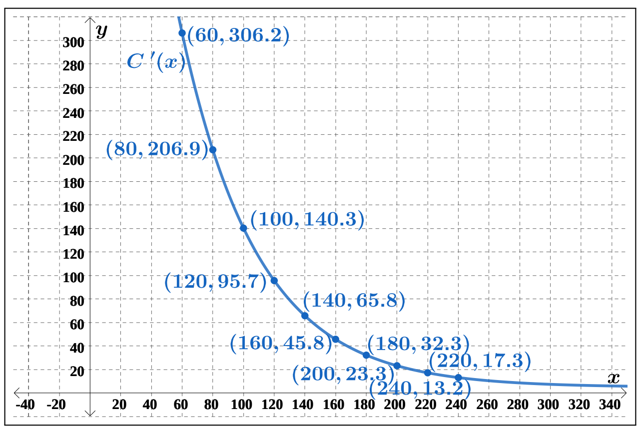
(a) Estimate the change in profit when the number of bracelets made and sold increases from 5 to 13 using a left-hand Riemann sum with 5 subintervals of equal width.

(b) Estimate the change in profit when the number of bracelets made and sold increases from 5 to 13 using a right-hand Riemann sum with 5 subintervals of equal width.

(c) Estimate the change in profit from making and selling the first 9 bracelets using a right-hand Riemann sum with 3 subintervals of equal width.

63. Sydnie walks in a straight line at a rate of $v(t) = \dfrac{10\ln(t+1)}{t+1}$ feet per second after t seconds have passed. Estimate the distance she walked during the first 30 seconds using a midpoint Riemann sum with five subintervals of equal width. Round your answer to two decimal places, if necessary.

64. The graph of an amusement park's marginal cost function, C', is shown below, where x is the number of tickets sold and $C'(x)$ is measured in dollars per ticket. Selected points are also shown on the graph.



(a) Estimate the change in cost when the number of tickets sold increases from 60 to 220 using a left-hand Riemann sum with 4 subintervals of equal width.

(b) Is the value of the sum you calculated in part (a) an overestimate or an underestimate of the change in cost?

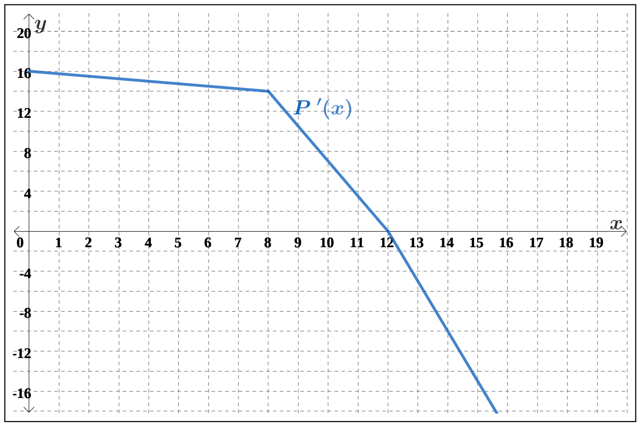
(c) Estimate the change in cost when the number of tickets sold increases from 60 to 220 using a right-hand Riemann sum with 4 subintervals of equal width.

(d) Is the value of the sum you calculated in part (c) an overestimate or an underestimate of the change in cost?

(e) Estimate the change in cost when the number of tickets sold increases from 60 to 220 using a midpoint Riemann sum with 4 subintervals of equal width.

65. After t months, a company's sales are changing at a rate of $s(t) = 0.08t^3 + 0.09t^2 + 0.3t + 3$ million dollars per month. Estimate the increase in the company's sales after 9 months using a right-hand Riemann sum with 3 subintervals of equal width.

66. The graph of a company's marginal profit function, P', is shown below, where x is the number of items sold and $P'(x)$ is measured in dollars per item.



(a) Find the change in profit when the first 8 items are sold.

(b) Find the change in profit when the number of items sold increases from 8 items to 14 items.

(c) If the company has a loss in profit of \$30 when it does not sell any items, find the profit from selling the first 12 items.

67. Particular values of the marginal profit function of a company are shown in the table below, where x is the number of items sold and $P'(x)$ is measured in dollars per item.

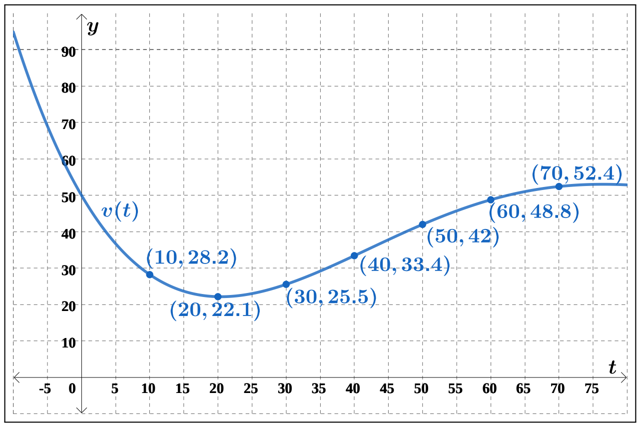
x	70	75	80	85	90	95	100	105	110	115
$P'(x)$	380	312.50	230	132.50	20	−107.50	−250	−407.50	−580	−767.50

(a) Estimate the change in profit when the number of items sold increases from 80 to 110 using a right-hand Riemann sum with 3 subintervals of equal width.

(b) Estimate the change in profit when the number of items sold increases from 80 to 110 using a left-hand Riemann sum with 3 subintervals of equal width.

(c) Estimate the change in profit when the number of items sold increases from 80 to 110 using a midpoint Riemann sum with 3 subintervals of equal width.

68. The velocity of an object after t seconds is $v(t)$ feet per second. The graph of v is shown below as well as selected points. Use the graph to estimate the distance traveled by the object after one minute using a left-hand Riemann sum with three subintervals of equal width.



69. Jessica invests money into an account, and the rate at which the balance of her account grows is given by $\frac{d}{dt}A(t) = 25e^{0.025t}$ dollars per year, where t is the numbers of years since Jessica invested her money. Estimate the increase in the balance of Jessica's account after 15 years using a left-hand Riemann sum with three subintervals of equal width.

70. The marginal cost and marginal revenue functions for a particular brand of graphing calculator are given by $C'(x) = 24$ and $R'(x) = 352 - 0.8x$ dollars per calculator when x calculators are produced and sold, respectively. Using geometric regions, calculate

    (a) the change in cost when the number of calculators produced increases from 40 to 90.

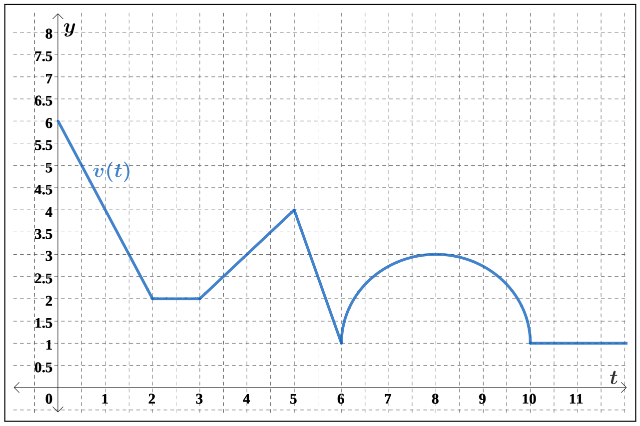
    (b) the change in revenue when the number of calculators sold increases from 400 to 440.

    (c) the change in profit when the number of calculators produced and sold increases from 500 to 600.

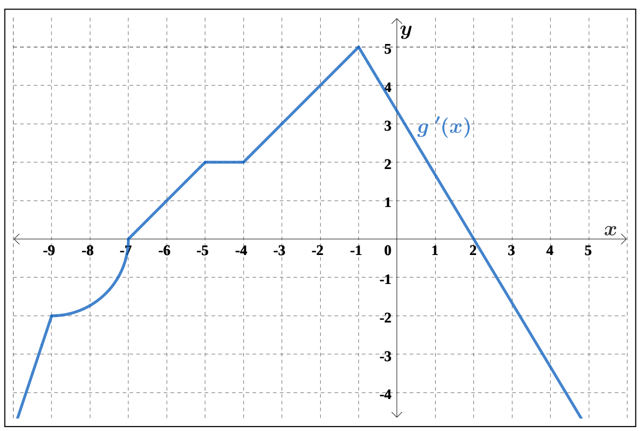
71. The marginal revenue function for Wally's Wacky World of Wallets is given by $R'(x) = 75\,(0.999^x)\,(x\ln(0.999) + 1)$ dollars per wallet when x wallets are sold.

    (a) Estimate the change in revenue when the number of wallets sold increases from 500 to 510 using a midpoint Riemann sum with 8 subintervals of equal width.

72. The velocity of an object after t seconds is $v(t)$ feet per second, and the graph of v is shown below. Find the distance the object traveled during the first 10 seconds. Round your answer to three decimal places, if necessary.



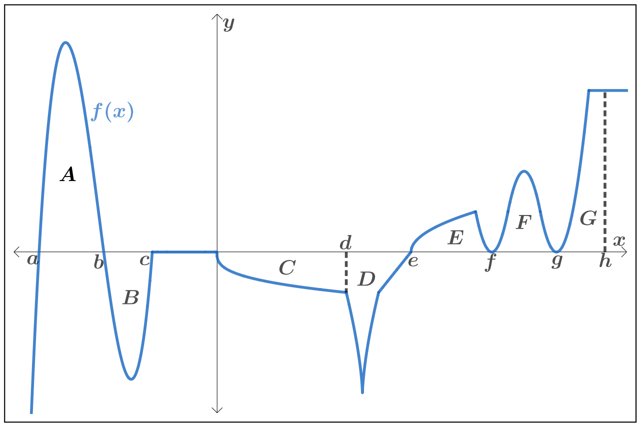
73. The graph of $g\,'$ is shown below. Use the graph to find each of the following.



(a) $\displaystyle\int_{-9}^{-4} g\,'(x)\,dx$

(b) $\displaystyle\int_{-5}^{2} g\,'(x)\,dx$

74. The graph of f is shown below. Use the indicated areas of the regions between the graph of f and the x-axis to find each of the following.



Area of A: 27
Area of B: 12
Area of C: 12
Area of D: 15
Area of E: 5
Area of F: 34
Area of G: 19

(a) $\int_0^c -3f(x)\,dx$

(b) $\int_a^c \frac{f(x)}{2}\,dx + \int_0^e -0.25f(x)\,dx$

(c) $\int_j^j \pi^2 f(x)\,dx - \int_h^f \frac{4}{5}f(x)\,dx$

(d) $\int_0^a f(x)\,dx - \int_d^f -2f(x)\,dx + \int_f^g 0.3f(x)\,dx$

75. Given $f(x)$ below, find the value of $\int_{-3}^{24} f(x)\,dx$ using geometric regions.

$$f(x) = \begin{cases} 4 & x < 2 \\ 2x & 2 \le x < 6 \\ -x+18 & x \ge 6 \end{cases}$$

76. If $\int_{-8}^{-5} f(x)\,dx = -10$, $\int_{4}^{-5} 3f(x)\,dx = 18$, $\int_{-8}^{-6} g(x)\,dx = 2$, $\int_{4}^{-6} g(x)\,dx = -7$, and $\int_{-8}^{4} 2h(x)\,dx = -16$, find each of the following.

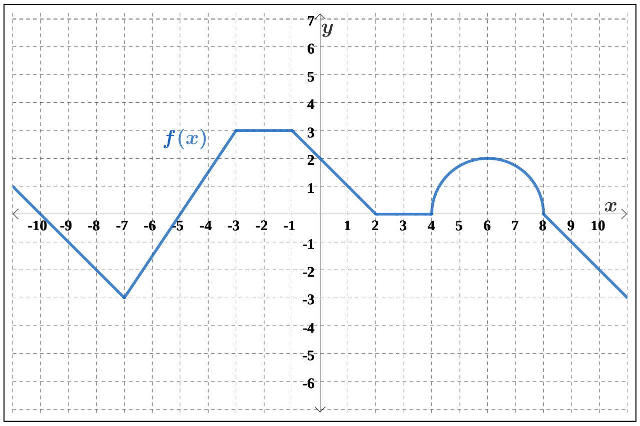
(a) $\displaystyle\int_{-8}^{4} f(x)\,dx$

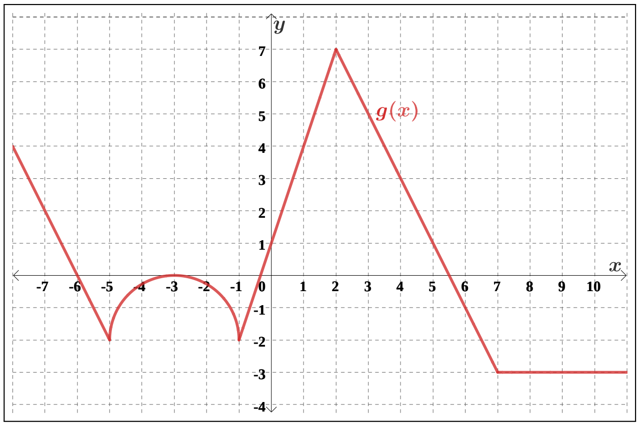
(b) $\displaystyle\int_{-8}^{4} h(x)\,dx - \int_{-8}^{-6} 3g(x)\,dx$

(c) $\displaystyle\int_{-5}^{4} -2f(x)\,dx + \int_{4}^{4}\left[\pi g(x) - \frac{h(x)}{7}\right]dx$

(d) $\displaystyle\int_{-8}^{4}\left[f(x) + 6g(x) - \frac{h(x)}{4}\right]dx$

77. Use the graphs of the functions f and g shown below to find each of the following.





(a) $\displaystyle\int_{-2}^{-7} 5f(x)\,dx$

(b) $\displaystyle\int_{-5}^{-1}\left[2f(x) - \frac{g(x)}{6}\right]dx$

(c) $\displaystyle\int_{2}^{9} [f(x) + 3g(x)]\,dx$

(d) $\displaystyle\int_{-7}^{-3} [g(x) - f(x)]\,dx$

(e) $\displaystyle\int_{-3}^{-1}\left[f(x) + \frac{g(x)}{4}\right]dx$

(f) $\displaystyle\int_{2}^{0} [g(x) + f(x)]\,dx$

78. If g is a continuous function such that $c \le g(t) \le d$ on the interval $[a,b]$, where a, b, c, and d are constants, find the smallest possible value of $\int_a^b g(t)\, dt$.

For Exercises 79 - 81, find the (a) largest and (b) smallest possible values of the definite integral.

79. $\int_{-2}^{3} \left[\left(x^2-1\right)^{2/3} - 5\right] dx$

80. $\int_1^4 \frac{\ln\left(x^2\right)}{x}\, dx$

81. $\int_{-\frac{1}{2}}^{1} xe^{-2x^2}\, dx$

82. Use sigma notation to represent the right-hand Riemann sum of a continuous function $f(x)$ on the interval $[a,b]$ that consists of n subintervals of equal width.

83. Use sigma notation to represent the left-hand Riemann sum of a continuous function $f(x)$ on the interval $[a,b]$ that consists of n subintervals of equal width.

84. Use sigma notation to represent the midpoint Riemann sum of a continuous function $f(x)$ on the interval $[a,b]$ that consists of ten subintervals of equal width.

85. When using a right-hand Riemann sum with 3 subintervals of equal width to estimate $\int_1^7 f(x)\, dx$, what is the value of the sample point x_1?

86. Use sigma notation to represent the sum of the areas of the fifth, sixth, and seventh rectangles of a left-hand Riemann sum of a continuous function $f(x)$ on the interval $[a,b]$ using n subintervals.

Communication Practice

87. When finding a left-hand Riemann sum of a continuous function f on the interval $[a,b]$ using n subintervals, explain the meaning of $f(x_1)\Delta x$.

88. When finding a right-hand Riemann sum of a continuous function f on the interval $[a,b]$ using n subintervals, explain the meaning of $\sum_{i=2}^{4} f(x_i)\, \Delta x$.

89. In order to get a better approximation of the area under a curve, should we increase or decrease the number of subintervals? Explain.

90. Will the value of a left-hand Riemann sum always be an underestimate of the area under a curve? Explain.

91. Does $\sum_{i=1}^{n} f\left(x_i^*\right)\Delta x = \int_a^b f(x)\,dx$? Explain.

92. When finding the limit of a Riemann sum as the number of subintervals goes to infinity, will we get the same answer whether we are taking the limit of a right-hand, left-hand, or midpoint Riemann sum? Explain.

93. Explain the difference between a definite and indefinite integral.

94. Explain under what circumstances a definite integral represents the area between the graph of a function f and the x-axis on some interval $[a,b]$.

95. If a company's marginal profit is $P'(x)$ dollars per item when x items are sold, interpret $\int_{100}^{150} P'(x)\,dx = 7000$.

96. If a company's cost is $C(x)$ dollars when x items are produced, does $\int_{45}^{65} C(x)\,dx$ give the change in the company's cost when production increases from 45 to 65 items? Explain.

97. If $v(t)$ gives the velocity of an object, in feet per second, after t seconds and $v(t) \geq 0$, interpret $\int_0^{60} v(t)\,dt$.

4.4 The Fundamental Theorem of Calculus

There is a deep connection between definite integrals we learned in **Section 4.3** and antiderivatives we learned in **Sections 4.1** and **4.2**. You might have already noticed they both involve an integral sign!

Isaac Newton and Gottfried Liebniz both discovered this connection independently in the late 1600s. It has been the basis of calculus ever since its discovery. That's why it is called the **Fundamental Theorem of Calculus**!

Learning Objectives:

In this section, you will learn the Fundamental Theorem of Calculus and use it to solve problems involving real-world applications. Upon completion you will be able to:

- State Parts 1 and 2 of the Fundamental Theorem of Calculus using mathematical notation.
- Evaluate a definite integral involving the Introductory Antiderivative Rules using Part 2 of the Fundamental Theorem of Calculus.
- Find the new definite integral in terms of u and du that results after applying the appropriate u-substitution when evaluating a definite integral using the method of substitution.
- Evaluate a definite integral involving the method of substitution using Part 2 of the Fundamental Theorem of Calculus.
- Use technology to approximate the value of a definite integral.
- Interpret the definite integral as the net change in a quantity.
- Apply Part 2 of the Fundamental Theorem of Calculus to find the change in a real-world quantity.
- Find the derivative of a definite integral using Part 1 of the Fundamental Theorem of Calculus.

The Fundamental Theorem of Calculus

Before we formally present the Fundamental Theorem of Calculus, let's investigate the relationship between definite integrals and antiderivatives to see if we can can discover the connection between the two!

Using our knowledge from the previous section, let's find the value of the definite integral $\int_0^{16}\left(-\frac{1}{2}x+8\right)dx$ exactly by interpreting the definite integral in terms of areas. The graph of $f(x)=-\frac{1}{2}x+8$ is shown in **Figure 4.4.1**, and the relevant region is shaded:

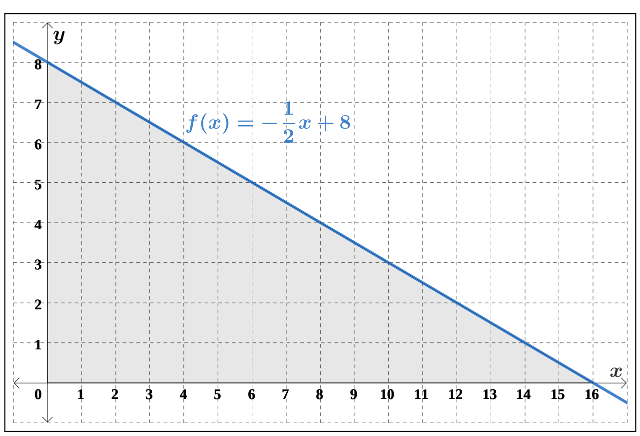


Figure 4.4.1: Graph of the function $f(x)=-\frac{1}{2}x+8$

Recall that the definite integral gives the net area between the graph of f and the x-axis on an interval $[a,b]$. In this case, the region between the graph of f and the x-axis on the interval $[0,16]$ is a triangle. Note that the area of the triangle will be counted positively because the region is above the x-axis.

The triangle has a base of 16 and a height of 8, so its area is $\frac{16\cdot 8}{2} = 64$. Thus, we know

$$\int_0^{16}\left(-\frac{1}{2}x+8\right)\,dx = 64$$

based on our knowledge of definite integrals representing the net area between the graph of a function and the x-axis.

Now, let's find the most general antiderivative of the same function. In other words, let's calculate $\int\left(-\frac{1}{2}x+8\right)\,dx$:

$$\begin{aligned}\int\left(-\frac{1}{2}x+8\right)\,dx &= -\frac{1}{2}\cdot\frac{1}{2}x^2+8x+C\\ &= -\frac{1}{4}x^2+8x+C\end{aligned}$$

Finally, let's substitute $x = 16$ and $x = 0$ (the endpoints of the interval for the definite integral we calculated above) into the most general antiderivative and subtract the quantities:

$$\begin{aligned}\left(-\frac{1}{4}(16)^2+8(16)+C\right)-\left(-\frac{1}{4}(0)^2+8(0)+C\right) &= -\frac{1}{4}(256)+128+C+0-0-C\\ &= -64+128\\ &= 64\end{aligned}$$

This is the same value we calculated for the definite integral $\int_0^{16}\left(-\frac{1}{2}x+8\right)\,dx$ when we interpreted it in terms of area! Thus, we have discovered the relationship between definite integrals and antiderivatives!

This connection is given by Part 2 of the Fundamental Theorem of Calculus Theorem, which has two parts. Specifically, Part 1 shows how to *differentiate* a function defined as a definite integral, while Part 2 shows how to *evaluate* a definite integral, assuming we can find an antiderivative of the integrand.

Theorem 4.2 The Fundamental Theorem of Calculus

Part 1: If f is continuous on an interval $[a,b]$ and the function F is defined by

$$F(x) = \int_a^x f(t)\,dt$$

where $a \le x \le b$, then $F'(x) = f(x)$ on the interval $[a,b]$.

Part 2: If f is continuous on the interval $[a,b]$ and F is *any* antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b)-F(a)$$

The proofs of both parts of the Fundamental Theorem of Calculus (sometimes abbreviated FTC) are beyond the scope of this textbook. The proof of the second part uses the result of the first part, hence the ordering of the two. In this textbook, we will focus on Part 2 of the Fundamental Theorem of Calculus first because we can apply it to relevant business applications in order to find the exact change in a quantity.

The Fundamental Theorem of Calculus, Part 2

Part 2 of the Fundamental Theorem of Calculus states that if we can find just one antiderivative of the integrand, then we can evaluate the definite integral by evaluating the antiderivative at the limits of integration and subtracting the quantities.

Recall that any antiderivative of a function will differ only by a constant (the "$+C$" from **Sections 4.1** and **4.2**). Because Part 2 of the Fundamental Theorem of Calculus says we can use *any* antiderivative to evaluate a definite integral, we can choose whatever value for the constant of integration, C, we want. The easiest choice is to let $C = 0$, so this is what we will do in all of the examples in this section.

Example 1 Evaluate each of the following definite integrals exactly.

a. $\displaystyle\int_0^1 x^2\,dx$

b. $\displaystyle\int_{-2}^3 \left(\frac{1}{3}x^3 + e^x\right)\,dx$

c. $\displaystyle\int_9^{12} (x-12)\left(x^3 - x^{-2}\right)\,dx$

d. $\displaystyle\int_{-4}^0 \frac{17e^x + e^{2x} - 4xe^x}{e^x}\,dx$

Solution:

a. The first step to evaluate $\displaystyle\int_0^1 x^2\,dx$ is to find an antiderivative of the integrand. One antiderivative of x^2 is $\frac{1}{3}x^3$ (remember we are assuming $C = 0$). Now, we substitute the limits of integration into this antiderivative and subtract the quantities. To indicate that we have found an antiderivative, $\frac{1}{3}x^3$, and we are about to substitute the limits of integration, we use the notation shown below. Also, we must remember to substitute the upper limit of integration first:

$$\begin{aligned}\int_0^1 x^2\,dx &= \frac{1}{3}x^3\Big|_0^1\\ &= \frac{1}{3}(1)^3 - \frac{1}{3}(0)^3\\ &= \frac{1}{3}\end{aligned}$$

Thus, $\displaystyle\int_0^1 x^2\,dx = \frac{1}{3}$. Note that because $y = x^2$ is a nonnegative function, we can say that the area between it and the x-axis on the interval $[0,1]$ is exactly $\frac{1}{3}$.

N *The notation $F(x)\Big|_a^b$ is shorthand for $F(b) - F(a)$, which is the last step in our work when evaluating a definite integral. Using this notation allows us to focus on finding an antiderivative of the integrand first, while still writing a valid equation, before substituting the limits of integration and subtracting.*

b. Recall that we must evaluate $\displaystyle\int_{-2}^3 \left(\frac{1}{3}x^3 + e^x\right)\,dx$. We can use the Introductory Antiderivative Rules we learned in **Section 4.1** first to find an antiderivative of the integrand. Then, we will substitute the limits of integration and subtract the quantities:

$$\begin{aligned}\int_{-2}^{3}\left(\frac{1}{3}x^3+e^x\right)\,dx &= \left(\frac{1}{3}\cdot\frac{1}{4}x^4+e^x\right)\Bigg|_{-2}^{3}\\ &= \left(\frac{1}{12}x^4+e^x\right)\Bigg|_{-2}^{3}\\ &= \left(\frac{1}{12}(3)^4+e^3\right)-\left(\frac{1}{12}(-2)^4+e^{-2}\right)\\ &= \frac{81}{12}+e^3-\frac{16}{12}-e^{-2}\\ &= \frac{65}{12}+e^3-e^{-2}\end{aligned}$$

Thus, $\int_{-2}^{3}\left(\frac{1}{3}x^3+e^x\right)\,dx$ is exactly $\frac{65}{12}+e^3-e^{-2}$, which is approximately 25.37.

c. Recall that we must evaluate $\int_{9}^{12}(x-12)\left(x^3-x^{-2}\right)\,dx$.

The first thing to remember is that *there is no product rule for antiderivatives!* Thus, there is no product rule for definite integrals, so we must algebraically manipulate the integrand before attempting to find an antiderivative.

We should also pay attention to the domain of the function in the integrand. It is defined everywhere except at $x=0$ (because the negative exponent implies division). In other words, the function is not continuous at $x=0$. However, because $x=0$ is not in the interval of the definite integral, $[9,12]$, there are no domain (or continuity) issues and we can proceed.

N *There are techniques of integration that allow us to consider functions that are not continuous, or in some cases, not fully defined on the interval of integration. These techniques are beyond the scope of this textbook, and as such, we will choose problems in which the integrand is continuous on the interval of integration.*

Algebraically manipulating the integrand so we can evaluate the definite integral gives

$$\begin{aligned}\int_{9}^{12}(x-12)\left(x^3-x^{-2}\right)\,dx &= \int_{9}^{12}\left(x\cdot x^3-x\cdot x^{-2}-12x^3+12x^{-2}\right)\,dx\\ &= \int_{9}^{12}\left(x^4-x^{-1}-12x^3+12x^{-2}\right)\,dx\\ &= \left(\frac{1}{5}x^5-\ln|x|-12\cdot\frac{1}{4}x^4+12\cdot\frac{1}{-1}x^{-1}\right)\Bigg|_{9}^{12}\\ &= \left(\frac{1}{5}x^5-\ln|x|-3x^4-12x^{-1}\right)\Bigg|_{9}^{12}\\ &= \left(\frac{1}{5}(12)^5-\ln|12|-3(12)^4-12(12)^{-1}\right)-\left(\frac{1}{5}(9)^5-\ln|9|-3(9)^4-12(9)^{-1}\right)\\ &= \frac{248,832}{5}-\ln(12)-62,208-1-\frac{59,049}{5}+\ln(9)+19,683+\frac{4}{3}\\ &= -\ln(12)+\ln(9)-\frac{68,521}{15}\end{aligned}$$

Thus, $\int_{9}^{12}(x-12)\left(x^3-x^{-2}\right)\,dx$ is exactly $-\ln(12)+\ln(9)-\frac{68,521}{15}$, which is approximately -4568.35.

d. Recall that we must evaluate $\displaystyle\int_{-4}^{0} \frac{17e^x + e^{2x} - 4xe^x}{e^x}\, dx$. Because *there is no quotient rule for antiderivatives*, we will again have to algebraically manipulate the integrand before we can evaluate the definite integral:

$$\begin{aligned}
\int_{-4}^{0} \frac{17e^x + e^{2x} - 4xe^x}{e^x}\, dx &= \int_{-4}^{0} \left(\frac{17e^x}{e^x} + \frac{e^{2x}}{e^x} - \frac{4xe^x}{e^x} \right) dx \\
&= \int_{-4}^{0} (17 + e^x - 4x)\, dx \\
&= \left(17x + e^x - 4 \cdot \frac{1}{2} x^2 \right) \Big|_{-4}^{0} \\
&= \left(17x + e^x - 2x^2 \right) \Big|_{-4}^{0} \\
&= \left(17(0) + e^0 - 2(0)^2 \right) - \left(17(-4) + e^{-4} - 2(-4)^2 \right) \\
&= 1 + 68 - e^{-4} + 32 \\
&= 101 - e^{-4}
\end{aligned}$$

Thus, $\displaystyle\int_{-4}^{0} \frac{17e^x + e^{2x} - 4xe^x}{e^x}\, dx$ is exactly $101 - e^{-4}$, which is approximately 100.98.

■

Part 2 of the Fundamental Theorem of Calculus is sometimes called the Evaluation Theorem because it allows us to evaluate definite integrals exactly if we can find an antiderivative. Sometimes, finding an antiderivative can be a difficult process. In fact, some fairly basic functions have complicated antiderivatives. Even so, the Evaluation Theorem is a powerful tool in mathematics, engineering, science, and many other fields.

Try It # 1:
Evaluate each of the following definite integrals exactly.

a. $\displaystyle\int_{1}^{32} \left(\frac{3}{x} + x^{2/5} \right) dx$

b. $\displaystyle\int_{2}^{5} \frac{6x^3 - x^2 + \sqrt{x}}{x^3}\, dx$

c. $\displaystyle\int_{-1}^{1} \left(4x^2 - 8x - 3 \right) x^7\, dx$

In the next example, we will need to use the method of substitution to find an antiderivative of the integrand. When we change variables from x to u, this will also change dx, which represents the change in x. It will also change the limits of integration, which are originally x-values!

There are two equal and equivalent methods for dealing with this issue. First, we can change the limits of integration to be in terms of u by using the substitution we define. The second method is to find the antiderivative using the entire method of substitution, including substituting back for u so the antiderivative is in terms of x, and then substituting the original limits of integration (which are x-values).

We will use the first method in this textbook when evaluating a definite integral that requires the method of substitution. In other words, we will convert the limits of integration from x-values to u-values based on the substitution. However, we will discuss the second method in the next example.

Example 2 Evaluate each of the following definite integrals exactly.

a. $\displaystyle\int_{-1}^{2} x^2 e^{x^3}\,dx$

b. $\displaystyle\int_{-10}^{-5} (2+t)\sqrt[3]{t-4}\,dt$

Solution:

a. To evaluate $\displaystyle\int_{-1}^{2} x^2 e^{x^3}\,dx$, we must use the method of substitution. We will let $u = x^3$ for the substitution. Thus, $du = 3x^2\,dx$, or more importantly, $\dfrac{1}{3}\,du = x^2\,dx$.

The limits of integration for this definite integral are in terms of x. As stated previously, we will change the limits of integration so they are in terms of u when using the method of substitution. To get the equivalent u-values for this definite integral, we substitute the lower limit of integration, $x = -1$, and the upper limit of integration, $x = 2$, into the substitution we defined for u ($u = x^3$):

$$u = (-1)^3 = -1$$
$$u = (2)^3 = 8$$

Now, we can rewrite the original definite integral entirely in terms of u and du:

$$\begin{aligned}\int_{-1}^{2} x^2 e^{x^3}\,dx &= \int_{-1}^{8} e^u \cdot \frac{1}{3}\,du\\ &= \int_{-1}^{8} \frac{1}{3}e^u\,du\end{aligned}$$

Finally, we evaluate this *new* definite integral in terms of u:

$$\begin{aligned}\int_{-1}^{8} \frac{1}{3}e^u\,du &= \frac{1}{3}e^u\Big|_{-1}^{8}\\ &= \frac{1}{3}e^8 - \frac{1}{3}e^{-1}\end{aligned}$$

Hence, $\displaystyle\int_{-1}^{2} x^2 e^{x^3}\,dx = \frac{1}{3}e^8 - \frac{1}{3}e^{-1}$.

Notice when evaluating this definite integral using the method of substitution, we did not go back and substitute for u like we did when finding indefinite integrals using the method of substitution. Because we changed the limits of integration from x-values to u-values, we are able to find the antiderivative and immediately substitute the new limits of integration to get the answer.

As stated earlier, there is another way we could have evaluated this definite integral, and we will demonstrate it now. Rather than changing the limits of integration from x-values to u-values when rewriting the new integral in terms of u and du, we could have instead used substitution to find the equivalent *indefinite* integral, and then after getting the antiderivative back in terms of x, substitute the original limits of integration in terms of x.

To work part **a** of this example using this technique, we start by finding

$$\int x^2 e^{x^3}\,dx$$

Using the same substitution, we have $du = 3x^2\,dx$, or more importantly, $\frac{1}{3}\,du = x^2\,dx$.

Now, we can rewrite the indefinite integral entirely in terms of u and du:

$$\begin{aligned}\int x^2 e^{x^3}\,dx &= \int e^u \cdot \frac{1}{3}\,du \\ &= \int \frac{1}{3}e^u\,du\end{aligned}$$

Finding the most general antiderivative gives

$$\int \frac{1}{3}e^u\,du = \frac{1}{3}e^u + C$$

Substituting back for u so that the most general antiderivative is in terms of the original variable, x, gives

$$\frac{1}{3}e^u + C = \frac{1}{3}e^{x^3} + C$$

Finally, we can substitute the original limits of integration in terms of x, $x = -1$ and $x = 2$, into this antiderivative and subtract (remember, we can choose $C = 0$):

$$\begin{aligned}\frac{1}{3}e^{x^3}\Big|_{-1}^{2} &= \frac{1}{3}e^{(2)^3} - \frac{1}{3}e^{(-1)^3} \\ &= \frac{1}{3}e^8 - \frac{1}{3}e^{-1}\end{aligned}$$

Although this is a valid method for evaluating a definite integral involving substitution, remember we will use the first method throughout this textbook (i.e., we will always change the limits of integration from x-values to u-values when using the method of substitution).

b. To evaluate $\int_{-10}^{-5} (2+t)\sqrt[3]{t-4}\,dt$, we will choose $u = t-4$ for the substitution. Thus, $du = dt$.

The limits of integration for this definite integral are in terms of t. To get the equivalent u-values, we substitute the lower limit of integration, $t = -10$, and the upper limit of integration, $t = -5$, into the substitution we defined for u ($u = t-4$):

$$\begin{aligned}u &= -10-4 = -14 \\ u &= -5-4 = -9\end{aligned}$$

Now, we can rewrite the original definite integral entirely in terms of u and du. Before doing so, however, notice that because $du = dt$, there will still be a "t" remaining in the integrand after we substitute u and du. Thus, we have to use the technique we learned previously in **Section 4.2** and solve the substitution $u = t-4$ for t, and substitute that quantity as well. Thus, we will substitute u for $t-4$, du for dt, and $u+4$ for t, and then we will algebraically manipulate the integrand so that we are able to find its antiderivative:

$$\begin{aligned}\int_{-10}^{-5} (2+t)\sqrt[3]{t-4}\,dt &= \int_{-14}^{-9} (2+(u+4))\sqrt[3]{u}\,du \\ &= \int_{-14}^{-9} (u+6)u^{\frac{1}{3}}\,du \\ &= \int_{-14}^{-9} \left(u^{\frac{4}{3}} + 6u^{\frac{1}{3}}\right)du\end{aligned}$$

Finally, we evaluate this *new* definite integral in terms of u:

$$\begin{aligned}\int_{-14}^{-9}\left(u^{\frac{4}{3}}+6u^{\frac{1}{3}}\right)\,du &= \left(\frac{3}{7}u^{\frac{7}{3}}+6\cdot\frac{3}{4}u^{\frac{4}{3}}\right)\Bigg|_{-14}^{-9}\\ &= \left(\frac{3}{7}u^{\frac{7}{3}}+\frac{18}{4}u^{\frac{4}{3}}\right)\Bigg|_{-14}^{-9}\\ &= \left(\frac{3}{7}(-9)^{\frac{7}{3}}+\frac{18}{4}(-9)^{\frac{4}{3}}\right)-\left(\frac{3}{7}(-14)^{\frac{7}{3}}+\frac{18}{4}(-14)^{\frac{4}{3}}\right)\\ &= \frac{3}{7}(-9)^{\frac{7}{3}}+\frac{18}{4}(-9)^{\frac{4}{3}}-\frac{3}{7}(-14)^{\frac{7}{3}}-\frac{18}{4}(-14)^{\frac{4}{3}}\end{aligned}$$

Hence, $\displaystyle\int_{-10}^{-5}(2+t)\sqrt[3]{t-4}\,dt = \frac{3}{7}(-9)^{\frac{7}{3}}+\frac{18}{4}(-9)^{\frac{4}{3}}-\frac{3}{7}(-14)^{\frac{7}{3}}-\frac{18}{4}(-14)^{\frac{4}{3}}$.

■

Try It # 2:

Evaluate each of the following definite integrals exactly.

a. $\displaystyle\int_{1}^{10}\frac{6x^2+12}{x^3+6x}\,dx$

b. $\displaystyle\int_{-5}^{5}\left(4(x-4)^2-16\right)\,dx$

■ **Example 3** The graph of $f\,'$ is shown in **Figure 4.4.2**. If $f(0)=7$, find each of the following.

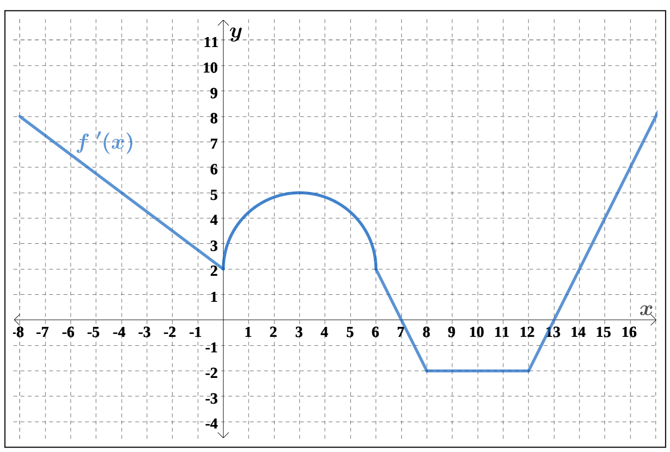


Figure 4.4.2: Graph of a rate of change function $f\,'$

a. $f(6)$

b. $f(10)$

Solution:

a. At first, it may appear that the information given for this problem is unrelated to the answer we want. We are given the graph of $f\,'$, but we need to find a function value of its antiderivative: $f(6)$. Thus, we need to determine the relationship between the two functions before we can proceed.

Their relationship is given by Part 2 of the Fundamental Theorem of Calculus. Although Part 2 of the Fundamental Theorem of Calculus is stated in terms of f and its antiderivative, F, on an interval $[a,b]$, the relationship holds no matter what notation we use. In other words, if the function in the integrand is f', then its antiderivative is f.

Therefore, the relationship we will use is given by Part 2 of the Fundamental Theorem of Calculus on the interval $[0,6]$:

$$\int_0^6 f'(x)\,dx = f(6) - f(0)$$

We know $f(0) = 7$, and using the graph of f', we can find $\int_0^6 f'(x)\,dx$. The value of this definite integral is the area of the region shaded in **Figure 4.4.3**:

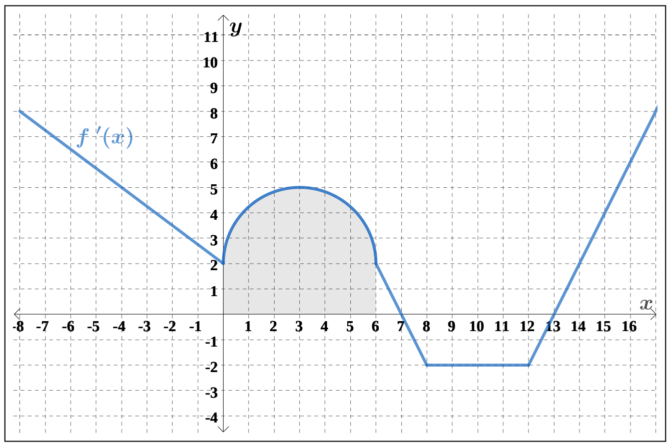


Figure 4.4.3: Shaded region between the graph of f' and the x-axis on the interval $[0,6]$

We can view this shaded region as a semicircle on top of a rectangle as shown in **Figure 4.4.4**:

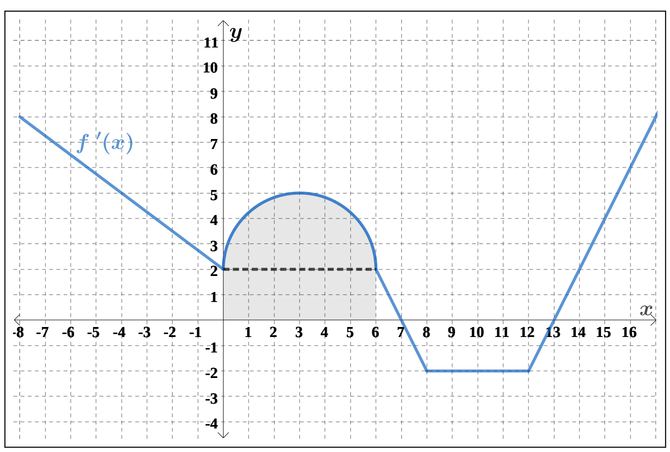


Figure 4.4.4: Graph of f' in which the regions of a semicircle and a rectangle are shaded

The rectangle has a length of 6 (from $x = 0$ to $x = 6$) and a height of 2, so its area is 12. The semicircle has a radius of 3, so its area is $\frac{\pi \cdot 3^2}{2} = \frac{9\pi}{2}$. The areas of both regions are counted positively because they are both above the x-axis. Thus, the definite integral $\int_0^6 f'(x)\,dx$ is equal to $12 + \frac{9\pi}{2}$.

Therefore,

$$\begin{aligned}\int_0^6 f'(x)\,dx &= f(6) - f(0) \implies \\ 12 + \frac{9\pi}{2} &= f(6) - 7 \\ 19 + \frac{9\pi}{2} &= f(6)\end{aligned}$$

Hence, $f(6) = 19 + \dfrac{9\pi}{2}$.

b. To find $f(10)$ given $f(0) = 7$, we will use Part 2 of the Fundamental Theorem of Calculus on the interval $[0, 10]$:

$$\int_0^{10} f'(x)\,dx = f(10) - f(0)$$

The definite integral on the left-hand side of this equation is equal to the net area of the regions shaded in **Figure 4.4.5**:

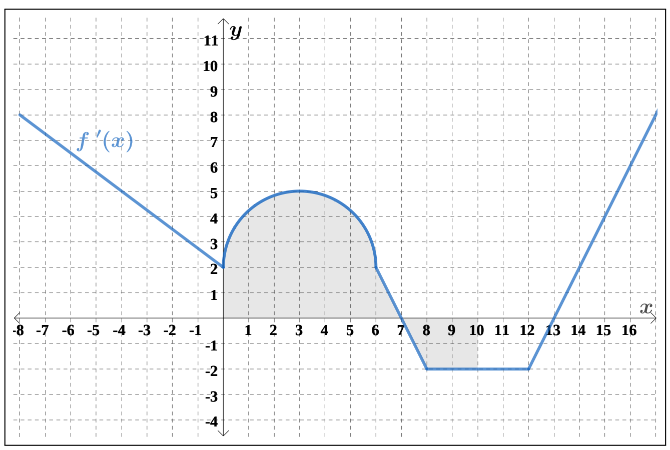


Figure 4.4.5: Shaded regions between the graph of f' and the x-axis on the interval $[0, 10]$

We will divide the regions into the five smaller regions of which we can calculate the areas. See **Figure 4.4.6**.

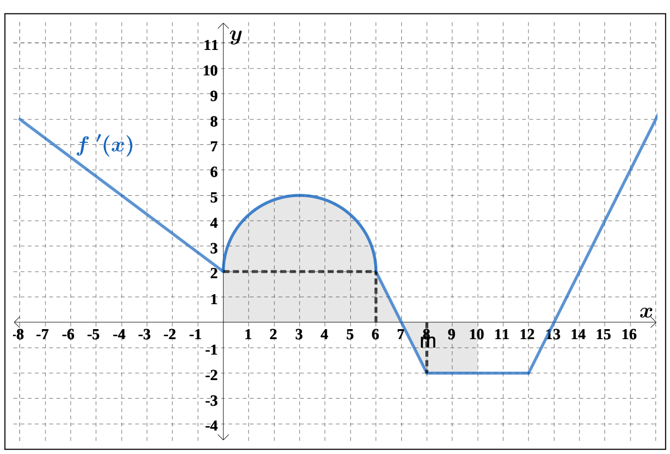


Figure 4.4.6: Graph of f' in which five regions are shaded

We found the area of the semicircle and the rectangle above the x-axis from $x = 0$ to $x = 6$ in part **a**. Recall that this area is $12 + \dfrac{9\pi}{2}$. So we will start with calculating the areas of the triangles.

The triangle above the x-axis has a base of length 1 (from $x = 6$ to $x = 7$) and a height of 2, so its area is 1. The area of this region will be counted positively because it is above the x-axis. The next triangle, which is below the x-axis, has a base of length 1 (from $x = 7$ to $x = 8$) and a height of 2, so its area is 1. The area of this region will be counted negatively because it is below the x-axis.

The last region is a rectangle below the x-axis, and it has a base of 2 (from $x = 8$ to $x = 10$) and a height of 2. Thus, its area is 4, which will be counted negatively. Hence,

$$\begin{aligned}\int_0^{10} f'(x)\,dx &= 12 + \frac{9\pi}{2} + 1 - 1 - 4 \\ &= 8 + \frac{9\pi}{2}\end{aligned}$$

Therefore,

$$\begin{aligned}\int_0^{10} f'(x)\,dx = f(10) - f(0) &\implies \\ 8 + \frac{9\pi}{2} &= f(10) - 7 \\ 15 + \frac{9\pi}{2} &= f(10)\end{aligned}$$

Thus, $f(10) = 15 + \dfrac{9\pi}{2}$.

▪

Try It # 3:
The graph of f' is shown in **Figure 4.4.7**. If $f(-6) = -2$, find each of the following.

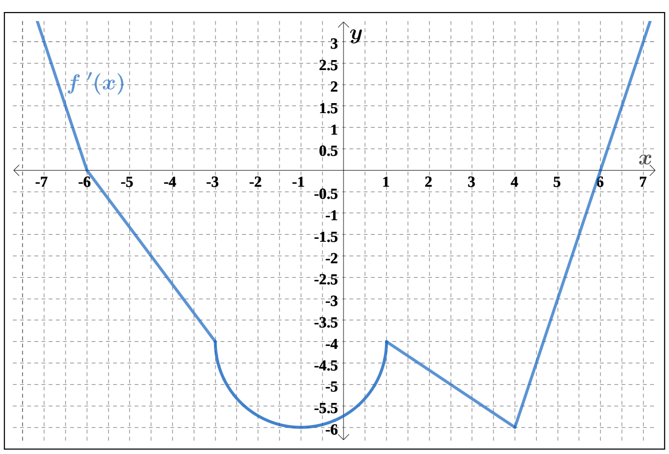


Figure 4.4.7: Graph of a rate of change function f'

a. $f(-3)$

b. $f(4)$

▪ **Example 4** Find each of the following definite integrals.

a. $\int_{-2}^{A} \left(4 - x^2\right)\, dx$, where $A > 0$

b. $\int_{3}^{A} \frac{t^2 + 4t - \sqrt{3}}{t^3}\, dt$, where $A > 3$

c. $\int_{B}^{e} \frac{12}{x \ln(x)}\, dx$, where $1 < B < e$

Solution:

a. Using Part 2 of the Fundamental Theorem of Calculus gives

$$\begin{aligned}\int_{-2}^{A} \left(4 - x^2\right)\, dx &= \left(4x - \frac{1}{3}x^3\right)\Bigg|_{-3}^{A} \\ &= \left(4A - \frac{1}{3}A^3\right) - \left(4(-3) - \frac{1}{3}(-3)^3\right) \\ &= 4A - \frac{1}{3}A^3 + 3\end{aligned}$$

N *This definite integral represents the net area between the graph of $f(x) = 4 - x^2$ and the x-axis from $x = -2$ to $x = A$! We can substitute any value for A (as long as $A > 0$) into the definite integral (or our answer above), and the result will be the net area of the region from $x = -2$ to $x = A$. Thus, the answer we found is a function in its own right (a function of A), and we'll explore functions of this type more in depth later in this section!*

b. Recall that we must find $\int_{3}^{A} \frac{t^2 + 4t - \sqrt{3}}{t^3}\, dt$, where $A > 3$. We need to algebraically manipulate the integrand before we can find the antiderivative and substitute the limits of integration:

$$\begin{aligned}\int_{3}^{A} \frac{t^2 + 4t - \sqrt{3}}{t^3}\, dt &= \left(\frac{t^2}{t^3} + \frac{4t}{t^3} - \frac{\sqrt{3}}{t^3}\right) dt \\ &= \int_{3}^{A} \left(t^{-1} + 4t^{-2} - \sqrt{3}t^{-3}\right) dt \\ &= \left(\ln|t| + 4 \cdot \frac{1}{-1}t^{-1} - \sqrt{3} \cdot \frac{1}{-2}t^{-2}\right)\Bigg|_{3}^{A} \\ &= \left(\ln|t| - 4t^{-1} + \frac{\sqrt{3}}{2}t^{-2}\right)\Bigg|_{3}^{A} \\ &= \left(\ln|A| - 4(A)^{-1} + \frac{\sqrt{3}}{2}(A)^{-2}\right) - \left(\ln|3| - 4(3)^{-1} + \frac{\sqrt{3}}{2}(3)^{-2}\right) \\ &= \ln|A| - 4(A)^{-1} + \frac{\sqrt{3}}{2}(A)^{-2} - \ln(3) + 4(3)^{-1} - \frac{\sqrt{3}}{2}(3)^{-2}\end{aligned}$$

c. To find $\int_{B}^{e} \frac{12}{x \ln(x)}\, dx$, where $1 < B < e$, we must use the method of substitution. We will let $u = \ln(x)$ because we see the derivative of $\ln(x)$, which is $\frac{1}{x}$, is in the integrand. Rewriting the definite integral may help to see this:

$$\int_{B}^{e} \frac{12}{x \ln(x)}\, dx = \int_{B}^{e} 12 \cdot \frac{1}{\ln(x)} \cdot \frac{1}{x}\, dx$$

Thus, we have $u = \ln(x)$ and $du = \frac{1}{x}\,dx$.

The limits of integration for this definite integral are in terms of x. To get the equivalent u-values, we substitute the lower limit of integration, $x = B$, and the upper limit of integration, $x = e$, into the substitution we defined for u ($u = \ln(x)$):

$$u = \ln(B)$$
$$u = \ln(e) = 1$$

Now, we can rewrite the original definite integral entirely in terms of u and du.

$$\begin{aligned}
\int_B^e \frac{12}{x\ln(x)}\,dx &= \int_B^e 12\cdot\frac{1}{\ln(x)}\cdot\frac{1}{x}\,dx \\
&= \int_{\ln(B)}^1 12\cdot\frac{1}{u}\,du \\
&= (12\ln|u|)\Big|_{\ln(B)}^1 \\
&= (12\ln|1|) - (12\ln|\ln(B)|) \\
&= 0 - 12\ln|\ln(B)| \\
&= -12\ln|\ln(B)|
\end{aligned}$$

Therefore, $\int_B^e \frac{12}{x\ln x}\,dx = -12\ln|\ln(B)|$.

▪

Try It # 4:
Find each of the following definite integrals.

a. $\int_0^B \left(e^x + x^8 - \pi x\right)\,dx$, where $B > 0$

b. $\int_A^{-1} \left(x^4 + 12x^2\right)\left(5x^{-2} + 10\right)\,dx$, where $A < -1$

Using Technology

Unfortunately, not every function has an antiderivative that we can calculate, even considering techniques beyond the scope of this textbook. This is why Riemann sums are important, as is using technology to calculate them with a large number of subintervals (i.e., rectangles).

Most graphing calculators can estimate a definite integral. We will learn how to use the graphing calculator to estimate a definite integral by checking our answers to two of the previous examples using the TI-84 Plus CE. Any version of the TI-83 and TI-84 has this functionality, but your screen and inputs may look a little different from the ones we show here.

Example 5 Use technology to estimate each of the following definite integrals. Round your answers to two decimal places.

a. $\displaystyle\int_0^1 x^2\,dx$

b. $\displaystyle\int_{-1}^2 x^2 e^{x^3}\,dx$

Solution:

a. To estimate, or approximate, the definite integral $\displaystyle\int_0^1 x^2\,dx$, we start by pressing the MATH button, and then we select command 9: fnInt(. See **Figure 4.4.8**.

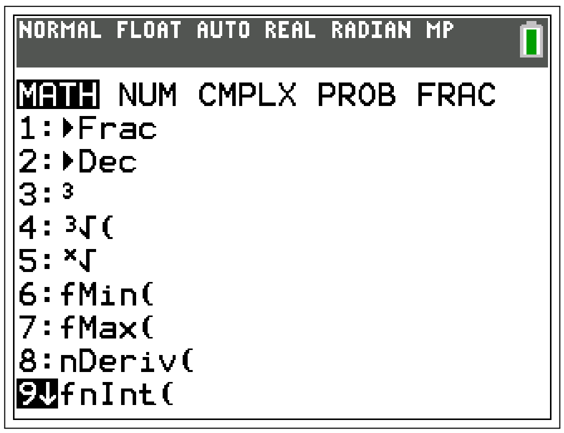


Figure 4.4.8: Location of the fnInt(command in the TI-84 Plus CE

With newer calculators like this one, we can see the inputs needed: the limits of integration, the integrand, and the variable. Inputting each of these into the correct location gives us the screen shown in **Figure 4.4.9**:

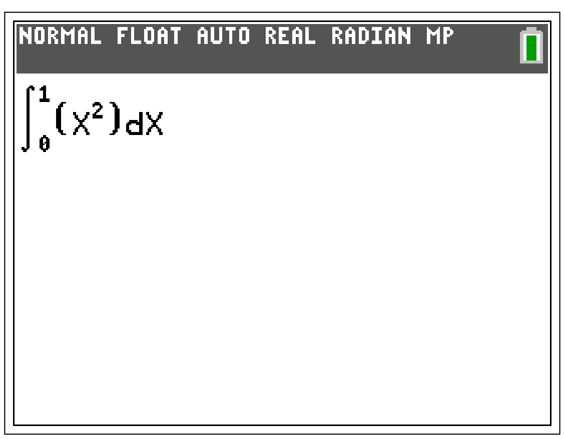


Figure 4.4.9: Inputs of the fnInt(command: limits of integration, integrand, and variable

Pressing ENTER will give us the same answer we obtained when working the problem in Example 1 (part **a**) using Part 2 of the Fundamental Theorem of Calculus. See **Figure 4.4.10**.

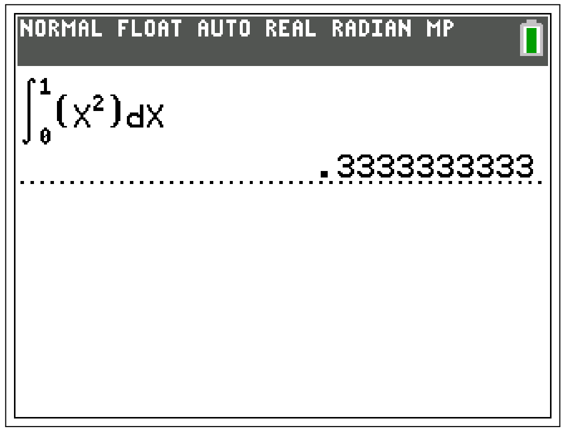


Figure 4.4.10: Approximation of the definite integral after pressing ENTER

Rounding this approximation to two decimal places gives 0.33.

b. To estimate, or approximate, the definite integral $\int_{-1}^{2} x^2 e^{x^3}\, dx$, we press the MATH button and then command 9: fnInt(. This will lead us to the same screen as before, and then we input the limits of integration, integrand, and variable. Pressing ENTER gives the screen shown in **Figure 4.4.11**:

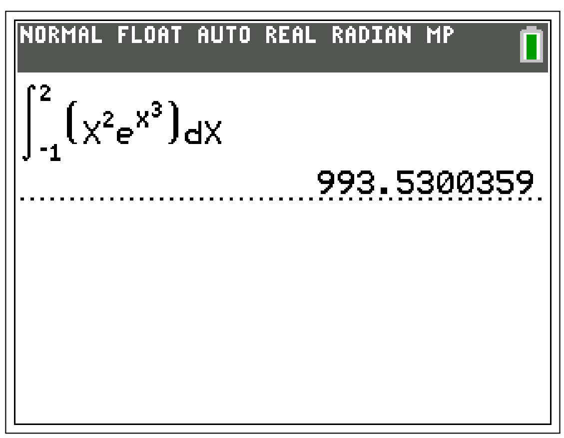


Figure 4.4.11: Using the fnInt(command to estimate $\int_{-1}^{2} x^2 e^{x^3}\, dx$

Rounding this approximation to two decimal places gives 993.53.

To verify this value is indeed equivalent to our previous answer in Example 2 (part **a**), which in exact form was $\frac{1}{3}e^8 - \frac{1}{3}e^{-1}$, we can use the calculator to find the approximation of this exact value. Doing so will also give 993.53 when rounding to two decimal places.

■

Try It # 5:
Use technology to estimate each of the following definite integrals. Round your answers to two decimal places.

a. $\int_{-2}^{3} \left(\frac{1}{3}x^3 + e^x\right)\, dx$

b. $\int_{-2}^{2} \left(4 - x^2\right)\, dx$

Part 2 of the Fundamental Theorem of Calculus is very useful for finding exact values of definite integrals, but it is also important mathematically. It shows us that if we take the derivative of a function and then find the integral of the result, we arrive, in essence, at the original function. Part 1 of the Fundamental Theorem of Calculus demonstrates the opposite: If we find the integral of a function and then take the derivative of the result, we arrive at the original function. Mathematically, these operations are inverses of each other! There are branches of higher mathematics based on this relationship, but they are beyond the scope of this textbook.

The Fundamental Theorem of Calculus, Part 1

Recall that Part 1 of the Fundamental Theorem of Calculus states that if f is continuous on an interval $[a,b]$ and the function F is defined by

$$F(x) = \int_{a}^{x} f(t)\, dt$$

where $a \le x \le b$, then $F'(x) = f(x)$ on the interval $[a,b]$.

Let's verify Part 1 by supposing $f(t) = 4 - t^2$ and defining $F(x)$ to be

$$F(x) = \int_{1}^{x} \left(4 - t^2\right)\, dt$$

Evaluating this definite integral gives

$$\begin{aligned} F(x) &= \int_{1}^{x} \left(4 - t^2\right)\, dt \\ &= 4t - \frac{1}{3}t^3 \Big|_{1}^{x} \\ &= \left(4x - \frac{1}{3}x^3\right) - \left(4(1) - \frac{1}{3}(1)^3\right) \\ &= 4x - \frac{1}{3}x^3 - \frac{11}{3} \end{aligned}$$

Thus, we know $F(x) = 4x - \frac{1}{3}x^3 - \frac{11}{3}$.

Part 1 of the Fundamental Theorem of Calculus says that the derivative of $F(x)$ should equal the integrand, $f(t) = 4 - t^2$, evaluated at x. In other words, $F'(x)$ should equal $f(x) = 4 - x^2$.

We can verify this result by actually finding $F'(x)$ using the Introductory Derivative Rules. If $F(x) = 4x - \frac{1}{3}x^3 - \frac{11}{3}$, then

$$\begin{aligned} F'(x) &= 4 - \frac{1}{3}(3)x^2 - 0 \\ &= 4 - x^2 \end{aligned}$$

Thus, we have verified Part 1 of the Fundamental Theorem of Calculus. Finding the derivative of a definite integral results in the original function in the integrand (evaluated at x).

■ **Example 6** Find the derivative of each of the following functions.

a. $F(x) = \int_0^x t^3\, dt$

b. $g(x) = \int_{-5}^x e^{-t^2}\, dt$

Solution:

a. Given $F(x) = \int_0^x t^3\, dt$, Part 1 of the Fundamental Theorem of Calculus tells us that $F'(x) = x^3$. At this point, we are done with the problem!

However, because we do know Part 2 of the Fundamental Theorem of Calculus, we can use it to verify that this is the correct answer by calculating the definite integral and then taking the derivative:

$$\begin{aligned} F(x) &= \int_0^x t^3\, dt \\ &= \frac{1}{4}t^4\Big|_0^x \\ &= \frac{1}{4}x^4 - \frac{1}{4}(0)^4 \\ &= \frac{1}{4}x^4 \end{aligned}$$

Now, we take the derivative:

$$\begin{aligned} F'(x) &= \frac{d}{dx}\left(\frac{1}{4}x^4\right) \\ &= \frac{1}{4}\cdot 4x^3 \\ &= x^3 \end{aligned}$$

We get the same answer either way!

b. Given $g(x) = \int_{-5}^x e^{-t^2}\, dt$, Part 1 of the Fundamental Theorem of Calculus tells us that

$$g'(x) = e^{-x^2}$$

N *Because we cannot find the antiderivative of the integrand* e^{-t^2} *using the integration techniques presented in this textbook, we are unable to verify this answer using Part 2 of the Fundamental Theorem of Calculus (i.e., by calculating the definite integral and then differentiating).*

■

Try It # 6:

Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of $F(x) = \int_8^x \left(\sqrt{t} + 5t^2\right) dt$. Verify your answer using Part 2 of the Fundamental Theorem of Calculus.

■ **Example 7** Find the derivative of each of the following functions.

a. $h(t) = \displaystyle\int_1^{t^2} \left(\frac{1}{x} + e^x\right) dx$

b. $f(x) = \displaystyle\int_7^{\ln(x)} \left(4t^4 - 2t^2\right)^{\frac{1}{3}}\ dt$

Solution:

a. The difference between this and the previous example is a subtle, but important one: The function $h(t) = \displaystyle\int_1^{t^2} \left(\frac{1}{x} + e^x\right) dx$ *does not have just the variable x as its upper limit of integration!*

Let's consider this in the light of Part 2 of the Fundamental Theorem of Calculus. Suppose $F(x)$ is an antiderivative of $f(x) = \frac{1}{x} + e^x$. Then,

$$\begin{aligned} h(t) &= \int_1^{t^2} \frac{1}{x} + e^x\, dx \\ &= F\left(t^2\right) - F(1) \end{aligned}$$

Now, when finding the derivative of $h(t) = F\left(t^2\right) - F(1)$, we will need to use the Chain Rule when finding the derivative of $F\left(t^2\right)$ because it is a composition of functions. The derivative of $F(1)$ will be zero because its value is a constant.

Combining this information yields the following when finding the derivative of $h(t)$, which is given by Part 1 of the Fundamental Theorem of Calculus:

$$\begin{aligned} h'(t) &= \frac{d}{dt}\left(\int_1^{t^2} \left(\frac{1}{x} + e^x\right) dx\right) \\ &= \frac{d}{dt}\left(F\left(t^2\right) - F(1)\right) \\ &= \frac{d}{dt}\left(F\left(t^2\right)\right) - \frac{d}{dt}(F(1)) \\ &= F'\left(t^2\right) \cdot \frac{d}{dt}\left(t^2\right) - 0 \\ &= F'\left(t^2\right) \cdot 2t \end{aligned}$$

Because we assumed $F(x)$ is an antiderivative of $f(x) = \frac{1}{x} + e^x$, we know $F'(x) = f(x)$. Thus,

$$\begin{aligned} h'(t) &= F'\left(t^2\right) \cdot 2t \\ &= f\left(t^2\right) \cdot 2t \\ &= \left(\frac{1}{t^2} + e^{t^2}\right) 2t \end{aligned}$$

Notice the implication of our answer: The derivative of a function defined as a definite integral is the integrand evaluated at the upper limit of integration and multiplied by its derivative ($2t$ in this case), minus the integrand evaluated at the lower limit of integration and multiplied by its derivative (0 in this case).

Thus, for these types of problems, we do not need to go through the previous analysis each and every time, but we should always keep in mind that we may need to incorporate the Chain Rule when using Part 1 of the Fundamental Theorem of Calculus to find the derivative of a function defined as a definite integral.

Because we can find the antiderivative of this particular integrand, $\frac{1}{x}+e^x$, using the Introductory Antiderivative Rules, let's verify our answer by calculating the definite integral using Part 2 of the Fundamental Theorem of Calculus and then differentiating:

$$\begin{aligned} h(t) &= \int_1^{t^2} \left(\frac{1}{x}+e^x\right) dx \\ &= \left(\ln|x|+e^x\right)\Big|_1^{t^2} \\ &= \left(\ln\left|t^2\right|+e^{t^2}\right)-\left(\ln|1|+e^1\right) \\ &= \ln\left|t^2\right|+e^{t^2}-0-e \\ &= \ln\left|t^2\right|+e^{t^2}-e \end{aligned}$$

Before we take the derivative of $h(t)$, notice that $t^2 \geq 0$ (because t is squared). So $\left|t^2\right| = t^2$. Hence, $h(t) = \ln\left(t^2\right)+e^{t^2}-e$. Taking the derivative of $h(t)$ gives

$$\begin{aligned} h'(t) &= \frac{d}{dt}\left(\ln\left(t^2\right)+e^{t^2}-e\right) \\ &= \frac{d}{dt}\left(\ln\left(t^2\right)\right)+\frac{d}{dt}\left(e^{t^2}\right)-\frac{d}{dt}(e) \\ &= \frac{\frac{d}{dt}\left(t^2\right)}{t^2}+e^{t^2}\cdot\frac{d}{dt}\left(t^2\right)-0 \\ &= \frac{2t}{t^2}+e^{t^2}\cdot 2t \\ &= \left(\frac{1}{t^2}+e^{t^2}\right)2t \end{aligned}$$

This is the same answer we got using Part 1 of the Fundamental Theorem of Calculus!

b. To find the derivative of $f(x) = \int_7^{\ln(x)} \left(4t^4-2t^2\right)^{\frac{1}{3}}\, dt$, we will use Part 1 of the Fundamental Theorem of Calculus and remember to incorporate the Chain Rule because the upper limit of integration is not just the variable x.

Recall from part **a** that this means we can (quickly) find the derivative of $f(x)$ by evaluating the integrand at the upper limit of integration, $\ln(x)$, and multiplying by its derivative, $\frac{1}{x}$, and then subtracting the integrand evaluated at the lower limit of integration, 7, and multiplied by its derivative, 0:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\int_7^{\ln(x)} \left(4t^4-2t^2\right)^{\frac{1}{3}}\, dt\right) \\ &= \left(4\left(\ln(x)\right)^4-2\left(\ln(x)\right)^2\right)^{\frac{1}{3}}\cdot\frac{1}{x}-\left(4(7)^4-2(7)^2\right)^{\frac{1}{3}}\cdot 0 \\ &= \left(4\left(\ln(x)\right)^4-2\left(\ln(x)\right)^2\right)^{\frac{1}{3}}\cdot\frac{1}{x} \end{aligned}$$

Let's verify our answer by working through the process we used in part **a** involving Part 1 of the Fundamental Theorem of Calculus. To do this, we will suppose $G(t)$ is an antiderivative of $g(t) = \left(4t^4 - 2t^2\right)^{\frac{1}{3}}$. Then,

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\int_7^{\ln(x)} \left(4t^4 - 2t^2\right)^{\frac{1}{3}}\, dt\right) \\ &= \frac{d}{dx}\left(G(\ln(x)) - G(7)\right) \\ &= \frac{d}{dx}\left(G(\ln(x))\right) - \frac{d}{dx}\left(G(7)\right) \\ &= G'(\ln(x)) \cdot \frac{d}{dx}(\ln(x)) - 0 \\ &= G'((\ln(x)) \cdot \frac{1}{x} \end{aligned}$$

Because we assumed $G(t)$ is an antiderivative of $g(t) = \left(4t^4 - 2t^2\right)^{\frac{1}{3}}$, we know $G'(t) = g(t)$. Thus,

$$\begin{aligned} f'(x) &= G'((\ln(x)) \cdot \frac{1}{x} \\ &= g(\ln(x)) \cdot \frac{1}{x} \\ &= \left(4(\ln(x))^4 - 2(\ln(x))^2\right)^{\frac{1}{3}} \cdot \frac{1}{x} \end{aligned}$$

This is the same answer we found previously!

N *Because we cannot find the antiderivative of the integrand* $\left(4t^4 - 2t^2\right)^{1/3}$ *using the integration techniques presented in this textbook, we are unable to verify this answer using Part 2 of the Fundamental Theorem of Calculus (i.e., by calculating the definite integral and then differentiating).*

▪

Try It # 7:

Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of $f(x) = \int_{0.2}^{e^x} \left(14y^7 - 3y^2 + y\right)\, dy$. Verify your answer using Part 2 of the Fundamental Theorem of Calculus.

▪ **Example 8** Find the derivative of each of the following functions.

a. $g(t) = \displaystyle\int_t^{e^{2t}} x^2\left(12\sqrt{x} - 13x\right)\, dx$

b. $f(x) = \displaystyle\int_{8x}^{15} \frac{e^u + \sqrt[3]{u^2}}{\ln(u)}\, du$

c. $P(x) = \displaystyle\int_{5^x}^{9x-3} z\log_3(z)\, dz$

Solution:

a. We will use Part 1 of the Fundamental Theorem of Calculus to find the derivative of $g(t) = \displaystyle\int_t^{e^{2t}} x^2\left(12\sqrt{x} - 13x\right)\, dx$. Before doing so, notice that there are variables in *both* limits of integration.

Recall that using Part 1 of the Fundamental Theorem of Calculus means we can (quickly) find the derivative of $g(t)$ by evaluating the integrand at the upper limit of integration, e^{2t}, and multiplying by its derivative, $2e^{2t}$, and then subtracting the integrand evaluated at the lower limit of integration, t, and multiplied by its derivative, 1:

$$\begin{aligned} g'(t) &= \frac{d}{dt}\left(\int_t^{e^{2t}} x^2\left(12\sqrt{x}-13x\right)\,dx\right) \\ &= \left(\left(e^{2t}\right)^2\left(12\sqrt{e^{2t}}-13e^{2t}\right)\right)\cdot 2e^{2t} - \left(t^2\left(12\sqrt{t}-13t\right)\right)\cdot 1 \end{aligned}$$

Let's verify our answer by working through all of the steps involving Part 1 of the Fundamental Theorem of Calculus. To do this, we will suppose $F(x)$ is an antiderivative of $f(x) = x^2\left(12\sqrt{x}-13x\right)$. Then,

$$\begin{aligned} g'(t) &= \frac{d}{dt}\left(\int_t^{e^{2t}} x^2\left(12\sqrt{x}-13x\right)\,dx\right) \\ &= \frac{d}{dt}\left(F\left(e^{2t}\right)-F(t)\right) \\ &= \frac{d}{dt}\left(F\left(e^{2t}\right)\right)-\frac{d}{dt}(F(t)) \\ &= F'\left(e^{2t}\right)\cdot\frac{d}{dt}\left(e^{2t}\right)-F'(t) \\ &= F'\left(e^{2t}\right)\cdot e^{2t}\cdot\frac{d}{dt}(2t)-F'(t) \\ &= F'\left(e^{2t}\right)\cdot e^{2t}\cdot 2-F'(t) \end{aligned}$$

Because we assumed $F(x)$ is an antiderivative of $f(x) = x^2\left(12\sqrt{x}-13x\right)$, we know $F'(x) = f(x)$. Thus,

$$\begin{aligned} g'(t) &= F'\left(e^{2t}\right)\cdot e^{2t}\cdot 2-F'(t) \\ &= f\left(e^{2t}\right)\cdot 2e^{2t}-f(t) \\ &= \left(\left(e^{2t}\right)^2\left(12\sqrt{e^{2t}}-13e^{2t}\right)\right)\cdot 2e^{2t}-t^2\left(12\sqrt{t}-13t\right) \end{aligned}$$

This is indeed equivalent to the expression for $g'(t)$ we found previously!

N *Because we can find the antiderivative of the integrand* $x^2\left(12\sqrt{x}-13x\right)$ *using the integration techniques presented in this textbook, we could also verify this answer using Part 2 of the Fundamental Theorem of Calculus (i.e., by calculating the definite integral and then differentiating). We will leave that for you to try!*

b. We will use Part 1 of the Fundamental Theorem of Calculus to find the derivative of $f(x) = \displaystyle\int_{8x}^{15} \frac{e^u+\sqrt[3]{u^2}}{\ln(u)}\,du$.

Recall that using Part 1 of the Fundamental Theorem of Calculus means we can (quickly) find the derivative of $f(x)$ by evaluating the integrand at the upper limit of integration, 15, and multiplying by its derivative, 0, and then subtracting the integrand evaluated at the lower limit of integration, $8x$, and multiplied by its derivative, 8:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\int_{8x}^{15} \frac{e^u + \sqrt[3]{u^2}}{\ln(u)}\,du\right) \\ &= \left(\frac{e^{15} + \sqrt[3]{(15)^2}}{\ln(15)}\right)\cdot 0 - \left(\frac{e^{8x} + \sqrt[3]{(8x)^2}}{\ln(8x)}\right)\cdot 8 \\ &= -\left(\frac{e^{8x} + \sqrt[3]{(8x)^2}}{\ln(8x)}\right)\cdot 8 \end{aligned}$$

Let's verify our answer by working through all of the steps involving Part 1 of the Fundamental Theorem of Calculus. To do this, we will suppose $G(u)$ is an antiderivative of $g(u) = \dfrac{e^u + \sqrt[3]{u^2}}{\ln(u)}$. Then,

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\int_{8x}^{15} \frac{e^u + \sqrt[3]{u^2}}{\ln(u)}\,du\right) \\ &= \frac{d}{dx}\left(G(15) - G(8x)\right) \\ &= \frac{d}{dx}(G(15)) - \frac{d}{dx}(G(8x)) \\ &= 0 - G'(8x)\cdot\frac{d}{dx}(8x) \\ &= -G'(8x)\cdot 8 \end{aligned}$$

Because we assumed $G(u)$ is an antiderivative of $g(u) = \dfrac{e^u + \sqrt[3]{u^2}}{\ln(u)}$, we know $G'(u) = g(u)$. Thus,

$$\begin{aligned} f'(x) &= -G'(8x)\cdot 8 \\ &= -g(8x)\cdot 8 \\ &= -\left(\frac{e^{8x} + \sqrt[3]{(8x)^2}}{\ln(8x)}\right)\cdot 8 \end{aligned}$$

This is indeed the same expression for $f'(x)$ we found previously!

N *We cannot verify this answer using Part 2 of the Fundamental Theorem of Calculus because we are unable to calculate the antiderivative of the integrand using the integration techniques presented in this textbook.*

c. We will use Part 1 of the Fundamental Theorem of Calculus to find the derivative of $P(x) = \displaystyle\int_{5^x}^{9x-3} z\log_3(z)\,dz$.

Recall that using Part 1 of the Fundamental Theorem of Calculus means we can (quickly) find the derivative of $P(x)$ by evaluating the integrand at the upper limit of integration, $9x-3$, and multiplying by its derivative, 9, and then subtracting the integrand evaluated at the lower limit of integration, 5^x, and multiplied by its derivative, $5^x\ln(5)$:

$$\begin{aligned} P'(x) &= \frac{d}{dx}\left(\int_{5^x}^{9x-3} z\log_3(z)\,dz\right) \\ &= ((9x-3)\log_3(9x-3))\cdot 9 - (5^x\log_3(5^x))\cdot 5^x\ln(5) \end{aligned}$$

Let's verify our answer by working through all of the steps involving Part 1 of the Fundamental Theorem of Calculus. To do this, we will suppose $H(z)$ is an antiderivative of $h(z) = z\log_3(z)$.

Then,

$$\begin{aligned} P'(x) &= \frac{d}{dx}\left(\int_{5^x}^{9x-3} z\log_3(z)\,dz\right) \\ &= \frac{d}{dx}\left(H(9x-3)-H(5^x)\right) \\ &= \frac{d}{dx}\left(H(9x-3)\right)-\frac{d}{dx}\left(H(5^x)\right) \\ &= H'(9x-3)\cdot\frac{d}{dx}(9x-3)-H'(5^x)\cdot\frac{d}{dx}(5^x) \\ &= H'(9x-3)\cdot 9-H'(5^x)\cdot 5^x\ln(5) \end{aligned}$$

Because we assumed $H(z)$ is an antiderivative of $h(z) = z\log_3(z)$, we know $H'(z) = h(z)$. Thus,

$$\begin{aligned} P'(x) &= H'(9x-3)\cdot 9-H'(5^x)\cdot 5^x\ln(5) \\ &= h(9x-3)\cdot 9-h(5^x)\cdot 5^x\ln(5) \\ &= ((9x-3)\log_3(9x-3))\cdot 9-(5^x\log_3(5^x))\cdot 5^x\ln(5) \end{aligned}$$

This is indeed the same expression for $P'(x)$ we found previously!

N *We cannot verify this answer using Part 2 of the Fundamental Theorem of Calculus because we are unable to calculate the antiderivative of the integrand using the integration techniques presented in this textbook.*

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Try It # 8:

Use Part 1 of the Fundamental Theorem of Calculus to find the derivative of $h(x) = \displaystyle\int_{(2x+1)^3}^{\log_4(x)} \frac{t^2-3t+7}{t}\,dt$. Verify your answer using Part 2 of the Fundamental Theorem of Calculus.

Applications

Recall in the last section, we learned that the definite integral of a rate of change function on an interval $[a,b]$ represents the *net change* in the relevant quantity on the interval.

In other words, $\displaystyle\int_a^b f'(x)\,dx$ gives the net change in f on the interval $[a,b]$.

However, in the previous section, we had to estimate the value of such a definite integral if the graph of the rate of change function was not defined in terms of geometric shapes of which we could calculate the areas. However, now we can calculate the exact value of any definite integral no matter what the graph of the rate of change function looks like, thanks to the Fundamental Theorem of Calculus! We formalize this interpretation below:

Interpreting the Definite Integral as Net Change

Given a rate of change function, f', the net change in its antiderivative, f, from $x = a$ to $x = b$ is given by

$$\int_a^b f'(x)\,dx = f(b)-f(a)$$

Now, we will work examples involving applications that are similar to the applications we encountered in the last section in which we were given a rate of change function. The big difference now is that we can find the *exact* value of the net change (or change), not just an estimation!

Example 9 The rate of change of price for high-end gaming computer mice sold by Better Purchase is given by $p'(x) = 300(0.997)^x \cdot (\ln(0.997))$ dollars per mice when x mice are sold. Find the change in price when the number of mice sold increases from 100 to 200.

Solution:
We must find the change (or net change) in price, and we are given the rate of change function of price, p'. This means we need to calculate $\int_{100}^{200} p'(x)\,dx$:

$$\begin{aligned}
\int_{100}^{200} p'(x)\,dx &= \int_{100}^{200} 300(0.997)^x \cdot (\ln(0.997))\,dx \\
&= \left(300(\ln(0.997)) \cdot \frac{1}{\ln(0.997)}(0.997)^x\right)\Bigg|_{100}^{200} \\
&= (300(0.997)^x)\Big|_{100}^{200} \\
&= \left(300(0.997)^{200}\right) - \left(300(0.997)^{100}\right) \\
&\approx -\$57.65
\end{aligned}$$

When the number of mice sold increases from 100 to 200, the price of each mouse decreases by \$57.65.

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Example 10 An object is shot straight up with a velocity given by $v(t) = -32t + 34$ feet per second, where t is time in seconds. Find and interpret $\int_0^1 v(t)\,dt$.

Solution:
We will first evaluate the definite integral:

$$\begin{aligned}
\int_0^1 v(t)\,dt &= \int_0^1 (-32t + 34)\,dt \\
&= \left(-32 \cdot \frac{1}{2}t^2 + 34t\right)\Bigg|_0^1 \\
&= \left(\frac{-32}{2}(1)^2 + 34(1)\right) - \left(\frac{-32}{2}(0)^2 + 34(0)\right) \\
&= 18
\end{aligned}$$

Velocity is the rate of change function of the position function, so the answer is the change in the position of the object during the first second. Because this velocity function is nonnegative on the interval $[0,1]$, this change in position is equivalent to the distance the object traveled during the first second.

Because the velocity function is given in feet per second and the time is in seconds, the answer is measured in feet.

Thus, $\int_0^1 v(t)\,dt$ tells us that the object traveled 18 feet upwards during the first second.

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Try It # 9:
The rate of change of sales for Twelfth Man Towels for Texas A&M University is given by $\frac{d}{dt}(S(t)) = 12{,}000(t+1)^{1/2} - 12{,}000$ towels per year, where t is time in years. How many towels were sold during the first twelve years? Round to the nearest integer, if necessary.

Try It Answers

1. **a.** $3\ln(3) + \frac{635}{7}$

   b. $18 - \ln(5) - \frac{2}{3}(5)^{-3/2} + \ln(2) + \frac{2}{3}(2)^{-3/2}$

   c. $-\frac{16}{9}$

2. **a.** $2\ln(1060) - 2\ln(7)$

   b. $\frac{2440}{3}$

3. **a.** -8

   b. $-39 - 2\pi$

4. **a.** $e^B + \frac{1}{9}B^9 - \frac{\pi}{2}(B)^2 - 1$

   b. $-\frac{311}{3} - \frac{125}{3}A^3 - 2A^5 - 60A$

5. **a.** 25.37

   b. 10.67

6. $F'(x) = \sqrt{x} + 5x^2$

7. $f'(x) = \left(14(e^x)^7 - 3(e^x)^2 + e^x\right)e^x$

8. $\left(\frac{(\log_4(x))^2 - 3\log_4(x) + 7}{\log_4(x)}\right)\left(\frac{1}{x\ln(4)}\right) - \left(\frac{\left((2x+1)^3\right)^2 - 3(2x+1)^3 + 7}{(2x+1)^3}\right)\left(3(2x+1)^2(2)\right)$

9. $222{,}977$ towels

EXERCISES

Basic Skills Practice

For Exercises 1 - 6, evaluate the definite integral exactly using Part 2 of the Fundamental Theorem of Calculus.

1. $\displaystyle\int_{-5}^{-1} \left(x^3 + 12x^2\right)\, dx$

2. $\displaystyle\int_{4}^{9} \left(-3x^2 + 6\sqrt{x} - 18\right)\, dx$

3. $\displaystyle\int_{0}^{1} \left(7e^x - 5x^4\right)\, dx$

4. $\displaystyle\int_{1}^{8} \left(\frac{x^3}{4} + x^{-5/3}\right)\, dx$

5. $\displaystyle\int_{-3}^{2} \left(-0.3x^2 + \frac{6}{5}x\right)\, dx$

6. $\displaystyle\int_{1}^{e} \left(8x + \frac{1}{x}\right)\, dx$

For Exercises 7 - 9, find $F'(x)$ using Part 1 of the Fundamental Theorem of Calculus.

7. $\displaystyle F(x) = \int_{0}^{x} \left(t^4 - 7\right)\, dt$

8. $\displaystyle F(x) = \int_{9}^{x} \frac{1}{t}(\ln(t))\, dt$

9. $\displaystyle F(x) = \int_{4}^{x} 2^{-t^2}\, dt$

10. The marginal cost function for a particular video game accessory is given by $C'(x) = 32 - 1.6x^2$ dollars per accessory when x accessories are made. Find the change in cost when production increases from 120 to 160 accessories.

11. The marginal profit function for Hardy-Daytona motorcycles is given by $P'(x) = 78 - 0.12x$ dollars per motorcycle when x motorcycles are sold. Find the change in profit when the number of motorcycles sold increases from 250 to 500.

12. The marginal revenue function for Wind Through the Pillows, a company that sells couch pillows, is given by $R'(x) = 7 - 0.92x$ dollars per pillow when x pillows are sold. Find the change in revenue when the number of pillows sold increases from 100 to 180.

13. An object moves at a rate of $f'(t) = 0.09t^2 + 0.18t$ meters per second after t seconds have passed. Find exactly how far the object has traveled during the first 5 seconds.

Intermediate Skills Practice

For Exercises 14 - 21, evaluate the definite integral exactly.

14. $\displaystyle\int_{-5}^{-3} \frac{-x^5 + 4x^3 + 2x^2}{x^2}\, dx$

15. $\displaystyle\int_{0}^{9} \left(\frac{5\sqrt{t}}{6} + 2e^t\right) dt$

16. $\displaystyle\int_{-2}^{2} -\frac{1}{9}x\left(3 - x^2\right)^4\, dx$

17. $\displaystyle\int_{-3}^{-1} \left(1 - 4x^3\right)\left(2x^4 - 7x\right) dx$

18. $\displaystyle\int_{2}^{4} -27t^2 e^{3t^3-2}\, dt$

19. $\displaystyle\int_{1}^{3} \left(\frac{10}{9x} - \frac{4}{x^5}\right) dx$

20. $\displaystyle\int_{5}^{8} -\frac{12x}{\sqrt{2x^2 - 1}}\, dx$

21. $\displaystyle\int_{-7}^{-5} \frac{45t^2 - 24t + 6}{5t^3 - 4t^2 + 2t}\, dt$

For Exercises 22 - 24, evaluate the definite integral.

22. $\displaystyle\int_{-2}^{A} \left(4x^3 - 6x + 1\right) dx$, where A is a constant and $A > -2$

23. $\displaystyle\int_{0}^{B} \left(8x^{3/5} - 5x^4 + 0.27x^2\right) dx$, where B is a constant and $B > 0$

24. $\displaystyle\int_{K}^{-1} \left(4e^x + \frac{2}{7x}\right) dx$, where K is a constant and $K < -1$

25. If $\int_1^7 f(x)\,dx = -48$ and $F(7) = -45$, where F is an antiderivative of f, find $F(1)$.

26. If $\int_{-3}^2 g(x)\,dx = 35$ and $G(-3) = -25$, where G is an antiderivative of g, find $G(2)$.

27. If $\int_{-8}^{-4} f'(x)\,dx = -9.6$ and $f(-8) = 13.8$, find $f(-4)$.

28. If $\int_5^{10} g'(x)\,dx = 225$ and $g(10) = 305$, find $g(5)$.

For Exercises 29 - 33, find $F'(x)$.

29. $F(x) = \int_{13}^{x} \frac{e^t}{t}\,dt$

30. $F(x) = \int_{x}^{-11} t\sqrt{9-t}\,dt$

31. $F(x) = \int_{x}^{5} (3\ln(6-t))\,dt$

32. $F(x) = \int_{2}^{x^2} \frac{1}{\sqrt{16-t^2}}\,dt$

33. $F(x) = \int_{-4}^{\sqrt[3]{x}} -9e^{-2t^2}\,dt$

34. The marginal revenue function for Hedgemony, a bush planting service, is given by $R'(x) = 20 - 0.35x$ dollars per bush planted, when x bushes are planted. Find the change in revenue for the first 75 bushes planted.

35. A company has a weekly marginal profit function given by $P'(x) = 40x - 0.3x^2 - 260$ dollars per item when x items are made and sold. Find the change in profit when the number of items made and sold each week increases from 120 to 140.

36. The marginal cost function for a street vendor selling tomatoes is given by $c(x) = 0.8x^{-1/3}$ dollars per tomato when x tomatoes are picked. Find the change in cost when the number of tomatoes picked increases from 3 to 30.

37. The marginal profit function for Boblin the Goblin dolls is given by $P'(x) = 200 + \frac{7}{x+1} + \frac{x^2}{7}$ dollars per doll when x dolls are sold.

    (a) Find the change in profit when the number of dolls sold increases from 50 to 75.

    (b) If the company loses \$15,000 in profit when it does not sell any dolls, find the profit from selling the first 100 dolls.

38. The velocity of an object is given by $v(t) = -t^2 + 6t + 8$ meters per second, where t is time in seconds. Find the distance traveled by the object during the first four seconds.

Mastery Practice

For Exercises 39 - 46, evaluate the definite integral exactly.

39. $\displaystyle\int_1^B (30+15x)\sqrt[4]{12x+3x^2}\,dx$, where B is a constant and $B > 1$

40. $\displaystyle\int_8^{27} \left(\frac{3}{7}e^t - \frac{3}{4\sqrt[3]{t}}\right) dt$

41. $\displaystyle\int_{-2}^{2} -x^2\left(3x^2-6x\right)\left(1-2x^4-x^5\right) dx$

42. $\displaystyle\int_{e^2}^{5} \frac{1}{2t\ln\left(\sqrt{t}\right)}\,dt$

43. $\displaystyle\int_K^{-4} \frac{-16x}{\sqrt[3]{4x+15}}\,dx$, where K is a constant and $K < -4$

44. $\displaystyle\int_{-1}^{0} \left(60t^3-15t\right)e^{-6t^4+3t^2}\,dt$

45. $\displaystyle\int_{-2}^{-1} \frac{x^2-2x}{0.2x^3-0.6x^2-1}\,dx$

46. $\displaystyle\int_3^A \frac{\frac{6}{7}x^{4/5}-2x^3-0.28x^2}{5x^3}\,dx$, where A is a constant and $A > 3$

For Exercises 47 - 52, find $F'(x)$.

47. $\displaystyle F(x) = \int_{10}^{2x+7} t\left(t^2-33\right)^{14}\,dt$

48. $\displaystyle F(x) = \int_x^{-1} \left(2t-8+4t^{-3}\right) dt$

49. $\displaystyle F(x) = \int_x^{x^2} \sqrt{9+t^2}\,dt$

50. $\displaystyle F(x) = \int_3^x (t\ln(t)-t)\,dt$

51. $F(x) = \int_{\sqrt{x}}^{9} \left(4t^3 + 10t\right) e^{t^4+5t^2+11}\, dt$

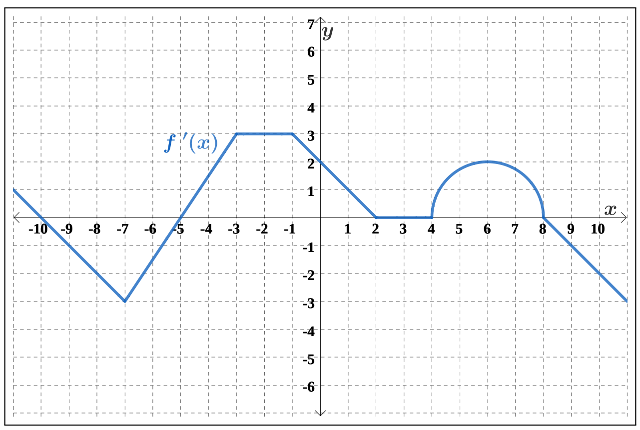
52. $F(x) = \int_{-3x}^{7x} \left(6t + \sqrt[3]{t}\right) dt$

53. If $\int_{-6}^{2} f'(x)\, dx = -8$ and $f(-6) = 12$, find $f(2)$.

54. If $\int_{15/4}^{8} h(x)\, dx = 122.825$ and $H(8) = 148.2$, where H is an antiderivative of h, find $H\left(\frac{15}{4}\right)$.

55. If $\int_{-7}^{-3} g''(x)\, dx = 24$, $g(-3) = 34$, $g'(-3) = -20$, and $g(-7) = 162$, find $g'(-7)$.

56. The graph of f' is shown below. Use the graph to answer each of the following.



(a) If $f(3) = 4.75$, find $f(-3)$.

(b) If $f(-5) = -8.75$, find $f(-10)$.

(c) If $f(-7) = -3.25$, find $f(2)$.

57. The marginal cost function for a company is given by $C'(x) = 20x + \dfrac{13}{2\sqrt{13x + 160}}$ dollars per item, where x is the number of items produced.

(a) Find the change in cost when production increases from 36 to 50 items.

(b) If the company has fixed costs of \$13,000, find the cost of producing the first 100 items.

58. One Awesome Rainbow, a company that sells custom rainbow headbands, has found the function $p'(x) = -0.75(0.997)^x$ dollars per headband models the rate of change of its price-demand function, where x is the number of headbands sold. How is the price per headband affected when the number of headbands sold increases from 200 to 300?

59. A walker moves around the track at $v(t) = \dfrac{10\ln(t+1)}{t+1}$ feet per second after t seconds. Calculate the distance walked during the first 30 seconds. Round your answer to two decimal places, if necessary.

60. Aerith sells flowers in Midgar. She has found that her daily marginal cost and revenue functions are given by $C'(x) = \dfrac{20}{3(10x+4)^{(1/3)}}$ and $R'(x) = 12 - 1.2x$ dollars per flower when x flowers are picked and sold, respectively.
    (a) Find the change in cost when the number of flowers picked increases from 10 to 20.
    (b) Find the change in revenue for the first 5 flowers Aerith sells.
    (c) Find the change in profit when the number of flowers picked and sold increases from 15 to 25.

61. Due to the limited availability of toilet paper during the COVID-19 pandemic, the Savage family tried to reduce the amount of toilet paper they used each month. The number of rolls of toilet paper they used each month decreased at a rate of $r(t) = -8e^{-t}$ rolls per month, where t is the number of months since the family began trying to use their toilet paper sparingly. Find the change in the number of rolls of toilet paper used after the first two months, and interpret your answer. Round your answer to two decimal places, if necessary.

62. The rate at which the balance of an account grows is given by $\dfrac{d}{dt}A(t) = 500e^{0.05t}$ dollars per year, where t is time in years. Find the increase in the balance of the account after eight years.

63. After t months, a company's sales are changing at a rate of $\dfrac{dS}{dt} = 0.08t^3 + 0.09t^2 + 0.3t + 3$ million dollars per month.
    (a) Find the increase in the company's sales after the first year.
    (b) Find the change in the company's sales during the second five month period.
    (c) Find the change in the company's sales during the third and fourth months.

Communication Practice

64. Explain the difference between Part 1 and Part 2 of the Fundamental Theorem of Calculus.

65. Given A is a constant and $A > 1$, does $\displaystyle\int_1^A \left(3x^2 + 2x\right)\, dx = 3A^2 + 2A - 5$? Explain.

66. Does $\displaystyle\int_{-3}^{7} \left(5x^2+3\right)^8\, dx = \int_{-3}^{7} \frac{1}{5}u^8\, du$? Explain.

67. If a company's marginal revenue is $R'(x)$ dollars per item when x is items are sold, explain how to find the change in revenue when the number of items increases from 200 to 250.

68. If $R'(x)$ and $C'(x)$, both measured in dollars per item, give a company's marginal revenue and cost, respectively, when x items are produced and sold, interpret $\displaystyle\int_{400}^{500} R'(x) - C'(x)\, dx = -12{,}000$.

69. If a company's profit is $P(x)$ dollars when x items are made and sold, does $\int_{50}^{75} P(x)\,dx$ give the change in the company's profit when the number of items made and sold increases from 50 to 75? Explain.

70. If a company's cost is $C(x)$ dollars when x items are produced, explain two different ways we can calculate the change in the company's cost when production increases from 1000 to 1200 items.

71. If the velocity of an object is $v(t)$ feet per second after t seconds and $v(t) \geq 0$, explain how to find the distance traveled by the object during the first 10 seconds.

72. Cody, a student in Professor Allen's calculus class, was helping a classmate with the following homework problem:

$$\text{"Find } g'(x) \text{ if } g(x) = \int_{x}^{4} t^2 \ln(t)\,dt.\text{"}$$

Cody told his classmate all they needed to do was substitute x for t in the integrand because the limits of integration only consisted of a constant and an "x".

(a) Which part of the Fundamental Theorem of Calculus applies (directly) to this homework problem?

(b) Was Cody's solution to the problem correct? Explain.

73. Explain when it is necessary to incorporate the Chain Rule when finding the derivative of a function defined as a definite integral using the Fundamental Theorem of Calculus, Part 1.

4.5 Average Value of a Function

Now that we know how to evaluate a definite integral using Part 2 of the Fundamental Theorem of Calculus, we will investigate more of its applications! In this section, we will turn to a concept familiar to many students: the *average* of a quantity.

We can calculate the average (or mean value) of n numbers by adding the numbers and dividing by how many numbers there are, n. In other words, if the numbers are $x_1,\ x_2,\ \ldots,\ x_n$, then the average of the numbers is given by

$$\frac{x_1+x_2+x_3+\cdots+x_n}{n}=\frac{\sum_{k=1}^{n} x_k}{n}$$

In a sense, computing the **average value** of a *function* requires us to do something similar: add all the function values and divide by how many function values there are. We will soon see that the computation to find the average value of a function involves a definite integral!

Learning Objectives:

In this section, you will learn how to find the average value of a function and solve problems involving real-world applications. Upon completion you will be able to:

- Calculate the average value of a function on an interval.
- Calculate the average value of a rate of change function on an interval.
- Calculate the average value of a function's rate of change function on an interval.
- Calculate the average value of a function's antiderivative on an interval.
- Calculate the average value of a function involving a real-world scenario.

Average Value of a Function

To estimate the average value (y-value) of a function f on the interval $[a,b]$, we can divide the interval into n subintervals of equal width $\Delta x=\dfrac{b-a}{n}$, choose an x-value, x_i^*, in each subinterval (left endpoint, right endpoint, midpoint, or some other x-value) and find its corresponding function value (y-value), and then find the average of these function values:

$$\begin{aligned}\text{average value of } f(x) &\approx \frac{f\left(x_1^*\right)+f\left(x_2^*\right)+\cdots+f\left(x_n^*\right)}{n}\\ &=\frac{\sum_{i=1}^{n} f\left(x_i^*\right)}{n}\end{aligned}$$

Rewriting this sum slightly gives

$$=\sum_{i=1}^{n} f\left(x_i^*\right)\cdot\frac{1}{n}$$

This resembles a Riemann sum, except we are multiplying by $\dfrac{1}{n}$ instead of the length of each subinterval, Δx. However, we can transform this sum into a Riemann sum! To do this, consider the formula for the length of each subinterval, Δx:

$$\Delta x = \frac{b-a}{n}$$
$$= \frac{1}{n}(b-a)$$

Solving for $\frac{1}{n}$ by dividing both sides by $b-a$ gives

$$\Delta x = \frac{1}{n}(b-a)$$
$$\frac{\Delta x}{b-a} = \frac{1}{n}$$

Substituting this quantity for $\frac{1}{n}$ in the sum gives

$$\sum_{i=1}^{n} f\left(x_i^*\right) \cdot \frac{1}{n} = \sum_{i=1}^{n} f\left(x_i^*\right) \cdot \frac{\Delta x}{b-a}$$

We almost have our Riemann sum! Continuing to rewrite the sum again gives

$$\sum_{i=1}^{n} f\left(x_i^*\right) \cdot \frac{\Delta x}{b-a} = \sum_{i=1}^{n} f\left(x_i^*\right) \cdot \Delta x \cdot \frac{1}{b-a}$$

Finally, moving the constant $\frac{1}{b-a}$ to the front, we have

$$\sum_{i=1}^{n} f\left(x_i^*\right) \cdot \Delta x \cdot \frac{1}{b-a} = \frac{1}{b-a} \sum_{i=1}^{n} f\left(x_i^*\right) \cdot \Delta x$$

Thus, our estimate for the average value of f on the interval $[a,b]$ is

$$\frac{1}{b-a} \sum_{i=1}^{n} f\left(x_i^*\right) \cdot \Delta x$$

Recall, however, if we take the limit as n goes to infinity of this Riemann sum (i.e., the number of subintervals increases without bound), we will arrive at an *exact* answer. Furthermore, we know such a limit of a Riemann sum is equivalent to the definite integral.

In other words, to obtain the exact average value of f on the interval $[a,b]$, we must calculate

$$\frac{1}{b-a}\left(\lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right) \cdot \Delta x\right) = \frac{1}{b-a} \int_a^b f(x)\, dx$$

Thus, we arrive at the following theorem:

Theorem 4.3 Average Value of a Function

The **average value** of a continuous function f on the interval $[a,b]$ is

$$\frac{1}{b-a} \int_a^b f(x)\, dx$$

If $f(x)$ is nonnegative on the interval, then we have a nice geometric interpretation of the formula for the average value of a function: Imagine that the area of the region shaded in **Figure 4.5.1** is liquid in a container in which the graph of f represents the "lid" of the container and the lines $x = a$ and $x = b$ are the sides of the container.

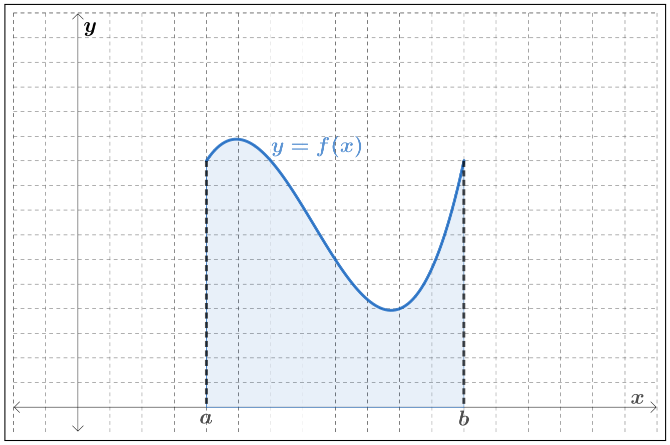


Figure 4.5.1: Region between the graph of $y = f(x)$ and the x-axis on the interval $[a,b]$ whose area is shaded

If we remove the "lid" (the graph of f), the liquid would settle into the shape of a rectangle as shown in **Figure 4.5.2**. Because the amount of liquid is the same, the areas of the regions shown in **Figures 4.5.1** and **4.5.2** are the same!

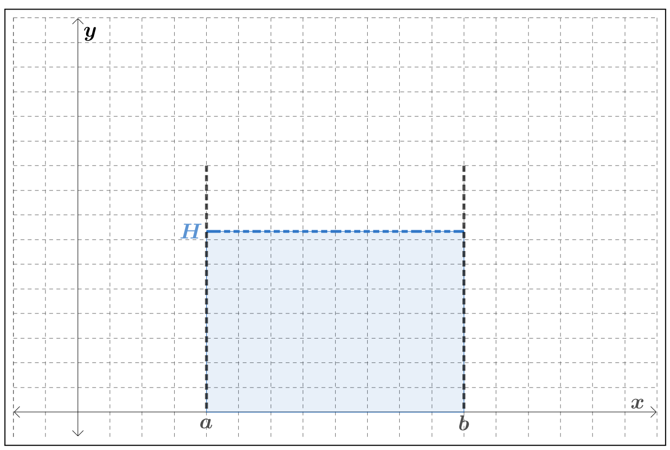


Figure 4.5.2: Rectangle with height H and width $b-a$ whose area is shaded

If the height of this rectangle is H, then the area of the rectangle is $H \cdot (b-a)$. Because $\int_a^b f(x)\,dx$ represents the area under the graph of f from $x = a$ to $x = b$ and this area is the same as the area of the rectangle whose area is $H \cdot (b-a)$, we have

$$H \cdot (b-a) = \int_a^b f(x)\,dx \implies$$
$$H = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Thus, the average value of a nonnegative function f on the interval $[a,b]$ is equal to the height of the rectangle whose area is the same as the area under the graph of f on the interval! The graph of f and its average value are shown in **Figure 4.5.3**:

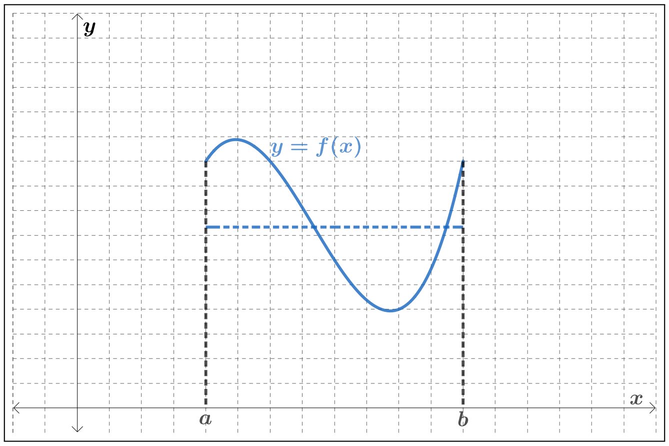


Figure 4.5.3: Graph of $y = f(x)$ and its average value on the interval $[a,b]$

■ **Example 1** Find the average value of $f(x) = x+1$ on the interval $[0,5]$.

Solution:
The average value of $f(x) = x+1$ on the interval $[0,5]$ is given by

$$\frac{1}{5-0}\int_0^5 (x+1)\,dx$$

Looking at the graph of f shown in **Figure 4.5.4**, we see that we could compute the average value by finding the area of the corresponding trapezoid (or areas of a rectangle and triangle added together) and then multiplying the result by $\frac{1}{5-0} = \frac{1}{5}$.

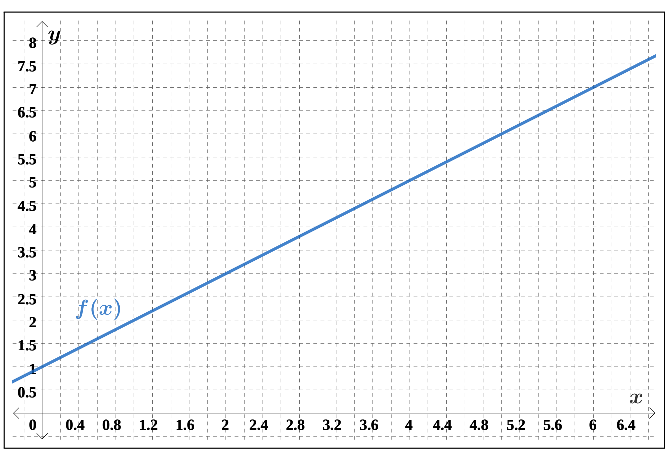


Figure 4.5.4: Graph of the function $f(x) = x+1$

However, we will get more practice using Part 2 of the Fundamental Theorem of Calculus and find an antiderivative to calculate the average value:

$$\begin{aligned}\frac{1}{5}\int_0^5 (x+1)\,dx &= \frac{1}{5}\left(\left(\frac{1}{2}x^2+x\right)\Bigg|_0^5\right)\\ &= \frac{1}{5}\left(\left(\frac{1}{2}(5)^2+5\right)-\left(\frac{1}{2}(0)^2+0\right)\right)\\ &= \frac{1}{5}\left(\frac{35}{2}\right)\\ &= \frac{7}{2}\end{aligned}$$

Thus, the average value of $f(x) = x+1$ on the interval $[0,5]$ is $\frac{7}{2}$.

The graph of f and its average value on the interval are shown in **Figure 4.5.5**:

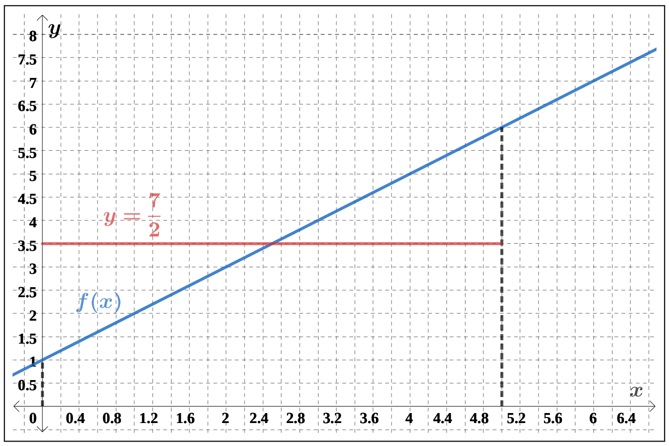


Figure 4.5.5: Graph of the function $f(x) = x+1$ and its average value, $y = \frac{7}{2}$, on the interval $[0,5]$

■

■ **Example 2** Find the average value of $f(x) = -x^2$ on the interval $[0,3]$.

Solution:
The average value of $f(x) = -x^2$ on the interval $[0,3]$ is given by

$$\frac{1}{3-0}\int_0^3 -x^2\,dx$$

We will start by calculating $\int_0^3 -x^2\,dx$, and then we will multiply the result by $\frac{1}{3}$ (i.e., divide by the length of the interval, 3). Calculating the definite integral gives

$$\begin{aligned}\int_0^3 -x^2\,dx &= -\int_0^3 x^2\,dx\\ &= -\left(\frac{1}{3}x^3\Bigg|_0^3\right)\\ &= -\left(\frac{1}{3}(3)^2-\frac{1}{3}(0)^3\right)\\ &= -9\end{aligned}$$

Now, to find the average value, we divide by the length of the interval: $\dfrac{-9}{3} = -3$. Note that multiplying -9 by $\frac{1}{3}$ instead will result in the same answer.

Therefore, the average value of $f(x) = -x^2$ on the interval $[0,3]$ is -3.

The graph of f and its average value on the interval are shown in **Figure 4.5.6**:

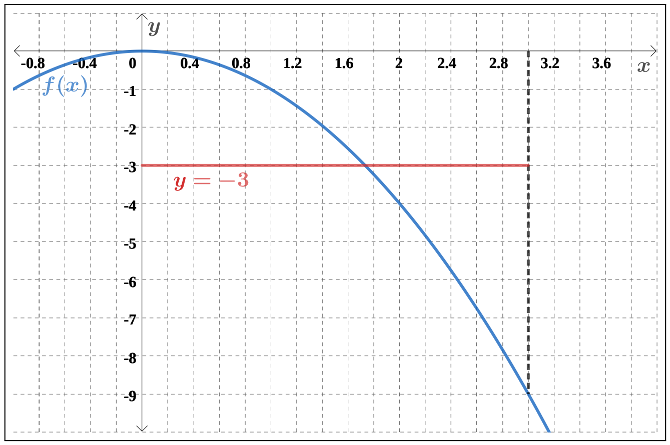


Figure 4.5.6: Graph of the function $f(x) = -x^2$ and its average value, $y = -3$, on the interval $[0,3]$

■

Try It # 1:
Find the average value of $f(x) = 8 - 2x$ on the interval $[0,4]$.

■ **Example 3** Find the average value of $f(x) = e^{-0.2x} - 2$ on the interval $[-2,2]$.

Solution:
The average value of $f(x) = e^{-0.2x} - 2$ on the interval $[-2,2]$ is given by

$$\frac{1}{2-(-2)} \int_{-2}^{2} \left(e^{-0.2x} - 2\right) dx = \frac{1}{4} \int_{-2}^{2} \left(e^{-0.2x} - 2\right) dx$$

Again, we will start by calculating $\int_{-2}^{2} \left(e^{-0.2x} - 2\right) dx$, and then we will multiply the result by $\frac{1}{4}$ (i.e., divide by the length of the interval, 4).

Notice that we must first use the Properties of the Definite Integral to rewrite the definite integral:

$$\int_{-2}^{2} \left(e^{-0.2x} - 2\right) dx = \int_{-2}^{2} e^{-0.2x}\, dx - \int_{-2}^{2} 2\, dx$$

We can use the Introductory Antiderivative Rules to find $\int_{-2}^{2} 2\, dx$, but to find $\int_{-2}^{2} e^{-0.2x}\, dx$, we need to use the method of substitution.

We will start by calculating $\int_{-2}^{2} e^{-0.2x}\,dx$ and let $u = -0.2x$ for the substitution. Thus, $du = -0.2\,dx$, or more importantly, $\frac{1}{-0.2}\,du = dx$.

The limits of integration for this definite integral are in terms of x. As stated previously, we will change the limits of integration so they are in terms of u. To get the equivalent u-values for this definite integral, we substitute the lower limit of integration, $x = -2$, and the upper limit of integration, $x = 2$, into the substitution we defined for u ($u = -0.2x$):

$$\begin{aligned} u &= -0.2(-2) = 0.4 \\ u &= -0.2(2) = -0.4 \end{aligned}$$

Now, we can rewrite the original definite integral entirely in terms of u and du:

$$\begin{aligned} \int_{-2}^{2} e^{-0.2x}\,dx &= \int_{0.4}^{-0.4} e^{u} \cdot \frac{1}{-0.2}\,du \\ &= \int_{0.4}^{-0.4} e^{u} \cdot (-5)\,du \\ &= \int_{0.4}^{-0.4} -5e^{u}\,du \end{aligned}$$

Finally, we evaluate this *new* definite integral in terms of u:

$$\begin{aligned} \int_{0.4}^{-0.4} -5e^{u}\,du &= -5e^{u}\Big|_{0.4}^{-0.4} \\ &= -5e^{-0.4} - \left(-5e^{0.4}\right) \\ &= -5e^{-0.4} + 5e^{0.4} \end{aligned}$$

Next, we will calculate the remaining definite integral, $\int_{-2}^{2} 2\,dx$, using the Introductory Antiderivative Rules:

$$\begin{aligned} \int_{-2}^{2} 2\,dx &= 2x\Big|_{-2}^{2} \\ &= 2(2) - 2(-2) \\ &= 8 \end{aligned}$$

Subtracting the values of the two definite integrals gives

$$\begin{aligned} \int_{-2}^{2} \left(e^{-0.2x} - 2\right)\,dx &= \int_{-2}^{2} e^{-0.2x}\,dx - \int_{-2}^{2} 2\,dx \\ &= -5e^{-0.4} + 5e^{0.4} - 8 \end{aligned}$$

Remember, our overall goal is to calculate the average value of the function $f(x) = e^{-0.2x} - 2$ on the interval $[-2, 2]$. Thus, we must multiply this result by $\frac{1}{4}$ (i.e., divide by the length of the interval, 4).

Hence, the average value of the function on the interval is

$$\frac{1}{4}\left(-5e^{-0.4} + 5e^{0.4} - 8\right)$$

The graph of f and its average value (approximately -0.97) on the interval are shown in **Figure 4.5.7**:

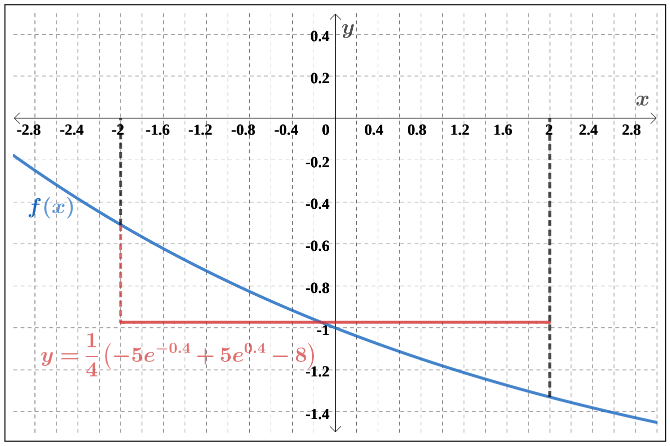


Figure 4.5.7: Graph of the function $f(x) = e^{-0.2x} - 2$ and its average value, $y \approx -0.97$, on the interval $[-2, 2]$

■

N *Recall that we can verify these answers using the calculator! See* ***Section 4.4*** *for more details.*

Try It # 2:
Find the average value of $g(x) = (x+1)\sqrt[3]{x^2+2x}$ on the interval $[-3, 4]$.

We can also compute the average value of more functions than just the one we are given. We can find the average value of a function's rate of change function (i.e., the average rate of change), the average value of a function's antiderivative, and more!

■ **Example 4** Given $f(x) = x^3 - 12x^2 + 4x$, find

a. the average value of f on the interval $[2, 8]$.

b. the average rate of change of f on the interval $[-5, -2]$.

c. the average value of the antiderivative of f whose graph passes through the point (0,8) on the interval $[3, 7]$.

Solution:

a. The average value of f on the interval $[2, 8]$ is given by

$$\frac{1}{8-2}\int_2^8 f(x)\,dx = \frac{1}{6}\int_2^8 \left(x^3 - 12x^2 + 4x\right)\,dx$$

We will start by calculating the definite integral:

$$\begin{aligned}
\int_2^8 \left(x^3 - 12x^2 + 4x\right)\,dx &= \frac{1}{4}x^4 - 12\cdot\frac{1}{3}x^3 + 4\cdot\frac{1}{2}x^2\Big|_2^8 \\
&= \left(\frac{1}{4}x^4 - 4x^3 + 2x^2\right)\Big|_2^8 \\
&= \left(\frac{1}{4}(8)^4 - 4(8)^3 + 2(8)^2\right) - \left(\frac{1}{4}(2)^4 - 4(2)^3 + 2(2)^2\right) \\
&= -896 - (-20) \\
&= -876
\end{aligned}$$

Multiplying -876 by $\frac{1}{6}$ (i.e., dividing by the length of the interval, 6) gives an average value of -146.

b. Recall that we must find the average rate of change of $f(x) = x^3 - 12x^2 + 4x$ on the interval $[-5, -2]$.

In **Section 2.1**, we learned that the average rate of change of f on the interval $[-5, -2]$ is given by

$$\begin{aligned} \frac{f(-2) - f(-5)}{-2 - (-5)} &= \frac{-64 - (-445)}{3} \\ &= \frac{381}{3} \\ &= 127 \end{aligned}$$

However, now we can also calculate the average rate of change of f by finding the average value of the rate of change function of f. In other words, we can apply the average value formula to the derivative of f (i.e., f').

Therefore, using this approach, we will calculate

$$\frac{1}{-2 - (-5)} \int_{-5}^{-2} f'(x)\, dx = \frac{1}{3} \int_{-5}^{-2} f'(x)\, dx$$

Before we can apply this average value formula, we must first find $f'(x)$:

$$f'(x) = 3x^2 - 24x + 4$$

Now, we will calculate the definite integral $\displaystyle\int_{-5}^{-2} \left(3x^2 - 24x + 4\right) dx$:

$$\begin{aligned} \int_{-5}^{-2} \left(3x^2 - 24x + 4\right) dx &= \left(3 \cdot \frac{1}{3}x^3 - 24 \cdot \frac{1}{2}x^2 + 4x\right)\Big|_{-5}^{-2} \\ &= \left(x^3 - 12x^2 + 4x\right)\Big|_{-5}^{-2} \\ &= \left((-2)^3 - 12(-2)^2 + 4(-2)\right) - \left((-5)^3 - 12(-5)^2 + 4(-5)\right) \\ &= -64 - (-445) \\ &= 381 \end{aligned}$$

Finally, we must multiply 381 by $\frac{1}{3}$ (i.e., divide by the length of the interval, 3) to get the average value. Because $\frac{381}{3} = 127$, we see that we arrive at the same answer for the average rate of change using this approach!

If the problem we are given is phrased this way, the first approach is the more efficient method to use to find the average rate of change. However, if we are only given the derivative function, then using the average value formula to find the average rate of change would be our only option.

c. Recall that we must find the average value of the antiderivative of $f(x) = x^3 - 12x^2 + 4x$ whose graph passes through the point (0,8) on the interval $[3, 7]$.

Thus, we need to apply the average value formula to the specific antiderivative of f whose graph passes through the point $(0, 8)$. Let's call this specific antiderivative F. Therefore, we must calculate

$$\frac{1}{7 - 3} \int_{3}^{7} F(x)\, dx = \frac{1}{4} \int_{3}^{7} F(x)\, dx$$

Before we can apply this average value formula, we need to find the specific antiderivative, F, first:

$$\begin{aligned}
F(x) &= \int f(x)\,dx \\
&= \int \left(x^3 - 12x^2 + 4x\right)\,dx \\
&= \frac{1}{4}x^4 - 12\cdot\frac{1}{3}x^3 + 4\cdot\frac{1}{2}x^2 + C \\
&= \frac{1}{4}x^4 - 4x^3 + 2x^2 + C
\end{aligned}$$

We will use the point $(0,8)$ to solve for the constant of integration, C:

$$\begin{aligned}
F(x) = \frac{1}{4}x^4 - 4x^3 + 2x^2 + C &\implies \\
F(0) = \frac{1}{4}(0)^4 - 4(0)^3 + 2(0)^2 + C = 8 &\implies \\
0 + C &= 8 \\
C &= 8
\end{aligned}$$

Therefore, $F(x) = \frac{1}{4}x^4 - 4x^3 + 2x^2 + 8$. Remember, our overall goal is to find the average value of this antiderivative on in the interval $[3,7]$. In other words, we must calculate

$$\frac{1}{4}\int_3^7 F(x)\,dx = \frac{1}{4}\int_3^7 \left(\frac{1}{4}x^4 - 4x^3 + 2x^2 + 8\right)\,dx$$

Again, we will start by calculating the definite integral:

$$\begin{aligned}
\int_3^7 \left(\frac{1}{4}x^4 - 4x^3 + 2x^2 + 8\right)\,dx &= \left(\frac{1}{4}\cdot\frac{1}{5}x^5 - 4\cdot\frac{1}{4}x^4 + 2\cdot\frac{1}{3}x^3 + 8x\right)\Big|_3^7 \\
&= \left(\frac{1}{20}x^5 - x^4 + \frac{2}{3}x^3 + 8x\right)\Big|_3^7 \\
&= \left(\frac{1}{20}(7)^5 - (7)^4 + \frac{2}{3}(7)^3 + 8(7)\right) - \left(\frac{1}{20}(3)^5 - (3)^4 + \frac{2}{3}(3)^3 + 8(3)\right) \\
&= \frac{-76,559}{60} - \left(-\frac{537}{20}\right) \\
&= -\frac{18,737}{15}
\end{aligned}$$

Finally, to get the average value, we multiply this number by $\frac{1}{4}$ (i.e., divide by the length of the interval, 4):

$$\begin{aligned}
\frac{1}{4}\left(-\frac{18,737}{15}\right) &= -\frac{18,737}{60} \\
&\approx -312.283
\end{aligned}$$

■

! *We must always be sure that the function we want to average is indeed the function in the integrand of the average value formula.*

Try It # 3:
Find the average rate of change of f on the interval $[1,3]$, where $f'(x) = \frac{1}{x} - 5x^2$.

Try It # 4:
Find the average value of f on the interval $[-5,0]$ if $f'(x) = 4e^x - 3x^2$ and the graph of f passes through the point $(0,-2)$.

Applications

The average of a finite group of numbers is a great way to condense data to a single statistic. However, finding the average value of a function over longer periods (or intervals), allows us to see the bigger picture of the function's behavior.

▪ **Example 5** The marginal profit function for RuneWorld Story, a multiplayer online game, is given by $P'(x) = 0.0001x^2 - 0.3336x + 200$ dollars per subscription when x gaming subscriptions are sold.

a. Find the average marginal profit when the number of subscriptions sold is between 2000 and 2500, and interpret the result.

b. If no subscriptions being sold results in a loss in profit of \$62,400, find the average profit when the number of subscriptions sold is between 2000 and 2500, and interpret the result.

Solution:

a. Notice first that we need to find the average value of the marginal profit function on the interval $[2000, 2500]$, and we are given the marginal profit function in the problem: $P'(x) = 0.0001x^2 - 0.3336x + 200$ dollars per subscription.

Thus, we must calculate

$$\frac{1}{2500-2000}\int_{2000}^{2500} P'(x)\,dx = \frac{1}{500}\int_{2000}^{2500}\left(0.0001x^2 - 0.3336x + 200\right)\,dx$$

To find this average value, we will first calculate the definite integral:

$$\begin{aligned}
\int_{2000}^{2500}\left(0.0001x^2 - 0.3336x + 200\right)\,dx &= \left(0.0001\cdot\frac{1}{3}x^3 - 0.3336\cdot\frac{1}{2}x^2 + 200x\right)\Bigg|_{2000}^{2500} \\
&= \left(\frac{0.0001}{3}(2500)^3 - \frac{0.3336}{2}(2500)^2 + 200(2500)\right) - \left(\frac{0.0001}{3}(2000)^3 - \frac{0.3336}{2}(2000)^2 + 200(2000)\right) \\
&= -\frac{65{,}000}{3} - \left(-\frac{1600}{3}\right) \\
&= -\frac{63{,}400}{3} \\
&\approx -\$21{,}133.33
\end{aligned}$$

Dividing this number by 500 yields the average value of $-\$42.27$ per subscription.

This means that when the number of subscriptions sold is between 2000 and 2500, profit is decreasing at an average rate of \$42.27 per subscription.

Our answer represents the average rate of change of profit on the interval $[2000, 2500]$ *because it is equivalent to the average value of the rate of change function.*

b. Recall that we must find the average profit when the number of subscriptions sold is between 2000 and 2500. We are also told that no subscriptions being sold results in a loss in profit of $62,400.

Before we can find the average value of the profit function, we must first find the profit function, P. In other words, we must find the specific antiderivative of P' whose graph passes through the point $(0, -62,400)$.

We start by finding the general antiderivative, which we can gather from our work in part **a**:

$$\begin{aligned} P(x) &= \int P'(x)\,dx \\ &= \int \left(0.0001x^2 - 0.3336x + 200\right)\,dx \\ &= \frac{0.0001}{3}x^3 - \frac{0.3336}{2}x^2 + 200x + C \end{aligned}$$

Now, we will use the point $(0, -62,400)$ to find the constant of integration, C:

$$\begin{aligned} P(x) = \frac{0.0001}{3}x^3 - \frac{0.3336}{2}x^2 + 200x + C \implies \\ P(0) = \frac{0.0001}{3}(0)^3 - \frac{0.3336}{2}(0)^2 + 200(0) + C &= -62,400 \\ 0 + C &= -62,400 \\ C &= -62,400 \end{aligned}$$

Therefore, the profit function is given by $P(x) = \frac{0.0001}{3}x^3 - \frac{0.3336}{2}x^2 + 200x - 62,400$ dollars when x subscriptions are sold.

Now, we will find the average value of this profit function on the interval [2000, 2500]. In other words, we will calculate

$$\frac{1}{2500-2000}\int_{2000}^{2500} P(x)\,dx = \frac{1}{500}\int_{2000}^{2500}\left(\frac{0.0001}{3}x^3 - \frac{0.3336}{2}x^2 + 200x - 62,400\right)\,dx$$

We start by finding the definite integral:

$$\begin{aligned} \int_{2000}^{2500}\left(\frac{0.0001}{3}x^3 - \frac{0.3336}{2}x^2 + 200x - 62,400\right)\,dx &= \left(\frac{0.0001}{3}\cdot\frac{1}{4}x^4 - \frac{0.3336}{2}\cdot\frac{1}{3}x^3 + 200\cdot\frac{1}{2}x^2 - 62,400x\right)\Bigg|_{2000}^{2500} \\ &= \left(\frac{0.0001}{12}x^4 - \frac{0.3336}{6}x^3 + 100x^2 - 62,400x\right)\Bigg|_{2000}^{2500} \\ &= \left(\frac{0.0001}{12}(2500)^4 - \frac{0.3336}{6}(2500)^3 + 100(2500)^2 - 62,400(2500)\right) \\ &\quad - \left(\frac{0.0001}{12}(2000)^4 - \frac{0.3336}{6}(2000)^3 + 100(2000)^2 - 62,400(2000)\right) \\ &= -\$37,962,500 \end{aligned}$$

Dividing this number by 500 yields an average value of $-\$75,925$.

This means that when the number of subscriptions sold is between 2000 and 2500, RuneWorld Story has an average profit of $-\$75,925$.

■

Try It # 5:
The marginal cost function for Reid's Cookie Shop is given by $C'(x) = 0.1x + 5$ dollars per cookie when x cookies are made. Find and interpret

a. the average marginal cost when the number of cookies made is between 10 and 20.

b. the average cost when the number of cookies made is between 15 and 40 if we know the cost of making 50 cookies is \$1350.

Try It # 6:
During a nine-hour workday at an automobile factory, the production rate t hours from the start of the shift was $r(t) = 5 + \sqrt{t}$ cars per hour. Find the average hourly production rate during the workday.

Enrichment: Moving Average Function

Function averages, as well as other techniques, are used to "smooth" data so that underlying patterns become more obvious. A moving average function reduces very rapid changes in a function and removes any high frequency "noise". To create a moving average function, the value of the original function at each x-value is replaced by the average value of the function on an interval that includes the x-value.

For example, the graph of f shown in **Figure 4.5.8** changes quite a bit over short intervals. We can replace f with a moving average function, g, whose function values are the average values of the function f on intervals of a specified length. In this case, we will let the length of each interval on which each average value of f is calculated be four. Note that this means for each x-value, the average value of f is calculated on the interval $[x-2, x+2]$ so that the x-value is included in the interval and the length of the interval is four. The resulting average values are the function values of the moving average function, g. In other words,

$$g(x) = \frac{1}{4}\int_{x-2}^{x+2} f(t)\,dt$$

The graph of g is shown in **Figure 4.5.9** for comparison:

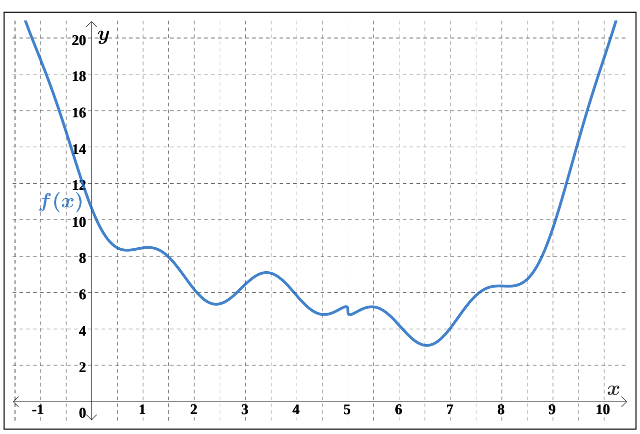


Figure 4.5.8: Graph of a function f whose values change rapidly over short intervals

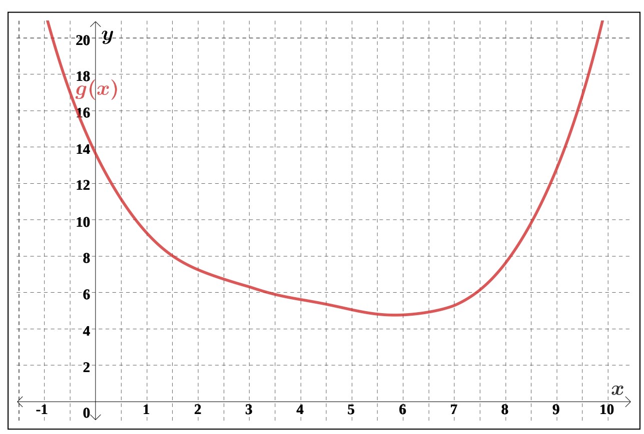


Figure 4.5.9: Graph of the moving average function, g, that corresponds to the function f

Notice that the moving average function, g, removes the "chaos" of the smaller intervals of f and exposes its general trend.

This technique can be used to find the general trend in temperatures over a long period of time (allowing climate scientists to make predictions), in signal processing (allowing clearer radio broadcasts), and in the finance sector (showing general trends and predictions in the stock market).

Try It Answers

1. 4
2. $\frac{1}{7}\left(\frac{3}{8}(24)^{4/3} - \frac{3}{8}(3)^{4/3}\right)$
3. $\frac{1}{2}\left(\ln(3) - \frac{130}{3}\right)$
4. $\frac{1}{5}\left(-4e^{-5} + \frac{625}{4} - 26\right)$
5. **a.** \$6.50 per cookie; When the number of cookies made is between 10 and 20, cost is increasing at an average rate of \$6.50 per cookie.

   b. \$1152.92; When the number of cookies made is between 15 and 40, Reid's Cookie Shop has an average cost of \$1152.92.
6. 7 cars per hour

EXERCISES

Basic Skills Practice

For Exercises 1 - 6, find the average value of the function f on the given interval.

1. $f(x) = 3x^2 - x + 1$ on the interval $[-6, -2]$

2. $f(x) = \frac{5}{4}x^4 + \frac{x^{-2/3}}{4}$ on the interval $[1, 8]$

3. $f(x) = -6x^2 - e^x$ on the interval $[0, 1]$

4. $f(x) = -4x^3 + 3\sqrt{x} + 10$ on the interval $[4, 9]$

5. $f(x) = 0.25x^4 - \frac{10}{9}x$ on the interval $[-3, 3]$

6. $f(x) = \frac{1}{x} + 2$ on the interval $[1, e]$

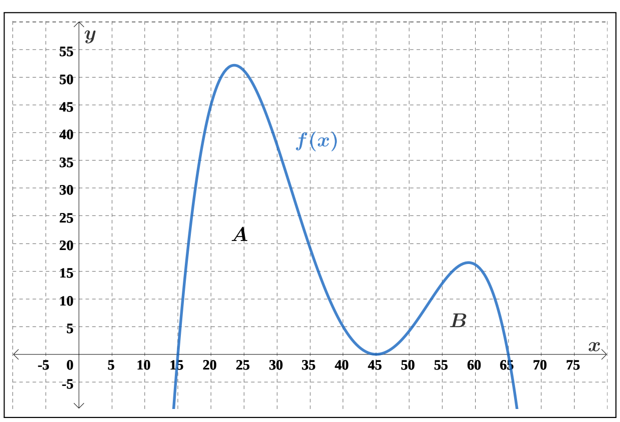
For Exercises 7 - 9, find the average value of the function f on the given interval.

7. $f(x) = \sqrt[3]{x - 4}$ on the interval $[-4, 3]$

8. $f(x) = -2e^{-2x}$ on the interval $[0, 1]$

9. $f(x) = (5x - 1)^4$ on the interval $[-2, 3]$

For Exercises 10 - 12, use the indicated areas of the regions between the graph of f and the x-axis to find the average value of f on the given intervals.

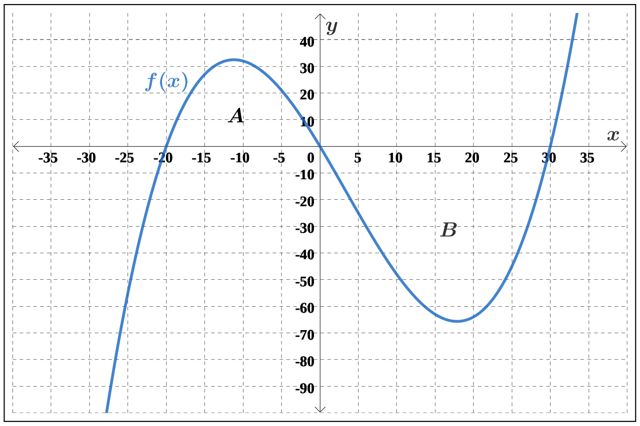
10. 

Area of A: 821
Area of B: 179

(a) $[15, 45]$

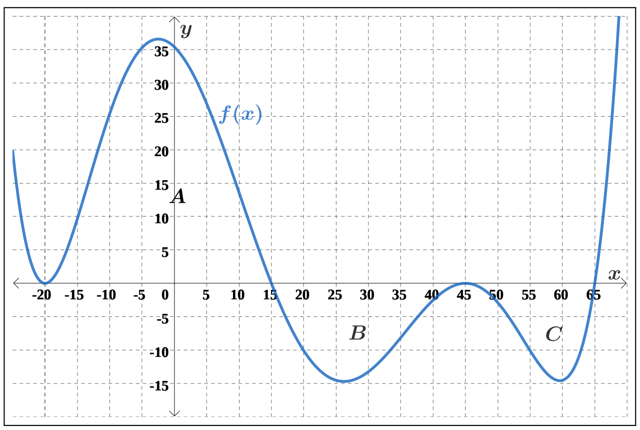
(b) $[45, 65]$

(c) $[15, 65]$

11. 

Area of A: 427
Area of B: 1260

 (a) $[-20, 0]$
 (b) $[0, 30]$
 (c) $[-20, 30]$

12. 

Area of A: 736
Area of B: 248
Area of C: 151

 (a) $[-20, 15]$
 (b) $[-20, 65]$
 (c) $[15, 45]$

13. The price-demand function for kids' Bright Lights sneakers is given by $p(x) = 207 - 0.2x$, where x is the number of pairs of sneakers sold at a price of \$$p$ each. Find the average price of a pair of sneakers when the number of pairs sold is between 300 and 400.

14. A small fast-food restaurant specializing in selling chicken sandwiches has a weekly revenue function given by $R(x) = 12.5x - 0.1x^2$ dollars, where x is the number of chicken sandwiches sold. Find the average revenue when the number of sandwiches sold each week is between 75 and 120.

15. The weekly marginal cost function for a company is given by $C'(x) = 90 - 0.1x$ dollars per item when x items are produced. Find the company's average marginal cost when the number of items produced each week is between 45 and 65.

16. The marginal profit function for a designer coat label is given by $P'(x) = -0.6x + 475$ dollars per coat when x coats are sold. Find the average marginal profit when the number of coats sold is between 650 and 785.

17. An object has a velocity given by $v(t) = t^2 - t + 8$ meters per second, where t is time in seconds. Find the object's average velocity during the first three seconds.

Intermediate Skills Practice

For Exercises 18 - 25, find the average value of the function on the given interval.

18. $f(x) = \dfrac{x^3 - 3x^5 + 4x^2}{x^2}$ on the interval $[-8, -2]$

19. $g(t) = \dfrac{3\sqrt{t}}{4} + 5e^t$ on the interval $[0, 4]$

20. $m(x) = -\dfrac{16}{9}x\left(1 - 2x^2\right)^3$ on the interval $[-2, 0]$

21. $h(x) = \left(6 - 5x^2\right)\left(2x^3 - 4x\right)$ on the interval $[-4, -1]$

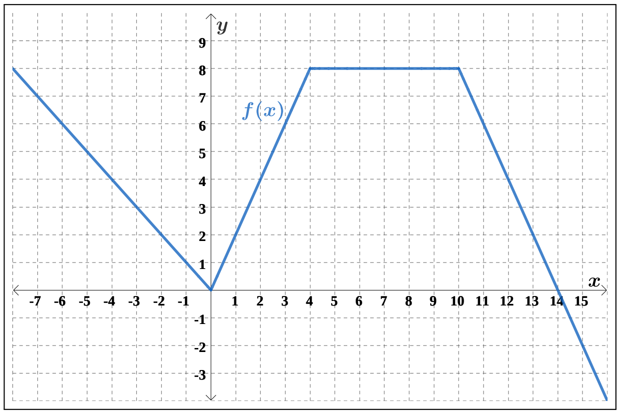
22. $r(t) = 5t^2 e^{-6t^3+1}$ on the interval $[-6, -1]$

23. $n(x) = \dfrac{7}{x} + \dfrac{9}{x^2}$ on the interval $[1, 3]$

24. $g(x) = \dfrac{7x}{\sqrt{x^2 - 3}}$ on the interval $[5, 8]$

25. $f(t) = \dfrac{4t^3 - 3t}{2t^4 - 3t^2 + 1}$ on the interval $[-6, -4]$

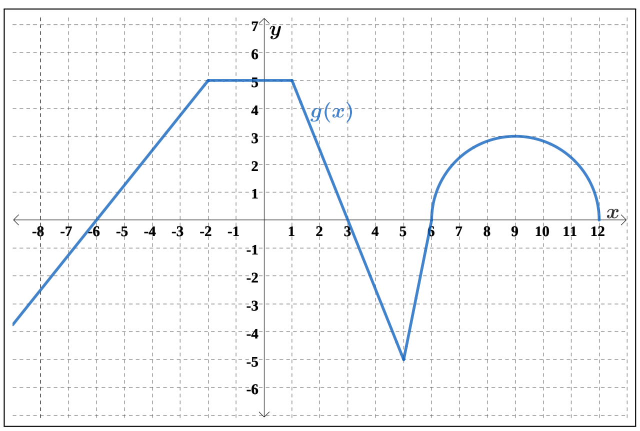
For Exercises 26 - 28, the graph of a function is shown. Find the average value of the function on the given intervals.

26. 

(a) $[-5,0]$

(b) $[4,14]$

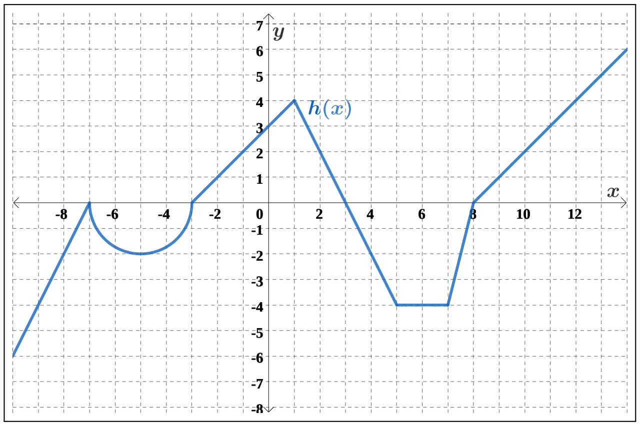
(c) $[-2,6]$

27. 

(a) $[-6,5]$

(b) $[6,12]$

(c) $[3,9]$

28. 

(a) $[-9,-3]$

(b) $[0,6]$

(c) $[2,12]$

For Exercises 29 - 31, f' and a function value of f are given. Use the information to find the average value of f on the given interval.

29. $f'(x) = 4x^3 - 3x^2 - 10$; $f(0) = -8$; $[-6,-2]$

30. $f'(x) = 2\sqrt{x} - \dfrac{6}{x^3}$; $f(1) = 5$; $[1,9]$

31. $f'(x) = \dfrac{2xe^{-x} - 4}{e^{-x}}$; $f(-3) = 12$; $[-1,0]$

32. Bootsie's Bandana Company sells designer bandanas for dogs. The price-demand function for the bandanas is given by $p(x) = 28 - 0.01x$, where x is the number of bandanas sold at a price of $\$p$ each. Find Bootsie's Bandana Company's average revenue when the number of bandanas sold is between 200 and 250.

33. A company has a marginal cost function given by $C'(x) = 15 - 12x + 3x^2$ dollars per item, where x is the number of items produced. Find the company's average marginal cost when the number of items produced is between 40 and 60.

34. Rockin' Roller Blades is a company that makes roller blades. The company's weekly marginal profit function is given by $P'(x) = 30x - 0.3x^2 - 250$ dollars per pair of roller blades when x pairs of roller blades are sold. The company has determined it breaks even when 50 pairs of roller blades are sold (i.e., the company's profit is \$0 when 50 pairs are sold). Find Rockin' Roller Blades average profit when the number of pairs of roller blades sold each week is between 60 and 80.

35. The rate of change of a company's total sales, in millions of dollars per month, after t months is given by $\dfrac{dS}{dt} = \sqrt{t}$. If the company's total sales after 9 months is 14 million dollars, find the average total sales during the first year. Round your answer to three decimal places.

36. Jason opens a bank account, and the rate at which the balance of his account grows is given by $A'(t) = 225e^{0.045t}$ dollars per year, where t is time in years. If the balance of his account is \$8,580.03 after 12 years,

    (a) find the average balance of the account during the first 15 years.

    (b) find the average rate at which the balance of the account grew during the first 10 years.

Mastery Practice

For Exercises 37 - 44, find the average value of the function on the given interval.

37. $m(x) = \dfrac{12x - 15}{(2x^2 - 5x + 1)^3}$ on the interval $[3, 3.5]$

38. $g(t) = \dfrac{5}{7\sqrt{t}} + 3e^t$ on the interval $[9, 16]$

39. $h(x) = x^2\left(5x^2 - 3\right)\left(7 - x^4 + 9x\right)$ on the interval $[-1, 1]$

40. $f(t) = \dfrac{4}{t\ln(t)}$ on the interval $[e, e^3]$

41. $g(x) = 7x\sqrt{x - 1}$ on the interval $[2, 10]$

42. $f(x) = \dfrac{x^5 + \sqrt{x^3} - \frac{2}{7}x^6}{4x^6}$ on the interval $[1, 4]$

43. $r(t) = t\left(0.8t^2 - 0.2\right)e^{0.2t^4 - 0.1t^2 + 6}$ on the interval $[-4, -2]$

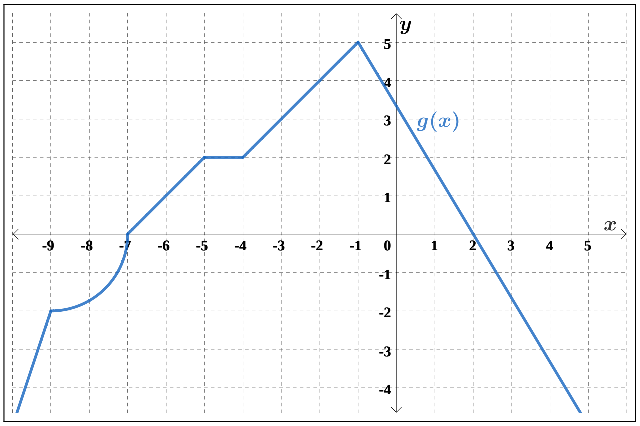
44. $n(x) = \dfrac{18x^4 - 32x^3 + 2}{9x^5 - 20x^4 + 5x}$ on the interval $[-3, -1]$

For Exercises 45 and 46, find the average value of the function on the given interval.

45. $f(x) = 2x^3 - 4x^7$ on the interval $[0, A]$, where $A > 0$

46. $f(x) = 9\sqrt{x} + 5e^x - 16$ on the interval $[4, B]$, where $B > 4$

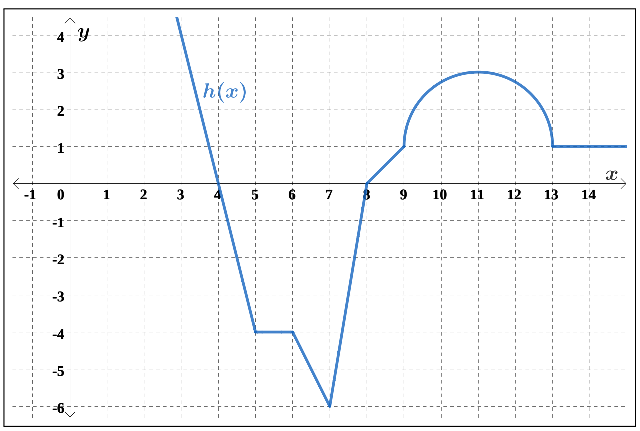
For Exercises 47 and 48, the graph of a function is shown. Find the average value of the function on the given intervals.

47. 

(a) $[-1,2]$

(b) $[-6,-2]$

(c) $[-9,-6]$

48. 

(a) $[4,8]$

(b) $[8,14]$

(c) $[3,11]$

49. Find the average value of y on the interval $[-2,0]$ if $\dfrac{dy}{dt} = \dfrac{t}{(t+5)^3}$ and $y(-6) = -2$.

50. The marginal cost function for Yummy Gummies, a company that makes packets of gummies in bulk, is given by $C'(x) = 3\sqrt{x}$ dollars per box when x boxes are made. If the cost of making 25 boxes of gummies is \$45, find the company's average cost when the number of boxes produced is between 115 and 185.

51. Mosquito Be Gone! sells cans of mosquito repellent. The company has determined its price-demand function is $p(x) = 25 - \sqrt{x}$, where x is the number of cans of repellent sold each week at a price of $\$p$ each. If it costs Mosquito Be Gone! \$2 to make each can of repellent, and the company has weekly fixed costs of \$700, find the company's average profit when it produces and sells between 200 and 250 cans of mosquito repellent each week.

52. A company that makes a popular coffee mug has a monthly cost function given by $C(x) = \sqrt{x} + 2x + 6000$ dollars, where x is the number of coffee mugs produced each month. Find the company's average marginal cost when it produces between 4500 and 5000 coffee mugs each month.

53. The weekly marginal revenue function for Space Jammas, a company that makes galaxy themed pajamas, is given by $r(x) = 50 - 10e^{-0.01x}$ dollars per pair of pajamas when x pairs of pajamas are sold each week.
    (a) Find the company's average revenue when it sells between 200 and 250 pairs of pajamas each week.
    (b) Find the average marginal revenue when the number of pairs of pajamas sold each week is between 300 and 350.

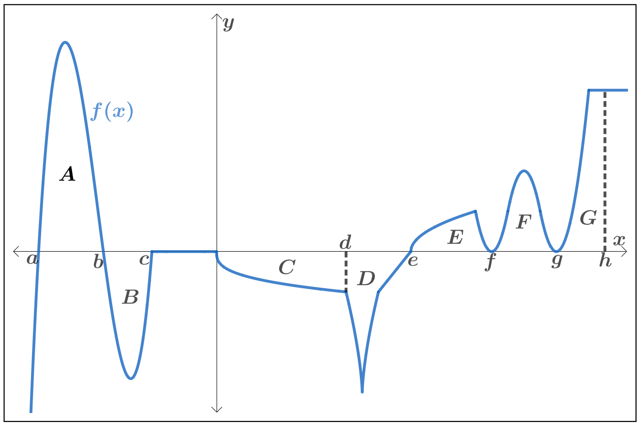
54. The marginal profit function from selling x packages of dry erase markers is given by $\frac{dP}{dx} = \frac{12x}{\sqrt{4x^2 + 29}}$ dollars per package. Find the average rate of change of profit when the number of packages sold is between 50 and 75.

55. Lesley invests \$2500 in an account that earns interest at a rate of 6.5% per year compounded continuously.
    (a) Find the average account balance during the second four year period of the investment.
    (b) Find the average rate at which the balance of the account grew during the fifth and sixth years.

56. The position of an object after t seconds is given by $f(t) = t^2 + 5t + 80$ feet. Find the average velocity of the object between 2 and 5 seconds.

57. After t months, a company's sales are changing at a rate of $\frac{dS}{dt} = 0.08t^3 + 0.09t^2 + 0.3t + 3$ million dollars per month. After 5 months, the company earned 10 million dollars in sales.
    (a) Find the company's average rate of change of sales during the first six months.
    (b) Find the company's average sales during the first 2 years.

58. Given the graph of f and the indicated areas of the regions between the graph of f and the x-axis shown below, find the average value of f on each of the following intervals.



Area of A: 27
Area of B: 12
Area of C: 12
Area of D: 15
Area of E: 5
Area of F: 34
Area of G: 19

(a) $[a, c]$

(b) $[b, d]$

(c) $[0, e]$

(d) $[d, h]$

59. Use the table of function values below to obtain the best possible estimate of the average value of

x	-2	-1	0	1	2	3	4
$f(x)$	-3	-2	-1	0	1	2	3
$g(x)$	0	-3	-4	-3	0	5	12

(a) f on the interval $[0, 4]$ using a left-hand Riemann sum.

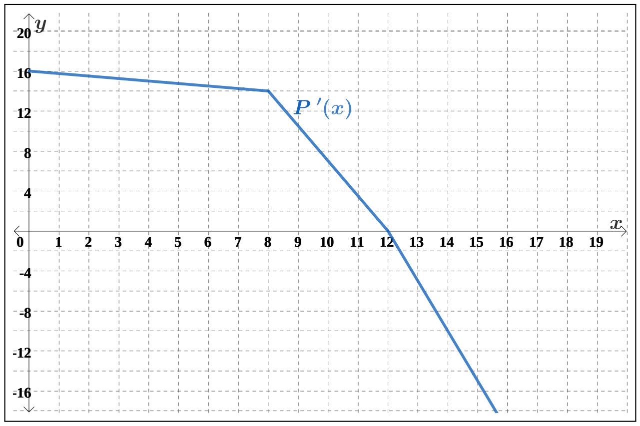
(b) g on the interval $[-2, 1]$ using a right-hand Riemann sum.

*Note: See **Section 4.3** for information about Riemann sums.*

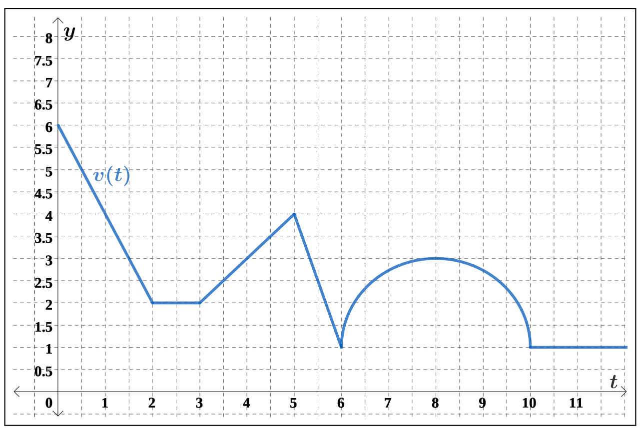
60. Find the value(s) of the constant B, where $B > 0$, such that the average value of $g(x) = -9x^2 + 8x + 6$ on the interval $[0, B]$ is 7.

61. Find the value(s) of the constant c such that the average value of $f(x) = (x-9)^2$ on the interval $[7, 10]$ equals $f(c)$.

62. The graph of a company's marginal profit function, P', is shown below, where x is the number of items sold and $P'(x)$ is measured in dollars per item. Find the company's average marginal profit when the number of items sold is between 10 and 15 items.



63. The velocity of an object after t seconds is $v(t)$ feet per second, and the graph of v is shown below. Find the average velocity of the object during the first eight seconds. Round to three decimal places, if necessary.



Communication Practice

64. April, a student in Professor Kilmer's calculus class, was helping a classmate with the following homework problem:

"Find the average value of $f(x) = x^3 - 2x + 6$ on the interval $[-2, 5]$."

April told her classmate all he needed to do was find the definite integral of f from $x = -2$ to $x = 5$. In other words, calculate $\int_{-2}^{5} f(x)\, dx$. Was April's solution to the problem correct? Explain.

65. If $C'(x)$ gives the marginal cost in dollars per item when x items are produced, interpret

(a) $\dfrac{\int_{50}^{100} C'(x)\,dx}{50} = -7.43.$

(b) $\dfrac{\int_{50}^{100} C(x)\,dx}{50} = 6320.$

66. Explain two different ways to calculate the average rate of change of a function f on the interval $[a,b]$.

4.6 Area Between Curves and Producers' and Consumers' Surplus

Most likely, you have known how to calculate the areas of certain shapes such as rectangles, circles, and triangles since grade school. But, what if the shape is more complex like the shape shown in **Figure 4.6.1**?

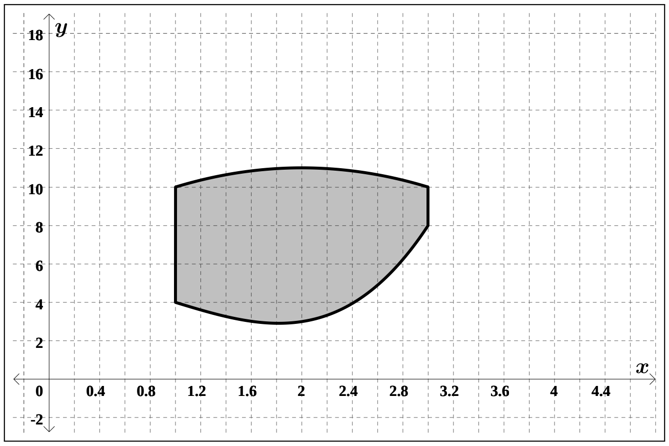


Figure 4.6.1: Shape whose area cannot be described exactly by an explicit formula such as that of a rectangle, circle, or triangle

Although we do not have a formula to calculate the area of the shape shown in **Figure 4.6.1** exactly, we can use our knowledge of definite integrals to develop one!

In particular, the boundaries of the shape are curves that can be described by the functions $f(x) = -x^2 + 4x + 7$ and $g(x) = x^3 - 3x^2 + x + 5$ between $x = 1$ and $x = 3$. This means we are looking for the **area between these curves** as shown in **Figure 4.6.2**, and we can use definite integrals to help us find the exact value of the area!

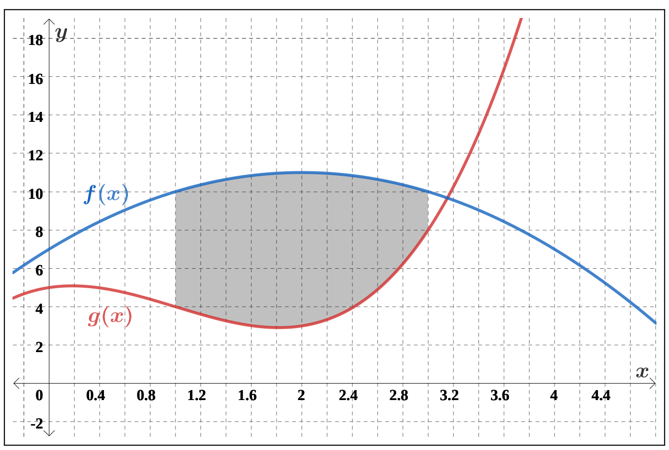


Figure 4.6.2: Shape whose boundaries are curves that can be described by the functions f and g between $x = 1$ and $x = 3$

Learning Objectives:

In this section, you will learn how to find the area between two curves and solve problems involving real-world applications, including consumers' and producers' surplus. Upon completion you will be able to:

- Calculate the area of a region between two curves on a given interval.
- Calculate the area of a region bounded by two curves.

- Calculate the area between two curves that intersect on a given interval.
- Distinguish between consumers' and producers' surplus by explaining the meaning of each.
- Calculate the consumers' surplus for a commodity given a demand equation and the equilibrium quantity and/or price.
- Calculate the producers' surplus for a commodity given a supply equation and the equilibrium quantity and/or price.
- Calculate the consumers' and producers' surplus for a commodity given the supply and demand equations.
- Interpret the area between two rate of change functions on an interval as the accumulated difference between their respective antiderivatives on the interval.
- Calculate the accumulated difference between two functions given their rate of change functions that involves a real-world scenario.

Area Between Curves

Let's return to the region between the curves $y = f(x)$ and $y = g(x)$ on the interval $[1,3]$ described in the introduction and explore how we might be able to derive a formula for the area of the region. The region is shown again in **Figure 4.6.3**:

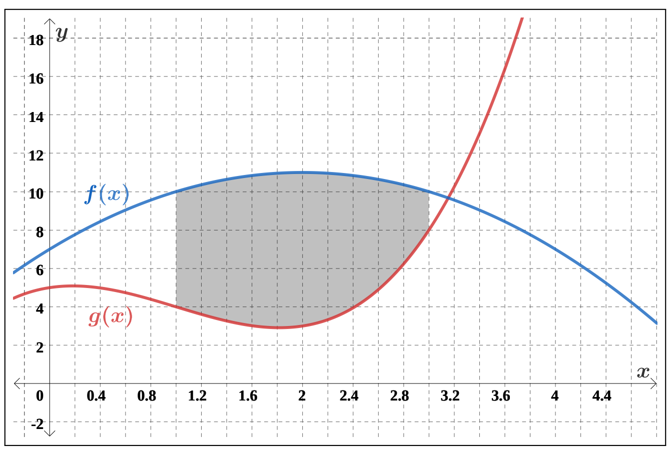


Figure 4.6.3: Shape whose boundaries are curves that can be described by the functions f and g between $x = 1$ and $x = 3$

Because $f(x)$ and $g(x)$ are both positive on the interval $[1,3]$, we can view the area of this shaded region as the area between the curve $y = f(x)$ and the x-axis minus the area between the curve $y = g(x)$ and the x-axis (i.e., the area under the curve $y = f(x)$ minus the area under the curve $y = g(x)$). See **Figures 4.6.4**, **4.6.5**, and **4.6.6**.

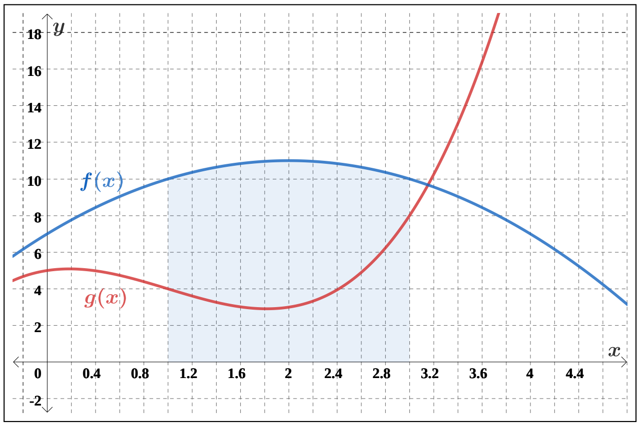


Figure 4.6.4: Area under the curve $y = f(x)$ on the interval $[1,3]$ is shaded

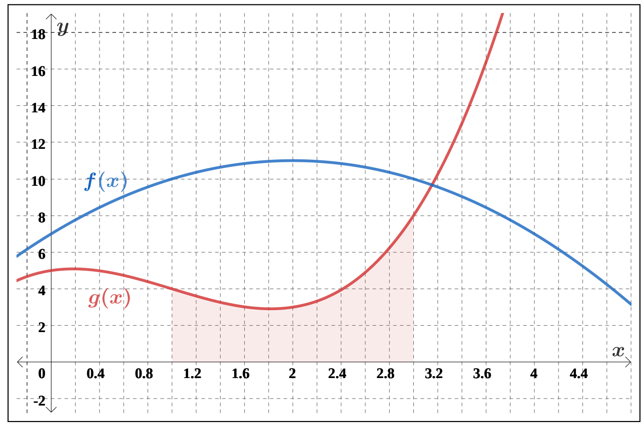


Figure 4.6.5: Area under the curve $y = g(x)$ on the interval $[1,3]$ is shaded

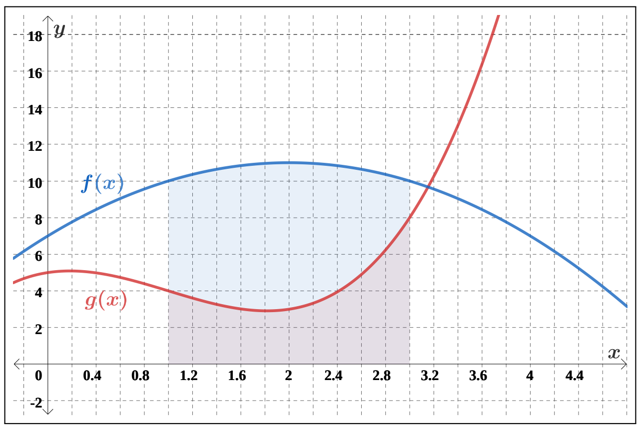


Figure 4.6.6: Areas under both the curves $y = f(x)$ and $y = g(x)$ on the interval $[1,3]$ are shaded

In other words,

$$\begin{aligned}\text{Area between the curves } y = f(x) \text{ and } y = g(x) \text{ on the interval } [1,3] &= \int_1^3 f(x)\,dx - \int_1^3 g(x)\,dx \\ &= \int_1^3 (f(x) - g(x))\,dx\end{aligned}$$

We now formally state this result with the following theorem:

Theorem 4.4 Let f and g be continuous functions and $f(x) \geq g(x)$ on the interval $[a,b]$. The region bounded by the curves $y = f(x)$ and $y = g(x)$ for $a \leq x \leq b$ has area equal to

$$\int_a^b (f(x) - g(x))\,dx$$

N *We may casually refer to this formula as "the definite integral of the top function minus the bottom function from $x = a$ to $x = b$."*

! *In this section, when we say the area between curves, we mean the physical area (not the net area that definite integrals represent). Remember, area itself is always a nonnegative value!*

Example 1 Find the area of the region between the curves $y = f(x) = x+4$ and $y = g(x) = 3 - \frac{x}{2}$ on the interval $[1,4]$.

Solution:
We will begin by graphing these functions to determine which is the "top" function and which is the "bottom" function. The curves of both functions are shown in **Figure 4.6.7**, and the area of the corresponding region is shaded:

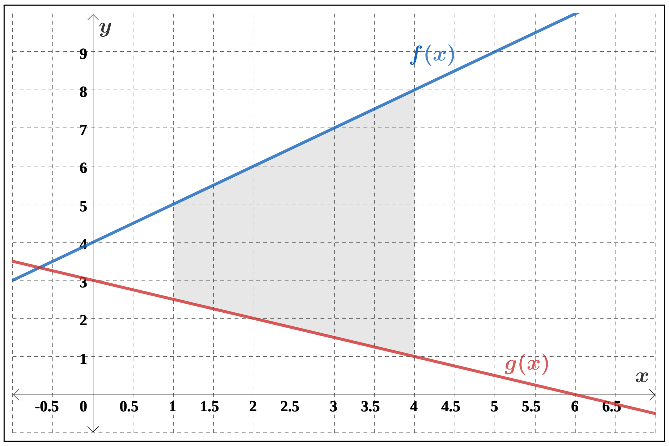


Figure 4.6.7: Area between the curves $y = f(x)$ and $y = g(x)$ on the interval $[1,4]$ is shaded

On the interval $[1,4]$, $f(x) > g(x)$ (i.e., f is the "top" function), so the integrand is $f(x) - g(x)$. Recalling that $y = f(x) = x+4$ and $y = g(x) = 3 - \frac{x}{2}$, we have

$$\begin{aligned}
\int_1^4 (f(x) - g(x))\ dx &= \int_1^4 \left((x+4) - \left(3 - \frac{x}{2}\right)\right) dx \\
&= \int_1^4 \left(x + 4 - 3 + \frac{x}{2}\right) dx \\
&= \int_1^4 \left(\frac{3}{2}x + 1\right) dx \\
&= \left(\frac{3}{2} \cdot \frac{1}{2}x^2 + x\right)\Bigg|_1^4 \\
&= \left(\frac{3}{4}x^2 + x\right)\Bigg|_1^4 \\
&= \left(\frac{3}{4}(4)^2 + 4\right) - \left(\frac{3}{4}(1)^2 + 1\right) \\
&= \frac{57}{4}
\end{aligned}$$

Thus, the area of the region between these curves on the interval $[1,4]$ is $\frac{57}{4}$.

Try It # 1:
Find the area of the region between the curves $y = f(x) = 5$ and $y = g(x) = 2x^2 + 5$ on the interval $[0,3]$.

Sometimes, we may not be given the x-values for the bounds of the region in question, but they will be implied by where the functions intersect. In this situation, before calculating the definite integral, we must find the relevant x-values (limits of integration) either graphically by intersecting the functions on the calculator or algebraically by setting the functions equal to each other and solving for x. We will see this in the next example.

▪ **Example 2** Find the area of the region bounded by the curves $y = f(x) = 6 - x$ and $y = g(x) = 9 - \frac{x^2}{4}$.

Solution:
Notice that we are not given an interval for the region bounded by the curves $y = f(x)$ and $y = g(x)$. Therefore, because we want the area *bounded* by the curves $y = f(x)$ and $y = g(x)$, we must find the region enclosed completely by the graphs of these functions. This region is shown in **Figure 4.6.8**:

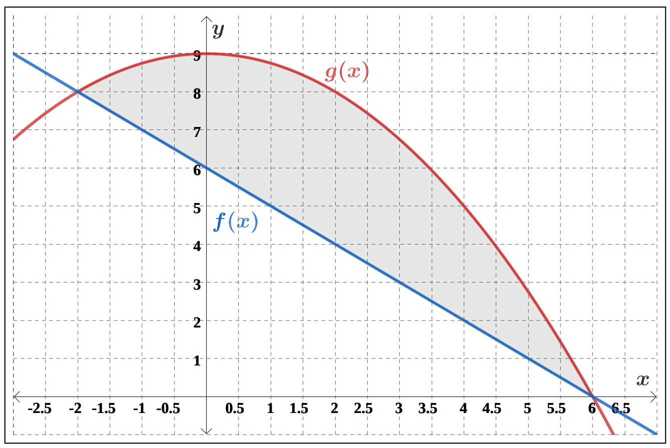


Figure 4.6.8: Region bounded by the curves $y = f(x)$ and $y = g(x)$ is shaded

The first thing to notice is that the enclosed region appears to correspond to the interval $[-2, 6]$. We will verify these x-values algebraically by setting the functions equal to each other and solving for x:

$$\begin{aligned} f(x) = g(x) &\implies \\ 6 - x &= 9 - \frac{x^2}{4} \\ \frac{x^2}{4} - x - 3 &= 0 \\ x^2 - 4x - 12 &= 0 \\ (x+2)(x-6) &= 0 \\ &\implies x = -2 \text{ and } x = 6 \end{aligned}$$

The second thing to notice is that $g(x) > f(x)$ on this interval, so we have

$$\begin{aligned}
\int_{-2}^{6} (g(x) - f(x))\ dx &= \int_{-2}^{6} \left(\left(9 - \frac{x^2}{4} \right) - (6 - x) \right)\ dx \\
&= \int_{-2}^{6} \left(9 - \frac{x^2}{4} - 6 + x \right)\ dx \\
&= \int_{-2}^{6} \left(-\frac{x^2}{4} + x + 3 \right)\ dx \\
&= \left(-\frac{1}{4} \cdot \frac{1}{3} x^3 + \frac{1}{2} x^2 + 3x \right) \Bigg|_{-2}^{6} \\
&= \left(-\frac{1}{12} x^3 + \frac{1}{2} x^2 + 3x \right) \Bigg|_{-2}^{6} \\
&= \left(-\frac{1}{12}(6)^3 + \frac{1}{2}(6)^2 + 3(6) \right) - \left(-\frac{1}{12}(-2)^3 + \frac{1}{2}(-2)^2 + 3(-2) \right) \\
&= 18 - \left(-\frac{10}{3} \right) \\
&= \frac{64}{3}
\end{aligned}$$

Thus, the area of the region bounded by these curves is $\frac{64}{3}$, which is approximately 21.3.

▪

Try It # 2:
Find the area of the region bounded by the curves $y = f(x) = \sqrt{x}$ and $y = g(x) = x^2$.

▪ **Example 3** Find the area between the curves $y = f(t) = t + 3$ and $y = g(t) = t^2 - 4t + 3$ for $-1 \le t \le 2$.

Solution:
We start by looking at the graphs of these functions to determine which is greater than the other on the interval $[-1, 2]$. Both curves are shown in **Figure 4.6.9**, as well as the corresponding regions:

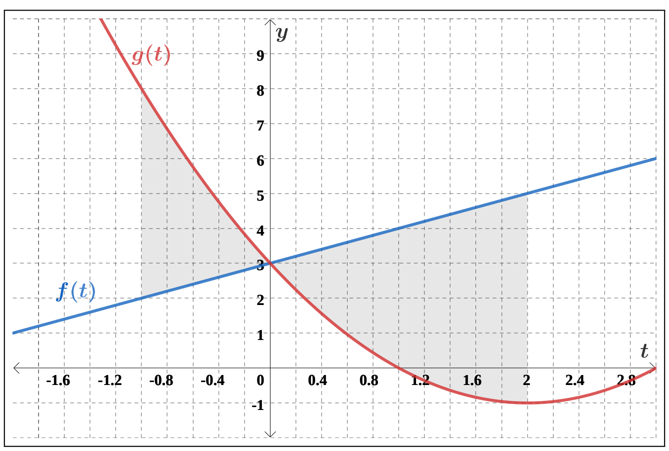


Figure 4.6.9: Area between the curves $y = f(t)$ and $y = g(t)$ on the interval $[-1, 2]$ is shaded

Notice that on the interval $[-1, 2]$, *neither of these curves is always greater than the other!* If we use $f(t) - g(t)$ or $g(t) - f(t)$ as the integrand, some of the area will "count" as negative, which is not what we want for this problem (recall that we are looking for the physical area between two curves in this section).

Thus, we need to find the areas of two separate regions and add them together to obtain the total area. On the interval $[-1,2]$, the two curves intersect at $t=0$ (you could verify this algebraically by setting the functions equal to each other and solving for t), so we will find the area between the curves $y=f(t)$ and $y=g(t)$ on the interval $[-1,0]$ and then add to it the area between the curves $y=f(t)$ and $y=g(t)$ on the interval $[0,2]$.

First, we will calculate the area of the region corresponding to the interval $[-1,0]$. On this interval, $g(t)>f(t)$, so we will use $g(t)-f(t)$ as the integrand:

$$\begin{aligned}
\int_{-1}^{0}(g(t)-f(t))\,dt &= \int_{-1}^{0}\left(\left(t^2-4t+3\right)-(t+3)\right)\,dt \\
&= \int_{-1}^{0}\left(t^2-4t+3-t-3\right)\,dt \\
&= \int_{-1}^{0}\left(t^2-5t\right)\,dt \\
&= \left(\frac{1}{3}t^3-5\cdot\frac{1}{2}t^2\right)\Big|_{-1}^{0} \\
&= \left(\frac{1}{3}t^3-\frac{5}{2}t^2\right)\Big|_{-1}^{0} \\
&= \left(\frac{1}{3}(0)^3-\frac{5}{2}(0)^2\right)-\left(\frac{1}{3}(-1)^3-\frac{5}{2}(-1)^2\right) \\
&= \frac{17}{6}
\end{aligned}$$

Be careful! This is not the total area. We still need to find the area of the other region on the interval $[0,2]$. Because $f(t)>g(t)$ on this interval, the integrand of this definite integral will be $f(t)-g(t)$:

$$\begin{aligned}
\int_{0}^{2}(f(t)-g(t))\,dt &= \int_{0}^{2}\left((t+3)-\left(t^2-4t+3\right)\right)\,dt \\
&= \int_{0}^{2}\left(t+3-t^2+4t-3\right)\,dt \\
&= \int_{0}^{2}\left(-t^2+5t\right)\,dt \\
&= \left(-1\cdot\frac{1}{3}t^3+5\cdot\frac{1}{2}t^2\right)\Big|_{0}^{2} \\
&= \left(-\frac{1}{3}t^3+\frac{5}{2}t^2\right)\Big|_{0}^{2} \\
&= \left(-\frac{1}{3}(2)^3+\frac{5}{2}(2)^2\right)-\left(-\frac{1}{3}(0)^3+\frac{5}{2}(0)^2\right) \\
&= \frac{22}{3}
\end{aligned}$$

Now that we have found the area of each region, we can find the total area by adding the areas of the regions together:

$$\frac{17}{6}+\frac{22}{3}=\frac{61}{6}$$

Notice that $\int_{-1}^{2} (f(t)-g(t))\,dt = \frac{9}{2}$ *and* $\int_{-1}^{2} (g(t)-f(t))\,dt = -\frac{9}{2}$*, neither of which is the correct answer! If the curves intersect in the given interval, we have to compute more than one definite integral!*

■

■ **Example 4** Find the area of the region bounded by the curves $y = f(x) = 4x^3$ and $y = g(x) = 16x$.

Solution:
Looking at the graphs of these two functions shown in **Figure 4.6.10**, we see two distinct regions enclosed by the graphs: one on the interval $[-2,0]$ and the other on the interval $[0,2]$. Note that you could verify algebraically that the graphs intersect at $x=-2$, $x=0$, and $x=2$ by setting the functions equal to each other and solving for x. Notice that the function which is greater switches at $x=0$:

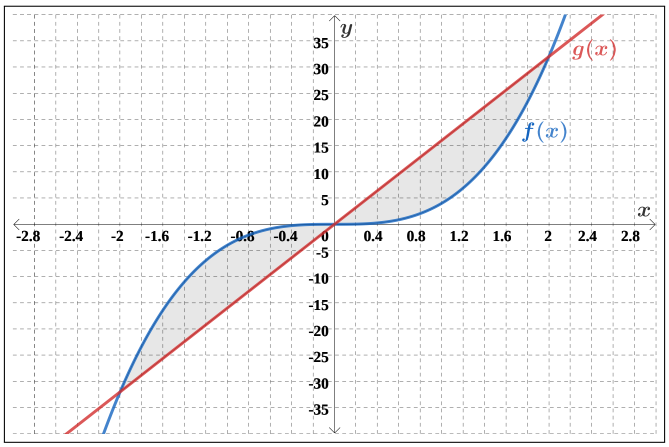


Figure 4.6.10: Regions bounded by the curves $y = f(x)$ and $y = g(x)$ are shaded

On the interval $[-2,0]$, $f(x) > g(x)$. On the interval $[0,2]$, $g(x) > f(x)$. Recalling that $y = f(x) = 4x^3$ and $y = g(x) = 16x$, this means we must calculate $\int_{-2}^{0}\left(4x^3-16x\right)dx$ and $\int_{0}^{2}\left(16x-4x^3\right)dx$ and then add the results to obtain the total area.

First, we will calculate $\int_{-2}^{0}\left(4x^3-16x\right)dx$:

$$\begin{aligned}\int_{-2}^{0}\left(4x^3-16x\right)dx &= \left(4\cdot\frac{1}{4}x^4 - 16\cdot\frac{1}{2}x^2\right)\Big|_{-2}^{0} \\ &= \left(x^4-8x^2\right)\Big|_{-2}^{0} \\ &= \left(0^4-8(0)^2\right)-\left((-2)^4-8(-2)^2\right) \\ &= 16\end{aligned}$$

Next, we calculate $\int_0^2 \left(16x - 4x^3\right)\, dx$:

$$\begin{aligned}\int_0^2 \left(16x - 4x^3\right)\, dx &= \left(16 \cdot \frac{1}{2}x^2 - 4 \cdot \frac{1}{4}x^4\right)\Big|_0^2 \\ &= \left(8x^2 - x^4\right)\Big|_0^2 \\ &= \left(8(2)^2 - (2)^4\right) - \left(8(0)^2 - (0)^4\right) \\ &= 16\end{aligned}$$

To find the total area, we add the areas of the separate regions: $16 + 16 = 32$.

■

Try It # 3:
Find the area of the region bounded by the curve $y = f(x) = x^3 + x^2 - 6x$ and the x-axis.

Producers' and Consumers' Surplus

One very useful application of finding the area between curves is **producers'** and **consumers' surplus**.

Recall that if p is the selling price per unit and x is the quantity supplied or demanded, then the market equilibrium point, (x_0, p_0), is where the price-supply curve, $p = S(x)$, and the price-demand curve, $p = D(x)$, intersect. x_0 is the equilibrium quantity, and p_0 is the equilibrium price. The equilibrium point gives the price per unit such that demand is met with no extra supply.

In any particular market, there are consumers who are willing to pay more than the equilibrium price to purchase an item. These consumers perceive themselves to be *saving* money because they are able to just pay the equilibrium price. The total amount of *savings* for all consumers in the relevant market is called the **consumers' surplus**.

Let's look at the graph of a demand function to help us find a formula for consumers' surplus. **Figure 4.6.11** shows the graph of the demand function $D(x) = -0.03x + 24$ dollars, which gives the price per item when x items are demanded, the equilibrium point $(400, 12)$, and the line p=12, which represents the equilibrium price:

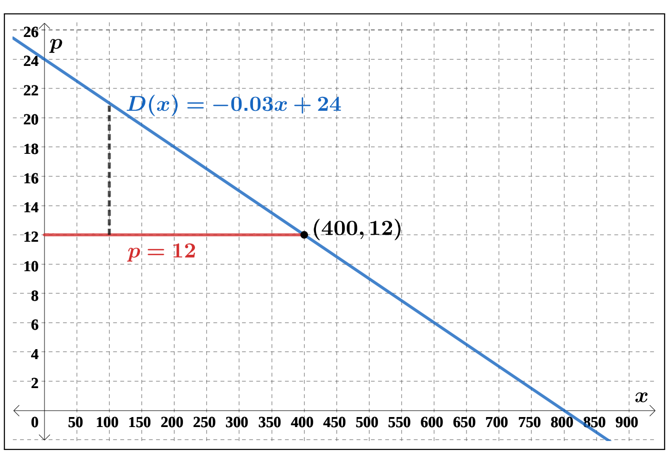


Figure 4.6.11: Graph of the demand function $D(x) = -0.03x + 24$, the equilibrium point $(400, 12)$, and the line $p = 12$, which represents the equilibrium price

To find the amount of *savings* for consumers in a market who are willing to pay a particular price greater than the equilibrium price, we take the difference between the price per unit they are willing to pay and the equilibrium price and then multiply by the number of units that would be demanded at the price consumers are willing to pay.

In this particular market, consumers who are willing to spend say \$21 on the commodity are able to just spend \$12. Thus, they have *saved* \$9 per unit. According to the price-demand equation $p = D(x) = -0.03x + 24$, if the price is \$21 per unit, the quantity demanded will be 100 units. Thus, the total *savings* for consumers who are willing to pay \$21 per unit is

$$\begin{aligned}(21-12)\cdot 100 &= 9\cdot 100\\ &= \$900\end{aligned}$$

If we calculate this savings for every possible price consumers are willing to pay *above* the equilibrium price, we arrive at the area between the curves $p = 12$ and $p = D(x)$ on the interval $[0, 400]$.

In symbols, this gives us the following definition:

Definition

The **consumers' surplus** is $\int_0^{x_0} (D(x) - p_0)\, dx$, where x_0 is the equilibrium quantity, p_0 is the equilibrium price, and $p = D(x)$ is the price-demand equation, which gives the price p per unit when x units are demanded. In words, this is the perceived *savings* for consumers who are willing to pay more for a commodity than the equilibrium price. ▪

N *Riemann sums can be used to provide a more rigorous justification for why the consumers' surplus is equivalent to the area between the demand curve, $p = D(x)$, and the line representing the equilibrium price, $p = p_0$, on the interval $[0, x_0]$, where x_0 is the equilibrium price. However, such an explanation is beyond the scope of this textbook.*

▪ **Example 5** Given the price-demand equation $p = D(x) = -0.03x + 24$ dollars, which gives the price per item when x items are demanded, and the equilibrium point $(400, 12)$, find the consumers' surplus for this commodity.

Solution:

The consumers' surplus is the area between the demand curve, $p = D(x)$, and the line representing the equilibrium price, $p = 12$, on the interval $[0, 400]$. This area is shaded in **Figure 4.6.12**:

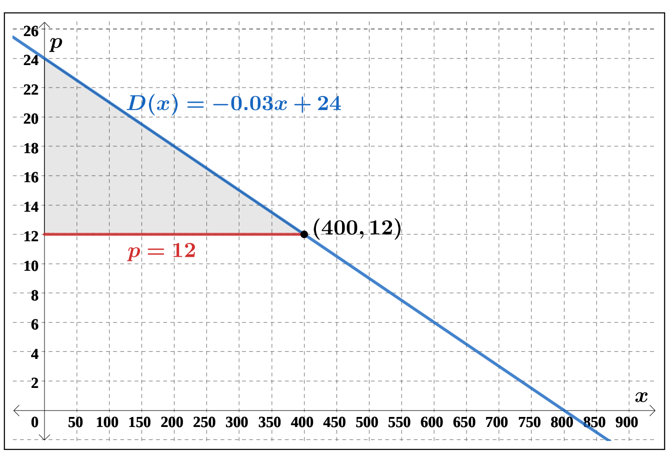


Figure 4.6.12: Region whose area represents the consumers' surplus is shaded

Viewing this as the area between the curves or applying the formula for consumers' surplus directly, we calculate the same quantity.

We will apply the formula for consumers' surplus:

$$\begin{aligned}\int_0^{x_0} (D(x)-p_0)\ dx &= \int_0^{400} ((-0.03x+24)-12)\ dx\\ &= \int_0^{400} (-0.03x+12)\ dx\\ &= \left(-0.03\cdot\frac{1}{2}x^2+12x\right)\Bigg|_0^{400}\\ &= \left(-\frac{0.03}{2}x^2+12x\right)\Bigg|_0^{400}\\ &= \left(-\frac{0.03}{2}(400)^2+12(400)\right)-\left(-\frac{0.03}{2}(0)^2+12(0)\right)\\ &= \$2400\end{aligned}$$

The consumers, as an entire market force, have a savings of \$2400. Thus, the consumers' surplus is \$2400.

■

■ **Example 6** For each of the following, find the consumers' surplus given the price-demand equation, which gives the price per item when x items are demanded, and relevant information.

a. $p = 300(0.997)^x$ dollars and the equilibrium point is $(440, 79.98)$.

b. $p = 400-0.0025x^2$ dollars and the equilibrium quantity is 200 items.

Solution:

a. We are given the price-demand equation, $p = 300(0.997)^x$ dollars, and the equilibrium point, $(440, 79.98)$. Using the formula for consumers' surplus, which is equivalent to finding the area between the curves, we have

$$\begin{aligned}\int_0^{x_0} (D(x)-p_0)\ dx &= \int_0^{440} (300(0.997)^x-79.98)\ dx\\ &= \left(300\cdot\frac{1}{\ln(0.997)}(0.997)^x-79.98x\right)\Bigg|_0^{440}\\ &= \left(\frac{300}{\ln(0.997)}(0.997)^x-79.98x\right)\Bigg|_0^{440}\\ &= \left(\frac{300}{\ln(0.997)}(0.997)^{(440)}-79.98(440)\right)-\left(\frac{300}{\ln(0.997)}(0.997)^0-79.98(0)\right)\\ &\approx \$38,038.15\end{aligned}$$

Thus, the consumers' surplus is \$38,038.15.

b. We are given the price-demand equation, $p = 400-0.0025x^2$ dollars, and the equilibrium quantity, 200 items. Even though we are not explicitly given the equilibrium price, because we know the demand curve must pass through the equilibrium point, we can find the equilibrium price by substituting $x_0 = 200$ items into the price-demand equation:

$$\begin{aligned}p_0 &= 400-0.0025(200)^2\\ &= \$300\end{aligned}$$

Thus, the equilibrium price is \$300.

Now, we can calculate the consumers' surplus:

$$\begin{aligned}
\int_0^{x_0} (D(x) - p_0)\ dx &= \int_0^{200} \left(400 - 0.0025x^2 - 300\right)\ dx \\
&= \int_0^{200} \left(100 - 0.0025x^2\right)\ dx \\
&= \left(100x - 0.0025 \cdot \frac{1}{3}x^3\right)\Bigg|_0^{200} \\
&= \left(100x - \frac{0.0025}{3}x^3\right)\Bigg|_0^{200} \\
&= \left(100(200) - \frac{0.0025}{3}(200)^3\right) - \left(100(0) - \frac{0.0025}{3}(0)^3\right) \\
&\approx \$13,333.33
\end{aligned}$$

Therefore, the consumers' surplus is $13,333.33.

▪

N *In part* **b** *of the previous example, we were given the equilibrium quantity and substituted it into the demand function to find the equilibrium price. If we had been given the equilibrium price instead, then we would have set the demand function equal to the equilibrium price and solved for x to determine the equilibrium quantity.*

Try It # 4:

The price-demand equation for a certain market is $p = 600 - \dfrac{x^2}{294}$ dollars, which gives the price per unit when x units are demanded. Find the consumers' surplus if

a. the equilibrium point is $(294, 306)$.
b. the equilibrium quantity is 120 units.
c. the equilibrium price is $150.

Similar to consumers, there are producers who are willing to sell their product for a lower price than the equilibrium price. These producers perceive themselves to be *gaining* money because they are able to sell their product for more than just the equilibrium price. The total amount of money *gained* by all producers in the relevant market is called the **producers' surplus**.

We will take the same approach as the consumers' surplus and look at the graph of the supply function $S(x) = 0.01x + 8$ dollars, which gives the price per item when x items are supplied, the equilibrium point $(400, 12)$, and the line p=12, which represents the equilibrium price. See **Figure 4.6.13**.

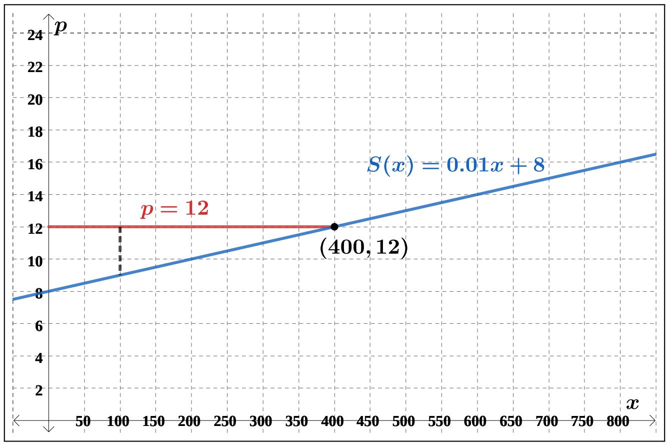


Figure 4.6.13: Graph of the supply function $S(x) = 0.01x+8$, the equilibrium point $(400,12)$, and the line $p = 12$, which represents the equilibrium price

To find the *gain* for producers in a market who are willing to sell their product for a particular price less than the equilibrium price, we take the difference between the price per unit for which they are willing to sell and the equilibrium price and then multiply by the number of units that would be supplied at the price for which producers are willing to sell.

In this particular market, producers who are willing to sell their commodity for say \$9 per unit are able to sell it for the equilibrium price of \$12 per unit. Thus, they have *gained* \$3 per unit. According to the price-supply equation, $p = S(x) = 0.01x+8$, if the price is \$9 per unit, the quantity supplied will be 100 units. Thus, the total *gain* for producers who are willing to sell their product for \$9 per unit is

$$\begin{aligned}(12-9)\cdot 100 &= 3\cdot 100\\ &= \$300\end{aligned}$$

If we calculate this gain for every possible price for which producers are willing to sell their product *below* the equilibrium price, we arrive at the area between the curves $p = 12$ and $p = S(x)$ on the interval $[0,400]$.

In symbols, this gives us the following definition:

Definition

The **producers' surplus** is $\int_0^{x_0} (p_0 - S(x))\ dx$, where x_0 is the equilibrium quantity, p_0 is the equilibrium price, and $p = S(x)$ is the price-supply equation, which gives the price p per unit when x units are supplied. In words, this is the perceived *gain* for producers who are willing to sell their commodity for less than the equilibrium price.
■

N *Similar to consumers' surplus, Riemann sums can be used to provide a more rigorous justification for why the producers' surplus is equivalent to the area between the line representing the equilibrium price, $p = p_0$, and the supply curve, given by $p = S(x)$, on the interval $[0, x_0]$, where x_0 is the equilibrium quantity. However, such an explanation is beyond the scope of this textbook.*

■ **Example 7** Given the price-supply equation $p = S(x) = 0.01x+8$ dollars, which gives the price per item when x items are supplied, and the equilibrium point $(400,12)$, find the producers' surplus for this commodity.

Solution:
The producers' surplus is the area between the supply curve, $p = S(x)$, and the line representing the equilibrium price, $p = 12$, on the interval $[0,400]$. This area is shaded in **Figure 4.6.14**:

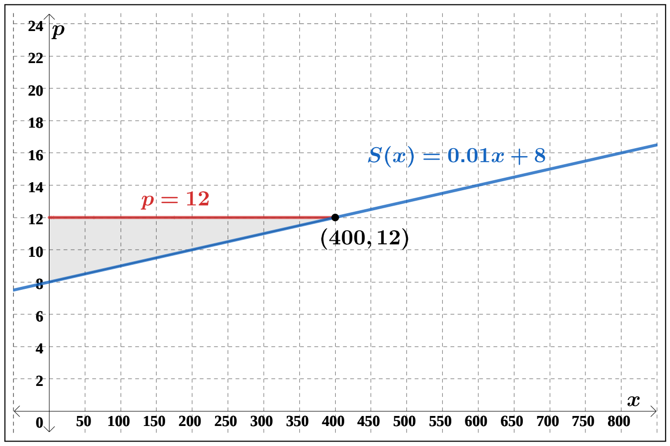


Figure 4.6.14: Region whose area represents the producers' surplus is shaded

Viewing this as the area between the curves or applying the formula for producers' surplus directly, we calculate the same quantity. We will apply the formula for producers' surplus:

$$\begin{aligned}
\int_0^{x_0} (p_0 - S(x))\,dx &= \int_0^{400} (12 - (0.01x + 8))\,dx \\
&= \int_0^{400} (12 - 0.01x - 8)\,dx \\
&= \int_0^{400} (4 - 0.01x)\,dx \\
&= \left(4x - 0.01 \cdot \frac{1}{2}x^2\right)\Bigg|_0^{400} \\
&= \left(4x - \frac{0.01}{2}x^2\right)\Bigg|_0^{400} \\
&= \left(4(400) - \frac{0.01}{2}(400)^2\right) - \left(4(0) - \frac{0.01}{2}(0)^2\right) \\
&= \$800
\end{aligned}$$

Thus, the producers' surplus is \$800. That is, the producers, collectively, have \$800 more than they expected from this market.

∎

Example 8 For each of the following, find the producers' surplus given the price-supply equation, which gives the price per item when x items are supplied, and relevant information.

a. $p = \frac{x}{3} + 208$ dollars and the equilibrium point is $(294, 306)$.

b. $p = 360(1.038)^x$ and the equilibrium quantity is 30 items.

Solution:

a. We are given the price-supply equation, $p = \frac{x}{3} + 208$ dollars, and the equilibrium point, $(294, 306)$. Using the formula for producers' surplus, which is equivalent to finding the area between the curves, we have

$$\begin{aligned}\int_0^{x_0} (p_0 - S(x))\ dx &= \int_0^{294} \left(306 - \left(\frac{x}{3} + 208\right)\right) dx \\ &= \int_0^{294} \left(306 - \frac{x}{3} - 208\right) dx \\ &= \int_0^{294} \left(98 - \frac{x}{3}\right) dx \\ &= \left(98x - \frac{1}{3}\cdot\frac{1}{2}x^2\right)\Bigg|_0^{294} \\ &= \left(98x - \frac{1}{6}x^2\right)\Bigg|_0^{294} \\ &= \left(98(294) - \frac{1}{6}(294)^2\right) - \left(98(0) - \frac{1}{6}(0)^2\right) \\ &= \$14,406\end{aligned}$$

Therefore, the producers' surplus for this market is $14,406.

> ! *One common mistake with producers' surplus (that we do not see with consumers' surplus) is forgetting to distribute the negative sign to the entirety of the supply function. If you leave off the parentheses around $S(x)$ when finding $p_0 - S(x)$, you may get the wrong answer!*

b. We are given the price-supply equation, $p = 360(1.038)^x$ dollars, and the equilibrium quantity, 30 items. Even though we are not explicitly given the equilibrium price, we can find it using the equilibrium quantity. In other words, we can find the equilibrium price by substituting $x_0 = 30$ items into the price-supply equation:

$$\begin{aligned}p_0 &= 360(1.038)^{30} \\ &\approx \$1102.11\end{aligned}$$

Thus, the equilibrium price is $1102.11 per item. Now, we can calculate the producers' surplus:

$$\begin{aligned}\int_0^{x_0} (p_0 - S(x))\ dx &= \int_0^{30} \left(1102.11 - 360(1.038)^x\right) dx \\ &= \left(1102.11x - 360\cdot\frac{1}{\ln(1.038)}(1.038)^x\right)\Bigg|_0^{30} \\ &= \left(1102.11x - \frac{360}{\ln(1.038)}(1.038)^x\right)\Bigg|_0^{30} \\ &= \left(1102.11(30) - \frac{360}{\ln(1.038)}(1.038)^{30}\right) - \left(1102.11(0) - \frac{360}{\ln(1.038)}(1.038)^0\right) \\ &= \$13,165.47\end{aligned}$$

Therefore, the producers have a surplus of $13,165.47.

■

Try It # 5:
The price-supply equation for a certain market is $p = 45 + 0.002x^3$ dollars, which gives the price per unit when x units are supplied. Find the producers' surplus if

a. the equilibrium point is $(50, 295)$.
b. the equilibrium quantity is 70 units.
c. the equilibrium price is $173.

■ **Example 9** If the price-supply equation is $p = 40 + \frac{x^2}{380}$ dollars and the price-demand equation is $p = 400(0.9943)^x$ dollars, and both equations represent the price per item when x items are supplied or demanded, respectively, find

a. the consumers' surplus.
b. the producers' surplus.

Solution:

a. We are given neither the equilibrium quantity, nor the equilibrium price. Thus, we will have to find the equilibrium point ourselves before we can proceed.

To find the equilibrium point, we must find the point where the demand and supply curves intersect. Although, in theory, we could do this algebraically by setting the functions equal to each other and solving for x, it is much easier in this case to use technology.

Most graphing calculators have a command that will approximate where two curves intersect. Using such a command, we find the supply and demand curves intersect at the point where $x \approx 190.0070$ and $p \approx 135.0070$. Because x is measured in number of items, we will round the x-value to the nearest integer. This means the equilibrium quantity is $x_0 = 190$ items. Because p is the price in dollars, we will round the p-value to two decimal places. This means the equilibrium price is $p_0 = \$135.01$.

The graphs of the supply and demand functions, as well as the line representing the equilibrium price, are shown in **Figure 4.6.15**, and the regions representing the consumers' and producers' surplus are shaded:

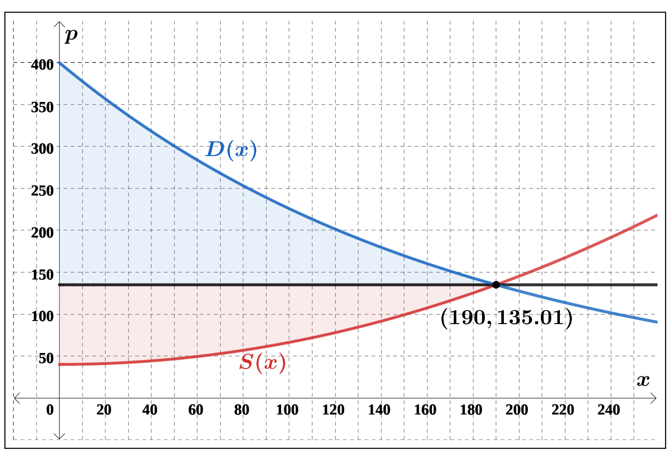


Figure 4.6.15: Regions whose areas represent the consumers' and producers' surplus

Now, we can calculate the consumers' surplus:

$$\begin{aligned}
\int_0^{x_0} (D(x) - p_0)\ dx &= \int_0^{190} (400(0.9943)^x - 135.01)\ dx \\
&= \left(400 \cdot \frac{1}{\ln(0.9943)}(0.9943)^x - 135.01x\right)\Bigg|_0^{190} \\
&= \left(\frac{400}{\ln(0.9943)}(0.9943)^x - 135.01x\right)\Bigg|_0^{190} \\
&= \left(\frac{400}{\ln(0.9943)}(0.9943)^{190} - 135.01(190)\right) - \left(\frac{400}{\ln(0.9943)}(0.9943)^0 - 135.01(0)\right) \\
&\approx \$20,704.54
\end{aligned}$$

Thus, the consumers' surplus is $20,704.54.

b. We know the equilibrium point is (190, 135.01) from part **a**, so we can immediately calculate the producers' surplus:

$$\begin{aligned}
\int_0^{x_0} (p_0 - S(x))\ dx &= \int_0^{190} \left(135.01 - \left(40 + \frac{x^2}{380}\right)\right)\ dx \\
&= \int_0^{190} \left(135.01 - 40 - \frac{x^2}{380}\right)\ dx \\
&= \int_0^{190} \left(95.01 - \frac{x^2}{380}\right)\ dx \\
&= \left(95.01x - \frac{1}{380} \cdot \frac{1}{3}x^3\right)\Bigg|_0^{190} \\
&= \left(95.01x - \frac{1}{1140}x^3\right)\Bigg|_0^{190} \\
&= \left(95.01(190) - \frac{1}{1140}(190)^3\right) - \left(95.01(0) - \frac{1}{1140}(0)^3\right) \\
&\approx \$12,035.23
\end{aligned}$$

Thus, the producers' surplus is $12,035.23.

▪

Try It # 6:
If the price-supply equation is $p = S(x) = 0.1x + 9.2$ dollars and the price-demand equation is $D(x) = 50 - 0.002x^2$ dollars, and both equations represent the price per item when x items are supplied or demanded, respectively, find

a. the consumers' surplus.
b. the producers' surplus.

Other Applications

There are other applications involving the area between curves in addition to consumers' and producers' surplus.

Recall from **Sections 4.3** and **4.4** that $\int_a^b f'(x)\,dx$ gives the net change (also the net area) in f on the interval $[a,b]$. This translates to the area between two derivative curves $y = f'(x)$ and $y = g'(x)$ on the interval $[a,b]$ being the accumulated difference between the functions f and g from $x = a$ to $x = b$, assuming the functions measure the same quantity.

Example 10 Two objects start from the same location and travel along the same path. After t seconds, object A has velocity $v_A(t) = t+5$ and object B has velocity $v_B(t) = t^2 - 4t + 5$, both measured in meters per second. Which object is farther ahead after three seconds, and by how much?

Solution:
The area between these velocity functions on the interval $[0,3]$ represents the accumulated difference between their positions. Because both velocity functions are positive on this interval, this translates to the area being the accumulated distance between them.

To find the area between the curves on the interval $[0,3]$, we must first determine which velocity function is greater (i.e., on "top") on the interval (and check to see if the curves intersect, in which case the greater function would switch and more than one definite integral would be needed).

We can do this by looking at the graphs of $v_A(t)$ and $v_B(t)$ on the same axes and examining the location of the interval $[0,3]$. The curves are shown in **Figure 4.6.16** with the relevant area shaded:

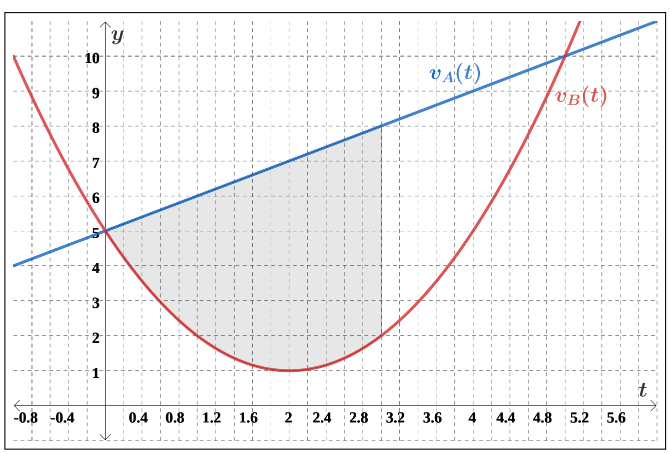


Figure 4.6.16: Region whose area represents the accumulated distance between two objects

First, notice $v_A(t) > v_B(t)$ on this interval, so object A is ahead after three seconds. To find how much farther ahead (i.e., the distance between them), we calculate the area given by $\int_0^3 (v_A(t) - v_B(t))\;dt$:

$$\begin{aligned}
\int_0^3 (v_A(t) - v_B(t))\ dt &= \int_0^3 \left((t+5) - \left(t^2 - 4t + 5\right)\right)\ dt \\
&= \int_0^3 \left(t + 5 - t^2 + 4t - 5\right)\ dt \\
&= \int_0^3 \left(-t^2 + 5t\right)\ dt \\
&= \left(-1 \cdot \frac{1}{3}t^3 + 5 \cdot \frac{1}{2}t^2\right)\Bigg|_0^3 \\
&= \left(-\frac{1}{3}t^3 + \frac{5}{2}t^2\right)\Bigg|_0^3 \\
&= \left(-\frac{1}{3}(3)^3 + \frac{5}{2}(3)^2\right) - \left(-\frac{1}{3}(0)^3 + \frac{5}{2}(0)^2\right) \\
&= \frac{27}{2} = 13.5 \text{ meters}
\end{aligned}$$

After three seconds, object A is 13.5 meters ahead of object B.

▪

Try It # 7:
Two objects start from the same location and travel along the same path. After t seconds, object A has velocity $v_A(t) = t + 4$ and object B has velocity $v_B(t) = -t^2 + 4t + 5$, both measured in meters per second. Which object is farther ahead after three seconds, and by how much?

▪ **Example 11** The Torn Tome, a medieval themed tavern, had a monthly marginal profit function of $P_N{}'(x) = -0.02x + 3.6$ dollars per mug of ginger ale for November 2019, where x is the number of mugs of ginger ale sold. During December 2019, the tavern had a monthly marginal profit function of $P_D{}'(x) = -0.06x + 8$ dollars per mug of ginger ale, where x is the number of mugs of ginger ale sold. If 110 mugs of ginger ale were sold during each of these months, in which month did The Torn Tome have a higher profit? How much higher was the profit?

Solution:
The area between these marginal profit functions on the interval $[0, 110]$ represents the accumulated difference in profit between the two months. Again, to find this area, we must first determine which function is greater on the interval (and check to see if the curves intersect, in which case the greater function would switch and more than one definite integral would be needed). **Figure 4.6.17** shows both curves on the interval $[0, 110]$ along with the relevant area shaded:

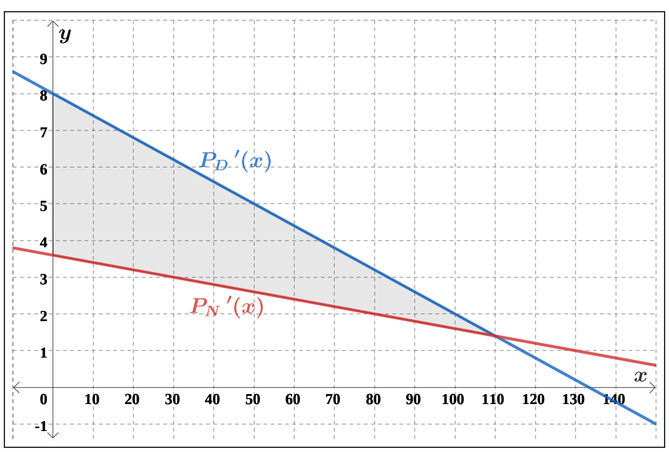


Figure 4.6.17: Region whose area represents the accumulated difference in profit between two months

Notice $P_D{}'(x) > P_N{}'(x)$ on the interval $[0, 110]$, so The Torn Tome had a higher profit in December. To find how much higher the profit was in December, we calculate the area:

$$\begin{aligned}
\int_0^{110} (P_D{}'(x) - P_N{}'(x))\; dx &= \int_0^{110} ((-0.06x+8)-(-0.02x+3.6))\; dx \\
&= \int_0^{110} (-0.06x+8+0.02x-3.6)\; dx \\
&= \int_0^{110} (-0.04x+4.4)\; dx \\
&= \left(-0.04\cdot\frac{1}{2}x^2+4.4x\right)\Big|_0^{110} \\
&= \left(-0.02x^2+4.4x\right)\Big|_0^{110} \\
&= \left(-0.02(110)^2+4.4(110)\right)-\left(-0.02(0)^2+4.4(0)\right) \\
&= \$242
\end{aligned}$$

Thus, The Torn Tome earned \$242 more in profit in December than it did in November.

▪

Try It # 8:
The marginal cost functions, in dollars per mug of ginger ale, for The Torn Tome for January and February 2020 are given by $C_J{}'(x) = 300 - 0.3x$ and $C_F{}'(x) = 340 - 0.001x^2$, respectively, where x is the number of mugs of ginger ale sold. If 250 mugs of ginger ale were sold during each of these months, in which month did The Torn Tome have higher costs? How much more did the tavern pay in costs that month?

Try It Answers

1. 18
2. $\frac{1}{3}$
3. $\frac{253}{12}$
4. **a.** \$57,624
   b. \$3918.42
   c. \$109,119.11
5. **a.** \$9375
   b. \$36,015
   c. \$3840
6. **a.** \$2304
   b. \$720

7. After three seconds, Object B is ahead by 7.5 meters.

8. The Torn Tome had higher costs during February. The tavern paid $\$14,166.67$ more in costs that month.

EXERCISES

Basic Skills Practice

For Exercises 1 - 6, find the area of the region between the curves on the given interval.

1. $y = x^3 + 2$ and $y = x - 2$ on the interval $[0, 2]$

2. $y = -x^2 + 4x - 1$ and $y = x + 5$ on the interval $[-1, 3]$

3. $y = 2x + 1$ and $y = -x - 8$ on the interval $[-7, -4]$

4. $y = -x + 1$ and $y = 3x + 6$ on the interval $[-3, 2]$

5. $y = \frac{1}{3}x^3 - 5$ and $y = -2x + 10$ on the interval $[1, 6]$

6. $y = -0.5x^5 + 16$ and $y = 0$ on the interval $[-2, 4]$

For Exercises 7 - 9, the price-demand equation and equilibrium point for a certain commodity are given. Use the information to find the consumers' surplus for the commodity.

7. $p = D(x) = -0.8x + 150$, and the equilibrium point is $(25, 130)$.

8. $p = D(x) = -3x + 1000$, and the equilibrium point is $(200, 400)$.

9. $p = D(x) = -0.5x + 200$, and the equilibrium point is $(75, 162.50)$.

For Exercises 10 - 12, the price-supply equation and equilibrium point for a certain commodity are given. Use the information to find the producers' surplus for the commodity.

10. $p = S(x) = 5.2x$, and the equilibrium point is $(25, 130)$.

11. $p = S(x) = 0.4x + 308$, and the equilibrium point is $(290, 424)$.

12. $p = S(x) = 0.03x + 50$, and the equilibrium point is $(45, 51.35)$.

13. The price-demand equation for a particular denim jacket is given by $p = 206 - 0.2x$ dollars, which gives the price per jacket when x jackets are demanded. If market equilibrium is reached when 500 jackets are sold at a price of \$106.00 each, find the consumers' surplus.

14. The price-supply equation for the denim jacket described in the previous example (Example #13) is given by $p = 0.04x + 86$ dollars, which gives the price per jacket when x jackets are supplied. Find the producers' surplus.

15. The supplier of a popular board game called *Who Dunnit?* has a price-supply equation given by $p = 0.05x + 10$ dollars, which gives the price per game when x games are supplied. If market equilibrium is reached when 300 games are sold at a price of \$25 each, find the producers' surplus.

16. The price-demand equation for the board game described in the previous example (Example #15) is given by $p = 28 - 0.01x$ dollars, which gives the price per game when x games are demanded. Find the consumers' surplus.

17. During 2018, a company that sells clothes dryers had a weekly marginal cost function given by $C_1{}'(x) = 12 + 0.4x$ dollars per dryer, where x is the number of dryers produced each week. During 2019, the company had a weekly marginal cost function given by $C_2{}'(x) = 10 + 0.3x$ dollars per dryer, where x is the number of dryers produced each week.
    (a) If 200 dryers were produced each week during both years, during which year did the company have higher weekly costs?
    (b) How much more did the company pay in weekly costs during that year?

18. Two rival dog treat companies, Daisy-licious and Tetra-tastic, have production rates given by $P_D{}'(t) = 50 + \frac{1}{8}t$ and $P_T{}'(t) = 50 + \frac{1}{3}t$ bags of treats per hour, respectively, t hours after they start making bags of treats.
    (a) After a 12 hour work shift, which company produces more bags of treats?
    (b) How many more bags of treats does this company produce than the other during the 12 hour work shift? Round your answer to the nearest integer, if necessary.

19. Super Rad Skateboardz specializes in selling glow in the dark skateboards. During March of 2019, the company had a monthly marginal revenue function given by $R_1{}'(x) = 200 - 1.6x$ dollars per skateboard, where x is the number of skateboards sold. During March of 2020, Super Rad Skateboardz had a monthly marginal revenue function given by $R_2{}'(x) = 90 - 2.4x$ dollars per skateboard, where x is the number of skateboards sold.
    (a) If the company sold 35 skateboards during March of both 2019 and 2020, during which year did the company have a higher monthly revenue?
    (b) How much greater was the monthly revenue?

20. Two objects start from the same location and travel along the same path. Object A has velocity $v_A(t) = 0.5t + 6$, and object B has velocity $v_B(t) = 0.3t + 8$, both measured in meters per second, after t seconds. How far ahead is object B after 5 seconds?

Intermediate Skills Practice

For Exercises 21 - 24, find the area of the region between the curves on the given interval.

21. $y = x^2$ and $y = 4$ on the interval $[-1, 5]$

22. $y = 6x^3 - 3x^2 + x - 4$ and $y = -x + 7$ on the interval $[-6, -2]$

23. $y = \sqrt{x}$ and $y = x^2$ on the interval $[0,4]$

24. $y = 2 - x^2$ and $y = -x$ on the interval $[-4,1]$

For Exercises 25 - 28, find the area of the region bounded by the curves.

25. $y = x^2$ and $y = 3x+4$

26. $y = x^3$ and $y = x^2$

27. $y = x^2 - 9$ and the x-axis

28. $y = -x^2 + 18x$ and $y = x^2$

For Exercises 29 - 32, the price-demand equation and either the equilibrium quantity, x_0, or equilibrium price, p_0, for a certain commodity are given. Use the information to find the consumers' surplus for the commodity.

29. $p = D(x) = 50 - x^2$; $x_0 = 5$

30. $p = D(x) = -0.2x + 175$; $x_0 = 400$

31. $p = D(x) = 670 - 0.01x^2$; $p_0 = 445$

32. $p = D(x) = -0.02x + 240$; $p_0 = 125$

For Exercises 33 - 36, the price-supply equation and either the equilibrium quantity, x_0, or equilibrium price, p_0, for a certain commodity are given. Use the information to find the producers' surplus for the commodity.

33. $p = S(x) = 0.3x + 100$; $x_0 = 90$

34. $p = S(x) = 0.002x^2 + 0.1x + 18$; $x_0 = 50$

35. $p = S(x) = 40x$; $p_0 = 600$

36. $p = S(x) = 0.01x + 25$; $p_0 = 35$

For Exercises 37 - 39, the price-demand equation, price-supply equation, and equilibrium point for a certain commodity are given. Use the information to find (a) the consumers' surplus and (b) the producers' surplus for the commodity.

37. $p = D(x) = -0.1x + 30$; $p = S(x) = 0.04x + 23$; $(x_0, p_0) = (50, 25)$

38. $p = D(x) = -0.03x + 490$; $p = S(x) = 0.0001x^2 + 30$; $(x_0, p_0) = (2000, 430)$

39. $p = D(x) = -0.02x^2 + 100$; $p = S(x) = 0.1x + 85$; $(x_0, p_0) = (25, 87.50)$

40. A popular three-story dollhouse has a price-demand equation given by $p = 210 - 0.8x$ dollars, which gives the price per dollhouse when x dollhouses are demanded. The price-supply equation for the dollhouse is given by $p = 0.2x + 135$ dollars, which gives the price per dollhouse when x dollhouses are supplied. Find (a) the consumers' surplus and (b) the producers' surplus.

41. The price-demand equation for a particular flashlight is given by $p = 150 - 0.001x$ dollars, which gives the price per flashlight when x flashlights are demanded. The price-supply equation for the flashlight is given by $p = 0.0004x + 52$ dollars, which gives the price per flashlight when x flashlights are supplied. Find (a) the consumers' surplus and (b) the producers' surplus.

42. A cordless leaf blower has a price-demand equation given by $p = -0.002x^2 + 164$ dollars, which gives the price per leaf blower when x leaf blowers are demanded. The price-supply equation for the leaf blower is given by $p = 0.001x^2 + 44$ dollars, which gives the price per leaf blower when x leaf blowers are supplied. Find (a) the consumers' surplus and (b) the producers' surplus.

43. Two rival trampoline companies, Jump Around and Jump Up, have production rates given by $P_A{}'(d) = \sqrt{d} + 10$ and $P_U{}'(d) = 15 + 0.15d$ trampolines per day, respectively, d workdays after they start producing trampolines.
    (a) After 60 workdays, which company produces more trampolines?
    (b) How many more trampolines does this company produce than the other after 60 workdays? Round your answer to the nearest integer, if necessary.

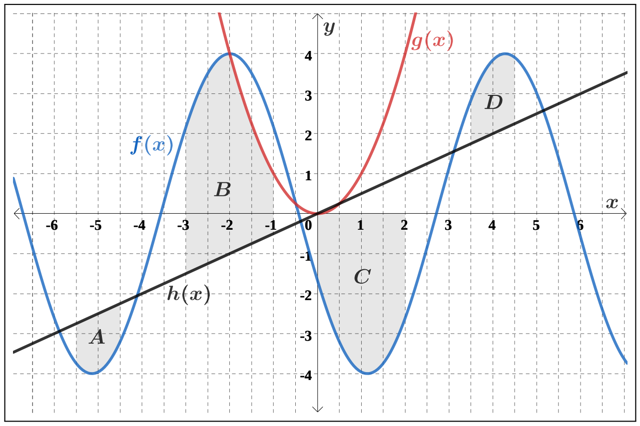
44. Company A has a marginal profit function given by $P_A{}'(x) = -0.002x^2 + 275$ dollars per item, where x is the number of items sold. Company B has a marginal profit function given by $P_B{}'(x) = -0.001x^2 + 305$ dollars per item, where x is the number of items of the same commodity sold.
    (a) If both companies sell 250 items, which company will have a higher profit?
    (b) How much greater will this company's profit be than the other company's?

45. During February of 2019, Mitochondria: Powerhouse of the Cell Phone, a company that produces cell phones, had a monthly marginal cost function given by $C_F{}'(x) = 0.01x^2 - 3x + 229$ dollars per cell phone, where x is the number of cell phones produced. During March of that same year, the company had a monthly marginal cost function of $C_M{}'(x) = 0.02x^2 - 4.2x + 229$ dollars per cell phone, where x is the number of cell phones produced.
    (a) If the company sold 120 cell phones during each of these months, during which month did the company have lower monthly costs?
    (b) How much lower were the monthly costs during this month?

46. Two objects start from the same location and travel along the same path. Object A has velocity $v_A(t) = -t^2 + 5t + 14$, and object B has velocity $v_B(t) = 0.8t^2 - 1.6t + 10.8$, both measured in feet per second, after t seconds.
    (a) Which object is ahead after 4 seconds?
    (b) How much farther ahead is the object?

Mastery Practice

47. The graphs of the functions f, g, and h are shown below. Write the definite integral(s) corresponding to the area of each of the following regions.



(a) region A
(b) region B
(c) region C
(d) region D

For Exercises 48 - 55, find the area of the region between the curves on the given interval.

48. $y = x^2 - 2x + 5$ and $y = 5x - 5$ on the interval $[6, 10]$

49. $y = x^2$ and $y = x$ on the interval $[0.5, 4]$

50. $y = x^6$ and $y = x^4$ on the interval $[-3, 1]$

51. $y = e^x$ and $y = x$ on the interval $[0, 2]$

52. $y = (x-1)^2$ and $y = x + 1$ on the interval $[-2, 5]$

53. $y = x$ and $y = \dfrac{1}{x}$ on the interval $[1, e]$

54. $y = 0$ and $y = x^2 - 16$ on the interval $[-5, 5]$

55. $y = e^x$ and $y = e^{-x}$ on the interval $[-1, 1]$

For Exercises 56 - 63, find the area of the region bounded by the curves.

56. $y = 2x^2 + 2x - 5$ and $y = x^2 + 3x + 7$

57. $y = x^3$ and $y = x$

58. $y = \frac{1}{x}$, $y = \frac{1}{x^2}$, and $x = 3$

59. $y = e^{2x-1}$, $y = e^x$, and $x = 0$

60. $y = \sqrt{x+4}$ and $y = \frac{1}{4}x + \frac{7}{4}$

61. $y = x+2$, $y = 4 - x^2$, and $x = 5$

62. $y = \sqrt[3]{x^2}$ and $y = \frac{x}{3}$

63. $y = x^3 - x^2 - 2x$, the x-axis, and $x = -3$

For Exercises 64 - 70, find the area of the region bounded by the curves using technology. Round any necessary values to three decimal places, including your answer.

64. $y = \sqrt{4 - x^2}$, $y = 2$, $x = -2$, and $x = 2$

65. $y = x^3$ and $y = x^2 + x$

66. $y = 3x^3 - 4x + 1$ and the x-axis

67. $y = \ln(x)$ and $y = x - 2$

68. $y = \sqrt{25 - x^2}$, $y = x^2 + 2x$, $x = -4$, and $x = 1$

69. $y = xe^x$ and $y = x^2 + 2x$

70. $y = -x^2 + 10x - 3$, $y = x^2 - 4x + 7$, $x = -0.5$, and $x = 8$

For Exercises 71 - 73, the price-demand equation, price-supply equation, and either the equilibrium quantity, price, or point for a certain commodity are given. Use the information to find (a) the consumers' surplus and (b) the producers' surplus for the commodity.

71. $p = D(x) = 250(0.998)^x$; $p = S(x) = 0.1x + 30.81$; $(x_0, p_0) = (540, 84.81)$

72. $p = D(x) = -0.004x^2 + 110$; $p = S(x) = 0.01x^2 + 50.85$; $p_0 = 93.10$

73. $p = D(x) = 50e^{-0.002x}$; $p = S(x) = 25e^{0.001x}$; $x_0 = 231$

74. Snappy Flash, a camera manufacturer, will not supply any cameras if the price of each camera is \$84 or lower. However, they will supply 90 cameras when the price is \$264 per camera. Snappy Flash has determined its price-demand equation to be $p = -0.04x^2 + 459$ dollars, which gives the price per camera when x cameras are demanded. Assuming the price-supply equation is linear and market equilibrium is reached when the price is \$234 per camera, find (a) the consumers' surplus and (b) the producers' surplus.

75. A popular sports watch has a price-supply equation given by $p = 39.936e^{0.01x}$ dollars, which gives the price per watch when x watches are supplied. The quantity of watches demanded is 250 when the price is \$39 each, but for each additional \$5 increase in price, the quantity demanded decreases by 4 watches. Assuming the price-demand equation is linear and market equilibrium is reached when 145 watches are sold, find (a) the consumers' surplus and (b) the producers' surplus.

76. The quantity demanded of a popular tablet is 300 per week when the price of each tablet is \$450. For each decrease in price of \$30, the quantity demanded increases by 100 tablets each week. The tablet's manufacturer will not supply any tablets if the price of each tablet is \$250 or lower. However, it will supply 375 tablets each week if the price is \$325 for each tablet. Assuming the price-supply and price-demand equations are linear, find (a) the consumers' surplus and (b) the producers' surplus.

77. Patrick invests money into an account that earns interest at a rate of 2.5% per year compounded continuously, and Angie invests her money into a different account that earns interest at a rate of 3.5% per year compounded continuously. The rate at which the balance of Patrick's account grows is given by $B_P = 25e^{0.025t}$ dollars per year, where t is the numbers of years since Patrick invested his money. The rate at which the balance of Angie's account grows is given by $B_A = 49e^{0.035t}$ dollars per year, where t is the numbers of years since she invested her money.

    (a) After 10 years, whose account balance will have increased the most?

    (b) By how much more will this account balance have increased after 10 years?

78. During 2018, a company that sells roller blades had a weekly marginal cost function given by $C_1\,'(x) = 15 + \dfrac{300}{0.2x+4}$ dollars per pair of roller blades, where x is the number of roller blades produced. During 2019, the same company had a weekly marginal cost function given by $C_2\,'(x) = 20 + \dfrac{100}{0.1x+2}$ dollars per pair of roller blades, where x is the number of roller blades produced.

    (a) If 60 pairs of roller blades were produced each week during both years, during which year did the company have higher weekly costs?

    (b) How much less did the company pay in weekly costs during the other year?

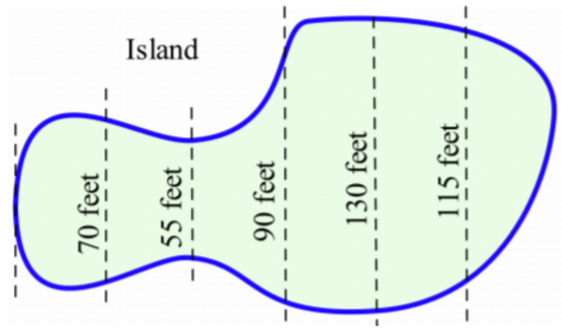
79. For the first three months after a new video game was released, Video Frenzy! had a rate of change of sales for the new video game given by $V_F(t) = 50\sqrt[3]{t}$ games per week, where t is the number of weeks after the game was released. A rival video game store, Video Bonanza!, had a rate of change of sales for the same video game given by $V_B(t) = 50\sqrt[2]{t}$ games per week, t weeks after the game was released.

    (a) During the first week after the new video game was released, which store sold more video games?

    (b) How many more of the new video games did this store sell during the first week? Round your answer to the nearest integer, if necessary.

80. A small amusement park has a marginal cost function given by $C\,'(x) = 1000e^{-0.02x} + 5$ dollars per ticket, where x represents the number of tickets sold. The marginal revenue function of the amusement park is given by $R\,'(x) = 60 - 0.1x$ dollars per ticket, where x is the number of tickets sold. Find the amusement park's profit if 150 tickets are sold.

81. Every 40 feet, the distance across an island was recorded as shown in the figure below. Find the best possible estimate for the area of the island.



82. Two objects start from the same location and travel along the same path. Object A has velocity $v_A(t) = t^3 - 2t + 5$, and object B has velocity $v_B(t) = -t^2 + 6t + 8$, both measured in meters per second, after t seconds.

   (a) Which object is ahead after 2 seconds?

   (b) How much farther ahead is this object after 2 seconds? Round your answer to three decimal places, if necessary.

   (c) Between 3 and 5 seconds, which object covers more distance?

   (d) How much more distance does this object cover between 3 and 5 seconds than the other? Round your answer to three decimal places, if necessary.

83. Write the definite integral(s) representing the area of the region bounded by $y = -x^2 - x + 6$ and the x-axis on the interval $[-1, A]$, where $A > 3$.

84. Grillstone, a company that manufactures grills, will supply 200 grills when the price is \$384 per grill. When the price of each grill is \$370, the quantity demanded is 150 grills. If the price-supply and price-demand equations are linear and the equilibrium point for the grills is $(160, 340)$, find (a) the consumers' surplus and (b) the producers' surplus.

85. Use the table of function values below to obtain the best possible estimate of the area

x	-2	-1	0	1	2	3	4
$f(x)$	-3	-2	-1	0	1	2	3
$g(x)$	0	-3	-4	-3	0	5	12
$h(x)$	-1	2	3	2	-1	-6	-13

   (a) between the curves $y = f(x)$ and $y = g(x)$ on the interval $[-1, 4]$ using a left-hand Riemann sum.

   (b) between the curves $y = g(x)$ and $y = h(x)$ on the interval $[-2, 2]$ using a left-hand Riemann sum.

   (c) between the curves $y = f(x)$ and $y = h(x)$ on the interval $[0, 3]$ using a right-hand Riemann sum.

   *Note: See **Section 4.3** for information about Riemann sums.*

86. Find the area of the region bounded by the curves $y = 12 - x$, $y = \sqrt{x}$, and $y = 1$.

87. Find the area of the region bounded by the curves $y = e$, $y = e^t$, and $y = e^{-t}$.

Communication Practice

88. When finding the area of a region between two curves, explain why it is important to graph the curves before performing any calculations.

89. Explain why multiple definite integrals may be necessary to calculate the area bounded by two curves.

90. $R_1\,'(x)$ and $R_2\,'(x)$ model the marginal revenue functions, both in dollars per item, of Company 1 and Company 2, respectively, where x is the number of items of the same commodity sold. If $R_1\,'(x) < R_2\,'(x)$, interpret $\int_0^{100} [R_2\,'(x) - R_1\,'(x)]\; dx = 25,000$.

91. Does the consumers' surplus represent the total gain to consumers? Explain.

92. Explain where the regions whose areas represent the consumers' and producers' surplus are located on the Cartesian plane.

V

Appendix

Exercise Answers

A. Exercise Answers

The answers to most communication exercises in this text may vary. The authors have provided a possible solution for each of these exercises.

Section 1.1

1. (a) 5
   (b) 5
   (c) 5
   (d) Undefined
2. (a) 6
   (b) 6
   (c) 6
   (d) −4
3. (a) −1
   (b) 1
   (c) Does Not Exist
   (d) 1
4. (a) Does Not Exist; $\lim_{x\to 3^-} f(x) \to \infty$
   (b) Does Not Exist; $\lim_{x\to 3^+} f(x) \to -\infty$
   (c) Does Not Exist; $\lim_{x\to 3^-} f(x) \to \infty$ and $\lim_{x\to 3^+} f(x) \to -\infty$
   (d) Undefined

5. (a)

x	$f(x) = 3x^2$
4.99	74.7003
4.999	74.9700
4.9999	74.9970

(b) 75

6. (a)

x	$f(x) = x^2 + 2x - 7$
−1.99	−7.0199
−1.999	−7.0020
−1.9999	−7.0002

(b) −7

7. (a)

x	$f(x) = \dfrac{x-7}{x+2}$
−3.01	9.9109
−3.001	9.9910
−3.0001	9.9991

(b) 10

8. (a)

x	$f(x) = \dfrac{2x+3}{3-x}$
6.01	−4.9900
6.001	−4.9990
6.0001	−4.9999

(b) −5

9. (a)

x	$f(x) = x^2 + 2x - 4$	x	$f(x) = x^2 + 2x - 4$
0.99	−1.0399	1.01	−0.9599
0.999	−1.0040	1.001	−0.9960
0.9999	−1.0004	1.0001	−0.9996

(b) −1

10. (a)

x	$f(x) = 2x^2 - 3x + 7$	x	$f(x) = 2x^2 - 3x + 7$
−2.01	21.1102	−1.99	20.8902
−2.001	21.0110	−1.999	20.9890
−2.0001	21.0011	−1.9999	20.9989

(b) 21

11. (a)

x	$f(x)=\frac{2x+1}{x+3}$	x	$f(x)=\frac{2x+1}{x+3}$
2.99	1.1653	3.01	1.1681
2.999	1.1665	3.001	1.1668
2.9999	1.1667	3.0001	1.1667

(b) $7/6$

12. (a)

x	$f(x)=\frac{x+2}{x-4}$	x	$f(x)=\frac{x+2}{x-4}$
−4.01	0.2509	−3.99	0.2491
−4.001	0.2501	−3.999	0.2499
−4.0001	0.2500	−3.9999	0.2500

(b) $1/4$

13. (a)

x	$f(x)=\frac{\sqrt{5-x}}{x}$	x	$f(x)=\frac{\sqrt{5-x}}{x}$
0.99	2.0227	1.01	1.9777
0.999	2.0023	1.001	1.9978
0.9999	2.0002	1.0001	1.9998

(b) 2

14. (a)

x	$f(x)=e^{x-3}$	x	$f(x)=e^{x-3}$
2.99	0.9900	3.01	1.0101
2.999	0.9990	3.001	1.0010
2.9999	0.9999	3.0001	1.0001

(b) 1

15. (a)

x	$f(x)=\ln(x+6)$	x	$f(x)=\ln(x+6)$
−5.01	−0.0101	−4.99	0.0100
−5.001	−0.0010	−4.999	0.0010
−5.0001	−0.0001	−4.9999	0.0001

(b) 0

16. (a) Does Not Exist; $\lim_{x\to-4^-} f(x) \to -\infty$

(b) Does Not Exist; $\lim_{x\to-4^+} f(x) \to \infty$

(c) Does Not Exist; $\lim_{x\to-4^-} f(x) \to -\infty$ and $\lim_{x\to4^+} f(x) \to \infty$

(d) 0

(e) 0

(f) 0

17. (a) 1

(b) 1

(c) 1

(d) Does Not Exist; $\lim_{x\to 4^-} f(x) \to \infty$

(e) Does Not Exist; $\lim_{x\to 4^+} f(x) \to \infty$

(f) Does Not Exist; $\lim_{x\to 4} f(x) \to \infty$

18. (a) Does Not Exist

(b) 4

(c) 2

(d) 4

(e) 4

(f) 2

19. (a) Does Not Exist; $\lim_{x\to -6} f(x) \to \infty$

(b) -4

(c) -7

(d) 4

(e) 0

(f) 4

20. (a)

x	$f(x)=\dfrac{x^3-x^2-10x-8}{x-4}$	x	$f(x)=\dfrac{x^3-x^2-10x-8}{x-4}$
3.99	29.8901	4.01	30.1101
3.999	29.9890	4.001	30.0110
3.9999	29.9989	4.0001	30.0011

(b) 30

21. (a)

x	$f(x)=\dfrac{x^3-5x^2+2x+8}{x+1}$	x	$f(x)=\dfrac{x^3-5x^2+2x+8}{x+1}$
-1.01	15.0801	-0.99	14.9201
-1.001	15.0080	-0.999	14.9920
-1.0001	15.0008	-0.9999	14.9992

(b) 15

22. (a)

x	$f(x)=\frac{\sqrt{x}-1}{x-1}$	x	$f(x)=\frac{\sqrt{x}-1}{x-1}$
0.99	0.5013	1.01	0.4988
0.999	0.5001	1.001	0.4999
0.9999	0.5000	1.0001	0.5000

(b) $1/2$

23. (a)

t	$f(t)=\frac{t+2}{t+7}$	t	$f(t)=\frac{t+2}{t+7}$
−7.01	501	−6.99	−499
−7.001	5001	−6.999	−4999
−7.0001	50001	−6.9999	−49999

(b) i. Does Not Exist; $\lim_{t\to -7^-} f(t)\to\infty$

ii. Does Not Exist; $\lim_{t\to -7^+} f(t)\to -\infty$

iii. Does Not Exist; $\lim_{t\to -7^-} f(t)\to\infty$ and $\lim_{t\to -7^+} f(t)\to -\infty$

24. (a)

t	$f(t)=\frac{2t-9}{(t-3)^2}$	t	$f(t)=\frac{2t-9}{(t-3)^2}$
2.99	−30200	3.01	−29800
2.999	−3002000	3.001	−2998000
2.9999	−300020000	3.0001	−299980000

(b) i. Does Not Exist; $\lim_{t\to 3^-} f(t)\to -\infty$

ii. Does Not Exist; $\lim_{t\to 3^+} f(t)\to -\infty$

iii. Does Not Exist; $\lim_{t\to 3} f(t)\to -\infty$

25. (a)

x	$f(x)=\frac{e^{-x}}{x+2}$	x	$f(x)=\frac{e^{-x}}{x+2}$
−2.01	−746.3317	−1.99	731.5534
−2.001	−7396.4489	−1.999	7381.6707
−2.0001	−73897.9504	−1.9999	73883.1723

(b) i. Does Not Exist; $\lim_{t\to -2^-} f(t)\to -\infty$

ii. Does Not Exist; $\lim_{t\to -2^+} f(t)\to\infty$

iii. Does Not Exist; $\lim_{t\to -2^-} f(t)\to -\infty$ and $\lim_{t\to -2^+} f(t)\to\infty$

26. (a)

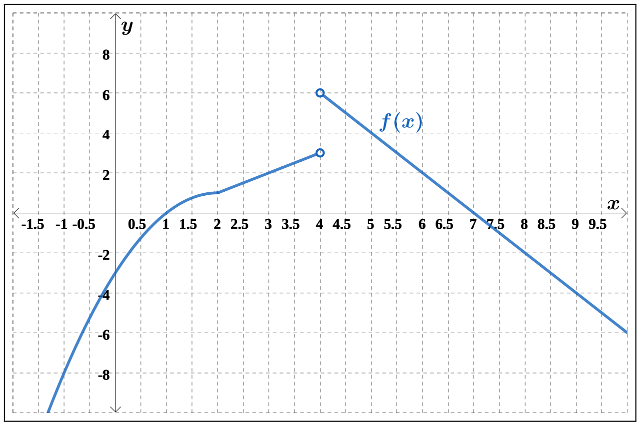
x	$f(x) = \dfrac{\ln(x+4)}{-x}$	x	$f(x) = \dfrac{\ln(x+4)}{-x}$
-0.01	138.3791	0.01	-138.8791
-0.001	1386.0443	0.001	-1386.5443
-0.0001	13862.6936	0.0001	-13863.1936

(b) i. Does Not Exist; $\lim_{t\to 0^-} f(t) \to \infty$

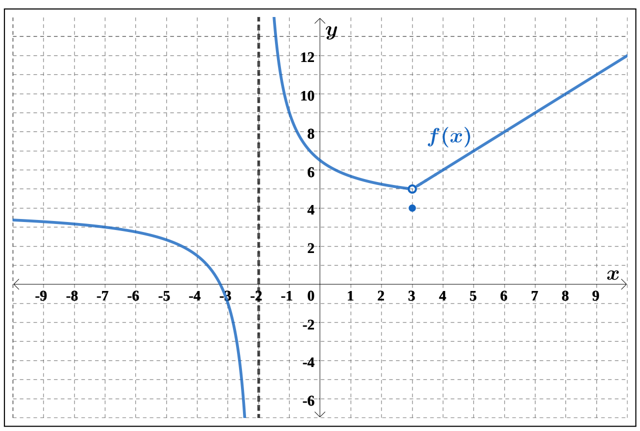
ii. Does Not Exist; $\lim_{t\to 0^+} f(t) \to -\infty$

iii. Does Not Exist; $\lim_{t\to 0^-} f(t) \to \infty$ and $\lim_{t\to 0^+} f(t) \to -\infty$

27.



28.



29. (a)

x	$f(x) = \dfrac{2x(3x-5)}{x^2-x-1}$
1.67	0.2809
1.667	0.0298
1.6667	0.0030

(b) 0

30. (a)

t	$f(t) = \frac{\|t-5\|}{t}$
0.01	499
0.001	4999
0.0001	49999

(b) Does Not Exist; $\lim_{t\to 0^+} f(t) \to \infty$

31. (a)

x	$f(x) = \frac{3x^2-9x+10}{10-7x}$	x	$f(x) = \frac{3x^2-9x+10}{10-7x}$
−2.01	1.6706	−1.99	1.6628
−2.001	1.6671	−1.999	1.6663
−2.0001	1.6667	−1.9999	1.6666

(b) 5/3

32. (a)

x	$f(x) = \frac{\sqrt{x+8}-2}{x+2}$
−2.01	−44.7448
−2.001	−449.2856
−2.0001	−4494.6933

(b) Does Not Exist; $\lim_{x\to -2^-} f(x) \to -\infty$

33. (a)

t	$f(t) = \frac{t^3-8t^2+16t}{2t^4-8t^3}$	t	$f(t) = \frac{t^3-8t^2+16t}{2t^4-8t^3}$
−0.01	−20050	0.01	−19950
−0.001	−2000500	0.001	−1999500
−0.0001	−200005000	0.0001	−199995000

(b) Does Not Exist; $\lim_{t\to 0} f(t) \to -\infty$

34. (a)

x	$f(t) = \frac{x^2-4}{\ln(3-x)}$	t	$f(t) = \frac{x^2-4}{\ln(3-x)}$
1.99	−4.0099	2.01	−3.9899
1.999	−4.0010	2.001	−3.9990
1.9999	−4.0001	2.0001	−3.9999

(b) −4

35. (a)

x	$f(x)=\dfrac{5x^2-9}{x+2}$	x	$f(x)=\dfrac{5x^2-9}{x+2}$
-2.01	-1120.05	-1.99	1080.05
-2.001	-11020.005	-1.999	10980.005
-2.0001	-110020.0005	-1.9999	109980.0005

(b) Does Not Exist; $\lim_{x\to -2^-} f(x) \to -\infty$ and $\lim_{x\to -2^+} f(x) \to \infty$

36. (a)

x	$f(x)=\dfrac{\sqrt{4-x}}{x^2+1}$
2.99	0.1011
2.999	0.1001
2.9999	0.1000

(b) 1/10

37. (a)

x	$f(x)=\dfrac{(x+1)^2-e^x}{x}$	x	$f(x)=\dfrac{(x+1)^2-e^x}{x}$
-0.01	0.9950	0.01	1.0050
-0.001	0.9995	0.001	1.0005
-0.0001	0.9999	0.0001	1.0000

(b) 1

38. As the x-values get closer to $x=c$ from both the left and right, the y-values of $f(x)$ get close to L, where L is a real, finite number.

39. The value of a function is the y-value obtained when we evaluate the function at the specified x-value, assuming it is defined. The value of a limit is the y-value the function approaches when x gets close to a specified x-value from the both the left and right, assuming it exists.

40. Finding a limit graphically means looking at the graph of the function to find the y-value it is heading toward as approaches the relevant x-value from both the left and the right. Finding a limit numerically means evaluating the function at several x-values, which are getting closer to the specified x-value from both the left and the right, to observe the behavior of the values of the function. We usually create a table to display the x-values and their associated y-values when finding a limit numerically.

41. First, a limit may fail to exist because the left- and right-hand limits are not equal. Second, a limit may fail to exist because it is infinite. In other words, the function approaches positive or negative infinity, which means it has a vertical asymptote at the relevant x-value.

42. Not necessarily. For the two-sided limit to exist, both the left- and right-hand limits must exist and be equal. We are only told the limit of $f(x)$ as x approaches 1 from the left equals 5. We are given no information about the limit from the right. Thus, we cannot conclude the two-sided limit exists, nor that it equals 5.

43. Not necessarily. Because the left- and right-hand limits of a function may differ, just because the limit of $f(x)$ as x approaches 1 from the left equals 5 that does not mean the limit from the right exists and equals 5.

44. Yes. The only way a two-sided limit can exist is if the corresponding one-sided limits from the left and right exist and are equal. Thus, if $\lim_{x\to 1} f(x) = 5$, we know $\lim_{x\to 1^-} f(x) = 5$ and $\lim_{x\to 1^+} f(x) = 5$.

45. If $L \neq M$, the function "jumps" at $x = c$ because it is approaching the y-value L from the left of $x = c$ and the y-value M from the right of $x = c$.

46. If $L = M$, the function approaches the same y-value from both the left and right of $x = c$, or in symbols, $\lim_{x\to c} f(x) = L = M$. Note this means there is no "jump" or vertical asymptote at $x = c$.

47. No. Even though the limits from both the left and the right of $x = 2$ tend to the same infinity (negative infinity), the limit as x approaches 2 does not exist because it is infinite. In other words, the function does not approach the same, finite value from both the left and the right of $x = 2$. In fact, recall that neither of the one-sided limits exist because they are both infinite as as well.

Section 1.2

1. 5
2. -5
3. 22
4. 0
5. -1
6. 13
7. (a) -5
   (b) 24
   (c) 11
   (d) 108
8. (a) -21
   (b) $-4/3$
   (c) 5
   (d) 256
9. Does Not Exist
10. Does Not Exist
11. Does Not Exist
12. Does Not Exist
13. 0
14. 54

15. 0

16. 2/3

17. Does Not Exist

18. Does Not Exist

19. Does Not Exist

20. Does Not Exist

21. -1

22. -3

23. 10

24. 41

25. Does Not Exist

26. 9

27. π

28. 70/27

29. 59

30. 4

31. e^3+1

32. (a) -2
    (b) $-1/2$
    (c) -27
    (d) $-1/10$
    (e) 5

33. (a) -9
    (b) 1
    (c) 8
    (d) 36
    (e) $-32/5$

34. -3

35. Does Not Exist; $\lim_{x\to 1^-} \frac{9x+2}{x-1} \to -\infty$ and $\lim_{x\to 1^+} \frac{9x+2}{x-1} \to \infty$

36. 8

37. Does Not Exist; $\lim\limits_{x\to -3^-} \frac{x^2+3x}{x^2+6x+9} \to \infty$ and $\lim\limits_{x\to -3^+} \frac{x^2+3x}{x^2+6x+9} \to -\infty$

38. -7

39. Does Not Exist; $\lim\limits_{x\to 2} \frac{x^2-4x+7}{(x-2)^2} \to \infty$

40. Does Not Exist; $\lim\limits_{x\to 2^-} \frac{x^2-x-2}{x^2-4x+4} \to -\infty$ and $\lim\limits_{x\to 2^+} \frac{x^2-x-2}{x^2-4x+4} \to \infty$

41. $1/6$

42. 1

43. 4

44. 25

45. 30

46. Does Not Exist

47. e^3

48. 3

49. Does Not Exist

50. $-1/2$

51. Does Not Exist; $\lim\limits_{x\to 4^-} f(x) \to -\infty$ and $\lim\limits_{x\to 4^+} f(x) \to \infty$

52. Does Not Exist; $\lim\limits_{x\to -6^-} f(x) \to \infty$

53. 5

54. $-1/2$

55. $-1/2$

56. -1

57. $-2/5$

58. $1/8$

59. $-1/2$

60. 10

61. -4

62. (a) -4
    (b) 21
    (c) 2/3
    (d) Does Not Exist

63. (a) 12
    (b) Does Not Exist
    (c) 30
    (d) -4

64. -1

65. Does Not Exist

66. Does Not Exist

67. $-1/12$

68. $-17/2$

69. 1013/8

70. 2/9

71. Does Not Exist

72. Does Not Exist; $\lim_{x\to -3^-} \dfrac{4e^{x-9}}{x^2-6x-27} \to \infty$

73. 1/10

74. 7

75. Does Not Exist; $\lim_{x\to 2^-} \dfrac{3x^2-11x+10}{x^3-4x^2+4x} \to -\infty$ and $\lim_{x\to 2^+} \dfrac{3x^2-11x+10}{x^3-4x^2+4x} \to \infty$

76. 1/32

77. 1/6

78. Does Not Exist; $\lim_{t\to -4} \dfrac{\ln(t-6)}{t^2+8t+16} \to \infty$

79. -1

80. $-7/32$

81. 7/24

82. Does Not Exist; $\lim_{x\to 0^-} \dfrac{x\cdot 2^x}{14x^3-35x^2} \to \infty$ and $\lim_{x\to 0^+} \dfrac{x\cdot 2^x}{14x^3-35x^2} \to -\infty$

83. 2

84. Does Not Exist

85. $-1/19$

86. $\dfrac{\ln(7)}{3}$

87. $-13/18$

88. $-1/20$

89. Does Not Exist; $\lim_{x\to 2^-} \dfrac{-4x^2}{\ln(3-x)} \to -\infty$

90. 6

91. Does Not Exist; $\lim_{x\to -4} \dfrac{7x-6}{(e^{2x+8}-1)^2} \to -\infty$

92. (a) 15
    (b) Undefined
    (c) $-5/2$

93. (a) $1/11$
    (b) 5
    (c) Does Not Exist; $\lim_{x\to \frac{5}{2}^-} f(x) \to -\infty$ and $\lim_{x\to \frac{5}{2}^+} f(x) \to \infty$

94. (a) Does Not Exist; $\lim_{x\to \frac{1}{4}^-} f(x) \to -\infty$ and $\lim_{x\to \frac{1}{4}^+} f(x) \to \infty$
    (b) 10
    (c) Does Not Exist

95. (a) $-5/3$
    (b) $-29/11$
    (c) Does Not Exist; $\lim_{x\to 4^-} f(x) \to -\infty$ and $\lim_{x\to 4^+} f(x) \to \infty$

96. (a) Does Not Exist; $\lim_{x\to -2^-} f(x) \to \infty$ and $\lim_{x\to -2^+} f(x) \to -\infty$
    (b) -29
    (c) 0

97. (a) e^2
    (b) Does Not Exist
    (c) Does Not Exist; $\lim_{x\to 4^-} f(x) \to -\infty$

98. (a) 1
    (b) Does Not Exist; $\lim_{x\to 0^-} f(x) \to \infty$
    (c) $5/3$

99. $\dfrac{2a(3a+7)}{a+2}$ if $a \neq -2$; Does Not Exist if $a = -2$

100. $\dfrac{-2a+7}{-2a}$

101. $\dfrac{-1}{a^2}$

102. (a) c
   (b) c
   (c) c
   (d) Undefined

103. (a) -1
   (b) 1
   (c) Does Not Exist
   (d) 0

104. Finding a limit graphically means looking at the graph of the function to find the y-value it is heading toward as approaches the relevant x-value from both the left and the right. Finding a limit numerically means evaluating the function at several x-values, which are getting closer to the specified x-value from both the left and the right, to observe the behavior of the values of the function. We usually create a table to display the x-values and their associated y-values when finding a limit numerically. Finding a limit algebraically involves first attempting direct substitution of the relevant x-value. If the result is of indeterminate form, we algebraically manipulate the function and attempt direct substitution again. One main difference between finding limits graphically or numerically versus finding limits algebraically is that the algebraic method is the only method that can result in an exact answer. The other two methods only give an approximation of a limit.

105. The first technique to try is direct substitution. Or, in other words, attempt to evaluate the function at the specified x-value of the limit.

106. If the limit is of indeterminate form, we must algebraically manipulate the function so that it is in a form in which we can try to use direct substitution successfully.

107. It is possible the limit exists, in which case there is a hole in the graph of the function at $x = a$. The only other possibility is that the limit does not exist, which indicates there is a vertical asymptote at $x = a$.

108. Yes. The only possibility is that there is a vertical asymptote at $x = 7$. This is exactly Case 2 in the section!

109. No. It is possible that f has a vertical asymptote at $x = a$! For instance, $\lim_{x\to 1} \dfrac{x-1}{(x-1)^2}$ is of the indeterminate form $\frac{0}{0}$, but the function $f(x) = \dfrac{x-1}{(x-1)^2}$ has a vertical asymptote at $x = 1$.

110. No. The functions are not equal. However, they are equal *almost* everywhere. The only x-value where they are not equal is $x = 2$ because $f(2)$ is undefined.

Section 1.3

1. (a) -4
   (b) -4

2. (a) 8
   (b) -2

3. (a) ∞
   (b) $-\infty$

4. $-\infty$

5. ∞

6. ∞

7. $-\infty$

8. 0

9. $-\infty$

10. ∞

11. $-1/4$

12. 0

13. ∞

14. $-\infty$

15. 0

16. (a) $x = 4$
    (b) $x = -5/2$

17. (a) $x = 9$
    (b) $x = 5/4$

18. (a) None
    (b) $x = -3/2$; $x = 6$

19. (a) None
    (b) $x = -1$; $x = 9/2$

20. None

21. None

22. $y = 0$

23. None

24. $y = -9/2$

25. None

26. $y = 0$

27. $y = 0$

28. ∞

29. $-\infty$

30. $-\infty$

31. ∞

32. 0

33. -7

34. 0

35. $-1/2$

36. ∞

37. 0

38. (a) $(-7/2, 0)$

    (b) $x = -3$, where $\lim_{x\to -3^-} f(x) \to \infty$ and $\lim_{x\to -3^+} f(x) \to -\infty$

39. (a) None

    (b) $x = -4$, where $\lim_{x\to -4^-} f(x) \to -\infty$ and $\lim_{x\to -4^+} f(x) \to \infty$; $x = 8/3$, where $\lim_{x\to \frac{8}{3}^-} f(x) \to \infty$ and $\lim_{x\to \frac{8}{3}^+} f(x) \to -\infty$

40. (a) $(4, 11/7)$

    (b) $x = -3$, where $\lim_{x\to -3^-} f(x) \to -\infty$ and $\lim_{x\to -3^+} f(x) \to \infty$

41. (a) $(1/2, -1)$

    (b) $x = 4$, where $\lim_{x\to 4^-} f(x) \to -\infty$ and $\lim_{x\to 4^+} f(x) \to \infty$

42. (a) $(0, 2)$

    (b) $x = -1/2$, where $\lim_{x\to -\frac{1}{2}^-} f(x) \to -\infty$ and $\lim_{x\to -\frac{1}{2}^+} f(x) \to \infty$; $x = 2$, where $\lim_{x\to 2^-} f(x) \to -\infty$ and $\lim_{x\to 2^+} f(x) \to \infty$

43. (a) None
    (b) $x = 0$, where $\lim_{x\to 0^-} f(x) \to \infty$ and $\lim_{x\to 0^+} f(x) \to -\infty$; $x = 1$, where $\lim_{x\to 1^-} f(x) \to -\infty$ and $\lim_{x\to 1^+} f(x) \to \infty$; $x = 3$, where $\lim_{x\to 3^-} f(x) \to \infty$ and $\lim_{x\to 3^+} f(x) \to -\infty$

44. $y = 7$

45. $y = 10/7$

46. None: $\lim_{x\to -\infty} f(x) \to \infty$ and $\lim_{x\to \infty} f(x) \to -\infty$

47. $y = 0$

48. $y = 0$; $y = 5$

49. $y = 0$; $y = 2$

50. $y = -3/8$; $y = 7/4$

51. $y = -5$; $y = -1/3$

52. (a) $y = 2/3$
    (b) None
    (c) $x = -5$; $x = 1/3$

53. (a) $y = 2$
    (b) $(3/2, 12/11)$
    (c) $x = -4$

54. (a) None
    (b) $(-4, -28/5)$
    (c) $x = 1$

55. (a) $y = 0$
    (b) $(-2, -5/18)$
    (c) $x = 0$; $x = 7$

56. ∞

57. 0

58. $-4/7$

59. ∞

60. ∞

61. $-9/2$

62. $-\infty$

63. $-\infty$

64. 0

65. $-\infty$

66. $-\infty$

67. $y = 0$

68. $y = \pi/2$

69. $y = 0$ when $x \to -\infty$; None when $x \to \infty$: $\lim_{x\to\infty} f(x) \to \infty$

70. None: $\lim_{x\to-\infty} f(x) \to -\infty$ and $\lim_{x\to\infty} f(x) \to -\infty$

71. None: $\lim_{x\to-\infty} f(x) \to \infty$ and $\lim_{x\to\infty} f(x) \to -\infty$

72. $y = 0$; $y = -4/3$

73. $y = 0$

74. (a) None: $\lim_{x\to-\infty} f(x) \to \infty$ and $\lim_{x\to\infty} f(x) \to \infty$
    (b) $(-9, 207)$
    (c) None

75. (a) $y = 8/5$
    (b) None
    (c) $x = -7/5$, where $\lim_{x\to-\frac{7}{5}^-} f(x) \to -\infty$ and $\lim_{x\to-\frac{7}{5}^+} f(x) \to \infty$; $x = 3$, where $\lim_{x\to3^-} f(x) \to -\infty$ and $\lim_{x\to3^+} f(x) \to \infty$

76. (a) $y = 0$
    (b) $(-1, -21/16)$
    (c) $x = 1$, where $\lim_{x\to1^-} f(x) \to \infty$ and $\lim_{x\to1^+} f(x) \to -\infty$

77. (a) $-\infty$
    (b) ∞

78. (a) 0
    (b) $1/8$

79. (a) 0
    (b) $-\infty$

80. (a) 0
    (b) -91

81. (a) $-\infty$
    (b) ∞

82. 0

83. $\dfrac{2}{a}$

84. $-\infty$

85. $-1/3$

86. (a) $x = -2$
    (b) $x = 0$

87. (a) $x = -1$ if $a = -1$; None if $a \neq -1$
    (b) $x = a$ if $a \neq -1$; $x = 1/2$ for all values of a

88. (a) $x = a$ if $a \neq 0$; None if $a = 0$
    (b) $x = 0$

89. (a) $x = a$ if $a \neq -1$; None if $a = -1$
    (b) $x = -1$

90. $y = \dfrac{5x-24}{x-5}$ (Answers may vary.)

91. $y = \dfrac{-2x^2+x+3}{x^2-1}$ (Answers may vary.)

92. ∞

93. ∞

94. $y = \dfrac{a}{b}$

95. $-\infty$

96. $\lim_{x\to\infty} g(x) \to -\infty$ and $\lim_{x\to-\infty} g(x) \to -\infty$

97. We find the limits at infinity of the function f. In other words, we find $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$.

98. Yes. A function has a horizontal asymptote if one or both of the limits at infinity exist(s). If neither of these limits exists, then there is no horizontal asymptote.

99. No. A function can have at most two horizontal asymptotes: one for the limit as x approaches positive infinity and one for the limit as x approaches negative infinity.

100. Not necessarily. A function may have the same infinite behavior near a vertical asymptote on both the left and right. For example, consider the function $f(x) = \dfrac{1}{(x-1)^2}$. The graph of f has a vertical asymptote at $x = 1$, and $\lim_{x\to 1^-} f(x) \to \infty$ and $\lim_{x\to 1^+} f(x) \to \infty$.

101. Yes. The graph will approach the line $x = 5$ because the function has infinite behavior (approaches positive infinity) as x approaches 5 (from both the left and right in this case).

102. $f(x)$ approaches negative infinity on the left of $x = 1$ and positive infinity on the right of $x = 1$. However, $g(x)$ approaches positive infinity on both the left and right of $x = 1$.

103. To find the horizontal asymptotes of a rational function, we calculate the limits at infinity by dividing the numerator and denominator by the highest power of x in the denominator and observing the behavior of each term in the resulting function.

104. An asymptote, either horizontal or vertical, is a line the graph of a function approaches. Horizontal asymptotes are lines of the form $y = L$, and vertical asymptotes are written as $x = c$. The concept of infinity underlies both of these ideas.

     For horizontal asymptotes, x approaches positive or negative infinity and the y-values approach a finite number L. However, for vertical asymptotes, x approaches a finite number c and the y-values approach positive or negative infinity. Another difference is that a function can have at most two horizontal asymptotes, but it may have infinitely many vertical asymptotes.

105. The graph of a function will have a hole at $x = a$ if the factor $x - a$ is in both the numerator and denominator of the function and it divides completely from the denominator (the factor $x - a$ may still remain in the numerator after dividing). However, the graph will have a vertical asymptote at $x = a$ if the factor $x - a$ remains in the denominator after dividing common factors.

106. The function $f(x) = x^2 + 2x - 4$ is a polynomial. Thus, its graph will not have any holes or vertical asymptotes because its domain is all real numbers. The graph of the function will not have any horizontal asymptotes because the limits at infinity (i.e., end behavior) of a polynomial tend to positive or negative infinity.

Section 1.4

1. $x = -3$, $f(-3)$ is undefined.

2. $x = 3$, $f(3)$ is undefined.

3. $x = 0$, $\lim_{x \to 0} f(x)$ does not exist.

4. $x = 5$, $\lim_{x \to 5} f(x) \neq f(5)$.

5. Continuous

6. Continuous

7. Continuous

8. Not Continuous, $f(5)$ is undefined.

9. Not Continuous, $f(7)$ is undefined.

10. Continuous

11. $(-\infty, \infty)$

12. $(-\infty, \infty)$

13. $(-\infty, 5) \cup (5, \infty)$

14. $(-\infty, -3) \cup (-3, \infty)$

15. $(-\infty, -3/2) \cup (-3/2, 3) \cup (3, \infty)$

16. $(-\infty, 1/7) \cup (1/7, 5/3) \cup (5/3, \infty)$

17. $(-\infty, \infty)$

18. $(-1, \infty)$

19. $(-\infty, \infty)$

20. $(-\infty, \infty)$

21. $(5/2, \infty)$

22. $(-\infty, 8/3)$

23. $x = -4$, $f(-4)$ is undefined; $x = 2$, $f(2)$ is undefined.

24. $x = -1$, $f(-1)$ is undefined; $x = 4$, $\lim_{x\to 4} f(x) \neq f(4)$.

25. $x = -3$, $\lim_{x\to -3} f(x)$ does not exist; $x = 4$, $\lim_{x\to 4} f(x) \neq f(4)$; $x = 6$, $f(6)$ is undefined.

26. $x = -6$, $f(-6)$ is undefined; $x = -3$, $\lim_{x\to -3} f(x)$ does not exist; $x = 3$, $\lim_{x\to 3} f(x) \neq f(3)$; $x = 6$, $\lim_{x\to 6} f(x)$ does not exist.

27. Continuous

28. Not Continuous, $f(7)$ is undefined.

29. Not Continuous, $f(8)$ is undefined.

30. Continuous

31. Not Continuous, $\lim_{x\to 2} f(x)$ does not exist.

32. Continuous

33. Not Continuous, $f(8)$ is undefined.

34. Not Continuous, $\lim_{x\to -2} f(x)$ does not exist.

35. $(4, \infty)$

36. $[-1, 1]$

37. $(7/2, \infty)$

38. $(1/9, \infty)$

39. $(-\infty, \infty)$

40. $(-\infty,-4)\cup(-4,6)\cup(6,\infty)$

41. $(-\infty,0)\cup(0,9]$

42. $(8,\infty)$

43. $(-\infty,\infty)$

44. $(-\infty,-3)\cup(-3,\infty)$

45. $(-\infty,\infty)$

46. $(-\infty,3)\cup(3,8)\cup(8,\infty)$

47. $(-\infty,1/2)\cup(1/2,\infty)$

48. $(-\infty,-5)\cup(-5,3)\cup(3,\infty)$

49. $(-\infty,-4)\cup(-4,\infty)$

50. $(-\infty,-2)\cup(-2,\infty)$

51. $(-\infty,6)\cup(6,\infty)$

52. $(-\infty,1)\cup(1,\infty)$

53. $(-\infty,\infty)$

54. $(-\infty,-7)\cup(-7,\infty)$

55. $k=9$

56. $k=-10$

57. $k=2$

58. $k=51/5$

59. $k=1/3$

60. $k=9/5$

61. $k=-2$

62. $k=-8$

63. $k=27/16$

64. $k=-10$

65. $(2,\infty)$

66. $(0,2/3)\cup(2/3,\infty)$

67. $(-\infty,\infty)$

68. $(-\infty,-4)\cup(-4,-3)\cup(-3,4)\cup(4,\infty)$

69. $[-5,1)\cup(1,2)$

70. $[-3/2,\infty)$

71. $(-3/7,-2/7)\cup(-2/7,\infty)$

72. $(-\infty,-5)\cup(-5,0)\cup(0,5)\cup(5,\infty)$

73. $[-4,5)\cup(5,13)$

74. $(3,628)\cup(628,\infty)$

75. $(-4,0)\cup(0,3)\cup(3,\infty)$

76. $(-\infty,-4)\cup(-4,-1)\cup(-1,0)\cup(0,\infty)$

77. $(-8,-23/3)\cup(-23/3,0)\cup(0,\infty)$

78. $(-\infty,-9)\cup(-9,-2)\cup(-2,9)$

79. $(-\infty,\infty)$

80. $(-\infty,-3)\cup(-3,-2)\cup(-2,\infty)$

81. $(-\infty,2)\cup(2,\infty)$

82. $(-\infty,-2)\cup(-2,5)\cup(5,\infty)$

83. $(-\infty,3)\cup(3,5)\cup(5,7)$

84. No such value of k exists. $\lim_{x\to 0^-} f(x) \neq f(0)$, so there is no way for the two-sided limit to equal $f(0)$.

85. $k = 9/16$

86. $k = 3$

87. $k = -13/16$

88. There is no such value of k because $\ln(2x+7)$ does not exist if $x < -7/2$.

89. k can be any value.

90. $k = -3$; $k = 2$

91. $k = 89/9$

92. $k = \dfrac{\ln(5)}{2}$

93. $k = 19/36$

94. $k = 12 - \ln(23)$

95. $k = 13/2$

96. $x = d$, $\lim_{x\to d} f(x)$ does not exist; $x = e$, $f(e)$ is undefined; $x = g$, $f(g)$ is undefined; $x = i$, $\lim_{x\to i} f(x) \neq f(i)$.

97. The function must be defined at $x = 5$, the limit as x approaches 5 must exist, and the value of the limit must equal the function value.

98. Yes. To be continuous, $\lim_{x\to c} f(x)$ must equal $f(c)$, which naturally implies the limit exists.

99. Yes. To be continuous, the two-sided limit, $\lim_{x\to c} f(x)$, must exist and be equal to $f(c)$. Because the two-sided limit exists, both the limit from the left and the limit from the right must also be equal to $f(c)$. In other words, $\lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = f(c)$.

100. To find where a function that is not piecewise-defined is continuous, we only need to find the domain of the function because it will be continuous on its domain.

101. First, we must find the domain of the piecewise-defined function. To do that, we find any domain restrictions for each rule and compare the restrictions to the x-values defined for that rule. Next, we check for continuity at each cutoff number using the three conditions of continuity at a point (the function is defined, the limit exists, and the value of the limit equals the function value). Finally, we find the solution that incorporates the continuity (domain) within each rule and the continuity (or not) of the cutoff numbers.

Section 2.1

1. (a) $8/5$
   (b) 0
   (c) $-19/6$

2. (a) 0
   (b) -0.25
   (c) 5.62

3. 4

4. -9

5. 0.453

6. -8

7. 8

8. 1

9. 7

10. -2

11. $y = 3x + 1$

12. $y = -7x + 98$

13. $y = 9$

14. (a) dollars per book
    (b) hit points per second
    (c) miles per gallon
    (d) donuts per dollar

15. (a) −$0.01 per chicken sandwich
    (b) −$0.01 per chicken sandwich

16. (a) −$6.50 per chicken sandwich
    (b) −$3.50 per chicken sandwich

17. (a) $75 per flux capacitor
    (b) $75 per flux capacitor

18. (a) 4 feet per second
    (b) 4 feet per second

19. (a) $3.06 per duck; When the first 350 hand carved ducks are made, cost is increasing at an average rate of $3.06 per duck.
    (b) $9.86 per duck; When the number of hand carved ducks made is between 500 and 700, cost is increasing at an average rate of $9.86 per duck.

20. (a) 1046.97 feet per second
    (b) 1076.45 feet per second

21. (a)
        i. $-10/3$
        ii. -2
        iii. -1
        iv. 9
        v. 4
        vi. 1
    (b) 0

22. (a)
        i. 12
        ii. 14.25
        iii. 16.41
        iv. 24
        v. 20.25
        vi. 17.61
    (b) 17

23. (a) i. 0.107
    ii. 0.104
    iii. 0.102
    iv. 0.096
    v. 0.098
    vi. 0.100
  (b) 0.101

24. (a) i. 0.746
    ii. 0.748
    iii. 0.750
    iv. 0.746
    v. 0.748
    vi. 0.750
  (b) 3/4

25. (a) i. −$1.90 per coffee cup
    ii. −$1.08 per coffee cup
    iii. −$0.42 per coffee cup
    iv. −$1.30 per coffee cup
    v. −$1.12 per coffee cup
    vi. −$0.61 per coffee cup
  (b) −$0.50 per coffee cup

26. (a) i. $0.504 per stick
    ii. $0.503 per stick
    iii. $0.501 per stick
    iv. $0.496 per stick
    v. $0.498 per stick
    vi. $0.499 per stick
  (b) $0.50 per stick

27. (a) i. −$3.26 per book
    ii. −$3.51 per book
    iii. −$3.75 per book
    iv. −$4.54 per book
    v. −$4.36 per book
    vi. −$4.17 per book
  (b) −$4.00 per book

28. (a) i. 1.067 meters per second squared
    ii. 0.933 meters per second squared
    iii. 0.827 meters per second squared
    iv. 0.533 meters per second squared
    v. 0.667 meters per second squared
    vi. 0.773 meters per second squared
   (b) 0.8 meters per second squared

29. 2

30. 3.9

31. 0.2485

32. $-1/3$

33. $-11/20$

34. 1

35. $-1/4$

36. $1/2$

37. $1/8$

38. $y = 12x - 40$

39. $y = 9x + 8$

40. $y = -7x + 7$

41. 29.2 thousand dollars per month; After one year, the amount of sales is increasing at a rate of 29.2 thousand dollars per month (or $29,200 per month).

42. −$5600 per year; After seven years, the car's value is decreasing at a rate of $5600 per year.

43. 20 thousand dollars per ghost; When 40 ghosts are busted, profit is increasing at a rate of 20 thousand dollars per ghost (or $20,000 per ghost).

44. 8 feet per second

45. $24 + 3h$

46. (a) blizzards per month
   (b) yo-yos per dollar
   (c) puzzles per hour
   (d) slime balls per bottle of glue

47. 0.68 feet per second

48. -0.2

49. \$10.36 per item; When 100 items are sold, revenue is increasing at a rate of \$10.36 per item.

50. 0.25 meters per second

51. -2.31

52. -0.0796

53. 0.8333

54. -7

55. 7

56. 20

57. -1

58. -16

59. $y = -\frac{43}{9}t + \frac{56}{3}$

60. $y = -\frac{6}{5}x + \frac{17}{5}$

61. −\$3.82 per item; When the number of items sold is between 600 and 675, revenue is decreasing at an average rate of \$3.82 per item.

62. (a) \$1.50 per dinner; When 75 dinners are sold, profit is increasing at a rate of \$1.50 per dinner.
    (b) \$0.00 per dinner; When 100 dinners are sold, profit is neither increasing nor decreasing.
    (c) −\$2.10 per dinner; When 135 dinners are sold, profit is decreasing at a rate of \$2.10 per dinner.

63. −390 feet per second

64. (a) \$10.56 per camera; When the number of cameras produced each week is between 20 and 25, cost is increasing at an average rate of \$10.56 per camera.
    (b) \$25.82 per camera; When the first 15 cameras are produced each week, cost is increasing at an average rate of \$25.82 per camera.
    (c) \$12.13 per camera; When 17 cameras are produced each week, cost is increasing at a rate of \$12.13 per camera.

65. −9 feet per second

66. \$10 per gift basket; When 10 gift baskets are made and sold, profit is increasing at a rate of \$10 per gift basket.

67. -3

68. (a) i. -4
    ii. -4
    iii. -4
    iv. 4
    v. 4
    vi. 4
  (b) Does Not Exist

69. (a) $2a$
  (b) $y = 2ax - 2a^2 + f(a)$

70. The slopes of secant lines are determined using two points on the graph of a function. The limit of these slopes is the slope of the tangent line, which is determined by a single point on the graph of a function.

71. No. The average rate of change is the slope of the secant line, or the slope of the line determined by two points on a curve. The limit of the slopes of the secant lines yields the slope of the tangent line, which is equivalent to the instantaneous rate of change.

72. When the number of items sold in a week is between 450 and 500, revenue is increasing at an average rate of $20,000 per item.

73. When 800 items are made in a week, cost is increasing at a rate of $650 per item.

74. No. The average rate of change of $85 per item is only applicable when the number of items sold is between 100 and 125.

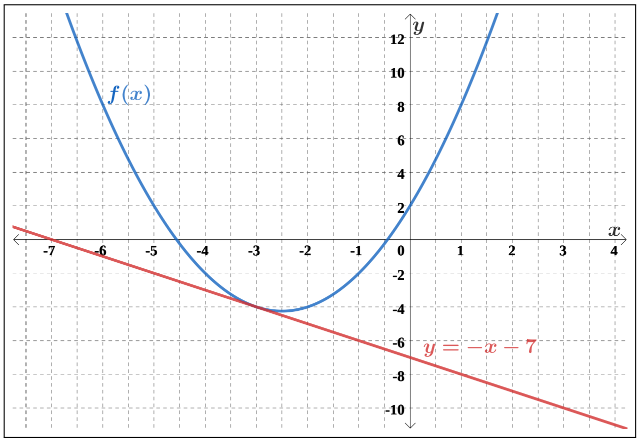
Section 2.2

1. $f'(x) = 4$
2. $f'(x) = -3$
3. $f'(x) = 2$
4. $\frac{dy}{dx} = 2x$
5. $\frac{dy}{dx} = 6x - 6$
6. $\frac{dy}{dx} = -10x + 2$
7. 9
8. 6
9. 5
10. $y = 5x - 13$

11. $y = -29x - 20$

12. $y = -6x + 7$

13. $y = -x - 7$;



14. $x = -3$; $x = 2$

15. $x = -1$; $x = 0$; $x = 5$

16. $x = \dfrac{-4 + \sqrt{40}}{4}$; $x = \dfrac{-4 - \sqrt{40}}{4}$

17. $-\$0.20$ per pair of sneakers

18. $-\$93$ per pair of sneakers

19. \$50 per handbag

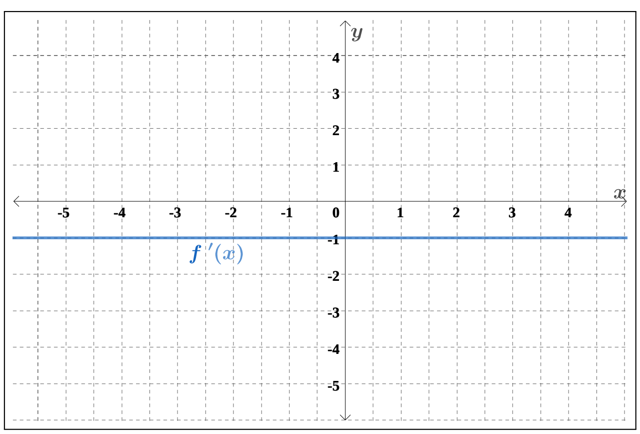
20. $v(t) = 8t - 5$ feet per second

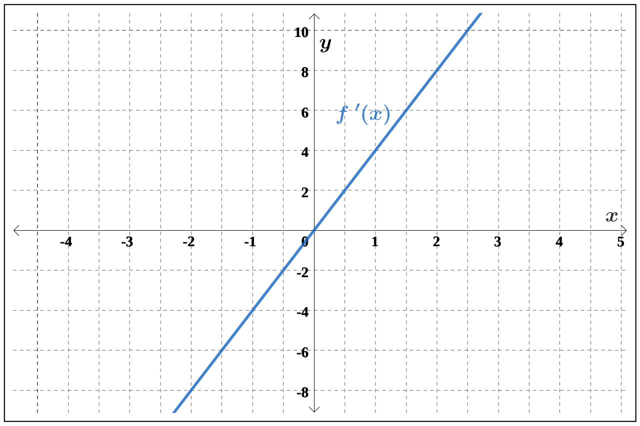
21. $x = -3$

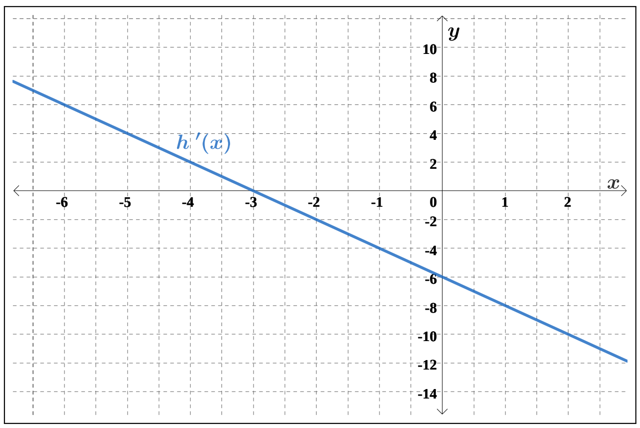
22. $x = 4$

23. $x = 5$

24.



25. 

26. 

27. $f'(x) = \dfrac{-2}{x^2}$

28. $f'(x) = \dfrac{-20}{(5x+3)^2}$

29. $f'(x) = \dfrac{8}{(8-x)^2}$

30. $\dfrac{dy}{dx} = \dfrac{-21}{(x-7)^2}$

31. $\dfrac{dy}{dx} = \dfrac{-1}{2\sqrt{5-x}}$

32. $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{2x+1}}$

33. $-6/25$

34. $1/20$

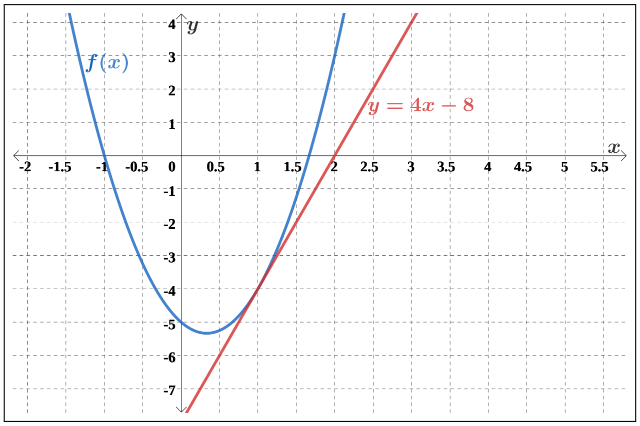
35. $1/8$

36. $y = -25x - 35$

37. $y = x - 6$

38. $y = 7x - 32$

39. $y = -\frac{1}{4}x + \frac{13}{4}$

40. $y = 4x - 8$; 

41. $x = 0$

42. $x = 5/6$

43. $x = 2$

44. $x = 8$

45. $x = -6$

46. $x = 4$

47. \$22 per board game; When 300 board games are sold, revenue is increasing at a rate of \$22 per board game.

48. \$192 per dryer; When 450 dryers are made, cost is increasing at a rate of \$192 per dryer.

49. −\$14 per microwave; When 900 microwaves are made and sold, profit is decreasing at a rate of \$14 per microwave.

50. 10 million dollars per month; After 7 months, sales are increasing at a rate of 10 million dollars per month (or \$10,000,000 per month).

51. 18 feet per second

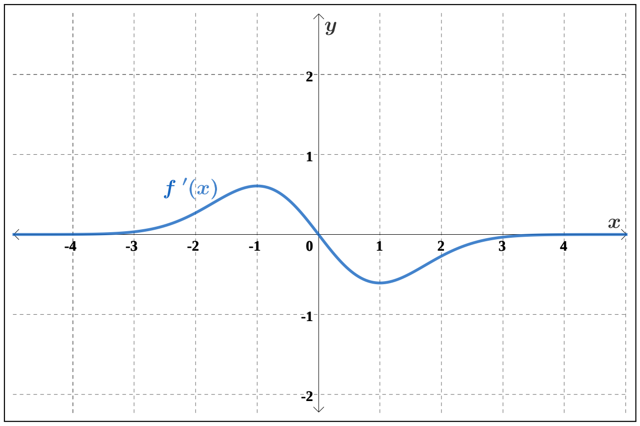
52. $x = 2$ because the graph has a cusp, so there is no tangent line; $x = 6$ because the graph has a corner, so there is no tangent line

53. $x = 2$ because the graph has a corner, so there is no tangent line

54. $x = -2$, $x = 1$, and $x = 2$ because the graph has discontinuities, so there are no tangent lines

55. $x = -5$ because the graph has a vertical tangent line; $x = 3$ because the graph has a discontinuity, so there is no tangent line; $x = 6$ because the graph has a corner, so there is no tangent line

56.



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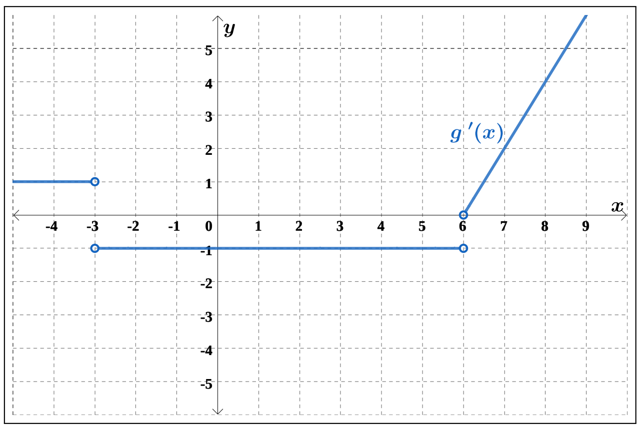

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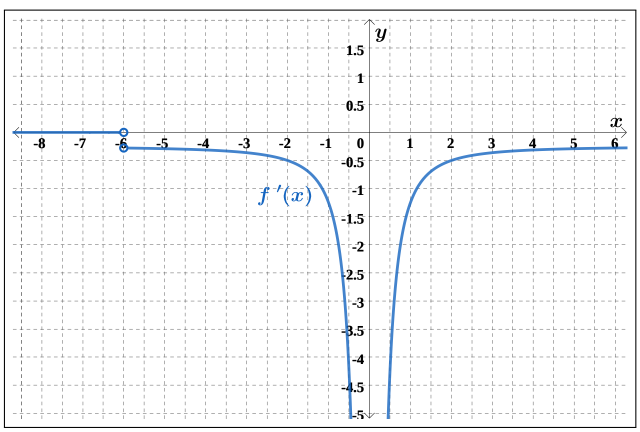
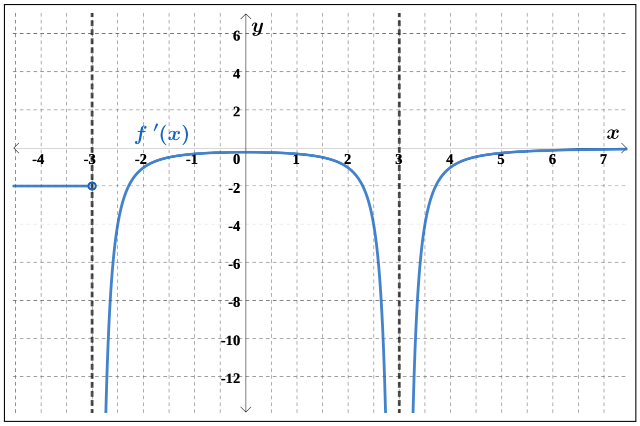


59. 

60. $\frac{df}{dx} = a$

61. $f'(t) = -6t - 0.5$

62. $y' = \frac{34}{(17-x)^2}$

63. $\frac{dy}{dx} = \frac{-46}{(8x+10)^2}$

64. $f'(t) = \frac{t^2-2t-4}{(t-1)^2}$

65. $q'(x) = \frac{-1}{2\sqrt{x}}$

66. $f'(x) = \frac{-1}{\sqrt{7-2x}}$

67. $\frac{dh}{dt} = \frac{5}{2\sqrt{5t+9}} - 4$

68. $y' = 3x^2 + 4$

69. $\frac{3}{2\sqrt{3a+50}}$

70. $y = -2x + 4$

71. $y = \frac{2}{5}x + \frac{17}{5}$

72. $y = 10t + 17$

73. $x = 0$; $x = 2/3$

74. $x = \frac{-b}{2a}$

75. 4

76. $t = 3$

77. (a) $-\$40$ per bike; When 150 bikes are sold, revenue is decreasing at a rate of \$40 per bike.
    (b) \$104 per bike; When 60 bikes are sold, revenue is increasing at a rate of \$104 per bike.
    (c) \$0.00 per bike; When 125 bikes are sold, revenue is neither increasing nor decreasing.

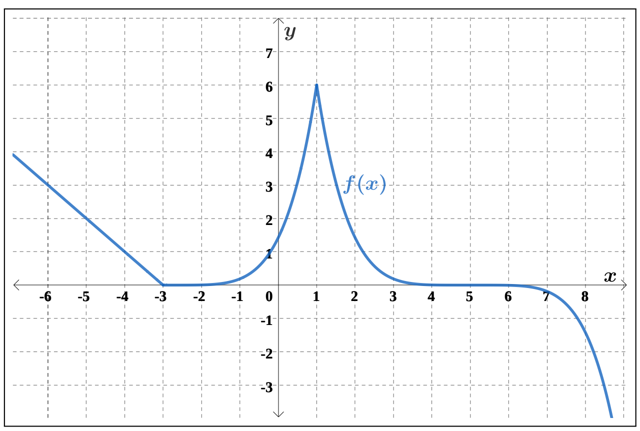
78. (a) $-\$3$ per curling iron; When 170 curling irons are made and sold, profit is decreasing at a rate of \$3 per curling iron.
    (b) The manufacturer should not increase production because $P'(170)$ is negative. Increasing production will decrease profit.

79. $v(t) = \dfrac{1-t^2}{(1+t^2)^2}$ feet per second

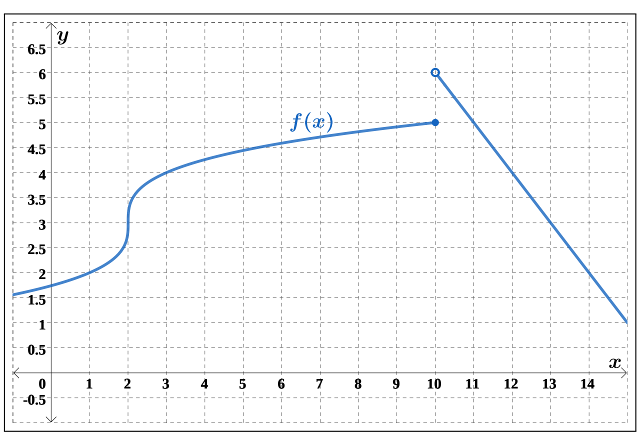
80. 0.28867513 million dollars per month; After one year, sales are increasing at a rate of 0.28867513 million dollars per month (or \$288,675.13 per month).

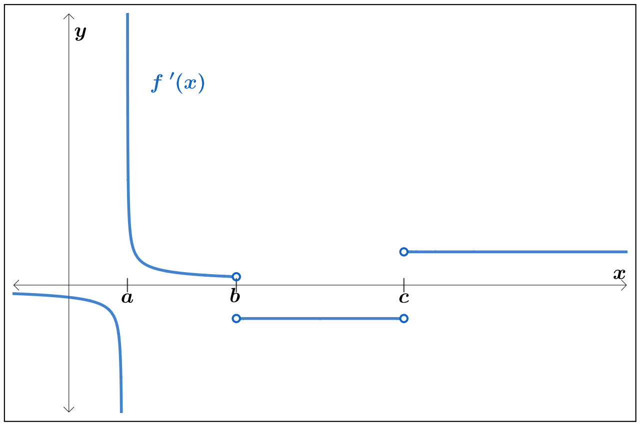
81. (a) 64 feet per second
    (b) $t = 4$ seconds

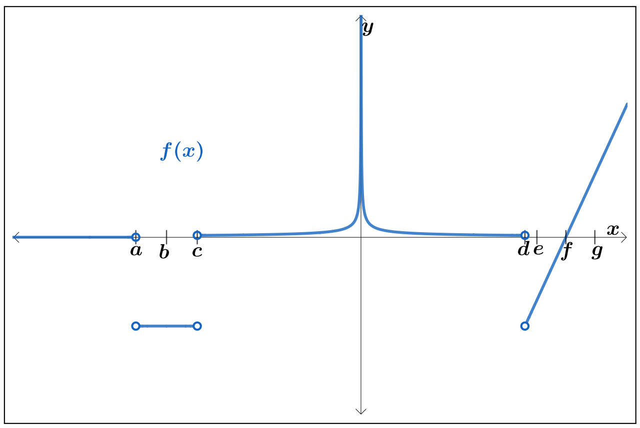
82.

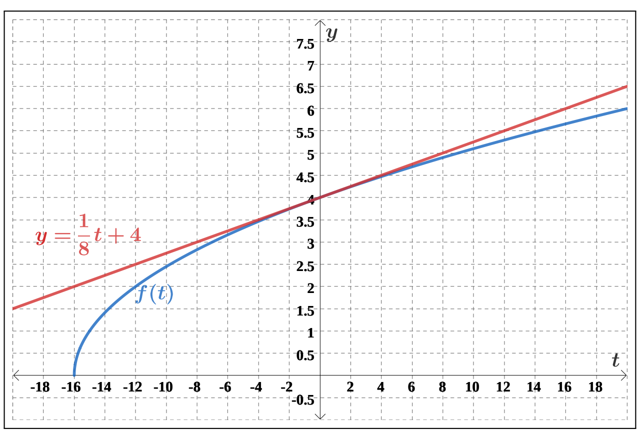


83.



84. 

85. 

86. $y = \frac{1}{8}t + 4$; 

87. No. The limit needs to have h approaching 0 to represent $f'(x)$, not x.

88. First, we find the derivative of the function. Then, we set the derivative equal to zero and solve for the x-value(s) of that equation.

89. When the company sells 40 items each week, profit is increasing at a rate of \$80 per item.

90. There is no tangent line at a corner on a graph because the slopes of the secant lines from the left and right do not approach the same value.

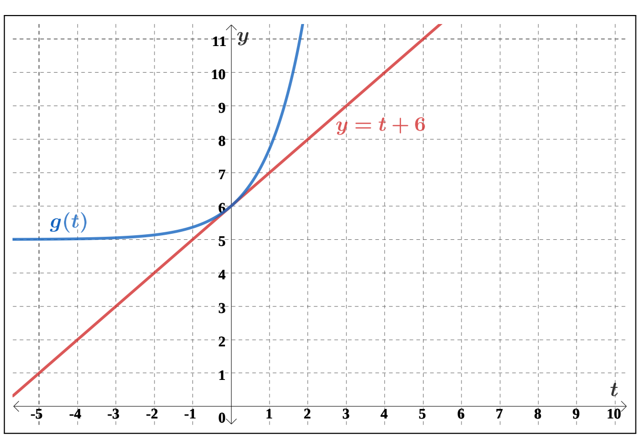
91. Not necessarily. It is possible that a function has a vertical tangent line, corner, or, cusp at $x = c$. In any of these situations, the function would be continuous at $x = c$, but not differentiable.

92. Yes. If f was not continuous at $x = 8$, then it could not be differentiable because there would be no tangent line at $x = 8$.

93. No. If g has a discontinuity at $x = 4$, then there is no tangent line.

Section 2.3

1. $f'(x) = 8x$
2. $f'(x) = -9x^2$
3. $f'(x) = -20x^{-5} - 9$
4. $y' = 12x - 4 + e^x$
5. $y' = \dfrac{1}{x} - 7x^6$
6. $y' = \dfrac{e^x}{5} - 17 - \dfrac{1}{x}$
7. 431.5
8. 1
9. 11
10. $y = \dfrac{637}{128}x + \dfrac{1}{8}$
11. $y = -161x - 221$
12. $y = -80x + 84$
13. $y = 26$
14. $y = 31x - 18$
15. $y = t + 6$;



16. $x = 1/3$; $x = 2$

17. $x = -3$; $x = 0$

18. $x = -1$; $x = 1$

19. \$5.60 per barbeque dinner

20. \$3.16 per calculator

21. (a) \$3.00 per flat iron
    (b) −\$5.00 per flat iron

22. 32 feet per second

23. (a) −2
    (b) 0
    (c) 34

24. (a) \$21.82 per camera
    (b) \$22.36 per camera
    (c) \$21.82
    (d) \$21.56
    (e) \$1559.40
    (f) \$1559.17

25. (a) \$322.16 per lawn mower
    (b) \$322.00 per lawn mower
    (c) \$322.00
    (d) \$322.04
    (e) \$116,955.48
    (f) \$116,955.84

26. (a) \$384.40 per blender
    (b) \$382.60 per blender
    (c) \$384.40
    (d) \$384.10
    (e) \$55,152.60
    (f) \$55,152.30

27. $f(x) = \frac{1}{4}x^{-3/4} - \frac{2}{7}x^{-6/7} + 10e^x - \frac{2}{x}$

28. $f(x) = \frac{6}{x} - 6x + 25e^x - \frac{15}{8}x^{-5/8}$

29. $f(x) = -\frac{5}{18}x^{-1/6} + 8x + 9^x \ln(9)$

30. $f(x) = -21x^2 + \frac{2}{3}x^{-1/3} - 5 \cdot 4^x \ln(4)$

31. $f(x) = 6x^2 - 7 + \frac{1}{x\ln(4)}$

32. $f(x) = -\frac{14}{55}x^{-7/5} - \frac{3}{x\ln(6)}$

33. $f(x) = 0.006x^{-0.8} - \frac{16}{3}x^{-29/3} + 6 \cdot 100^x \ln(100)$

34. $\frac{dy}{dx} = -12x^{-5} + 4x^{-3} + \frac{12}{7}x^{-5}$

35. $\frac{dy}{dx} = \frac{1}{2}x^{-1/2} + \frac{8}{x\ln(5)} - \frac{36}{10}x^{-19/10}$

36. $\frac{dy}{dx} = \frac{10}{x} - \frac{2}{3}x^{-1/3} + 15x^4$

37. $\frac{dy}{dx} = \frac{24}{5}x^{-2/5} + 9 \cdot 30^x \ln(30) - \frac{27}{4}x^{-3/4}$

38. $f'(x) = 24x^7 + 60x^4 - 45x^2 + 12x$

39. $g'(x) = 4x^3 - 18x^2 + 1$

40. $h'(t) = -6t^{-2} - 4t^{-3} + 24t - 17$

41. $y' = -2x^{-2} - 6x^5 + 84x^{-5}$

42. $f'(t) = -\frac{3}{5}t^{-2} - \frac{16}{5}t^{-9} + 4t^{-6} - \frac{48}{5}t^{-7}$

43. $h(x) = 20x^5 + 3x^{-2} + 10x$

44. 41

45. $3e + 18$

46. 599/4

47. 87/2

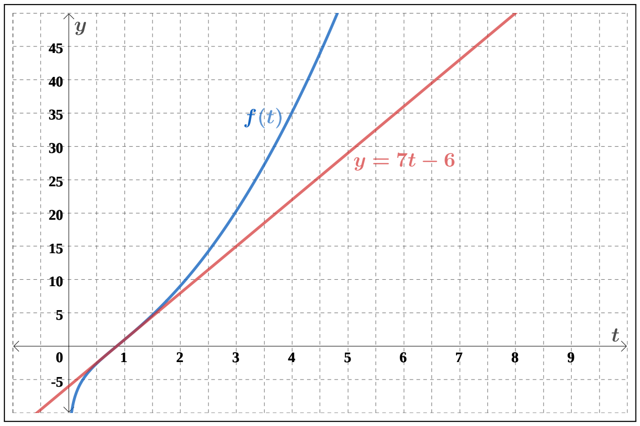
48. $y = 18x - 17$

49. $y = (2\ln(10))x + 5$

50. $y = \frac{307}{2}x + \frac{571}{2}$

51. $y = -\frac{22}{3}x + 6$

52. $y = 7t - 6$;



53. $x = \ln(5/2)$

54. $x = -1/2$; $x = 1/3$

55. $x = 1/3$; $x = 1$

56. $x = 8/13$

57. $x = -7$; $x = 1/3$

58. $x = -2$; $x = 2$

59. (a) $97/3$

    (b) -336

    (c) $-2\sqrt{2} + 5$

60. (a) \$95 per karaoke machine; When 225 karaoke machines are sold, revenue is increasing at a rate of \$95 per karaoke machine.

    (b) −\$15 per karaoke machine; When 500 karaoke machines are sold, revenue is decreasing at a rate of \$15 per karaoke machine.

    (c) \$127.40

    (d) \$127.20

    (e) \$37,500.20

    (f) \$37,500

61. (a) −\$110 per pair of shoes; When 975 pairs of shoes are made and sold, profit is decreasing at a rate of \$110 per pair of shoes.

    (b) \$100 per pair of shoes; When 625 pairs of shoes are made and sold, profit is increasing at a rate of \$100 per pair of shoes.

    (c) \$55.60

    (d) \$55.30

    (e) \$13,093.10

    (f) \$13,092.80

62. (a) \$214 per dishwasher; When 500 dishwashers are produced each week, cost is increasing at a rate of \$214 per dishwasher.
    (b) \$294 per dishwasher; When 700 dishwashers are produced each week, cost is increasing at a rate of \$294 per dishwasher.
    (c) \$270.80
    (d) \$271.00
    (e) \$29,174.80
    (f) \$29,175.00

63. 52.12 million dollars per month; After 8 months, sales are increasing at a rate of 52.12 million dollars per month (or \$52,120,000 per month).

64. (a) $R'(x) = 210 - 1.6x$ dollars per dollhouse
    (b) \$90 per dollhouse; When 75 dollhouses are sold, revenue is increasing at a rate of \$90 per dollhouse.

65. \$39.70

66. \$9.10

67. (a) \$12.60
    (b) \$12.30
    (c) \$380.30
    (d) \$380.00

68. \$4250.10

69. -5 feet per second

70. $f'(t) = \frac{1}{36}t^{-5/6} + t^{-7/4}$

71. $f'(x) = -\frac{1}{2} + \frac{51}{4}x^{13/4} + \frac{5}{6}x^{1/4} - 18x^{7/2}$

72. $\frac{dy}{dx} = \frac{4}{3}x^{1/3} + \frac{112}{15}x^{1/15} - \frac{85}{12}x^{5/12}$

73. $y' = -6(0.43)^x \ln(0.43) + \frac{9}{10x}$

74. $f'(t) = -4e^t + \frac{14}{3}t^{-1/3}$

75. $q'(x) = -1824x^{16/3} + 2772x^{10} + 828x^{17/6} - \frac{3213}{2}x^{15/2}$

76. $f'(t) = -\frac{2}{3}t^{-2} + 18t^2 + 182t^{-3} + 18t + 60t^{-7}$

77. $\frac{dh}{dx} = -\frac{7}{4}x^{-3} - 6 \cdot 4^x \ln(4)$

78. $y' = \frac{3}{2}x^{-1/2} - 60x^4 + 18x^{-11/2}$

79. $y = 168x - \frac{423}{2}$

80. $y = (4+5e)t - 4$

81. $x = \frac{1}{4\ln(3)}$

82. $x = \ln(1/21)$

83. $t = 2$

84. (a) $-11/3$
    (b) $1/8$
    (c) $-\frac{3}{\ln(5)}$
    (d) $\frac{9}{25}\ln(0.6) - 4$

85. $a = 2$; $b = -3$; $c = 5$

86. $-\$0.04$ per headband; When 600 headbands are sold, price is decreasing at a rate of 4 cents per headband.

87. (a) $C'(x) = 45$ dollars per cordless drill
    (b) \$166,000.03
    (c) \$35.03
    (d) \$38 per cordless drill; When 1000 cordless drills are produced and sold, profit is increasing at a rate of \$38 per cordless drill.

88. (a) \$0.10 per golf ball; When 25 golf balls are produced, cost is increasing at a rate of \$0.10 per golf ball.
    (b) \$0.05 per golf ball; When 100 golf balls are produced, cost is increasing at a rate of \$0.05 per golf ball.

89. (a) \$14.50
    (b) \$14.45

90. $-\$1.75$ per item; When 144 items are made and sold, profit is decreasing at a rate of \$1.75 per item.

91. (a) 27 meters per second
    (b) $t = 1$ second

92. 6.13597263 million dollars per month; After two years, the company's total sales are increasing at a rate of 6.13597263 million dollars per month (or \$6,135,972.63 per month).

93. \$712.60

94. \$390 per item; When 200 items are sold, revenue is increasing at a rate of \$390 per item.

95. (a) \$14.20
    (b) \$14.10
    (c) \$200.10
    (d) \$200.00

96. The rule used here only applies for power functions (i.e., when the function is a variable raised to a numerical power). e^t is an exponential function and has a different derivative rule.

97. Because π^5 is a constant (approximately 306.02), its derivative is 0.

98. The approximate profit from selling 101 items in a week is \$15,000.

99. No. We must first use algebraic manipulation to rewrite the function so we can apply the Introductory Derivative Rules. We cannot take the derivative of the numerator and divide by the derivative of the denominator.

100. The approximate cost of making the 26^{th} item in a day is \$78. Or, when the company makes 25 items in a day, the daily cost is increasing at a rate of \$78 per item.

101. The exact revenue earned from selling the 94^{th} item is \$125.

102. Because the marginal profit, or derivative, gives us the change in profit when one more item is sold, the x-value we substitute into the derivative needs to be one less than the item number we are trying to approximate. In other words, the n^{th} item is the "one more", so we must substitute $n-1$.

Section 2.4

1. $f'(x) = xe^x + e^x$

2. $f'(x) = (5+x)\frac{1}{x} + \ln(x)$

3. $f'(x) = 8x\left(\frac{1}{x\ln(2)}\right) + (\log_2(x)) \cdot 8$

4. $f'(x) = \frac{(x-2)7-7x}{(x-2)^2}$

5. $f'(x) = \frac{(3x+5)12-36x}{(3x+5)^2}$

6. $f'(x) = \frac{(e^x-8)(10x-2)-\left(5x^2-2x+3\right)e^x}{(e^x-8)^2}$

7. 64

8. $-1/6$

9. $-1/162$

10. -14

11. $x = -1$

12. $x = -3$; $x = 3$

13. $x = e^{-1/2}$

14. (a) $\overline{C}(x) = \dfrac{4000}{x} + 32 + 0.5x$ dollars per item

(b) $\overline{C}'(x) = \dfrac{-4000}{x^2} + 0.5$ dollars per item per item

15. (a) $\overline{R}(x) = 300x(0.997)^x$ dollars per item

(b) $\overline{R}'(x) = 300x(0.997)^x \ln(0.997) + (0.997)^x 300$ dollars per item per item

16. (a) $\overline{P}(x) = \dfrac{100}{\sqrt{x}} - \dfrac{600}{x}$ dollars per item

(b) $\overline{P}'(x) = \dfrac{-50}{x^{3/2}} + \dfrac{600}{x^2}$ dollars per item per item

17. $f'(x) = 5x \cdot 8^x \ln(8) + 8^x \cdot 5 + 4x$

18. $f'(x) = \left(14x^{7/3} + 90x\right)\left(45x^2 - \dfrac{1}{x\ln(4)}\right) + \left(15x^3 - \log_4(x)\right)\left((98/3)x^{4/3} + 90\right)$

19. $f'(x) = \dfrac{\left(7x^{-2} + 15x\right)\left((1/2)x^{-1/2} + 15x^4\right) - \left(\sqrt{x} + 3x^5\right)\left(-14x^{-3} + 15\right)}{\left(7x^{-2} + 15x\right)^2}$

20. $\dfrac{dy}{dx} = \dfrac{\left(16x^4 - e^x\right)\left(12.6x^{-0.1} + 13.5x^{-1.9}\right) - \left(14x^{0.9} - 15x^{-0.9}\right)\left(64x^3 - e^x\right)}{\left(16x^4 - e^x\right)^2}$

21. $\dfrac{dy}{dx} = \dfrac{(x-7)(xe^x + e^x) - xe^x}{(x-7)^2}$

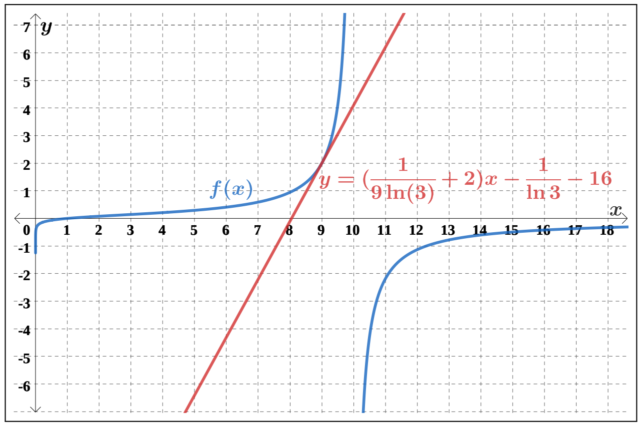
22. $\dfrac{dy}{dx} = \left(\dfrac{5^x}{x^2+1}\right)\left(50x^{49} + 420x^{69}\right) + \left(x^{50} + 6x^{70}\right)\dfrac{\left(x^2+1\right)(5^x \ln(5)) - 5^x(2x)}{\left(x^2+1\right)^2}$

23. $y = -2$

24. $y = 15x - 15$

25. $y = 1024x + 2048$

26. $y = \left(\frac{1}{9\ln(3)} + 2\right)x - \frac{1}{\ln(3)} - 16$;



27. $x = -2$; $x = 2$

28. $x = 0$

29. $x = 1$

30. $x = -4$; $x = 0$

31. $S'(t) = \frac{\left(t^2 - 14t - 15\right)(136t - 952) - \left(68t^2 - 952t - 142\right)(2t - 14)}{\left(t^2 - 14t - 15\right)^2}$ thousand dollars per month

32. -0.059503 ten thousand dollars per year; After 12 years have passed, the value of the yacht will be decreasing at a rate of 0.059503 ten thousand dollars per year (or \$595.03 per year).

33. 0.2916 hundred dollars per seashell; When 40 seashells are sold, profit is increasing at a rate of 0.2916 hundred dollars per seashell (or \$29.16 per seashell).

34. (a) $R'(x) = 10x^2(0.878)^x \ln(0.878) + (0.878)^x(20x)$ dollars per package of incense sticks
    (b) $-\$12.11$ per package of incense sticks; When 25 packages of incense sticks are sold, revenue is decreasing at a rate of \$12.11 per package.
    (c) $-\$5.92$ per package of incense sticks; When 18 packages of incense sticks are sold, revenue is decreasing at a rate of \$5.92 per package.
    (d) \$4.04
    (e) \$12.98
    (f) \$181.44
    (g) \$311.49

35. $C'(x) = \frac{\left(x^3 + 125\right)\left(6900x^2 + 180x\right) - \left(2300x^3 + 90x^2 + 21250\right)\left(3x^2\right)}{\left(x^3 + 125\right)^2}$ thousand dollars per thousand consoles

36. $v(t) = \frac{\left(t^2 + 1\right)(80t + 40) - \left(40t^2 + 40t + 67\right)(2t)}{\left(t^2 + 1\right)^2}$ feet per second

37. (a) 476

 (b) 598/75

 (c) $\dfrac{195e^{16}-3627}{\left(e^{16}-9\right)^2}$

38. (a) 8

 (b) 1157/128

39. (a) $\overline{C}(323)$ = \$85.89 per dryer; When 323 dryers are produced, the average cost per dryer is \$85.89.

 (b) $\overline{C}\,'(227)$ = \$0.14 per dryer per dryer; When 227 dryers are produced, the average cost per dryer is increasing at a rate of 14 cents per dryer.

 (c) \$18,8293.80

40. (a) $\overline{R}(60)$ = \$19.19 per desk; When 60 desks are sold, the average revenue per desk is \$19.19.

 (b) $\overline{R}\,'(237)$ = \$0.05 per desk per desk; When 237 desks are sold, the average revenue per desk is increasing at a rate of 5 cents per desk.

 (c) \$2692.60

41. (a) $\overline{P}(20)$ = \$92.87 per hairbrush; When 20 hairbrushes are made and sold, the average profit per hairbrush is \$92.86.

 (b) $\overline{P}\,'(16) = -\$6.23$ per hairbrush per hairbrush; When 16 hairbrushes are made and sold, the average profit per hairbrush is decreasing at a rate of \$6.23 per hairbrush.

 (c) \$1794.73

42. $f'(x) = \left(\sqrt{x}+18x\right)\cdot 2(10)^x\ln(10) + 2(10)^x\left((1/2)x^{-1/2}-18\right)$

43. $f'(x) = \left(\dfrac{9x^3+1}{\ln(x)+19x^6}\right)\cdot(e^x) + (e^x-7)\left(\dfrac{\left(\ln(x)+19x^6\right)\left(27x^2\right)-\left(9x^3+1\right)\left(\dfrac{1}{x}+114x^5\right)}{\left(\ln(x)+19x^6\right)^2}\right)$

44. $f'(x) = x^2\log_5(x)\cdot e^x + e^x\left(x^2\left(\dfrac{1}{x\ln(5)}\right)+\log_5(x)\cdot 2x\right)$

45. $f'(x) = \dfrac{\left(4x^3-8x\right)(18(0.5)^x\ln(0.5)) - \left(18(0.5)^x+37\left(12x^2-8\right)\right)}{\left(4x^3-8x\right)^2}$

46.

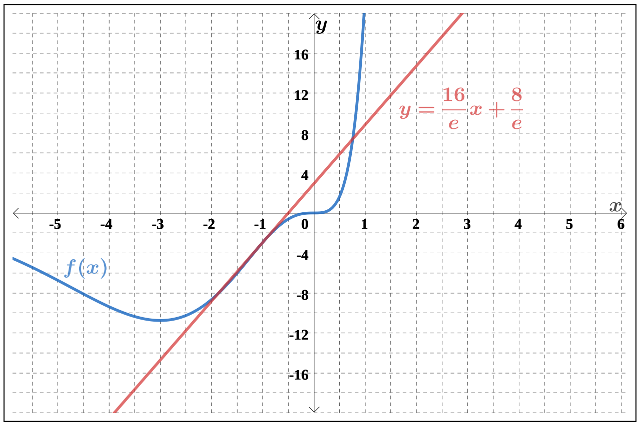
$$f'(x) = \frac{\left(\left(\sqrt[3]{x^2}+8e^x\right)(2\log(x)-9.7)\right)\left(x^{-3/4}\right)-\left(4x^{1/4}+\pi^3\right)\left(\left(\sqrt[3]{x^2}+8e^x\right)\left(\dfrac{2}{x\ln(10)}\right)+(2\log(x)-9.7)\left(\tfrac{2}{3}x^{-1/3}+8e^x\right)\right)}{\left(\left(\sqrt[3]{x^2}+8e^x\right)(2\log(x)-9.7)\right)^2}$$

47. (a) -2

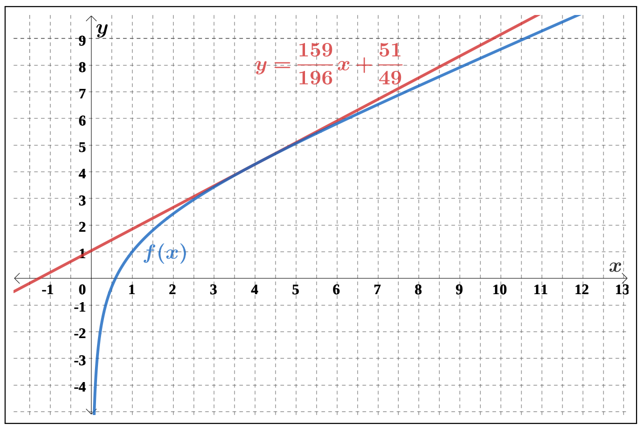
 (b) 42

 (c) $-24\ln(4)-24$

48. $y = \frac{16}{e}x + \frac{8}{e}$;



49. $y = \frac{159}{196}x + \frac{51}{49}$;



50. $x = 0$

51. $x = -3$; $x = 1$

52. $x = -3$; $x = 7$

53. $x = 0$; $x = 6$

54. 0.954 feet per second squared

55. \$2.57 per hot dog; When 20 hot dogs are sold, profit is increasing at a rate of \$2.57 per hot dog.

56. \$8.79 per week

57. $\overline{C}\,'(300) = -\$0.04$ per pound per pound; When 300 pounds of product are made, the average cost per pound is decreasing at a rate of \$0.04 per pound.

58. (a) \$13.33 per shovel; When 50 shovels are sold, revenue is increasing at a rate of \$13.33 per shovel.
    (b) \$13.38
    (c) \$13.36
    (d) \$775.28
    (e) \$775.26

59. (a) $\overline{P}(234) = \$25.77$ per dishwasher; When 234 dishwashers are made and sold, the average profit per dishwasher is \$25.77.

    (b) $\overline{P}'(209) = -\$0.80$ per dishwasher per dishwasher; When 209 dishwashers are made and sold, the average profit per dishwasher is decreasing at a rate of \$0.80 per dishwasher.

    (c) \$12,660.41

60. (a) 0

    (b) -15

61. No. To find the derivative of a product of two functions, we must use the Product Rule. We cannot just multiple the derivatives of the functions.

62. No. Because there is a quotient of functions, we must use the Quotient Rule to find the derivative. We cannot just divide the derivatives of the functions.

63. No, Vanessa will not get the correct answer. The numerator of the formula Kathryn wrote is incorrect. The formula for the Quotient Rule is $\dfrac{BT' - TB'}{B^2}$.

Section 2.5

1. $f'(x) = 24(4x+12)^5$

2. $g'(x) = \frac{4}{3}\left(12x^3 + 33x\right)^{1/3}\left(36x^2 + 33\right)$

3. $f'(t) = 5\left(8t + e^t\right)^4\left(8 + e^t\right)$

4. $\dfrac{dy}{dx} = e^{x^2}(2x)$

5. $\dfrac{dy}{dx} = e^{\sqrt{x}-(13/x)}\left(\dfrac{1}{2}x^{-1/2} + 13x^{-2}\right)$

6. $\dfrac{dy}{dx} = 8^{9x^5 - 12x^{-5}}\left(45x^4 + 60x^{-6}\right)(\ln(8))$

7. $\dfrac{dy}{dx} = 10^{2^x + 14}(2^x \ln(2))(\ln(10))$

8. $f'(x) = \dfrac{1}{4+x}$

9. $f'(x) = \dfrac{27x^2 + \dfrac{39}{2}x^{-5/2}}{9x^3 - 13x^{-3/2} + 17}$

10. $f'(x) = \dfrac{-22x^{-3} + 110x}{\left(\dfrac{11}{x^2} + 55x^2\right)(\ln(3))}$

11. $f'(x) = \dfrac{396x^{10} - 84x^6 + 3x^2}{(36x^{11} - 12x^7 + x^3 - 1)(\ln(5))}$

12. 756

13. $12\ln(6)$

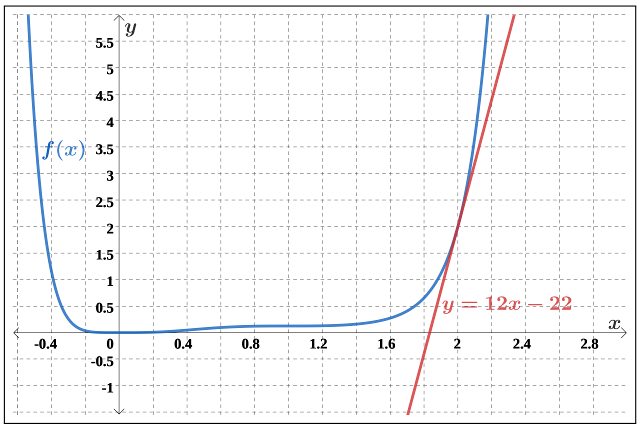
14. $\dfrac{6}{5\ln(11)}$

15. $y = 35x - 36$

16. $y = 12x + 1$

17. $y = x + 1$

18. $y = 12x - 22$;



19. $x = -2$; $x = 0$; $x = 2$

20. $x = 3$

21. $x = -11/2$

22. $\dfrac{dy}{dx} = \dfrac{1}{2}\left(8x^3 - 8\right)^{-1/2}\left(24x^2\right)$

23. $\dfrac{dy}{dx} = 24\left(\sqrt{x}\right)^2\left(\dfrac{1}{2}x^{-1/2}\right)$

24. $\dfrac{dy}{dx} = \left(\dfrac{1}{x^9 - 3x^3 + 1} - 13\right)\left(9x^8 - 9x^2\right)$

25. $\dfrac{dy}{dx} = \dfrac{1}{(e^x + 7x + 1)^2}\,(e^x + 7)$

26. $-\$8.64$ per frame

27. \$600.10 per toaster

28. $3/8$ inch per second

29. $f'(x) = 9\,(xe^x + 8x)^8\,(xe^x + e^x + 8)$

30. $f'(x) = \left(x^3 - 17\log_2(x)\right)^8(2) + (2x + 14)\left(8\left(x^3 - 17\log_2(x)\right)^7\left(3x^2 - \dfrac{17}{x\ln(2)}\right)\right)$

31. $f'(x) = \sqrt[5]{\left(12x^7 - e^x + \pi\right)^4}\,(9^x \ln(9)) + 9^x \left(\frac{4}{5}\left(12x^7 - e^x + \pi\right)^{-1/5}\left(84x^6 - e^x\right)\right)$

32. $f'(x) = \dfrac{(3x+15)^5\left(10^x \ln(10) + 10x^9\right) - \left(10^x + x^{10}\right)\left(5(3x+15)^4(3)\right)}{(3x+15)^{10}}$

33. $f'(x) = 6\left(\dfrac{\ln(x) + 17x^4}{x^2 - 3}\right)^5 \left(\dfrac{\left(x^2-3\right)\left(\frac{1}{x} + 68x^3\right) - \left(\ln(x) + 17x^4\right)(2x)}{\left(x^2-3\right)^2}\right)$

34. $f'(x) = \dfrac{\left(4x^5+x\right)^6\left(93x^2+7\right) - \left(31x^3+7x-3\right)\left(6\left(4x^5+x\right)^5\left(20x^4+1\right)\right)}{\left(4x^5+x\right)^{12}}$

35. $f'(x) = x^2 e^{x^3+21}\left(3x^2\right) + e^{x^3+21}(2x)$

36. $f'(x) = e^{x^3 4^x}\left(x^3\left(4^x \ln(4)\right) + 4^x\left(3x^2\right)\right)$

37. $f'(x) = 7^{x/(x-7)}(\ln(7))\left(\dfrac{(x-7)-x}{(x-7)^2}\right)$

38. $f'(x) = \dfrac{(17x+17)\left(-e^{x^2-1}(2x)\right) - \left(1 - e^{x^2-1}\right)(17)}{(17x+17)^2}$

39. $f'(x) = -\left(x^4 e^{12x^{-3}} - 9\right)^{-2}\left(x^4 e^{12x^{-3}}\left(-36x^{-4}\right) + e^{12x^{-3}}\left(4x^3\right)\right)$

40. $f'(x) = \left(8x^2\right)\left(\dfrac{2x}{x^2+49}\right) + \left(\ln\left(x^2+49\right)\right)(16x)$

41. $f'(x) = (e^x - 1)\left(\dfrac{-\frac{1}{2}x^{-1/2}}{\left(9 - \sqrt{x}\right)(\ln(10))}\right) + \left(\log\left(9 - \sqrt{x}\right)\right)e^x$

42. $f'(x) = \dfrac{(\ln(7x+14))\left(15x^2 - 125\right) - \left(5x^3 - 125x\right)\left(\frac{7}{7x+14}\right)}{(\ln(7x+14))^2}$

43. $f'(x) = \dfrac{\left(8x^{0.2} - 2x^{0.8}\right)\left(\frac{2x}{(x^2+36)(\ln(4))}\right) - \left(\log_4\left(x^2+36\right)\right)\left(1.6x^{-0.8} - 1.6x^{-0.2}\right)}{(8x^{0.2} - 2x^{0.8})^2}$

44. $f'(x) = \dfrac{\left(x^3-1\right)e^x\left(\frac{12x+13}{6x^2+13x-2}\right) - \left(\ln\left(6x^2+13x-2\right)\right)\left(\left(x^3-1\right)e^x + e^x\left(3x^2\right)\right)}{\left(\left(x^3-1\right)e^x\right)^2}$

45. (a) $f(x) = 7\ln(x+4) + 14\ln(x-10)$

(b) $f'(x) = \dfrac{7}{x+4} + \dfrac{14}{x-10}$

46. (a) $f(x) = 6\log_5(5x-77) - 3\log_5(18x+9)$

(b) $f'(x) = \dfrac{30}{(5x-77)(\ln(5))} - \dfrac{54}{(18x+9)(\ln(5))}$

47. (a) $f(x) = 8\ln(x) + 5\ln\left(15-22x^3\right) - 33\ln\left(3x^2-11\right) - 12\ln\left(16-x^2\right)$

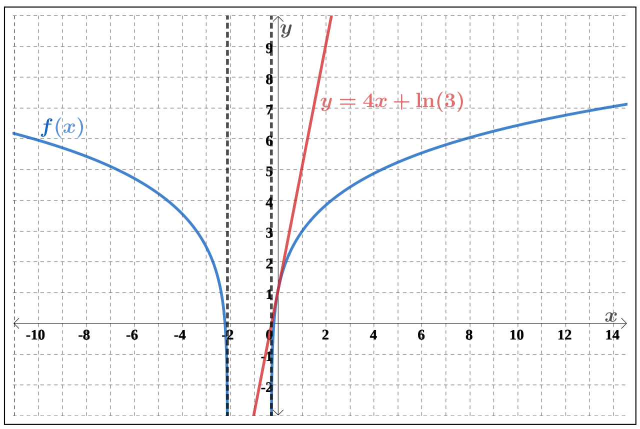
(b) $f'(x) = \dfrac{8}{x} - \dfrac{330x^2}{15-22x^3} - \dfrac{198x}{3x^2-11} + \dfrac{24x}{16-x^2}$

48. 73

49. $17e$

50. $-7/4$

51. $y = 4x + \ln(3)$;



52. $x=-1$; $x=0$

53. $x=0$

54. $x=0$; $x=7$

55. $x=-7$; $x=-4$

56. $\dfrac{dy}{dx} = \left(2\left(4x^9\right)+1\right)^{-1/2}\left(36x^8\right)$

57. $\dfrac{dy}{dx} = \left(5^{\ln\left(x^2+36\right)}(\ln(5)) - 39\left(\ln\left(x^2+36\right)\right)^2\right)\left(\dfrac{2x}{x^2+36}\right)$

58. $\dfrac{dy}{dx} = \left(\left(\left(2x^3+5\right)^2+1\right)\left(e^{2x^3+5}\right) + e^{2x^3+5}\left(2\left(2x^3+5\right)\right)\right)\left(6x^2\right)$

59. $\dfrac{dy}{dx} = 3^{\left(2x^2+32\right)^2-2\left(2x^2+32\right)+1}(\ln(3))\left(2\left(2x^2+32\right)-2\right)(4x)$

60. \$11.04 per crepe; When 33 crepes are sold, profit is increasing at a rate of \$11.04 per crepe.

61. (a) \$0.25 per box; When 160 boxes of sticky notes are produced, cost is increasing at a rate of \$0.25 per box.

    (b) \$0.20

    (c) \$0.23

    (d) \$217.19

    (e) \$243.10

62. \$2,609.37 per year

63. \$38.18 per year

64. 25.64 feet per second

65. (a) 0

    (b) -1

    (c) 12

66. (a) -1

    (b) 2

    (c) $9\ln(2)-(3/2)$

67. $\frac{dy}{dx}=\frac{112-112x}{(14x-7x^2)(\ln(2))}+\frac{6x^2}{2(2x^3-1)(\ln(2))}-\frac{7}{(x-33)(\ln(2))}-4$

68. $\frac{dy}{dx}=\frac{(x\ln(x))\left(7\left(11x^5+6^{4x^2-16}\right)^6\left(55x^4+6^{4x^2-16}(\ln(6))(8x)\right)\right)-\left(11x^5+6^{4x^2-16}\right)^7\left(x\left(\frac{1}{x}\right)+\ln(x)\right)}{(x\ln(x))^2}$

69. $\frac{dy}{dx}=\left(\frac{\left(\ln\left((4x-12)^{3/2}+1\right)\right)\left(24\left((4x-12)^{3/2}\right)^2\right)-8\left((4x-12)^{3/2}\right)^3\left(\frac{1}{(4x-12)^{3/2}+1}\right)}{\left(\ln\left((4x-12)^{3/2}+1\right)\right)^2}\right)\left(\frac{3}{2}(4x-12)^{1/2}(4)\right)$

70. $\frac{dy}{dx}=\frac{1}{5}\left(4x^3-3x^{-4}\right)^{-4/5}\left(12x^2+12x^{-5}\right)$

71. $\frac{dy}{dx}=\frac{25^{\sqrt{x^2-3x+4}}\left(-\frac{4}{3}x^{1/3}\right)-\left(9e^7-\sqrt[3]{x^4}\right)\left(25^{\sqrt{x^2-3x+4}}(\ln(25))\left(\frac{1}{2}\left(x^2-3x+4\right)^{-1/2}(2x-3)\right)\right)}{\left(25^{\sqrt{x^2-3x+4}}\right)^2}$

72. $\frac{dy}{dx}=\left(8-x^3\right)\left(4^{3x^2-64}(\ln(4))(6x)\right)+4^{3x^2-64}\left(-3x^2\right)$

73. $\frac{dy}{dx}=\left(1258\left(\left(14\ln(x)+9x^{-1}\right)^3\right)^{16}+245\left(\left(14\ln(x)+9x^{-1}\right)^3\right)^6-41\right)\left(3\left(14\ln(x)+9x^{-1}\right)^2\left(\frac{14}{x}-9x^{-2}\right)\right)$

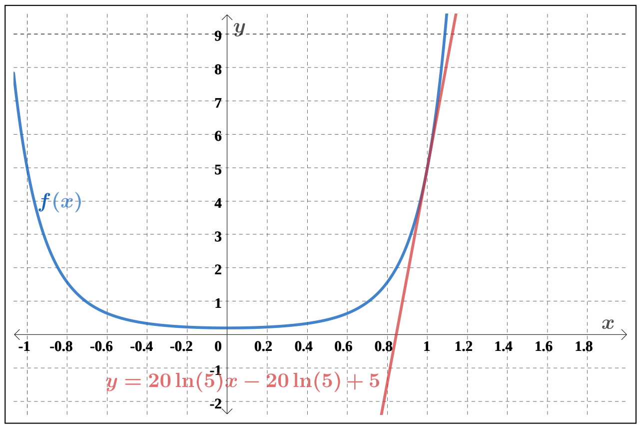
74. $\frac{dy}{dx}=\frac{-2x}{3(x^2+16)}-\frac{10\cdot 3^x\ln(3)}{3^x+19}$

75. $\dfrac{dy}{dx} = e^{2x^{11}+44}\left(22x^{10}\right)$

76. $\dfrac{dy}{dx} = \dfrac{1}{2}\left(\left(\pi x^{-2}\right)\left(\log_4\left(9x^2-5e^x\right)\right)\right)^{-1/2}\left(\left(\pi x^{-2}\right)\left(\dfrac{18x-5e^x}{\left(9x^2-5e^x\right)(\ln(4))}\right)+\left(\log_4\left(9x^2-5e^x\right)\right)\left(-2\pi x^{-3}\right)\right)$

77. $\dfrac{dy}{dx} = \dfrac{1}{2}\left(e^{\left(\ln\left(x^2+1\right)\right)^2}+5\right)^{-1/2}\left(e^{\left(\ln\left(x^2+1\right)\right)^2}\right)\left(2\ln\left(x^2+1\right)\right)\left(\dfrac{2x}{x^2+1}\right)$

78. $y = (20\ln(5))\,x - 20\ln(5) + 5$;



79. $x = -32/3$; $x = -2$

80. $x = -1$; $x = 0$; $x = 3$

81. $x = 0$

82. $x = 0$

83. $x = -4$; $x = 4$

84. $x = 0$; $x = 2$

85. 1.253 inches per second

86. 0.034 thousand Wakandan dollars per hour; When the mine has been open for 50 hours, cost is increasing at a rate of 0.034 thousand Wakandan dollars per hour (or 34 Wakandan dollars per hour).

87. 6.5030 hundred dollars per move; When 3299 moves are completed, profit is increasing at a rate of 6.5030 hundred dollars per move (or \$650.30 per move).

88. (a) −\$104.31 per frame; When 150 diploma frames are sold, revenue is decreasing at a rate of \$104.31 per frame.

    (b) −\$1.72 per frame per frame; When 150 diploma frames are sold, the average revenue per frame is decreasing at a rate of \$1.72 per frame.

89. (a) \$1.04 per doodad; When 116 doodads are made, cost is increasing at a rate of \$1.04 per doodad.
    (b) \$1.04
    (c) \$1.04
    (d) \$88.87
    (e) \$88.87

90. \$16.26 per year

91. \$947.21 per year

92. (a) $76e$
    (b) Does Not Exist
    (c) $\dfrac{\dfrac{40}{40+e^5}-16\ln\left(40+e^5\right)}{25}$

93. (a) 0
    (b) 4
    (c) -2

94. Because this function is a composite function of the form $y = f(g(x))$, where $f(x) = x^{1/3}$ and $g(x) = \dfrac{6x^4 - x^2 + 7}{5x^3 - x + 1}$ is a quotient, the Chain Rule should be used first (and then the Quotient Rule). Note that we should look at the "outside" function of a composition when determining which derivative rule to use first.

95. Using the Properties of Logarithms to expand a function first will generally make the process of taking the derivative much easier than using the Generalize Logarithm (base b) Rule or the Generalized Natural Logarithm Rule first.

96. No. The derivative of the "outside" function should be multiplied by the derivative of the "inside" function:

$$\frac{d}{dx}\left(\left(x^2+4x\right)^5\right) = 6\left(x^2+4x\right)^6(2x+4)$$

97. The Alternate Form of the Chain Rule should be used when finding the derivative of a function that is a composition of functions, but the functions have not yet been composed.

98. After using the Alternate Form of the Chain Rule, we must substitute to ensure the derivative is entirely in terms of the independent variable the question asked us to find the derivative with respect to.

99. Rather than using the Quotient Rule, we can move the denominator to the numerator as long as we raise it to the power of -1 (or change the sign of the power when moving it to the numerator if it is already a power function). Then, we will have a product of functions where at least one of them will require the Chain Rule because it will be raised to a power.

100. Yes. Both forms of the Chain Rule are mathematically equivalent. Although, the results from using each form may look very different.

Section 2.6

1. $\dfrac{dy}{dx} = \dfrac{-4x^3}{6y}$

2. $\dfrac{dy}{dx} = \dfrac{\frac{3}{2}x^{-1/2} + 5x^{-2}}{e^y}$

3. $\dfrac{dy}{dx} = \dfrac{-2}{x^9\left(2y^{-1}+1\right)}$

4. $\dfrac{dy}{dx} = \dfrac{-18x^5 + 10}{2\left(4^y \ln(4)\right) - \frac{2}{3}y^{-1/3}}$

5. $\dfrac{dy}{dx} = \dfrac{\frac{1}{x}}{0.04y^{-0.6} - \dfrac{7}{y\ln(2)}}$

6. 9

7. $-\dfrac{1}{3e}$

8. $\dfrac{5\ln(9)}{2}$

9. $y = \dfrac{1}{54}x + 4$

10. $y = -\dfrac{120}{13}x + \dfrac{253}{13}$

11. $y = \dfrac{3}{\ln(5)}x - \dfrac{18}{\ln(5)}$

12. -106

13. $\dfrac{10\ln(6)+8}{25}$

14. 54

15. \$7410 per week

16. \$17,550 per month

17. \$3575 per week

18. 28 square feet per second

19. 0.9π square miles per day

20. 2.1 feet per second

21. $\frac{dy}{dx} = \frac{2x + y^3}{6y - 3xy^2}$

22. $\frac{dy}{dx} = \frac{e^y}{\frac{1}{4y} - xe^y + 8y^{-2}}$

23. $\frac{dy}{dx} = \frac{\frac{6}{x} + 21yx^{-8} + \sqrt{y}\,(5^x \ln(5))}{3x^{-7} + \frac{1}{y\ln(10)} - 5^x\left(\frac{1}{2}y^{-1/2}\right)}$

24. $\frac{dy}{dx} = \frac{2x^5 - 2y^{-3} - \sqrt[4]{7}}{-6xy^{-4}}$

25. $\frac{dy}{dx} = \frac{4e^x - \frac{1}{x\ln(2)} + \frac{1}{2}x^{-3/2}e^y}{x^{-1/2}e^y}$

26. $\frac{dy}{dx} = \frac{(\ln(y))(8x) - 3x^2(\ln(y))^2}{2(\ln(y))^2 - 4x^2y^{-1}}$

27. $\frac{dy}{dx} = \frac{10x}{6y^2e^{2y^3} + \frac{9}{2}y^{-1/2}}$

28. $\frac{dy}{dx} = \frac{2(6^x \ln(6))}{42y\left(7y^2 - 31\right)^2 + 45y^{-4}}$

29. $\frac{dy}{dx} = \frac{-12x^3 - x^{-1}}{8y^{-3} - \frac{2y}{y^2 + 36}}$

30. $\frac{dy}{dx} = \frac{-0.04x^{-7/5}}{1 - 2^{\sqrt{y}}(\ln(2))\left(\frac{1}{2}y^{-1/2}\right)}$

31. $\frac{dy}{dx} = \frac{\left(3y^5 - y\right)(x - 6)}{-\left(0.5x^2 - 6x\right)\left(15y^4 - 1\right)}$

32. $\frac{dy}{dx} = \frac{-\left(3y^2 + 14y\right)\left(4.5x^{-1/2} + 2\right)}{\left(9\sqrt{x} + 2x\right)(6y + 14)}$

33. $\frac{dy}{dx} = \frac{35x^4 - \left(4y^3 - 2y^2 + y\right)\left(5x^4 - 3\right)}{\left(x^5 - 3x\right)\left(12y^2 - 4y + 1\right)}$

34. $\frac{dy}{dx} = \frac{5y^4\left(4x - x^2\right)}{\left(2x^2 - \frac{1}{3}x^3\right)\left(20y^3\right)}$

35. $\dfrac{dy}{dx} = \dfrac{18\sqrt{y} - 42y^{0.2} - 16x}{-9xy^{-1/2} + 8.4xy^{-0.8}}$

36. $\dfrac{dy}{dx} = 25 + 30x^{0.2} - 5x^{-2} - 0.4x^{-1.8}$

37. $\dfrac{dy}{dx} = \dfrac{-2e^{4x^2+\sqrt{y}}(8x)}{e^{4x^2+\sqrt{y}}\left(y^{-1/2}\right)}$

38. $\dfrac{dy}{dx} = \dfrac{1}{6y^2}\left(\dfrac{3}{2\left(2y^3 - x^4\right)^7} + 4x^3\right)$

39. $\dfrac{dy}{dx} = \dfrac{12x^3}{5}$

40. $26/7$

41. $-\dfrac{8}{3\ln(7)}$

42. $-4/3$

43. $y = \dfrac{5}{3}x - \dfrac{40}{3}$

44. $y = -\dfrac{1}{16}x - \dfrac{9}{16}$

45. $y = 4x + 11$

46. $25/66$

47. -12

48. $-60\ln(10)$

49. $3/10$

50. $2/3$

51. -24

52. \$950 per month; When 5000 night lights are sold each month, revenue is increasing at a rate of \$950 per month.

53. −\$1950 per week; When 180 life vests are sold each week, profit is decreasing at a rate of \$1950 per week.

54. \$3312 per week; When 315 freezers are produced each week, cost is increasing at a rate of \$3312 per week.

55. −\$12 per week; When 100 dog beds are sold each week, the price per dog bed is decreasing at a rate of \$12 per week.

56. (a) 6 square inches per hour

    (b) 3/5 inch per hour

    (c) 3/5 inch per hour

57. 14 cm^2/min

58. (a) 2 feet per second

    (b) 0 square feet per second

59. 40π square feet per second

60. 2 ft/sec

61. −390 mi/h (This means the distance between the planes is decreasing at a rate of 390 mi/h.)

62. (a) 30.594 ft/min

    (b) 36.056 ft/min

    (c) 301.496 ft/min

63. 2 miles per hour

64. −0.9 units per second

65. $$\frac{dy}{dx} = \frac{\left(y^2 - 611y^3\right)\left(-0.02x^{-0.8} + \dfrac{1}{x\ln(5)}\right)}{\left(-0.1x^{0.2} + \log_5(x)\right)\left(2y - \dfrac{18}{11}y^2\right)}$$

66. $$\frac{dy}{dx} = \frac{\left(4\ln(y) - 3y^2 + y\right)\left(0.5x^{-0.5} + 3\right)}{6y^2 - \left(\sqrt{x} + 3x\right)\left(4y^{-1} - 6y + 1\right)}$$

67. $$\frac{dy}{dx} = -\frac{0.5\,(10^y - e^y)^2\,x^{-1/2} - (10^y - e^y)\left(4x + 5x^{1/4}\right)}{\left(2x^2 + 4x^{5/4}\right)(10^y\ln(10) - e^y)}$$

68. $$\frac{dy}{dx} = \frac{\dfrac{2x}{\left(x^2+y^2\right)(\ln(2))} + 3x^2y^4}{\dfrac{-2y}{\left(x^2+y^2\right)(\ln(2))} - 4x^3y^3 - \dfrac{5}{3}y^{2/3}}$$

69. $$\frac{dy}{dx} = \frac{3e^{4y^2-\pi}}{x\left(5y^4 - 8\left(3\ln(x) - y^5\right)e^{4y^2-\pi}y - 14y\right)}$$

70. $$\frac{dy}{dx} = \frac{12x^{-1/4}y^{-1/9} + 8x^{3/4}\left(7^x\ln(7)\right) + 6x^{-1/4}\left(7^x\right)}{\dfrac{5}{y} + \dfrac{16}{9}x^{3/4}y^{-10/9}}$$

71. $\dfrac{dy}{dx} = \dfrac{2ye^{2xy}}{\dfrac{5}{y\ln(6)} - 2xe^{2xy}}$

72. $\dfrac{dy}{dx} = \dfrac{-5\,(4^y)\,x^4 + 10x^5 - 2\ln(y) + 2x^5 - 3}{16y^{-3} - 4^y y^{-1} + 2xy^{-1} - (\ln(y))\,(4^y\ln(4)) + x^5\,(4^y\ln(4))}$

73. $\dfrac{dy}{dx} = \dfrac{2xy - \dfrac{1}{x}}{-x^2 + \frac{3}{2}y^{1/2}}$

74. $55/84$

75. $466/207$

76. $\dfrac{\frac{3}{2}\ln(5) - 4}{-\frac{1}{4} + 48\ln(5)}$

77. $y = \dfrac{1}{8}x + \dfrac{7}{8}$

78. $y = \dfrac{3}{20}x + 5$

79. $y = \dfrac{1}{2}x - \dfrac{1}{2}$

80. 168

81. -4

82. $-\$1.28$ per month; When the company sells 2500 microphones each month, price is decreasing at a rate of \$1.28 per month.

83. $-\$210$ per week; When the company makes and sells 200 lamps each week, profit is decreasing at a rate of \$210 per week.

84. $-\$7969.18$ per month; When the jeweler sells 2500 necklaces each month, revenue is decreasing at a rate of \$7969.18 per month.

85. \$5625.00 per week; When 375 toasters are manufactured each week, cost is increasing at a rate of \$5625.00 per week.

86. \$886.12 per week; When 450 items are produced and sold each week, profit is increasing at a rate of \$886.12 per week.

87. -18 items per month; When the price is \$50 per item, demand is decreasing at a rate of 18 items per month.

88. \$4.90 per month; When 200 items are demanded each month, price is increasing at a rate of \$4.90 per month.

89. $\$90,300$ per week; After 2 weeks, cost will be increasing at a rate of \$90,300 per week.

90. 30.426 feet per minute

91. 3.402 feet per second

92. 5.25 feet per second

93. 26.917 feet per second

94. -3 cm/s

95. 47.170 miles per hour

96. $1/3$ ft/s

97. $-1/3$ ft/s

98. 24 m^3/s

99. 640π cubic centimeters per minute

100. $-5/6$ m/s

101. 2400π cubic meters per second

102. (a) 0.79577 cm/s
   (b) 0.00796 cm/s
   (c) 0.00008 cm/s

103. 0.282 cm

104. We must use implicit differentiation to find $\frac{dy}{dx}$ when we are given an equation relating x and y which cannot be written in the form $y = f(x)$. In other words, we must use implicit differentiation to find $\frac{dy}{dx}$ when the equation cannot be solved explicitly for y.

105. No. We can solve the equation explicitly for y:

$$y = -3x^4 + e^x + 7 - \ln(x)$$

106. When using implicit differentiation to find $\frac{dy}{dx}$, we treat y as though it is a function of x.

107. To find $\frac{dy}{dx}$ implicitly, we apply the Implicit Differentiation Method:

   1. Take the derivative of both sides of the equation, and multiply by $\frac{dy}{dx}$ when taking the derivative of y terms.
   2. Move all the $\frac{dy}{dx}$ terms to one side of the equation and all the remaining terms to the other side.
   3. Factor the $\frac{dy}{dx}$ term and solve for $\frac{dy}{dx}$ by dividing.

108. A related-rates problem is one in which quantities are related and all but one of the quantities have a known rate of change. Thus, we are able to determine the rate of change of the other quantity.

109. The Related-Rates Strategy consists of the following steps:
    1. Assign variables and draw a picture, if applicable.
    2. State, in terms of the variables, the information that is given and the rate to be determined.
    3. Find an equation relating the variables introduced in step 1, if necessary.
    4. Differentiate both sides of the equation found in step 3 with respect to time using implicit differentiation.
    5. Substitute all known values into the equation found in step 4, and then solve for the unknown rate of change.

Section 3.1

1. (a) $x = 3$
   (b) $f'(x)$ exists for all x.
   (c) f is increasing on $(-\infty, 3)$, and f is decreasing on $(3, \infty)$.
   (d) f has a local maximum at $x = 3$ and no local minima.
2. (a) $x = -1$; $x = 1$
   (b) $f'(x)$ exists for all x.
   (c) f is increasing on $(-\infty, -1)$ and $(1, \infty)$, and f is decreasing on $(-1, 1)$.
   (d) f has a local maximum at $x = -1$ and a local minimum at $x = 1$.
3. (a) $f'(x)$ is never equal to zero.
   (b) $f'(x)$ does not exist at $x = 2$ and $x = 6$.
   (c) f is increasing on $(2, 6)$, and f is decreasing on $(-\infty, 2)$ and $(6, \infty)$.
   (d) f has a local maximum at $x = 6$ and a local minimum at $x = 2$.
4. (a) $x = -3$; $x = 3$
   (b) f is increasing on $(-\infty, -3)$ and $(3, \infty)$, and f is decreasing on $(-3, 3)$.
5. (a) $x = 0$; $x = 5$
   (b) f is increasing on $(0, 5)$ and $(5, \infty)$, and f is decreasing on $(-\infty, 0)$.
6. (a) $x = -4$; $x = 0$; $x = 8$
   (b) f is increasing on $(-4, 0)$ and $(8, \infty)$, and f is decreasing on $(-\infty, -4)$ and $(0, 8)$.
7. (a) $x = -3$; $x = -1/2$; $x = 0$
   (b) f is increasing on $(-\infty, -3)$ and $(-1/2, 0)$, and f is decreasing on $(-3, -1/2)$ and $(0, \infty)$.
8. $x = -9$; $x = -1$
9. $x = -7$; $x = 0$; $x = 3$
10. $t = -1$; $t = 0$; $t = 1$

11. $x=-2/3$; $x=0$; $x=5/2$

12. (a) $x=-5$; $x=2$; $x=8$
 (b) $x=-5$; $x=2$; $x=8$
 (c) f is increasing on $(-\infty,-5)$, $(-5,2)$, and $(8,\infty)$, and f is decreasing on $(2,8)$.
 (d) f has a local maximum at $x=2$ and a local minimum at $x=8$.

13. (a) $x=-4$; $x=0$; $x=7$
 (b) $x=-4$; $x=0$; $x=7$
 (c) f is increasing on $(-4,0)$ and $(7,\infty)$, and f is decreasing on $(-\infty,-4)$ and $(0,7)$.
 (d) f has a local maximum at $x=0$ and local minima at $x=-4$ and $x=7$.

14. (a) $x=-2$; $x=1$; $x=6$
 (b) $x=-2$; $x=1$
 (c) f is increasing on $(-\infty,-2)$, $(1,6)$, and $(6,\infty)$, and f is decreasing on $(-2,1)$.
 (d) f has a local maximum at $x=-2$ and a local minimum at $x=1$.

15. (a) $x=-8$; $x=-1$; $x=0$
 (b) $x=-8$; $x=0$
 (c) f is increasing on $(-\infty,-8)$, $(-8,-1)$, and $(-1,0)$, and f is decreasing on $(0,\infty)$.
 (d) f has a local maximum at $x=0$ and no local minima.

16. (a) $x=-9$; $x=3$; $x=5$
 (b) $x=-9$; $x=3$
 (c) f is increasing on $(-\infty,-9)$ and $(5,\infty)$, and f is decreasing on $(-9,3)$ and $(3,5)$.
 (d) f has a local maximum at $x=-9$ and no local minima.

17. (a) $x=2$; $x=4$
 (b) $x=2$; $x=4$
 (c) f is increasing on $(-\infty,2)$ and $(4,\infty)$, and f is decreasing on $(2,4)$.
 (d) f has a local maximum of 15 at $x=2$ and a local minimum of 11 at $x=4$.

18. (a) $x=-2$; $x=0$; $x=6$
 (b) $x=-2$; $x=0$; $x=6$
 (c) f is increasing on $(-\infty,-2)$ and $(6,\infty)$, and f is decreasing on $(-2,0)$ and $(0,6)$.
 (d) f has a local maximum of 148 at $x=-2$ and a local minimum of -2924 at $x=6$.

19. (a) $x=-1$; $x=0$; $x=8/3$
 (b) $x=-1$; $x=0$; $x=8/3$
 (c) f is increasing on $(-1,0)$ and $(8/3,\infty)$, and f is decreasing on $(-\infty,-1)$ and $(0,8/3)$.
 (d) f has a local maximum of 0 at $x=0$ and local minima of $-19/3$ at $x=-1$ and $-7168/81$ at $x=8/3$.

20. (a) $x = 2$; $x = 8$
    (b) $x = 2$; $x = 8$
    (c) f is increasing on $(2, 8)$, and f is decreasing on $(-\infty, 2)$ and $(8, \infty)$.
    (d) f has a local maximum at $x = 8$ and a local minimum at $x = 2$.

21. (a) $x = -5$; $x = 0$; $x = 4$
    (b) $x = -5$; $x = 0$; $x = 4$
    (c) f is increasing on $(-5, 0)$ and $(4, \infty)$, and f is decreasing on $(-\infty, -5)$ and $(0, 4)$.
    (d) f has a local maximum at $x = 0$ and local minima at $x = -5$ and $x = 4$.

22. (a) $x = -4$; $x = 0$; $x = 8$
    (b) $x = -4$; $x = 0$; $x = 8$
    (c) f is increasing on $(0, 8)$, and f is decreasing on $(-\infty, -4)$, $(-4, 0)$, and $(8, \infty)$.
    (d) f has a local maximum at $x = 8$ and a local minimum at $x = 0$.

23. (a) Revenue is increasing on $(0, 465)$ and decreasing on $(465, \infty)$.
    (b) $x = 465$ microphones

24. (a) Profit is increasing on $(0, 920)$ and decreasing on $(920, \infty)$.
    (b) $x = 920$ coats

25. (a) Cost is increasing on $(65, \infty)$ and decreasing on $(0, 65)$.
    (b) $x = 65$ vacuum cleaners

26. f is increasing on $(-1, \infty)$, and f is decreasing on $(-\infty, -1)$.

27. f is increasing on $(-\infty, -6)$ and $(0, \infty)$, and f is decreasing on $(-6, -3)$ and $(-3, 0)$.

28. f is increasing on $\left(e^{-1/2}, \infty\right)$, and f is decreasing on $\left(0, e^{-1/2}\right)$.

29. f is increasing on $(-5, 0)$ and $(5, \infty)$, and f is decreasing on $(-\infty, -5)$ and $(0, 5)$.

30. $x = 0.2$

31. $x = 0$

32. $t = -2$

33. $x = e^{-1}$

34. $t = 0$

35. $x = 2$

36. (a) $x = 3$
    (b) f is increasing on $(3, \infty)$, and f is decreasing on $(-\infty, 3)$.
    (c) f has no local maxima, but it has a local minimum of $-e^3$ at $x = 3$.

37. (a) $x = e^{-1/3}$

(b) f is increasing on $\left(e^{-1/3}, \infty\right)$, and f is decreasing on $\left(0, e^{-1/3}\right)$.

(c) f has no local maxima, but it has a local minimum of $-\dfrac{1}{3e}$ at $x = e^{-1/3}$.

38. (a) None

(b) f is increasing nowhere and decreasing on $(-\infty, -1)$, $(-1, 1)$, and $(1, \infty)$.

(c) f has no local extrema.

39. (a) $x = 0$; $x = 3/2$; $x = 3$

(b) f is increasing on $(0, 3/2)$ and $(3, \infty)$, and f is decreasing on $(-\infty, 0)$ and $(3/2, 3)$

(c) f has a local maximum of $(-9/4)^{4/3}$ at $x = 3/2$ and local minima at both $x = 0$ and $x = 3$ of 0.

40. (a) $x = e^{1/2}$

(b) f is increasing on $\left(0, e^{1/2}\right)$, and f is decreasing on $\left(e^{1/2}, \infty\right)$.

(c) f has a local maximum of $\dfrac{1}{2e}$ at $x = e^{1/2}$ and no local minima.

41. (a) $x = -3$; $x = 3$

(b) f is increasing on $(-\infty, -3)$ and $(3, \infty)$, and f is decreasing on $(-3, 3)$.

(c) f has a local maximum of e^{54} at $x = -3$ and a local minimum of $\dfrac{1}{e^{54}}$ at $x = 3$.

42. (a) $x = -\sqrt{27}$; $x = 0$; $x = \sqrt{27}$

(b) f is increasing on $\left(-\infty, -\sqrt{27}\right)$ and $\left(\sqrt{27}, \infty\right)$, and f is decreasing on $\left(-\sqrt{27}, -3\right)$, $(-3, 0)$, $(0, 3)$, and $\left(3, \sqrt{27}\right)$.

(c) f has a local maximum of $\dfrac{\left(-\sqrt{27}\right)^3}{18}$ at $x = -\sqrt{27}$ and a local minimum of $\dfrac{\left(\sqrt{27}\right)^3}{18}$ at $x = \sqrt{27}$.

43. (a) $x = 0$

(b) f is increasing on $(0, \infty)$, and f is decreasing on $(-\infty, 0)$.

(c) f has no local maxima, but it has a local minimum of $\ln(4)$ at $x = 0$.

44. (a) $x = 0$

(b) f is increasing on $(0, \infty)$, and f is decreasing on $(-\infty, 0)$.

(c) f has no local maxima, but it has a local minimum of 2 at $x = 0$.

45. (a) $x = -4$; $x = 4$

(b) f is increasing on $(-\infty, -4)$ and $(4, \infty)$, and f is decreasing on $(-4, 0)$ and $(0, 4)$.

(c) f has a local maximum of -8 at $x = -4$ and a local minimum of 8 at $x = 4$.

46. (a) $x = -8$; $x = -3$; $x = 2$; $x = 5$

    (b) $x = -8$; $x = -3$; $x = 2$; $x = 5$

    (c) f is increasing on $(-\infty, -8)$, $(-3, 2)$, and $(5, \infty)$, and f is decreasing on $(-8, -3)$ and $(2, 5)$.

    (d) f has local maxima of 3 at $x = -8$ and 1 at $x = 2$, and it has local minima of -2 at $x = -3$ and 0 at $x = 5$.

47. (a) $x = a$; $x = b$; $x = c$

    (b) $x = a$; $x = c$

    (c) f is increasing on (c, ∞), and f is decreasing on $(-\infty, a)$, (a, b), and (b, c).

    (d) f has no local maxima, but it has a local minimum at $x = c$.

48. (a) $x = a$; $x = b$; $x = c$; $x = 0$; $x = d$; $x = e$

    (b) $x = a$; $x = c$; $x = 0$

    (c) f is increasing on (a, b), $(c, 0)$, and (d, e), and f is decreasing on $(-\infty, a)$, (b, c), $(0, d)$, and (e, ∞).

    (d) f has a local maximum at $x = 0$ and local minima at $x = a$ and $x = c$.

49. (a) $x = a$; $x = b$; $x = c$; $x = d$; $x = e$

    (b) $x = a$; $x = b$; $x = d$

    (c) f is increasing on $(-\infty, a)$, (c, d), and (e, ∞), and f is decreasing on (a, b), (b, c), and (d, e).

    (d) f has local maxima at $x = a$ and $x = d$ and no local minima.

50. (a) $x = -2$; $x = 0$; $x = 1$

    (b) $x = -2$; $x = 0$; $x = 1$

    (c) f is increasing on $(-2, 0)$ and $(1, \infty)$, and f is decreasing on $(-\infty, -2)$ and $(0, 1)$.

    (d) f has a local maximum at $x = 0$ and local minima at $x = -2$ and $x = 1$.

51. (a) $x = b$; $x = d$

    (b) $x = b$; $x = d$

    (c) f is increasing on $(-\infty, b)$ and (d, ∞), and f is decreasing on (b, d).

    (d) f has a local maximum at $x = b$ and a local minimum at $x = d$.

52. (a) $x = a$; $x = b$; $x = c$

    (b) $x = b$; $x = c$

    (c) f is increasing on (b, c) and (c, ∞), and f is decreasing on $(-\infty, a)$ and (a, b).

    (d) f has no local maxima, but it has a local minimum at $x = b$.

53. (a) $x = b$; $x = c$; $x = d$; $x = e$; $x = f$

    (b) $x = b$; $x = c$; $x = e$

    (c) f is increasing on (b, c), (d, e), and (f, ∞), and f is decreasing on $(-\infty, b)$, (c, d), and (e, f).

    (d) f has local maxima at $x = c$ and $x = e$ and a local minimum at $x = b$.

54. P is increasing on $(0, 50)$ and decreasing on $(50, \infty)$. The manufacturer should make 50 lamps to maximize profit.

55. C is increasing on $(40,\infty)$ and decreasing on $(0,40)$. The company should make 40 cases to minimize costs.

56. R is increasing on $(0,125)$ and decreasing on $(125,\infty)$. Feeling Fabulous should sell 125 spa packages to maximize revenue.

57. (a) R is increasing on $(0,6)$ and decreasing on $(6,\infty)$. The company should make 600 items to maximize revenue.
    (b) $\overline{R}(x) = -25x^2 + 225x$ dollars per hundred items
    (c) 450 items

58. f is increasing on (e,∞), and f is decreasing on $(0,e)$.

59. f is increasing on $(-1,1)$, and f is decreasing on $(-\infty,-1)$ and $(1,\infty)$.

60. f is increasing on $(-2,0)$, and f is decreasing on $(-\infty,-2)$, $(0,1)$, and $(1,\infty)$.

61. $x = 3$; $x = 24/5$; $x = 6$

62. $x = 5$

63. $t = 0.5e^{-1}$

64. (a) $x = -1$; $x = 1$
    (b) f is increasing on $(-1,0)$ and $(1,\infty)$, and f is decreasing on $(-\infty,-1)$ and $(0,1)$.
    (c) f has no local maxima, but it has local minima at $x = -1$ and $x = 1$.

65. (a) $x = -6$; $x = -2$
    (b) f is increasing on $(-\infty,-6)$, $(-6,-2)$, and $(7,\infty)$, and f is decreasing on $(-2,7)$.
    (c) f has a local maximum at $x = -2$ and no local minima.

66. (a) $x = 0$; $x = 1$
    (b) f is increasing on $(-\infty,0)$ and $(1,\infty)$, and f is decreasing on $(0,1)$.
    (c) f has a local maximum at $x = 0$ and a local minimum at $x = 1$.

67. (a) $x = 3$; $x = 4$
    (b) f is increasing on $(3,4)$, and f is decreasing on $(-\infty,3)$ and $(4,\infty)$.
    (c) f has a local maximum at $x = 4$ and a local minimum at $x = 3$.

68. (a) $x = 0$; $x = 9$
    (b) f is increasing on $(-\infty,-4)$, $(-4,0)$, and $(0,9)$, and f is decreasing on $(9,\infty)$.
    (c) f has a local maximum at $x = 9$ and no local minima.

69. (a) $x = e^{1/2}/3$
    (b) f is increasing on $\left(0, e^{1/2}/3\right)$, and f is decreasing on $\left(e^{1/2}/3, \infty\right)$.
    (c) f has a local maximum at $x = e^{1/2}/3$ and no local minima.

70. (a) $x = -5$; $x = 19$

 (b) f is increasing on $(-\infty, -5)$ and $(19, \infty)$, and f is decreasing on $(-5, 1)$ and $(1, 19)$.

 (c) f has a local maximum at $x = -5$ and a local minimum at $x = 19$.

71. (a) $x = e$

 (b) f is increasing on $(0, e)$, and f is decreasing on (e, ∞).

 (c) f has a local maximum at $x = e$ and no local minima.

72. (a) $x = -5$; $x = -1.4$; $x = 1$

 (b) f is increasing on $(-5, -1.4)$ and $(1, \infty)$, and f is decreasing on $(-\infty, -5)$ and $(-1.4, 1)$.

 (c) f has a local maximum at $x = -1.4$ and local minima at $x = -5$ and $x = 1$.

73. (a) $x = 0$

 (b) f is increasing on $(-\infty, 0)$, and f is decreasing on $(0, \infty)$.

 (c) f has a local maximum at $x = 0$ and no local minima.

74. (a) $x = -1/2$; $x = 0$; $x = 8/3$

 (b) f is increasing on $(-1/2, 0)$ and $(8/3, \infty)$, and f is decreasing on $(-\infty, -1/2)$ and $(0, 8/3)$.

 (c) f has a local maximum at $x = 0$ and local minima at $x = -1/2$ and $x = 8/3$.

75. (a) $x = -e^{-1}$; $x = e^{-1}$

 (b) f is increasing on $\left(-\infty, -e^{-1}\right)$ and $\left(e^{-1}, \infty\right)$, and f is decreasing on $\left(-e^{-1}, 0\right)$ and $\left(0, e^{-1}\right)$.

 (c) f has a local maximum at $x = -e^{-1}$ and a local minimum at $x = e^{-1}$.

76. (a) $x = 0$

 (b) f is increasing on $(0, 7)$, and f is decreasing on $(-\infty, 0)$ and $(7, \infty)$.

 (c) f has no local maxima, but it has a local minimum at $x = 0$.

77. (a) $x = -\sqrt{4 - (1/\ln(2))}$; $x = 0$; $x = \sqrt{4 - (1/\ln(2))}$

 (b) f is increasing on $\left(-\sqrt{4 - (1/\ln(2))}, 0\right)$ and $\left(\sqrt{4 - (1/\ln(2))}, \infty\right)$, and f is decreasing on $\left(-\infty, -\sqrt{4 - (1/\ln(2))}\right)$ and $\left(0, \sqrt{4 - (1/\ln(2))}\right)$

 (c) f has a local maximum at $x = 0$ and local minima at $x = -\sqrt{4 - (1/\ln(2))}$ and $x = \sqrt{4 - (1/\ln(2))}$.

78. (a) $t = -a$; $t = 0$; $t = 1$; $t = b$

 (b) $t = -a$; $t = 0$; $t = b$

 (c) g is increasing on $(-\infty, -a)$, $(-a, 0)$, and $(1, b)$, and g is decreasing on $(0, 1)$ and (b, ∞).

 (d) g has local maxima at $t = 0$ and $t = b$ and no local minima.

79. (a) $x = a$; $x = b$; $x = 0$; $x = d$; $x = e$; $x = g$; $x = j$; $x = k$

    (b) $x = a$; $x = b$; $x = 0$; $x = d$; $x = g$; $x = j$; $x = k$

    (c) g is increasing on $(b,0)$, (d,e), (g, j), and (k,∞), and g is decreasing on $(-\infty,a)$, (a,b), $(0,d)$, (e,g), and (j,k).

    (d) g has local maxima at $x = 0$ and $x = j$ and a local minimum at $x = b$.

80. (a) $x = a$; $x = c$; $x = d$; $x = e$; $x = g$; $x = h$; $x = i$; $x = j$; $x = k$

    (b) $x = c$; $x = d$; $x = e$; $x = g$; $x = h$; $x = i$; $x = k$

    (c) f is increasing on $(-\infty,a)$, (c,d), (d,e), (g,h), (i, j), and (j,k), and f is decreasing on (a,c), (e,g), (h,i), and (k,∞).

    (d) f has local maxima at $x = e$, $x = h$, and $x = k$ and local minima at $x = c$, $x = g$, and $x = i$.

81. (a) Revenue is increasing on $(0,7500)$ and decreasing on $(7500,\infty)$.

    (b) \$7.50 per flashlight

82. Profit is increasing on $(0,1700)$ and decreasing on $(1700,\infty)$. Summer Splashin' should sell 1700 pools to maximize profit.

83. 60 floats

84. The marginal revenue function, $R\,'$, is decreasing on $(0,\infty)$.

85. \$960

86. 500 items

87. To determine where a function f is increasing/decreasing, we find the domain of f, calculate the partition numbers of $f\,'$, and create a sign chart of $f\,'(x)$ to map the sign of $f\,'(x)$ to the behavior of f.

88. Not necessarily. If $f(x) > 0$, that means the graph of f is above the x-axis (it could be increasing or decreasing). f is only increasing on an interval if $f\,'(x) > 0$ on that interval.

89. Both the partition numbers of $f\,'$ and the critical values of f occur at the x-values where $f\,'(x) = 0$ or $f\,'(x)$ does not exist. However, critical values of f must be in the domain of f. Thus, the critical values of f are a subset of the partition numbers of $f\,'$.

90. Not necessarily. To be a critical value, $x = -2$ must also be in the domain of f.

91. The First Derivative Test is a test to find any local extrema of f, but we can also use the information when creating a sign chart of $f\,'(x)$ to find where f is increasing/decreasing.

92. For a function f to have a local extremum at $x = c$, $f\,'(c) = 0$ or $f\,'(c)$ does not exist, $f(c)$ is defined (i.e., $x = c$ is in the domain of f), and $f\,'(x)$ changes sign at $x = c$.

93. To determine where a function f has local extrema and the intervals where it is increasing/decreasing, we find the domain of f, calculate the partition numbers of $f\,'$, determine which partition numbers of $f\,'$ are also critical values of f, create a sign chart of $f\,'(x)$ to map the sign of $f\,'(x)$ to the behavior of f, and apply the First Derivative Test to find any local extrema.

94. Not necessarily. For f to have a local extremum at $x = 5$, $f(5)$ must be defined and $f'(x)$ must change sign at $x = 5$.

95. It is important to indicate on the sign chart of $f'(x)$ which partition numbers are in the domain of the function because if $f'(x)$ changes sign at a partition number, but that partition number is not in the domain of the function (meaning, it is not a critical value), then it cannot be a local extremum.

96. No. Local extrema can only occur at critical values. Thus, if a function does not have any critical values, it cannot have any local extrema. (But remember, just because a function has a critical value, that does not mean there is a local extremum at that critical value.)

Section 3.2

1. $f''(x) = 42x - 8x^{-2} - 30x^{-7}$

2. $f''(x) = 288(2x+13)^7$

3. $f''(x) = e^x + 24x^{-4}$

4. $f''(x) = 5e^{5x+9}$

5. $f''(x) = 44x^{43} - 3.6x^{11} + 0.5x^{-1/2} + 0.8x^{-9/5}$

6. $f''(x) = \dfrac{24x^2}{8x^3 - 22}$

7. (a) f is concave up on $(4, \infty)$, and f is concave down on $(-\infty, 4)$.
   (b) $x = 4$

8. (a) f is concave up nowhere, and f is concave down on $(-\infty, \infty)$.
   (b) None

9. (a) f is concave up on $(-\infty - 2)$ and $(2, \infty)$, and f is concave down on $(-2, 2)$.
   (b) $x = -2$; $x = 2$

10. (a) $x = -6$; $x = 3$; $x = 7$
    (b) f is concave up on $(-\infty, -6)$, $(-6, 3)$, and $(7, \infty)$, and f is concave down on $(3, 7)$.
    (c) $x = 3$; $x = 7$

11. (a) $x = -4$; $x = 0$; $x = 8$
    (b) f is concave up on $(-4, 0)$ and $(8, \infty)$, and f is concave down on $(-\infty, -4)$ and $(0, 8)$.
    (c) $x = -4$; $x = 0$; $x = 8$

12. (a) $x = -5$; $x = -2$; $x = 10$
    (b) f is concave up on $(-5, -2)$ and $(10, \infty)$, and f is concave down on $(-\infty, -5)$ and $(-2, 10)$.
    (c) $x = -2$; $x = 10$

13. (a) None
    (b) f is concave up on $(-\infty, \infty)$, and f is concave down nowhere.
    (c) None

14. (a) $x = -2$
    (b) f is concave up on $(-2, \infty)$, and f is concave down on $(-\infty, -2)$.
    (c) $(-2, 835)$

15. (a) $x = -3$; $x = 3$
    (b) f is concave up on $(-\infty, -3)$ and $(3, \infty)$, and f is concave down on $(-3, 3)$.
    (c) $(-3, -405)$; $(3, -405)$

16. (a) $x = 2$
    (b) f is concave up on $(2, \infty)$, and f is concave down on $(-\infty, 2)$.
    (c) $x = 2$

17. (a) $x = 3$; $x = 6$
    (b) f is concave up on $(3, 6)$, and f is concave down on $(-\infty, 3)$ and $(6, \infty)$.
    (c) $x = 3$; $x = 6$

18. (a) $x = -1$; $x = 0$; $x = 1$; $x = 4$
    (b) f is concave up on $(-1, 0)$ and $(1, 4)$, and f is concave down on $(-\infty, -1)$, $(0, 1)$, and $(4, \infty)$.
    (c) $x = -1$; $x = 0$; $x = 1$; $x = 4$

19. \$774.60

20. \$9.26

21. \$44.44

22. f has a local minimum at $x = 1$.

23. f has a local maximum at $x = -8$.

24. f has a local maximum at $x = 14$.

25. f has a local minimum at $x = 12.5$.

26. f has a local maximum at $x = -127$.

27. $\dfrac{d^2y}{dx^2} = 42x^5 + \left(\dfrac{8}{81}\right)x^{-17/9} + 12x^{-5}$

28. $\dfrac{d^2y}{dx^2} = xe^x + 2e^x$

29. $\dfrac{d^2y}{dx^2} = \dfrac{(x+14)^2\left(12x^3 + 168x^2\right) - 2\left(3x^4 + 56x^3\right)(x+14)}{(x+14)^4}$

30. $\frac{d^2y}{dx^2} = \frac{-64}{(\ln(7))(8x+5)^2} - 60x$

31. $\frac{d^2y}{dx^2} = 13^{x^5+4x^2}(\ln(13))\left(20x^3+8\right) + \left(5x^4+8x\right)^2 13^{x^5+4x^2}(\ln(13))^2$

32. (a) f is concave up on $(-\infty,-6)$, and f is concave down on $(-6,\infty)$.
    (b) None

33. (a) f is concave up on $(-\infty,1)$, and f is concave down on $(1,\infty)$.
    (b) None

34. (a) f is concave up on $(1,\infty)$, and f is concave down on $(-\infty,1)$.
    (b) None

35. (a) f is concave up on $(2,\infty)$, and f is concave down on $(-\infty,2)$.
    (b) $(2,0)$

36. (a) f is concave up $(-\infty,-10)$ and $(10,\infty)$, and f is concave down $(-10,10)$.
    (b) $\left(-10, 46e^{-10}\right)$; $\left(10,-34e^{10}\right)$

37. (a) f is concave up on $(-\infty,-1/2)$ and $(1/2,\infty)$, and f is concave down on $(-1/2,1/2)$.
    (b) $\left(-1/2, e^{127/2}\right)$; $\left(1/2, e^{127/2}\right)$

38. (a) f is concave up on $(0,2)$, and f is concave down on $(-\infty,0)$ and $(2,\infty)$.
    (b) $(0,0)$; $(2,12)$

39. (a) f is concave up on (a,b) and (c,∞), and f is concave down on $(-\infty,a)$ and (b,c).
    (b) $x=a$; $x=c$

40. (a) f is concave up on $(-\infty,b)$ and $(b,0)$, and f is concave down on (d,e) and (e,∞).
    (b) None

41. (a) $x=-2$; $x=0$; $x=2$
    (b) f is concave up on $(-\infty,-2)$ and $(2,\infty)$, and f is concave down on $(-2,0)$ and $(0,2)$.
    (c) $x=-2$; $x=2$

42. (a) $x=b$; $x=d$
    (b) f is concave up on $(-\infty,b)$ and (d,∞), and f is concave down on (b,d).
    (c) $x=b$; $x=d$

43. (a) $x=a$; $x=d$; $x=e$; $x=g$; $x=h$
    (b) f is concave up on (a,d), (e,g), and (h,∞), and f is concave down on $(-\infty,a)$, (d,e), and (g,h).
    (c) $x=a$; $x=d$; $x=g$; $x=h$

44. (a) $x = 2$

(b) f is concave up on $(2, \infty)$, and f is concave down on $(-\infty, 2)$.

(c) $x = 2$

45. (a) $x = a$; $x = 0$; $x = c$

(b) f is concave up on $(a, 0)$ and (c, ∞), and f is concave down on $(-\infty, a)$ and $(0, c)$.

(c) $x = 0$; $x = c$

46. (a) $x = a$; $x = 0$; $x = d$; $x = f$

(b) f is concave up on $(-\infty, a)$, $(a, 0)$, and (f, ∞), and f is concave down on $(0, d)$ and (d, f).

(c) $x = 0$

47. $x = 22.90871$; Tom Nook's returns diminish when he spends \$22,908.71 per month on advertising.

48. $x = 32.66667$; Greyt Toys, Inc.'s returns diminish when it spends \$32,666.67 each quarter on advertising.

49. $x = 91.67$; Elga's returns diminish when she spends \$91.67 on advertising in September.

50. f has a local minimum at $x = 7$.

51. It is not possible to determine if f has a local extremum at $x = 100$ because the Second Derivative Test is inconclusive (i.e., the Second Derivative Test fails).

52. f has a local maximum at $x = -37$.

53. f has a local maximum of 97 at $x = -3$ and a local minimum of -28 at $x = 2$.

54. f has a local maximum of $\dfrac{10}{e}$ at $x = -1$ and local minima of $\dfrac{-74}{e^7}$ at $x = -7$ and 3 at $x = 0$.

55. $x = 0$ is the only critical value of f, but the Second Derivative Test fails at $x = 0$ because $f''(0) = 0$.

56. (a) g is concave up on (a, b), (i, j), and (j, ∞), and g is concave down on $(-\infty, a)$, (d, e), (e, g), and (g, i).

(b) $x = a$; $x = i$

57. $f''(x) = 132x^{10} - 270x^8 + 120x^4$

58. $f''(x) = \dfrac{2(\ln(10))\left(x^2+7\right) - 4(\ln(10))x^2}{\left(x^2+7\right)^2(\ln(10))^2}$

59. $f''(x) = \dfrac{-24(x-3)^5 - 5(-24x-80)(x-3)^4}{(x-3)^{10}}$

60. $f''(x) = 9^{7x^8-33x}(\ln(9))\left(392x^6\right) + \left(56x^7-33\right)^2 9^{7x^8-33x}$

61. $f''(x) = \dfrac{2\sqrt{44x^3-8x^2+1}(264x-16) - 0.5\left(132x^2-16x\right)^2\left(44x^3-8x^2+1\right)^{-1/2}}{\left(2\sqrt{44x^3-8x^2+1}\right)^2}$

62. $f''(x) = -2x^{-2} + \dfrac{\left(4x^2+7\right)(72)-(72x)(8x)}{\left(4x^2+7\right)^2} + \dfrac{32}{(13-2x)^2}$

63. (a) $x=a$; $x=c$; $x=d$; $x=e$; $x=g$; $x=h$; $x=i$; $x=j$; $x=k$

    (b) f is concave up on $(-\infty,a)$, (c,d), (d,e), (g,h), (i,j), and (j,k), and f is concave down on (a,c), (e,g), (h,i), and (k,∞).

    (c) $x=c$; $x=e$; $x=g$; $x=h$; $x=i$; $x=k$

64. The amount spent by Big Box Store at the point of diminishing returns is \$14,666.67.

65. (a) f is concave up on $(-\infty,0)$ and $(9/10,\infty)$, and f is concave down on $(0,9/10)$.

    (b) $x=0$; $x=9/10$

66. (a) f is concave up on $(-\infty,-7)$, $(-7,-2)$, and $(3,\infty)$, and f is concave down on $(-2,3)$.

    (b) $x=-2$

67. (a) f is concave up on $(-\infty,\infty)$, and f is concave down nowhere.

    (b) None

68. (a) f is concave up on $(-\infty,0)$ and $(2,\infty)$, and f is concave down on $(0,2)$.

    (b) $x=0$; $x=2$

69. (a) f is concave up on $(17,\infty)$, and f is concave down on $(-\infty,17)$.

    (b) $x=17$

70. (a) f is concave up on $(-\infty,-6)$ and $(1,\infty)$, and f is concave down on $(-6,-2)$ and $(-2,1)$.

    (b) $x=-6$; $x=1$

71. (a) f is concave up on $\left(e^{3/2},\infty\right)$, and f is concave down on $\left(0,e^{3/2}\right)$.

    (b) $x=e^{3/2}$

72. (a) f is concave up on $(-1,1)$, and f is concave down on $(-\infty,-1)$ and $(1,\infty)$.

    (b) $x=-1$; $x=1$

73. (a) f is concave up on $(-2,4)$ and $(4,\infty)$, and f is concave down on $(-\infty,-2)$.

    (b) $x=-2$

74. (a) f is concave up $(0,\infty)$, and f is concave down $(-\infty,0)$.

    (b) None

75. (a) f is concave up on $(-\infty,-1)$ and $(5/3,\infty)$, and f is concave down on $(-1,0)$ and $(0,5/3)$.

    (b) $x=-1$; $x=5/3$

76. (a) $x = a$; $x = b$; $x = 0$; $x = d$; $x = e$; $x = g$; $x = j$; $x = k$

(b) f is concave up on (b, c), (d, e), (g, j), and (k, ∞), and f is concave down on $(-\infty, a)$, (a, b), (c, d), (e, g), and (j, k).

(c) $x = b$; $x = c$; $x = e$; $x = j$; $x = k$

77. (a) $t = -b$; $t = a$; $t = c$

(b) g is concave up on $(-\infty, -b)$, $(-b, a)$, and (c, ∞), and g is concave down on (a, c).

(c) $t = a$

78. f has a local maximum of 28 at $x = 3$ and a local minimum of -4 at $x = -1$.

79. f has a local maximum of $\dfrac{216}{e^3}$ at $x = 6$, but the Second Derivative Test fails at $x = 0$ because $f''(0) = 0$.

80. f has a local maximum of -2 at $x = -4$ and a local minimum of 0 at $x = 0$.

81. (a) $x = -3$

(b) $x = 0$; $x = 2$

(c) f' is positive on $(-\infty, -3)$ and $(-3, 0)$, and f' is negative on $(0, 2)$ and $(2, \infty)$.

(d) f'' is positive on $(-3, -2)$ and $(2, \infty)$, and f'' is negative on $(-\infty, -3)$.

(e) f' has a local minimum at $x = -3$, but f' has no local maxima.

82. (a) f is increasing on $(-\infty, -1)$ and $(4, \infty)$, and f is decreasing on $(-1, 1)$ and $(1, 4)$.

(b) f has a local maximum at $x = -1$ and a local minimum at $x = 4$.

(c) f'' is positive on $(1, 3.5)$ and $(3.5, \infty)$, and f'' is negative on $(-\infty, -1)$ and $(-1, 1)$.

(d) $x = -1$

(e) $x = 1$; $x = 3.5$

(f) $x = 1$

83. (a) f' is increasing on $(-\infty, -3)$, $(-1, 1)$, and $(3, \infty)$, and f' is decreasing on $(-3, -1)$ and $(1, 3)$.

(b) f' has local maxima at $x = -3$ and $x = 1$ and local minima at $x = -1$ and $x = 3$.

(c) f is concave up on $(-\infty, -3)$, $(-1, 1)$, and $(3, \infty)$, and f is concave down on $(-3, -1)$ and $(1, 3)$.

(d) $x = -2.25$; $x = 0$; $x = 2.25$

(e) $x = -3$; $x = -1$; $x = 1$; $x = 3$

84. $x = 191.91919$; Staples Depot will have diminishing returns when they spend \$191,919.19 on advertising each year.

85. $x = 55.55556$; Civet Coffee, Inc will have diminishing returns when they spend \$55,555.56 on advertising each year.

86. f has a local maximum at $x = -24$.

87. f does not have a local extremum at $x = 0$ because $x = 0$ is not a critical value of f due to the fact that $f'(0) = 5$.

88. The Second Derivative Test fails at $x = 3$ because $f''(3) = 0$.

89. f has a local maximum at $x = 11$.

90. We find the partition numbers of f'' by finding the x-values where $f''(x) = 0$ or $f''(x)$ does not exist.

91. The Concavity Test allows to find the intervals of concavity of a function as well as where the function has inflection points.

92. To apply the Concavity Test, we find the domain of f, the partition numbers of f'', and create a sign chart of $f''(x)$.

93. The three conditions necessary for f to have an inflection point at $x = c$ are: $f''(c) = 0$ or $f''(c)$ does not exist, $f(c)$ is defined, and $f''(x)$ changes sign at $x = c$.

94. Not necessarily. For f to have an inflection point at $x = -3$, $x = -3$ must be in the domain of f and $f''(x)$ must change sign at $x = -3$.

95. "f is concave up" is equivalent to stating that the slopes of the graph of f are increasing, f' is increasing, and $f''(x) > 0$.

96. The Second Derivative Test is an alternative test to find local extrema.

97. We can only attempt to use the Second Derivative Test for critical values such that $f'(x) = 0$.

98. For a critical value $x = c$ in which $f'(c) = 0$, if we perform the Second Derivative Test and get $f''(c) = 0$, then the test fails, and we cannot determine whether or not there is a local extremum at $x = c$. Similarly, if $f''(c)$ does not exist, we cannot determine whether or not there is a local extremum at $x = c$ because the Second Derivative Test assumes the function is twice-differentiable at $x = c$.

99. No. If the Second Derivative Test fails (or cannot be used at a critical value), we do not know whether or not the function has a local extremum at the critical value.

100. If the Second Derivative Test fails (or cannot be used at a critical value), we must go back and apply the First Derivative Test to determine if the function has a local extremum at the critical value.

101. (a) f is increasing on (a,b).
     (b) We cannot determine anything regarding the graph of f'' on (a,b).

102. (a) We cannot determine anything regarding the graph of f' on (a,b).
     (b) We cannot determine anything regarding the graph of f'' on (a,b).

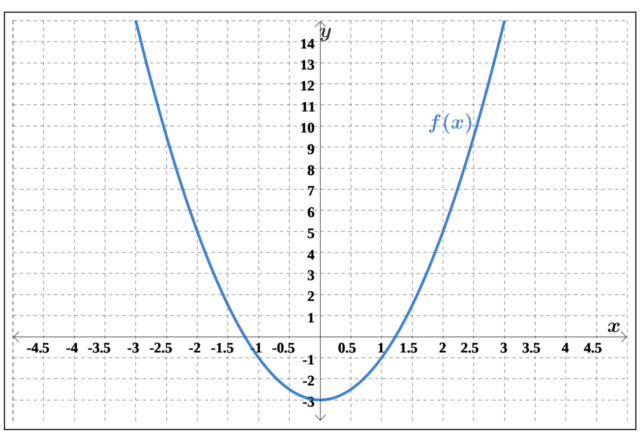
103. (a) f is concave up on (a,b).
     (b) f' is increasing on (a,b).

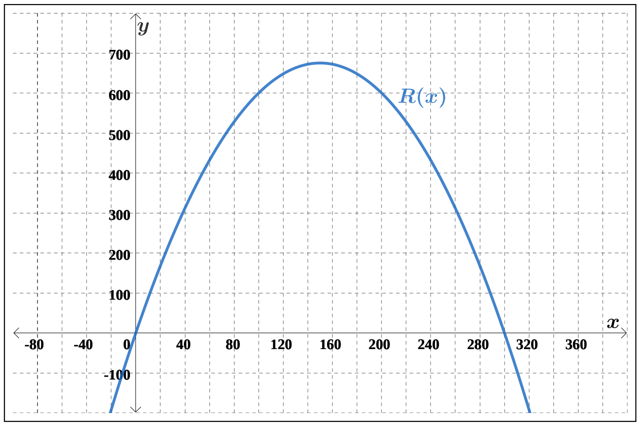
Section 3.3

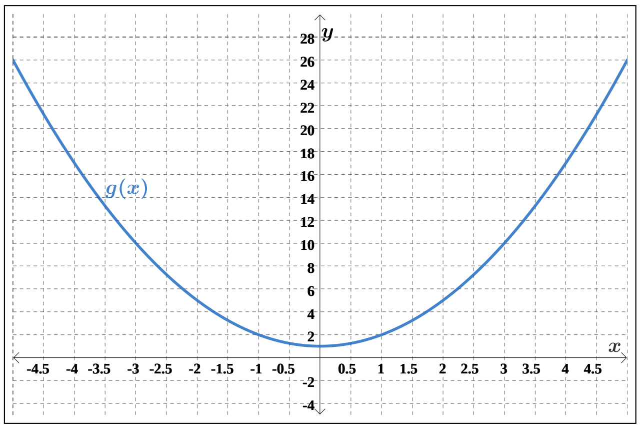
1. (a) $(-\infty,\infty)$
   (b) $(-3,0)$; $(3,0)$; $(0,9)$
   (c) None
   (d) f is increasing on $(-\infty,0)$, and f is decreasing on $(0,\infty)$.
   (e) f has a local maximum at $x=0$ and no local minima.
   (f) f is concave up nowhere, and f is concave down on $(-\infty,\infty)$.
   (g) None

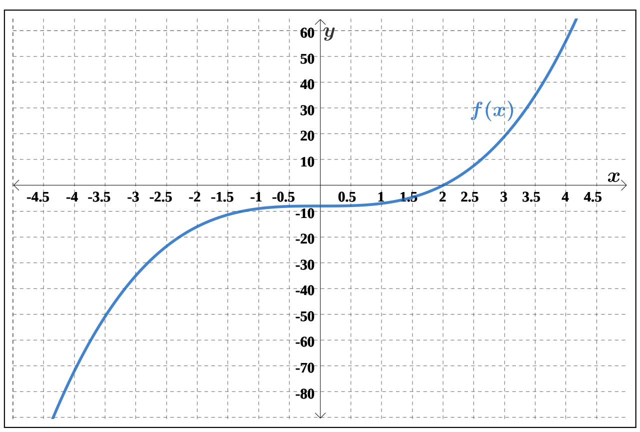
2. (a) $(-\infty,-1)\cup(-1,0)\cup(0,\infty)$
   (b) $(0.5,0)$
   (c) Hole at $x=-1$; Asymptotes: $x=0$; $y=-2$; $y=0$
   (d) f is increasing on $(-\infty,-1)$ and $(-1,0)$, and f is decreasing on $(0,\infty)$.
   (e) None
   (f) f is concave up on $(-\infty,-1)$, $(-1,0)$, and $(0,\infty)$, and f is concave down nowhere.
   (g) None

3. (a) $(-\infty,2)\cup(2,\infty)$
   (b) $(0,0)$; $(0.5,0)$; $(1.5,0)$
   (c) No holes; Asymptotes: $x=0$; $x=2$; $y=0$
   (d) f is increasing on $(0,1)$, and f is decreasing on $(-\infty,0)$, $(1,2)$, and $(2,\infty)$.
   (e) f has a local maximum at $x=1$ and no local minima.
   (f) f is concave up on $(-\infty,0)$ and $(2,\infty)$, and f is concave down on $(0,2)$.
   (g) $x=0$

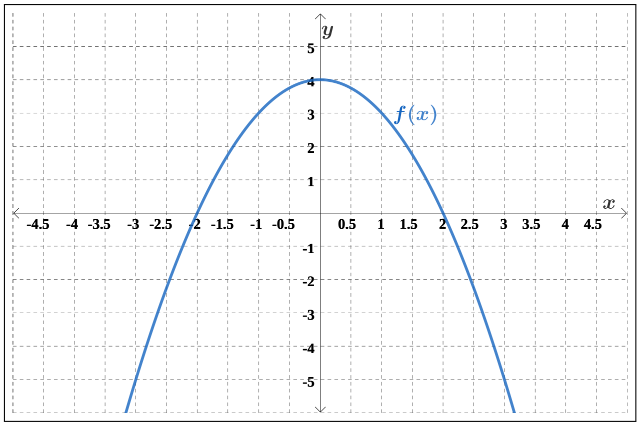
4. 

5. 
y
700
600
500
400
300
200
100
-100
$R(x)$
x
-80 -40 0 40 80 120 160 200 240 280 320 360

6. 
y
28 26 24 22 20 18 16 14 12 10 8 6 4 2 -2 -4
$g(x)$
x
-4.5 -4 -3.5 -3 -2.5 -2 -1.5 -1 -0.5 0.5 1 1.5 2 2.5 3 3.5 4 4.5

7. 
y
60 50 40 30 20 10 -10 -20 -30 -40 -50 -60 -70 -80
$f(x)$
x
-4.5 -4 -3.5 -3 -2.5 -2 -1.5 -1 -0.5 0.5 1 1.5 2 2.5 3 3.5 4 4.5

8.



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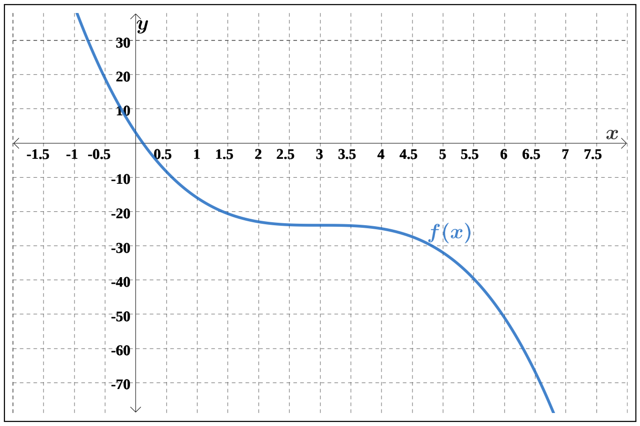

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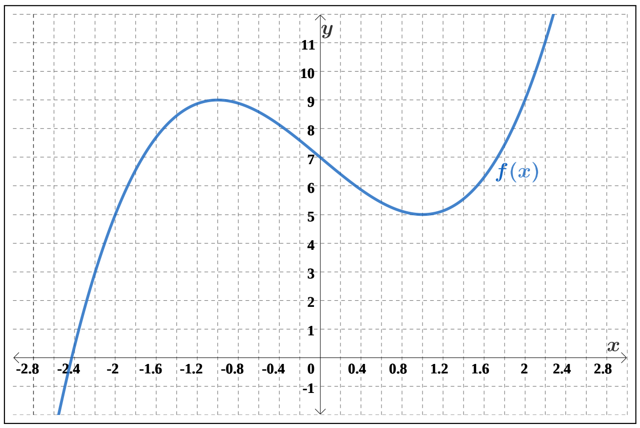
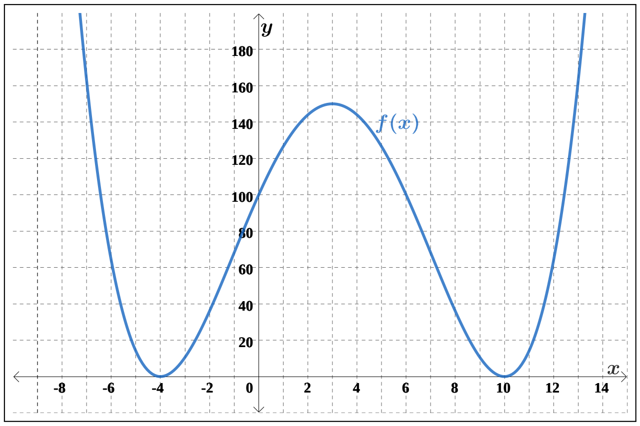
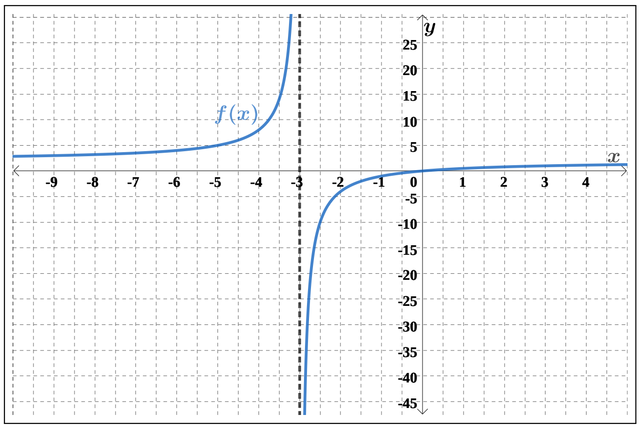


11.



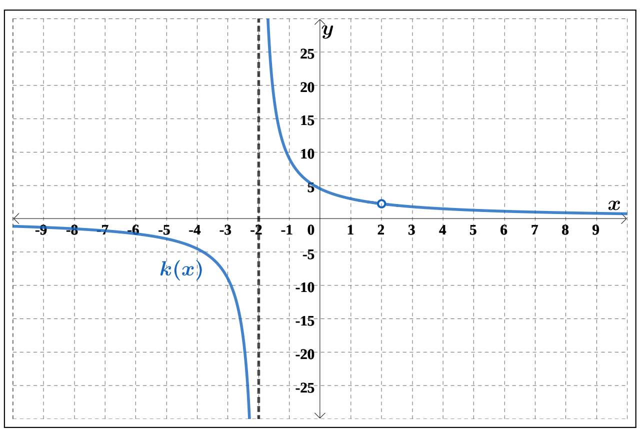
y
x
f(x)
180
160
140
120
100
80
60
40
20
-8 -6 -4 -2 0 2 4 6 8 10 12 14

12.

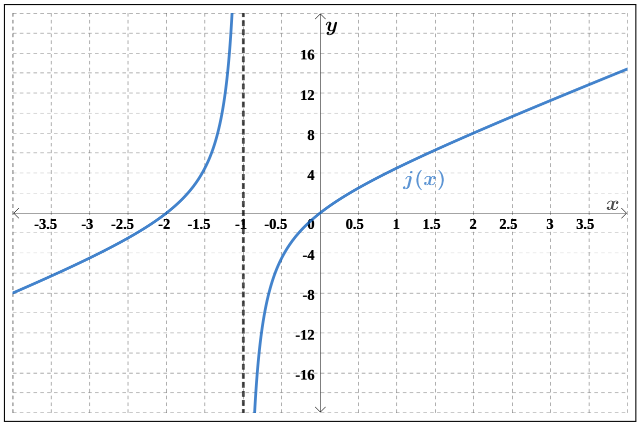


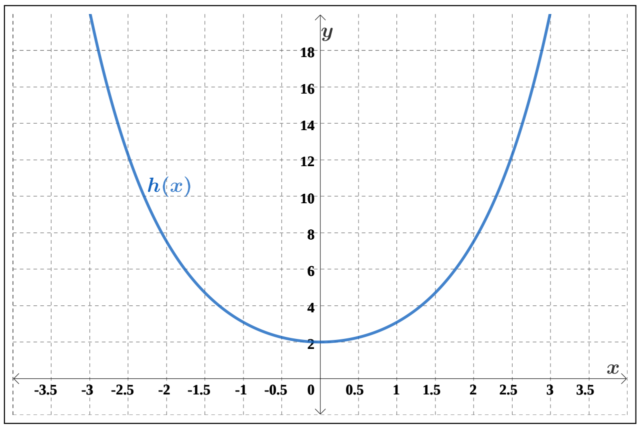
y
x
f(x)
25
20
15
10
5
-9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4
-5
-10
-15
-20
-25
-30
-35
-40
-45

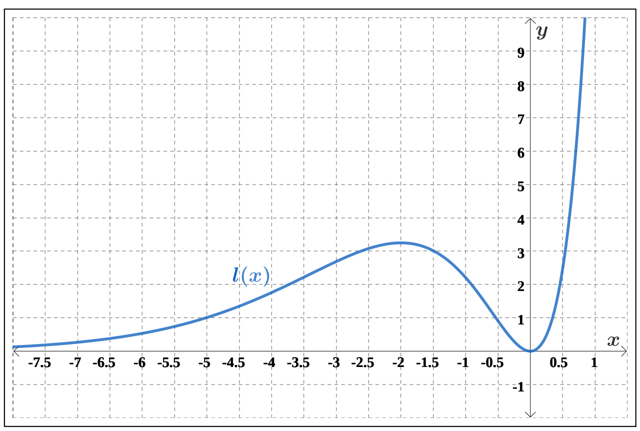
13.



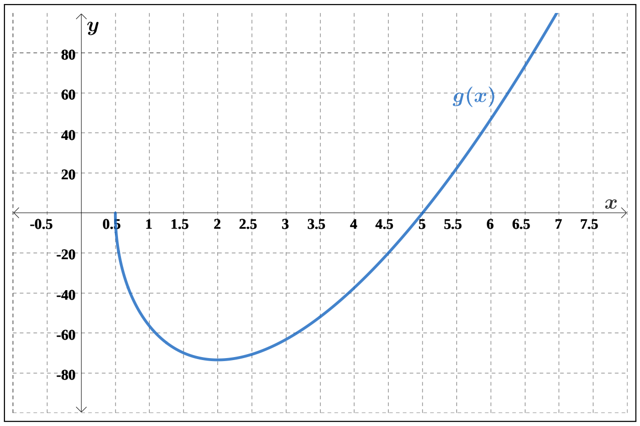
y
x
k(x)
25
20
15
10
5
-9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9
-5
-10
-15
-20
-25

14. 
y
16
12
8
4
j(x)
x
-3.5 -3 -2.5 -2 -1.5 -1 -0.5 0 0.5 1 1.5 2 2.5 3 3.5
-4
-8
-12
-16

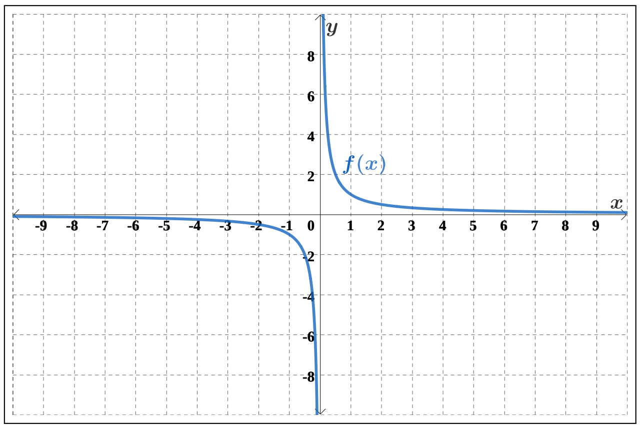
15. 
y
18
16
14
12
h(x)
10
8
6
4
2
x
-3.5 -3 -2.5 -2 -1.5 -1 -0.5 0 0.5 1 1.5 2 2.5 3 3.5

16. 
y
9
8
7
6
5
4
3
l(x)
2
1
x
-7.5 -7 -6.5 -6 -5.5 -5 -4.5 -4 -3.5 -3 -2.5 -2 -1.5 -1 -0.5 0.5 1
-1

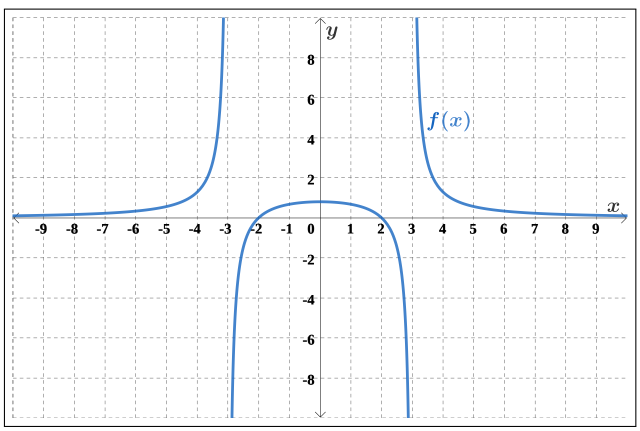
17.

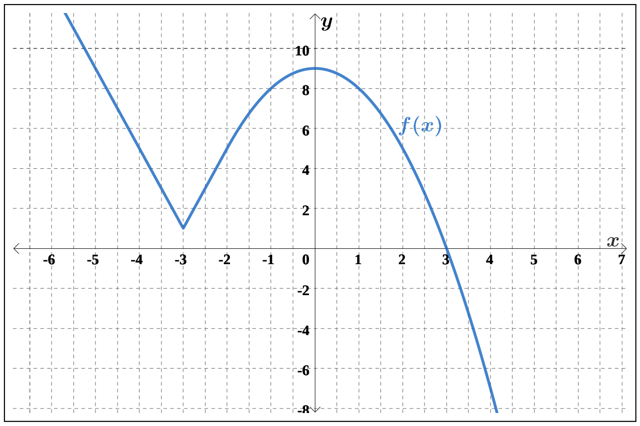

y
80
60
40
20
-20
-40
-60
-80
x
-0.5
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
5.5
6
6.5
7
7.5
$g(x)$

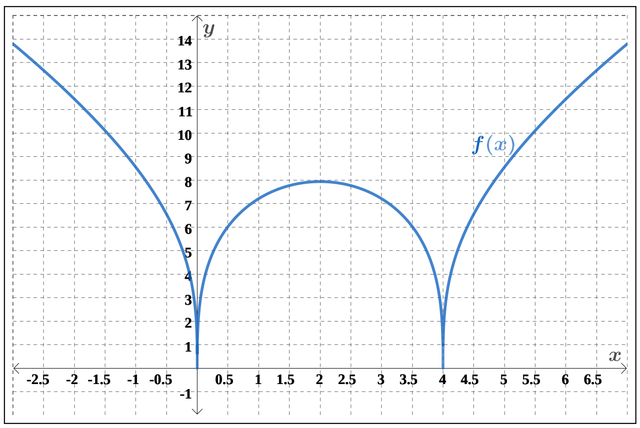
18.

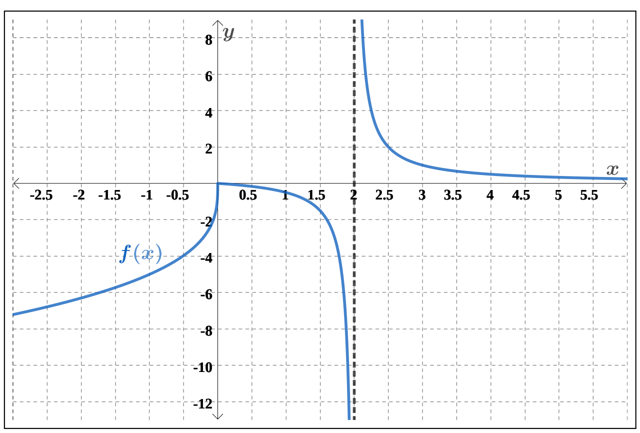

y
8
6
4
2
0
-2
-4
-6
-8
x
-9 -8 -7 -6 -5 -4 -3 -2 -1
1 2 3 4 5 6 7 8 9
$f(x)$

19.


y
8
6
4
2
0
-2
-4
-6
-8
x
-9 -8 -7 -6 -5 -4 -3 -2 -1
1 2 3 4 5 6 7 8 9
$f(x)$

20. 

21. 

22. 

23. (a) $(-\infty,-1)\cup(-1,2)\cup(2,\infty)$

(b) $(-1.5,0)$; $(0,0)$; $(3,0)$

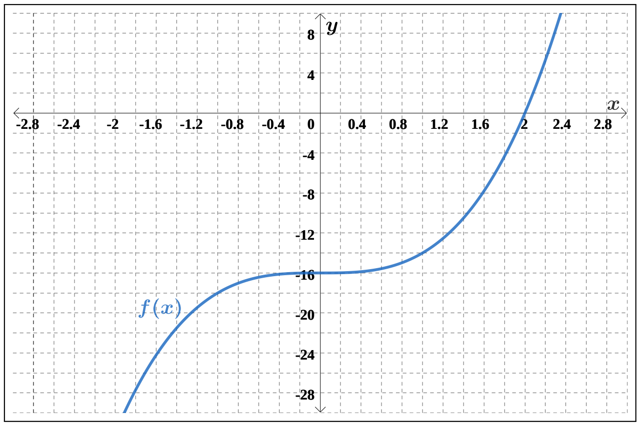
(c) Hole at $x=2$; Asymptotes: $x=-1$; $y=2$

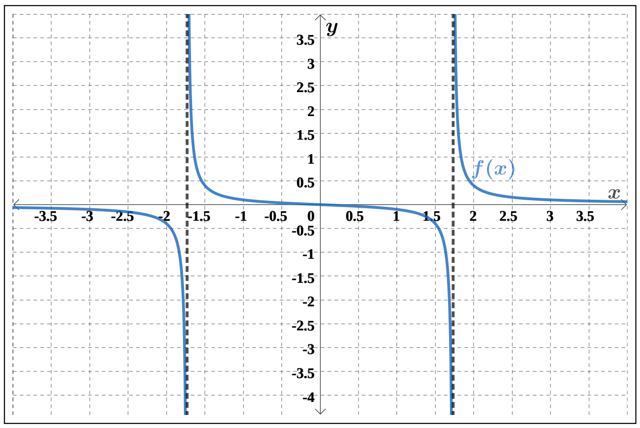
(d) f is increasing on $(-1,0)$ and $(0,2)$, and f is decreasing on $(-\infty,-1)$ and $(2,\infty)$.

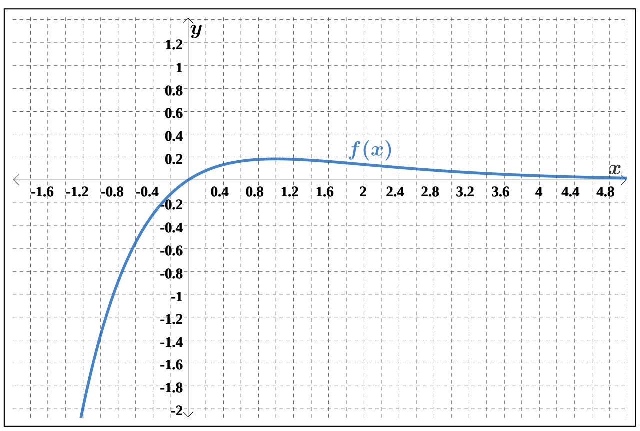
(e) f has no local extrema.

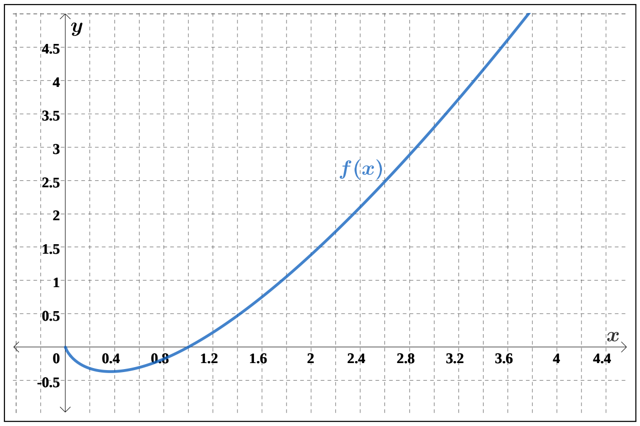
(f) f is concave up on $(0.5,2)$, and f is concave down on $(-\infty,-1)$, $(-1,0.5)$, and $(2,\infty)$.

(g) $x=0.5$

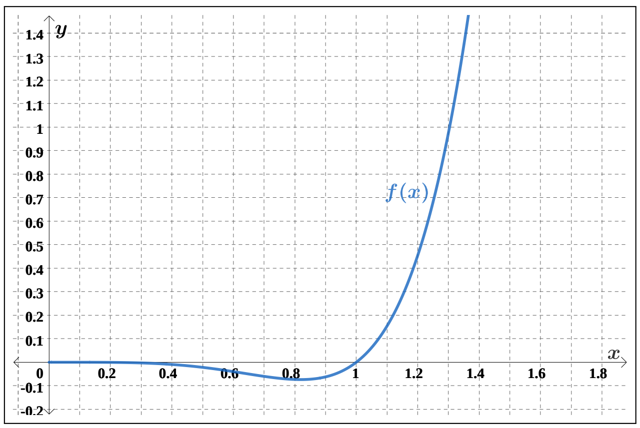
24. 

25. 

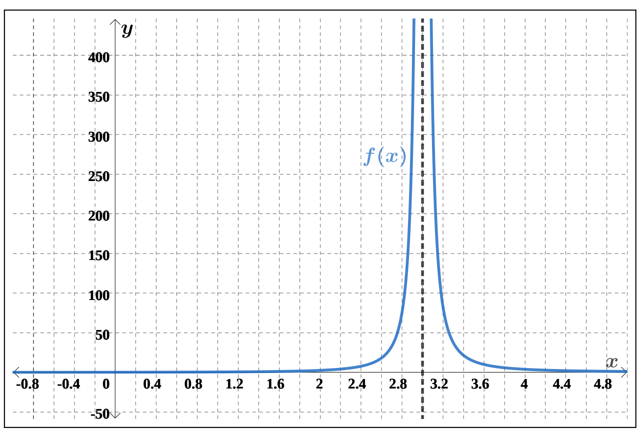
26. 

27. 

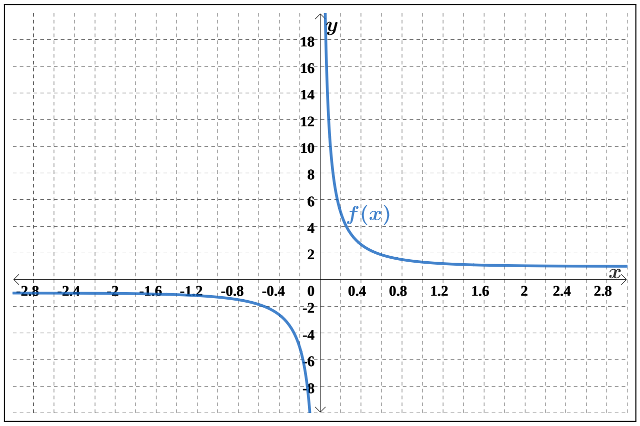
y
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
0.4
0.8
1.2
1.6
2
2.4
2.8
3.2
3.6
4
4.4
x
f(x)

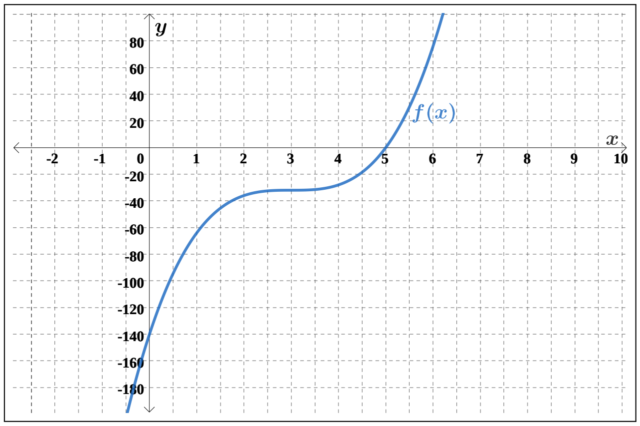
28. 

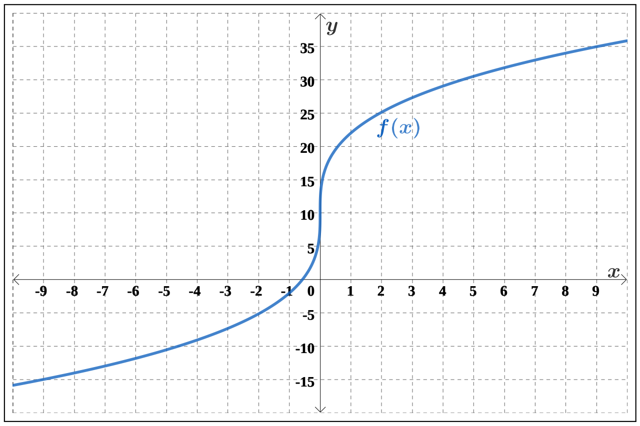
y
1.4
1.3
1.2
1.1
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
x
f(x)

29. 

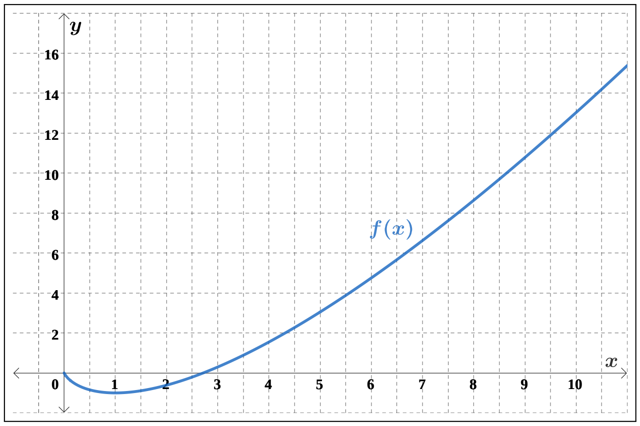
y
400
350
300
250
200
150
100
50
-50
-0.8
-0.4
0
0.4
0.8
1.2
1.6
2
2.4
2.8
3.2
3.6
4
4.4
4.8
x
f(x)

30. 
y
x
f(x)
18
16
14
12
10
8
6
4
2
-2
-4
-6
-8
-2.8 -2.4 -2 -1.6 -1.2 -0.8 -0.4 0 0.4 0.8 1.2 1.6 2 2.4 2.8

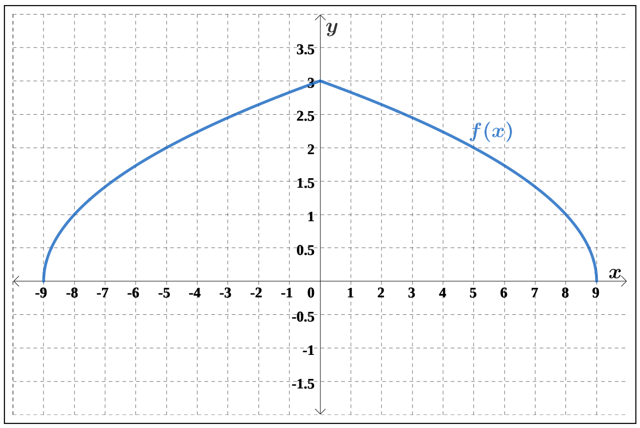
31. 
y
x
f(x)
80
60
40
20
-20
-40
-60
-80
-100
-120
-140
-160
-180
-2 -1 0 1 2 3 4 5 6 7 8 9 10

32. 
y
x
f(x)
35
30
25
20
15
10
5
-5
-10
-15
-9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9

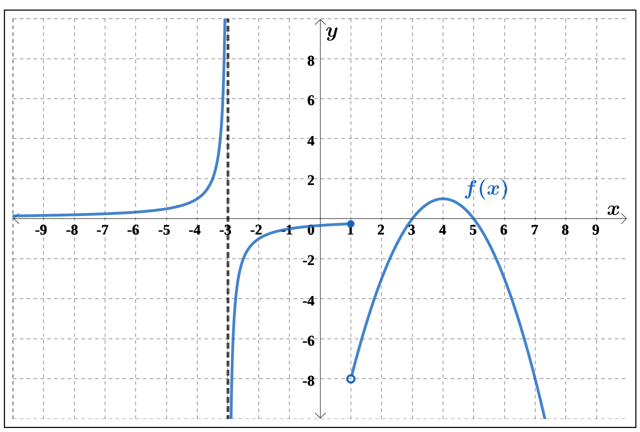
33.



34.



35.



36. To apply the graphing strategy, we must analyze f (domain, intercepts, holes, and asymptotes), analyze $f\,'$ (increasing/decreasing and local extrema), analyze $f\,''$ (concavity and inflection points), create a shape chart using the sign charts of $f\,'(x)$ and $f\,''(x)$, and then sketch the graph of the function.

37. The graph of f has a horizontal asymptote, $y = 5$.

38. The graph of f has a vertical asymptote, $x = -2$, and the function tends to positive infinity on both the left- and right-hand sides of the vertical asymptote.

39. The possible shapes of f include increasing and concave up, increasing and concave down, decreasing and concave up, and decreasing and concave down.

Section 3.4

1. (a) Local maximum at $x = 3$; No local minima
   (b) Absolute maximum at $x = 3$; No absolute minimum
2. (a) Local maximum at $x = -1$; Local minimum at $x = 1$
   (b) No absolute maximum; No absolute minimum
3. (a) No local maxima; No local minima
   (b) No absolute maximum; No absolute minimum
4. (a) Local maximum at $x = -3$; No local minima
   (b) No absolute maximum; No absolute minimum
5. (a) Local maximum at $x = 0$; Local minima at $x = -1.4$ and $x = 1.4$
   (b) No absolute maximum; Absolute minimum at $x = -1.4$ and $x = 1.4$
6. (a) Absolute maximum: 12; Absolute minimum: 0
   (b) Absolute maximum: 12; Absolute minimum: -4
   (c) Absolute maximum: 10; Absolute minimum: 6
7. (a) Absolute maximum: 1; Absolute minimum: -1
   (b) Absolute maximum: 5; Absolute minimum: -2
   (c) Absolute maximum: 2; Absolute minimum: -2
8. (a) Absolute maximum: 3; Absolute minimum: None
   (b) Absolute maximum: None; Absolute minimum: 1
   (c) Absolute maximum: None; Absolute minimum: None
9. (a) Absolute maximum: 2; Absolute minimum: -1
   (b) Absolute maximum: None; Absolute minimum: -1
   (c) Absolute maximum: 6; Absolute minimum: None
10. (a) Absolute maximum: None; Absolute minimum: None
    (b) Absolute maximum: 6; Absolute minimum: None
    (c) Absolute maximum: 6; Absolute minimum: 4
11. (a) Absolute maximum: 0; Absolute minimum: None
    (b) Absolute maximum: 0; Absolute minimum: None
    (c) Absolute maximum: None; Absolute minimum: -4
12. (a) Absolute maximum: 16807; Absolute minimum: -729
    (b) Absolute maximum: 64; Absolute minimum: -297
    (c) Absolute maximum: 7203; Absolute minimum: -297

13. (a) Absolute maximum: 65; Absolute minimum: 11
    (b) Absolute maximum: 11; Absolute minimum: −309
    (c) Absolute maximum: 15; Absolute minimum: −5

14. (a) Absolute maximum: 32; Absolute minimum: 0
    (b) Absolute maximum: 81/7; Absolute minimum: 0
    (c) Absolute maximum: −40; Absolute minimum: −72

15. (a) Absolute maximum: $-1/2$; Absolute minimum: −1
    (b) Absolute maximum: 4; Absolute minimum: 4/5
    (c) Absolute maximum: $-2/5$; Absolute minimum: $-4/5$

16. (a) Absolute maximum: 16; Absolute minimum: 49/4
    (b) Absolute maximum: 0; Absolute minimum: −4
    (c) Absolute maximum: 16; Absolute minimum: 12

17. (a) Absolute maximum: 242; Absolute minimum: −14
    (b) Absolute maximum: 242; Absolute minimum: −183
    (c) Absolute maximum: 242; Absolute minimum: −14

18. (a) Absolute maximum: 2148; Absolute minimum: −2924
    (b) Absolute maximum: 148; Absolute minimum: −2924
    (c) Absolute maximum: 148; Absolute minimum: −924

19. (a) Absolute maximum: 481; Absolute minimum: −2126
    (b) Absolute maximum: $-58,270$; Absolute minimum: $-77,575$
    (c) Absolute maximum: 50; Absolute minimum: $-77,575$

20. (a) Absolute maximum: 4/3; Absolute minimum: None
    (b) Absolute maximum: 1; Absolute minimum: None
    (c) Absolute maximum: 0; Absolute minimum: None

21. (a) Absolute maximum: None; Absolute minimum: None
    (b) Absolute maximum: None; Absolute minimum: None
    (c) Absolute maximum: None; Absolute minimum: None

22. (a) Local maxima of 24 at $x = -4$ and 4 at $x = 12$; No local minima
    (b) Absolute maximum is 24 at $x = -4$; No absolute minimum

23. (a) No local maxima; Local minima of −16 at $x = 3$ and 4 at $x = 6$
    (b) No absolute maximum; Absolute minimum is −16 at $x = 3$

24. (a) Local maximum of 8 at $x = -6$; No local minima
    (b) No absolute maximum; No absolute minimum

25. (a) Absolute maximum: 2; Absolute minimum: None
    (b) Absolute maximum: 6; Absolute minimum: None
    (c) Absolute maximum: None; Absolute minimum: 2

26. (a) Absolute maximum: 4; Absolute minimum: None
    (b) Absolute maximum: None; Absolute minimum: None
    (c) Absolute maximum: None; Absolute minimum: 1

27. (a) Absolute maximum: 0; Absolute minimum: $-e$
    (b) Absolute maximum: $-7e^{-5}$; Absolute minimum:-2
    (c) Absolute maximum: e^3; Absolute minimum: $-e$

28. (a) Absolute maximum: $248,832$; Absolute minimum: -243
    (b) Absolute maximum: 3125; Absolute minimum: -1024
    (c) Absolute maximum: $4,084,101$; Absolute minimum: -1024

29. (a) Absolute maximum: $\frac{1}{2e}$; Absolute minimum: 0
    (b) Absolute maximum: $\frac{1}{2e}$; Absolute minimum: $4\ln(1/2)$
    (c) Absolute maximum: $\frac{\ln(2)}{4}$; Absolute minimum: $\frac{\ln(4)}{16}$

30. (a) Absolute maximum: 4; Absolute minimum: 0
    (b) Absolute maximum: $\sqrt[3]{225}$; Absolute minimum: 0
    (c) Absolute maximum: 1; Absolute minimum: 0

31. (a) Absolute maximum: e^{-9}; Absolute minimum: e^{-49}
    (b) Absolute maximum: 1; Absolute minimum: e^{-1}
    (c) Absolute maximum: 1; Absolute minimum: $e^{-9/4}$

32. (a) Absolute maximum: $36\ln(3)$; Absolute minimum: 0
    (b) Absolute maximum: $100\ln(5)$; Absolute minimum: $16\ln(2)$
    (c) Absolute maximum: $-\frac{1}{25}\ln(10)$; Absolute minimum: $-\frac{2}{e}$

33. (a) Absolute maximum: $-\frac{1000}{91}$; Absolute minimum: $-\frac{125}{8}$
    (b) Absolute maximum: $\frac{8}{5}$; Absolute minimum: $-\frac{1}{8}$
    (c) Absolute maximum: $\frac{(27)^{3/2}}{18}$; Absolute minimum: $\frac{64}{7}$

34. (a) Absolute maximum: 1; Absolute minimum: $5-5\ln(5)$
    (b) Absolute maximum: $2-5\ln(2)$; Absolute minimum: $4-5\ln(4)$
    (c) Absolute maximum: $20-5\ln(20)$; Absolute minimum: $15-5\ln(15)$

35. (a) Absolute maximum: None; Absolute minimum: 3/2
    (b) Absolute maximum: None; Absolute minimum: $-1/2$
    (c) Absolute maximum: None; Absolute minimum: None

36. (a) Absolute maximum: $(28)^{4/3}$; Absolute minimum: 0
    (b) Absolute maximum: None; Absolute minimum: 0
    (c) Absolute maximum: $(-9/4)^{4/3}$; Absolute minimum: 0

37. (a) Absolute maximum: 3/4; Absolute minimum: $-3/4$
    (b) Absolute maximum: 3/4; Absolute minimum: $-3/4$
    (c) Absolute maximum: 3/4; Absolute minimum: 0

38. (a) Absolute maximum: None; Absolute minimum: None
    (b) Absolute maximum: $(-5/4)e^2$; Absolute minimum: $(-5/16)e^4$
    (c) Absolute maximum: None; Absolute minimum: $(-5/64)e^8$

39. (a) Absolute maximum: $\ln(1/2)-(1/2)$; Absolute minimum: None
    (b) Absolute maximum: None; Absolute minimum: None
    (c) Absolute maximum: $\ln(1/2)-(1/2)$; Absolute minimum: None

40. (a) Absolute maximum: $-1/25$; Absolute minimum: $-17/9$
    (b) Absolute maximum: None; Absolute minimum: None
    (c) Absolute maximum: None; Absolute minimum: None

41. (a) No local maxima; Local minimum of -3 at $x = 2$
    (b) No absolute maximum; Absolute minimum is -5 at $x = -2$

42. (a) Absolute maximum: None; Absolute minimum: -4
    (b) Absolute maximum: 7; Absolute minimum: None
    (c) Absolute maximum: 0; Absolute minimum: None
    (d) Absolute maximum: None; Absolute minimum: None
    (e) Absolute maximum: 4; Absolute minimum: None
    (f) Absolute maximum: None; Absolute minimum: None

43. (a) Absolute maximum: $2500e^{-2}$; Absolute minimum: None
    (b) Absolute maximum: $2500e^{-2}$; Absolute minimum: 0
    (c) Absolute maximum: None; Absolute minimum: None

44. (a) Absolute maximum: $5/24$; Absolute minimum: -5
    (b) Absolute maximum: None; Absolute minimum: $-5/8$
    (c) Absolute maximum: None; Absolute minimum: None

45. (a) Absolute maximum: None; Absolute minimum: $-2e^{-1}$
    (b) Absolute maximum: 0; Absolute minimum: $-2e^{-1}$
    (c) Absolute maximum: None; Absolute minimum: None

46. (a) Absolute maximum: $\dfrac{-e}{2}$; Absolute minimum: $\dfrac{-e^4}{4}$
    (b) Absolute maximum: $2e^{1/16}$; Absolute minimum: $\dfrac{e^{1/2}}{\sqrt{2}}$
    (c) Absolute maximum: None; Absolute minimum: None

47. (a) Absolute maximum: None; Absolute minimum: -25
    (b) Absolute maximum: 623; Absolute minimum: 88
    (c) Absolute maximum: None; Absolute minimum: 168

48. (a) Absolute maximum: None; Absolute minimum: None
    (b) Absolute maximum: 2401; Absolute minimum: None
    (c) Absolute maximum: 0; Absolute minimum: -625

49. (a) Absolute maximum: 59,049; Absolute minimum: 0
    (b) Absolute maximum: 1,048,576; Absolute minimum: 0
    (c) Absolute maximum: 1,048,576; Absolute minimum: 0

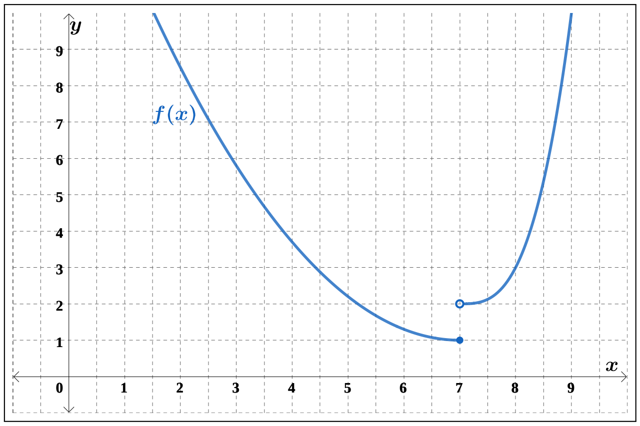
50. (a) Absolute maximum: 16/9; Absolute minimum: 7/18
    (b) Absolute maximum: 16/9; Absolute minimum: None
    (c) Absolute maximum: None; Absolute minimum: 0

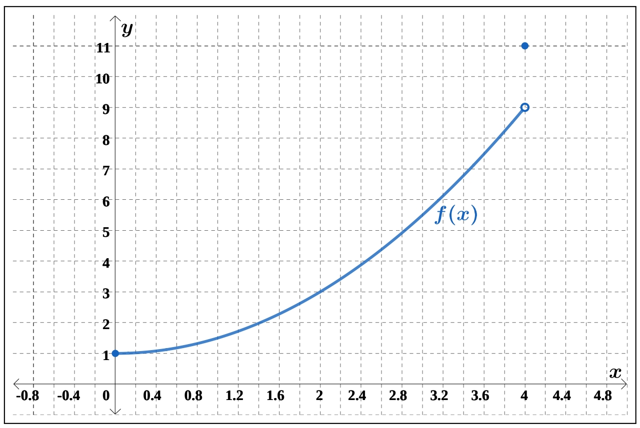
51. (a) Absolute maximum: $\dfrac{8}{e}$; Absolute minimum: None
    (b) Absolute maximum: None; Absolute minimum: None
    (c) Absolute maximum: None; Absolute minimum: 0

52. (a) Absolute maximum: $(-18)^{4/5}$; Absolute minimum: 0
    (b) Absolute maximum: None; Absolute minimum: None
    (c) Absolute maximum: $(2)^{4/5}$; Absolute minimum: None

53. (a) Local maxima at $x = 0$, $x = d$, and $x = j$; Local minima at $x = b$ and $x = i$
    (b) Absolute maximum at $x = j$; No absolute minimum

54. (a) $x = 8$
    (b) None

55. 

56. 

57. Yes. It is possible for a function to have a local maximum, but not have any absolute extrema. For example, a function may have a vertical asymptote in which it tends to positive infinity on one side of the vertical asymptote and negative infinity on the other side, so the function has no absolute extrema. However, the function can still have local extrema in other areas of the graph.

58. Yes. It is possible for a function to have a local extremum at an x-value even if it is discontinuous at that x-value. A local extremum must occur in a neighborhood of the function (i.e., it must have function on both sides), but the function need not be continuous at the local extremum.

59. The Extreme Value Theorem guarantees that a function that is continuous on a closed interval, $[a,b]$, will have both an absolute minimum and absolute maximum on the interval.

60. Absolute extrema can occur simultaneously with local extrema inside the graph of a function or at an endpoint. In other words, absolute extrema can occur at critical values or at an endpoint.

61. The three steps of the Closed Interval Method include finding the domain of the function and checking that it is continuous on the closed interval, finding the critical values of the function, and evaluating the function at the critical values in the interval and the endpoints of the interval.

62. The Closed Interval Method is based on the Extreme Value Theorem, which only guarantees that a function that is continuous on a closed interval will have absolute extrema on the interval.

63. No. A function cannot have more than one absolute maximum (y-value). However, an absolute maximum may occur at more than one x-value.

64. No. We cannot use the Closed Interval Method to find the absolute minimum of $f(x) = \dfrac{x^2}{x+5}$ on the interval $[-7,-1]$ because, even though the interval is closed, the function is not continuous on the interval. The function is continuous on its domain of $(\infty,-5)\cup(-5,\infty)$.

65. If we cannot use the Closed Interval Method, we can find the critical values of the function and use technology to graph the function to help us locate any absolute extrema. Finding the critical values will allow us to determine the absolute extrema *exactly*.

66. Local extrema can only occur inside a function (not at an endpoint). Absolute extrema can occur anywhere (inside the function or at an endpoint). There can be multiple local maximum and minimum values (y-values) of a function, but there can only be at most one absolute maximum value and one absolute minimum value (although each could occur at multiple x-values).

Section 3.5

1. The numbers are 20 and 20.

2. The numbers are -8 and 8.

3. The numbers are $\sqrt{30}$ and $\sqrt{30}$.

4. 520 pairs of sneakers

5. 12 pizzas

6. (a) 4,375 lawnmowers

   (b) \$765,625

7. (a) 750 blenders

   (b) \$158,250

8. 100 ft by 100 ft ($x = 100$; $y = 100$)

9. 40 ft by 40 ft ($x = 40$; $y = 40$)

10. (a) 21 ft by 35 ft ($x = 21$; $y = 35$)

    (b) $3,675$ ft^2

11. 30 ft by 20 ft ($x = 30$; $y = 20$)

12. 10.954 m by 18.257 m ($x = 10.954$; $y = 18.257$)

13. (a) $x = 105$ ft; $y = 35$ ft

    (b) 1837.5 ft^2

14. $x = 25$; $y = 10$

15. $x = 5$; $y = 5$

16. $x = -1/2$; $y = 31/2$

17. 125 bikes

18. (a) 155 curling irons
    (b) \$402.50

19. (a) \$109
    (b) \$138,750

20. \$15 per ticket

21. (a) $\overline{C}(x) = 64x^{-1} + 1.5 + 0.01x$ dollars per oven mitt
    (b) 80 oven mitts

22. 100 ft by 100 ft

23. 150 ft by 100 ft (fence divider 100 ft)

24. 80,000 ft^2

25. 20-ft reinforced fence

26. (a) north/south: 10 ft; east/west: 20 ft
    (b) \$120

27. 7.560 in by 7.560 in by 3.780 in ($x = 7.560$; $y = 3.780$)

28. 1.474 ft by 1.474 ft by 1.842 ft ($x = 1.474$; $h = 1.842$)

29. (a) 25 cm by 25 cm by 12.5 cm ($x = 25$; $y = 12.5$)
    (b) 7,812.5 cm^3

30. \$384 per container

31. 3 ft by 6 ft by 4 ft ($x = 3$; $h = 4$)

32. (a) 6.076 in by 11.076 in by 1.962 in
    (b) 132.038 in^3

33. 36 in by 24 in by 12 in ($x = 12$; $y = 36$)

34. (a) 5.848 in by 5.848 in by 2.924 in
    (b) 102.599 in^3

35. \$3.65

36. The numbers are 18 and 18.

37. 1600 ft^2

38. 1.521 ft^3

39. 2 units by 2 units
40. (a) \$93 per microphone
    (b) \$43,245
41. \$360.05
42. (a) 2 in
    (b) 144 in^3
43. $x = 1$; $y = 9$
44. (a) 6000 flashlights
    (b) \$26,000
45. 35.713 cm by 17.856 cm
46. 351,562.5 ft^2
47. \$4.12 per danglie
48. \$637.60
49. 25 ft
50. 1
51. \$2,860
52. $x = 5/3$ in
53. 17 passengers
54. \$52,490
55. (a) $t = 26$
    (b) $t = 35$
    (c) $t = 41$
56. north/south: $\frac{T}{4A}$ ft; east/west: $\frac{T}{4B}$
57. To apply the five steps of the optimization process, we translate the problem into mathematical form (create variables, an objective function, and a constraint equation, if applicable), state the objective function in terms of one variable, find the interval on which the objective function must be optimized, use calculus to find the optimal solution (absolute maximum or minimum), and reread and answer the question.
58. For geometric optimization problems, you should draw a picture and label it with variables in order to help you create the objective function and constraint equation, if applicable.

59. If an objective function has two variables, a constraint equation (with the same two variables) is needed in order to write the objective function in terms of one variable. To do this, we solve the constraint equation for one of the variables and then substitute the result into the objective function. Then, the objective function will be in terms of one variable.

60. If the interval is closed and the objective function is continuous on the interval, the Closed Interval Method should be used to find the absolute maximum or minimum. If the interval is open, the objective function has only one critical value in the interval, and the objective function is continuous on the interval, either the First Derivative Test (sign chart) or the Second Derivative Test can be used to find the optimal solution, if it exists (assuming the conditions for these tests are met).

61. If a continuous function has only one critical value and a local extremum occurs at that critical value, the local extremum will also be an absolute extremum because without another critical value, the function cannot "turn around" and get larger (or smaller) than the local extremum. Thus, if a continuous function has only one critical value, $x = 4$, and a local maximum at $x = 4$, the function also has an absolute maximum at $x = 4$ because the function cannot turn around and attain a y-value greater than that of the local maximum.

Section 4.1

1. 19^x
2. xe^x
3. $\dfrac{\ln(x)}{x^2}$
4. $f(x) = (1/8)x^8 + C$
5. $g(x) = e^x + (1/2)x^2 - 2x + C$
6. $\ln|x| - x^{-1} + C$
7. $(3/5)x^{5/3} - (3/5)x^5 + e^x + C$
8. $(1/5)x^5 - 3\ln|x| + (6/13)x^{13/6} + 13x + C$
9. $C = -2$
10. $C = 22$
11. $C = -113$
12. $h(x) = 2.5x^{1.2} + 3.75x^{0.8} - 0.25$
13. $f(t) = 2\ln|t| + (1/3)t^{-3} + (1/3)$
14. $g(x) = (1/4)x^4 + (1/3)x^3 + (1/2)x^2 + x + 1$
15. $C(x) = 90x - 0.05x^2 + 450$ dollars
16. $\$16,500$
17. $P(x) = -0.3x^2 + 475x - 10,500$ dollars

18. $f(t) = 0.03t^3 + 0.09t^2 + 20$ meters

19. $-84x^{-3/14} - (1/3)x^{-3} - (1/\ln(10))10^x + C$

20. $-19e^x - (21/8)x^{8/3} + 22x + C$

21. $(2/3)x^{12} - 12\ln|x| - (5/2)x^2 + C$

22. $f(x) = (1/4)x^4 - (1/3)x^3 + C$

23. $j(x) = x^2 + x - 9\ln|x| + C \ (x \neq 0)$

24. $g(t) = (1/3)t^3 + 2t^2 - 96t + C$

25. $h(x) = 8x^2 - (20/13)x^{13} + C \ (x \neq 0)$

26. $k(t) = (3/2)t^2 + 39t + C \ (t \neq 16)$

27. $C = 20/e$

28. $C = 800$

29. $C = 1$

30. $f(x) = (2/5)x^5 - (5/2)x^4 - (2/3)x^3 + 5x^2 - (73/30)$

31. $g(u) = e^u - (1/4)u^4 + 4$

32. $R(t) = -0.07t^{-1} + 5.93\ln|t| - 0.256t^5 + 0.326$

33. $R(x) = 0.04x^2 + 350x$ dollars

34. $C(x) = 6x^{1/2} + 5$ dollars

35. \$25,907.90

36. $d(t) = 3t^3 - 9t$ meters

37. The derivative of $\dfrac{x\ln(x) - x}{\ln(7)} + 2x^5 + C$ is $\log_7(x) + 10x^4$.

38. $f(u) = \dfrac{(9/4)^u}{\ln(9/4)} - 48u^{1/4} + C$

39. $H(x) = (3/4)x^{-4} + 33\ln|x| - 2x^3 + (66/7)x^7 + C$

40. $g(t) = 7\ln|t| + (1/9)t^6 + (3/7)t^{-7/3} + \left(2^5\right)t + C$

41. $F(x) = (5/(\ln 7))7^x - (1/4)x^4 + 26\ln|x| + C$

42. $9e^t + 7t + C$

43. $-(13/7)x^{21} + (147/4)x^{4/3} - 19x + C$

44. $-(4/7)u^{-14} + (18/13)u^{13} + C$

45. $8x - (43/\ln(9))\,9^x + (2/5)\,x^4 + C$

46. $C = 0$

47. $C = (1/3) - e$

48. $f(x) = (5/14)\,x^{2.8} - (5/4)\,x^{-0.8} + (25/28)$

49. $g(t) = 54t - 4t^{3/2} + (18/7)\,t^{7/2} - (1/4)\,t^4 - 18$

50. $R(x) = (20/\ln(5))\,5^x + 3x$

51. $f(x) = x^3 + 6x^2 + 12x + 1$

52. (a) $C(x) = 5x + 2000$ dollars
    (b) $R(x) = 38x - 0.1x^2$ dollars
    (c) $P(x) = -0.1x^2 + 33x - 2000$ dollars

53. 2.515 seconds

54. $288.97

55. $S(t) = 0.5t^{2/3} + 0.2t^{1/2} + 6t - 40 - 0.2\sqrt{8}$ million dollars

56. 8 feet to the left of its initial position

57. Not necessarily. F and G could each be antiderivatives of the same function with different constants of integration.

58. We can check that our antiderivative is correct by finding its derivative.

59. (a) integral sign
    (b) integrand
    (c) antiderivative
    (d) constant of integration

60. No. The antiderivative of the second term in the integrand, $-3x^2$, is $-x^3$ (not $-6x$, which is its derivative).

61. We can only find a specific antiderivative if we are given a point through which the graph of the most general antiderivative passes. We can use the point to solve for the constant of integration, C, to obtain the specific antiderivative.

62. The right-hand side of the inequality is the result of algebraically manipulating the left-hand side so the antiderivative can be found. The antiderivative of the integrand has not been found (yet).

63. No. The antiderivative of a product does not equal the product of the antiderivatives. Also, because there is no Product (or Quotient) Rule for antiderivatives, we must algebraically manipulate the function first (using FOIL in this case), before we can find the antiderivative using the Introductory Antiderivative Rules.

Section 4.2

1. $u = x^2 + 4$
2. $u = \ln(x)$
3. $u = 8x^3 - 7$
4. $u = x^9 - 13$; $\int u^{0.7}\, du$
5. $u = 4x$; $\int 2e^u\, du$ or $u = e^{4x}$; $\int 2\, du$
6. $u = x^3 + 4$; $\int \sqrt{u}\, du$
7. $f(x) = (100/3)(5x - 13)^{0.6} + C$
8. $g(x) = 3(\ln(x))^{10} + C$
9. $h(t) = 8e^{t^4} + C$
10. $j(x) = 12/(5 - x) + C$
11. $m(x) = 2\ln\left|9x^4 - 15\right| + C$
12. $k(t) = (22/3)\left(t^3 - 9\right)^{3/2} + C$
13. $C = 12$
14. $C = 2$
15. $C = -3e^7$
16. $f(x) = -6(1 - 7x)^{5/3} + 392$
17. $g(t) = -4e^{-t^2} + 14$
18. $h(x) = \ln\left|x^2 + 16\right|$
19. \$53,293.15
20. $P(x) = 10\left(x^2 - 1000\right)^{1/3}$ dollars
21. $R(x) = 500\ln|x + 2| - 500\ln(2)$ dollars
22. \$3,990.64
23. $d(t) = 45\ln\left|t^2 + 4\right| - 45\ln(4)$ feet
24. $u = x^2 + x + 18$; $\int 4u^{9/4}\, du$

25. $u = x^3 + 12x - 2;\ \int (1/3)\, u^{-5}\, du$

26. $u = \ln(x);\ \int 4u^{-2}\, du$

27. $f(x) = -4e^{1/x} + C$

28. $g(x) = -3\left(112 + 8x - x^4\right)^{-1} + C$

29. $h(t) = (1/5)\left(15t^2 - 240\right)^{5/3} + C$

30. $f(x) = (7/5)(\ln(x))^5 + C$

31. $m(x) = (1/2)\ln\left|8x^3 - 4x + 5\right| + C$

32. $R(t) = (2/3)\left(2t^3 - 9t^2 + 24t + 22\right)^{3/2} + C$

33. $C = 19$

34. $C = 0$

35. $C = 5/6$

36. $h(x) = 6\left(5 - x^2\right)^{-2/3} - 16$

37. $f(t) = (1/8)(t+1)^8 + 7/8$

38. $g(x) = (-1/2)\, e^{-x^2-4x-3} + 2$

39. \$350.07

40. 12 items

41. \$449.95

42. $d(t) = -\left(t^2 + 1\right)^{-3} + 3$ meters

43. $u = \ln(x)$

44. $u = 2x^2 - 8x + 14$

45. $u = e^{5x} + 9;\ \int u^{-1}\, du$

46. $u = 3x^4 + 4x^2 - 4x - 6;\ \int (5/4)\, e^u\, du$

47. $u = 4 - x;\ \int \left(-12u^{-2/5} + 3u^{3/5}\right) du$

48. $h(x) = (16/3)\, x^3 + (1/4)\, e^{8x} + C$

49. $P(t) = 2 - 3t - 2\ln|2 - 3t| + C$

50. $g(x) = (1/2)\ln\left|e^{2x} - e^{-2x}\right| + C$

51. $f(x) = 2\ln\left|\ln\left(\sqrt{x}\right)\right| + C$

52. $j(t) = \left(\frac{1}{2\ln(7)}\right)7^{2t} + C$

53. $C = -27$

54. $C = -\frac{(\ln(5))^3}{24}$

55. $f(x) = (-1/20)(2e^{-x} - 3)^{10} + (1/10)$

56. $g(t) = -3\left(t^{-1} + 20\right)^{4/3} + 61 + 3(19)^{4/3}$

57. $h(x) = (2/5)(x-7)^{5/2} + (14/3)(x-7)^{3/2} - (1/15)$

58. $j(t) = (1/9)\ln\left|e^{9t} + 6\right|$

59. \$1202.58

60. $R(x) = 300x + 26,000e^{-0.01x} - 26,000$ dollars

61. \$247,651.62

62. $C(x) = 100x^2 + \sqrt{15x + 169} + 1987$ dollars

63. 58.961 feet

64. If we are able to rewrite the integral completely in terms of u and du and find the antiderivative using the Introductory Antiderivative Rules, then we have selected the correct substitution (i.e., u).

65. If the substitution we selected does not work, we have to go back and select a different substitution (i.e., u).

66. No. The new integrand in terms of u and du is missing a factor of $-1/3$. The new integral in terms of u and du is $\int -\frac{1}{3}e^u \, du$.

67. Yes. $u = 4x^2 - x + 5$ is an appropriate substitution because its derivative, $8x - 1$, is a factor in the integrand and the integral can be rewritten entirely in terms of u and du.

68. No. $u = 12x^2 - 22$ is not an appropriate substitution because its derivative, $24x$, is not a factor in the integrand. In addition, there would still be another factor that would remain in the integrand with x's, $\left(4x^3 - 22x + 18\right)^{2/7}$, so the integral could not be rewritten entirely in terms of u and du.

69. If there is still an x in the integrand after performing the substitution, we can try to use the original substitution "equation" to solve for x and substitute the resulting expression in terms of u into the integrand.

Section 4.3

1. 36
2. 10
3. 42
4. $\sum_{k=0}^{6} \frac{k^3}{4}$
5. $\sum_{k=1}^{4} \left(5k^2 + 2k\right)$
6. $\sum_{k=0}^{6} \left(k^4 + 4\right)$
7. (a) 29
   (b) 26
   (c) 6
   (d) 18
   (e) 20
8. (a) 22
   (b) 20
   (c) 72
   (d) 48
   (e) 63
9. 68
10. 5505
11. 88
12. (a) miles
    (b) kerfuffles
    (c) charges
    (d) kernels
13. (a) \$1200
    (b) \$480
    (c) \$660
14. (a) \$950
    (b) \$440

15. (a) \$1321
    (b) \$1403.50

16. (a) \$686.85
    (b) −\$295.95

17. (a) 26 feet
    (b) We cannot determine whether the sum is an overestimate or an underestimate. For the first interval, we are underestimating, but for the next two intervals, we are overestimating.

18. 92.6 feet

19. 7.65 meters

20. (a) 16
    (b) 8
    (c) 48

21. 20

22. −2.5

23. (a) \$54
    (b) \$154
    (c) \$210

24. 6 feet

25. (a) 1708
    (b) 416.5

26. (a) −1.5
    (b) −42
    (c) 8
    (d) 3
    (e) −11
    (f) 65

27. (a) 56
    (b) 21

28. (a) 4
    (b) −20

29. (a) −80
    (b) −240

30. (a) 168
    (b) 40

31. (a) 50
    (b) 5

32. (a) 6
    (b) −6

33. (a) 4
    (b) 7
    (c) −4
    (d) 18
    (e) 3

34. (a) 0.005
    (b) 14.005
    (c) 10.01
    (d) −28
    (e) 17.505

35. 15

36. 6

37. 18.764

38. (a) $430
    (b) Because R' is decreasing, $430 is an overestimate.
    (c) $100
    (d) Because R' is decreasing, $100 is an underestimate.
    (e) $273

39. (a) $282.40
    (b) $302.40

40. (a) −$0.33
    (b) −$0.53
    (c) $5428.67

41. (a) $1539.33
    (b) $548.76

42. 45 meters

43. (a) 10
    (b) $2.5+\frac{9\pi}{2}$
44. -9
45. 66
46. (a) \$25
    (b) \$78.50
    (c) \$182.50
    (d) \$47
47. 28 feet
48. (a) 1056
    (b) 856.3
49. (a) 16
    (b) -8
    (c) 7
50. (a) 86,436
    (b) -3564
51. (a) 54
    (b) 48
52. (a) $-12e^{-2}$
    (b) -8
53. 2.45
54. $\sum_{i=0}^{5}\frac{1}{2}f(x_i)$
55. -78
56. 0.119
57. 0
58. (a) 1.9
    (b) 1.15
    (c) 4

59. (a) -6
    (b) -4
    (c) -16
    (d) 5

60. (a) cats
    (b) hit points
    (c) dollars

61. (a) 131 feet
    (b) 81 feet

62. (a) \$17.61
    (b) \$13.97
    (c) \$25.41

63. 60.58 feet

64. (a) \$21,784
    (b) It is an overestimate because C' is decreasing.
    (c) \$10,228
    (d) It is an underestimate because C' is decreasing.
    (e) \$14,868

65. 310.5 million dollars

66. (a) \$120
    (b) \$18
    (c) \$118

67. (a) $-\$8100$
    (b) \$0
    (c) $-\$3825$

68. 2110 feet

69. \$427.15

70. (a) \$1200
    (b) \$640
    (c) $-\$11,200$

71. (a) $-\$5.53$
    (b) \$223.89

72. 28.783 feet

73. (a) $4-\pi$
    (b) 20

74. (a) 0
    (b) 14.25
    (c) 42.4
    (d) -24.8

75. 142

76. (a) -16
    (b) -14
    (c) 12
    (d) 40

77. (a) -15
    (b) $\frac{58-\pi}{3}$
    (c) $2\pi+\frac{23}{2}$
    (d) $\pi-3$
    (e) $5+\frac{\pi}{4}$
    (f) -10

78. $c(b-a)$

79. (a) -5
    (b) -25

80. (a) $6/e$
    (b) 0

81. (a) $\frac{3}{4\sqrt{e}}$
    (b) $-\frac{3}{4\sqrt{e}}$

82. $\sum_{i=1}^{n} f(x_i)\Delta x$, or $\sum_{i=1}^{n} f(x_i)\left(\frac{b-a}{n}\right)$

83. $\sum_{i=0}^{n-1} f(x_i)\Delta x$, or $\sum_{i=0}^{n-1} f(x_i)\left(\frac{b-a}{n}\right)$

84. $\sum_{i=1}^{10} f\left(x_i^*\right)\Delta x$, or $\sum_{i=1}^{10} f\left(x_i^*\right)\left(\frac{b-a}{10}\right)$

85. $x_1 = 3$

86. $\sum_{i=4}^{6} f(x_i)\Delta x$, or $\sum_{i=4}^{6} f(x_i)\left(\frac{b-a}{n}\right)$

87. $f(x_1)\Delta x$ represents the area of the second rectangle. The height of the rectangle is $f(x_1)$ and its width is given by Δx.

88. The summation represents the total area of the second, third, and fourth rectangles.

89. Increasing the number of rectangles will give a better approximation of the area under a curve because the widths of the rectangles will become increasingly smaller (and so the heights of the rectangles will be determined by all of the function values).

90. No. A left-hand Riemann sum gives an underestimate of the area under the curve of a function only if the function is strictly increasing.

91. No. The Riemann sum gives an approximation of the net area between the graph of f and the x-axis. The definite integral gives the exact net area between the graph of f and the x-axis.

92. Yes. When taking the limit of a Riemann sum, we obtain an exact answer (represented by the definite integral). We will get the same (exact) answer no matter if we are taking the limit of a right-hand, left-hand, or midpoint Riemann sum because the area under the curve will be the area regardless of the type of sum.

93. A definite integral has limits of integration and is of the form $\int_a^b f(x)\,dx$. It represents the net area between the graph of f and the x-axis on the interval $[a,b]$ (or the change in f), so it will have a numeric value.

    An indefinite integral does not have limits of integration and is of the form $\int f(x)\,dx$. It represents the family of antiderivatives of f, so it will not have a numeric value (it will be a function with the variable x).

94. A definite integral represents the actual area between a function and the x-axis on an interval if the function is nonnegative on the interval.

95. When the number of items sold increases from 100 to 150, profit increases by \$7000.

96. No. The change in cost is given by the definite integral of the rate of change function, C'. In other words, $\int_{45}^{65} C'(x)\,dx$ will give the change in cost when production increases from 45 to 65 items.

97. The definite integral gives the distance traveled, in feet, by the object during the first 60 seconds.

Section 4.4

1. 340

2. -679

3. $7e-8$

4. $\frac{4113}{16}$

5. -6.5

6. $4e^2 - 3$

7. $F'(x) = x^4 - 7$

8. $F'(x) = \dfrac{1}{x}(\ln(x))$

9. $F'(x) = 2^{-x^2}$

10. $-\$1,261,653.33$

11. $8250

12. $-\$9744$

13. 6 meters

14. 108

15. $13 + 2e^9$

16. 0

17. 8040

18. $-3e^{190} + 3e^{22}$

19. $(10/9)\ln(3) - (80/81)$

20. $-6\sqrt{127} + 42$

21. $3\ln(735) - 3\ln(1925)$

22. $A^4 - 3A^2 + A - 2$

23. $5B^{8/5} - B^5 + 0.09B^3$

24. $4e^{-1} - 4e^K - \dfrac{2}{7}\ln|K|$

25. $F(1) = 3$

26. $G(2) = 10$

27. $f(-4) = 23.4$

28. $g(5) = 80$

29. $F'(x) = \dfrac{e^x}{x}$

30. $F'(x) = -x\sqrt{9 - x}$

31. $F'(x) = -3\ln(6-x)$

32. $F'(x) = \dfrac{2x}{\sqrt{16-x^4}}$

33. $F'(x) = -3e^{-2x^{2/3}}x^{-2/3}$

34. \$515.63

35. −\$2800

36. \$9.09

37. (a) \$19,139.70

(b) \$52,651.35

38. 176/3 meters

39. $2\left(12B+3B^2\right)^{5/4} - 2(15)^{5/4}$

40. $\frac{3}{7}e^{27} - \frac{3}{7}e^{8} - \frac{45}{8}$

41. $-192/5$

42. $\ln\left|\ln\left(\sqrt{5}\right)\right|$

43. $\frac{231}{10} - \frac{45}{2}(4K+15)^{2/3} + \frac{3}{5}(4K+15)^{5/3}$

44. $\frac{5}{2}e^{-3} - \frac{5}{2}$

45. $\frac{5}{3}\ln(1.8) - \frac{5}{3}\ln(5)$

46. $-\frac{1}{7}A^{-6/5} - 2A - 0.056\ln|A| + \frac{1}{7}3^{-6/5} + 6 + 0.056\ln(3)$

47. $F'(x) = 2(2x+7)\left((2x+7)^2 - 33\right)^{14}$

48. $F'(x) = -2x + 8 - 4x^{-3}$

49. $F'(x) = -\sqrt{9+x^2} + 2x\sqrt{9+x^4}$

50. $F'(x) = x\ln(x) - x$

51. $F'(x) = -\left(4\left(\sqrt{x}\right)^3 + 10\sqrt{x}\right)e^{x^2+5x+11}\left(\frac{1}{2}x^{-1/2}\right)$

52. $F'(x) = \left(42x + \sqrt[3]{7x}\right) + 3\left(18x + \sqrt[3]{-3x}\right)$

53. $f(2) = 4$

54. $H\left(\frac{15}{4}\right) = 25.375$

55. $g'(7) = -44$

56. (a) $f(-3) = -5.75$
    (b) $f(-10) = -1.25$
    (c) $f(2) = 7.25$

57. (a) $\$12,043.40$
    (b) $\$113,025.56$

58. $-\$35.52$

59. 58.96 feet

60. (a) $12.54
    (b) $45
    (c) -131.38

61. -6.92 rolls of toilet paper. The Savage family used 6.92 rolls less than their usual two month usage.

62. $4918.25

63. (a) 524.16 million dollars
    (b) 240 million dollars
    (c) 14.28 million dollars

64. Part 1 of the Fundamental Theorem of Calculus states how to differentiate a function defined as a definite integral, whereas Part 2 states how to evaluate a definite integral.

65. No. We have to find the antiderivative of the function in the integrand before substituting the limits of integration.

66. No. When using the method of substitution to evaluate a definite integral, the limits of integration must also be in terms of the new variable, u.

67. We can find the change in revenue by using Part 2 of the Fundamental Theorem of Calculus to evaluate the definite integral of the marginal revenue function, R', on the interval $[200, 250]$. In other words, the change in revenue on this interval is given by $\displaystyle\int_{200}^{250} R'(x)\,dx$.

68. When the number of items produced and sold increases from 400 to 500, profit decreases by $12,000.

69. No. The change in profit is given by the definite integral of the marginal profit function, P', on the interval $[50, 75]$. In other words, the change in profit is given by $\displaystyle\int_{50}^{75} P'(x)\,dx$.

    Alternatively, if we are given $P(x)$, we can calculate the change in profit by finding $P(75) - P(50)$, which is exactly what Part 2 of the Fundamental Theorem of Calculus states.

70. One option is to find the derivative of $C(x)$, which is $C'(x)$, and then calculate the definite integral. In other words, we can find $\int_{1000}^{2000} C'(x)\,dx$. Or, we can simply find $C(2000)-C(1000)$, which is exactly what Part 2 of the Fundamental Theorem of Calculus states.

71. Because the velocity function is nonnegative, the distance traveled is given by the change in the object's position. In other words, to find the distance, we calculate the definite integral of the velocity function, v, on the interval $[0,10]$: $\int_0^{10} v(t)\,dt$.

72. (a) Part 1

    (b) No. The solution is incorrect because the lower limit of integration is x and the upper limit of integration is the constant. Thus, we cannot just substitute x for t in the integrand. If the limits of integration were switched, then we could substitute x for t.

73. If the lower limit of integration is a constant and the upper limit of integration is just x (or t, etc.), then we do not have to incorporate the Chain Rule when differentiating the definite integral. Otherwise, we must incorporate the Chain Rule.

Section 4.5

1. 57
2. $\frac{16385}{14}$
3. $-1-e$
4. $-1,243.4$
5. $81/20$
6. $\frac{2e-1}{e-1}$
7. $-45/28$
8. $e^{-2}-1$
9. 5,591
10. (a) $821/30$
    (b) $179/20$
    (c) 20
11. (a) $427/20$
    (b) -42
    (c) $-833/50$

12. (a) $736/35$
    (b) $337/85$
    (c) $-124/15$
13. \$137 per pair
14. \$251.25
15. \$84.50 per item
16. \$44.50 per coat
17. 9.5 meters per second
18. 509
19. $\dfrac{5e^4-1}{4}$
20. $-1200/9$
21. 1655
22. $\dfrac{-e^7+e^{1297}}{18}$
23. $\dfrac{7\ln(3)+6}{2}$
24. $(7/3)\left(\sqrt{61}-\sqrt{22}\right)$
25. $\dfrac{\ln(465)-\ln(2485)}{4}$
26. (a) $5/2$
    (b) $32/5$
    (c) $17/4$
27. (a) $25/11$
    (b) $(3\pi)/4$
    (c) $(3\pi-10)/8$
28. (a) $(-2-\pi)/3$
    (b) $-1/12$
    (c) $-1/2$
29. $2496/5$
30. $257/15$
31. $4e^{-1}+4e^{-3}-(2/3)$

32. \$5971.67

33. \$7015 per item

34. \$9000

35. 7.085 million dollars

36. (a) \$7140.99

    (b) \$284.16 per year

37. $9/64$

38. $\dfrac{10+21e^{16}-21e^{9}}{49}$

39. $-8/63$

40. $\dfrac{4\ln(3)}{e^3-e}$

41. $1498/15$

42. $\dfrac{1}{3}\left(\dfrac{1}{4}\ln(4)-\dfrac{1}{14}(4)^{-7/2}-\dfrac{1}{7}\right)$

43. $\dfrac{e^{8.8}-e^{55.6}}{2}$

44. $\dfrac{\ln(34)-\ln(3822)}{5}$

45. $\dfrac{A^4-A^8}{2A}$

46. $\dfrac{6B^{3/2}+5e^{B}-16B-5e^{4}+16}{B-4}$

47. (a) $5/2$

    (b) $19/8$

    (c) $\dfrac{1-2\pi}{6}$

48. (a) $-7/2$

    (b) $\dfrac{11+4\pi}{12}$

    (c) $\dfrac{2\pi-19}{16}$

49. $\dfrac{3(\ln 3)-3(\ln 5)-20}{6}$

50. \$3494.29

51. \$1094.79

52. \$2.01 per mug

53. (a) \$10,356.50
    (b) \$49.61 per pair of pajamas

54. \$5.99 per package

55. (a) \$3702.86
    (b) \$225.06 per year

56. 12 feet per second

57. (a) 9.3 million dollars per month
    (b) 1470.584 million dollars

58. (a) $15/(c-a)$
    (b) $-24/(d-b)$
    (c) $-27/e$
    (d) $43/(h-d)$

59. (a) $1/2$
    (b) $-10/3$

60. $B = 1/3$; $B = 1$

61. $c = 8$; $c = 10$

62. −\$3.10 per item

63. 2.955 feet per second

64. No. After evaluating the definite integral, he needs to divide by the length of the interval, which is 7 in this case.

65. (a) The average rate of change of cost on the interval $[50,100]$ is −\$7.43 per item. In other words, when the number of items produced is between 50 and 100, cost is decreasing at an average rate of \$7.43 per item.
    (b) The average cost on the interval $[50,100]$ is \$6320.00. In other words, when the number of items produced is between 50 and 100, the average cost is \$6320.00.

66. To find the average rate of change of f on the interval $[a,b]$, we can either use the formula for the slope of the secant line

$$\frac{f(b)-f(a)}{b-a}$$

or we can use the average value formula and calculate

$$\frac{1}{b-a}\int_a^b f'(x)\,dx$$

Note that using the average value formula would require us to find the derivative function, f', first.

Section 4.6

1. 10
2. 64/3
3. 45/2
4. 109/4
5. 1183/12
6. 368
7. $250
8. $60,000
9. $1406.25
10. $1625
11. $15,200
12. $30.38
13. $25,000
14. $5000
15. $2250
16. $450
17. (a) 2018
    (b) $2400 each week
18. (a) Tetra-tastic
    (b) 15 bags
19. (a) 2019
    (b) $4340
20. 7.5 meters
21. 36
22. 2204
23. 50/3
24. 155/6

25. 125/6

26. 1/12

27. 36

28. 243

29. \$83.33

30. \$16,000

31. \$22,500

32. \$330,625

33. \$1215

34. \$291.67

35. \$4500

36. \$5000

37. (a) \$125
    (b) \$50

38. (a) \$60,000
    (b) \$533,333.33

39. (a) \$208.33
    (b) \$31.25

40. (a) \$2250
    (b) \$562.50

41. (a) \$2,450,000
    (b) \$980,000

42. (a) \$10,666.67
    (b) \$5333.33

43. (a) Jump Up
    (b) 260 trampolines

44. (a) Company B
    (b) \$12,708.33

45. (a) March
    (b) \$2880

46. (a) Object A

(b) 27.2 feet

47. (a) $\int_{-5.5}^{-4.5} (h(x) - f(x))\ dx$

(b) $\int_{-3}^{-2} (f(x) - h(x))\ dx + \int_{-2}^{-1} (g(x) - h(x))\ dx$

(c) $\int_{0}^{2} (-f(x))\ dx$

(d) $\int_{3.5}^{4.5} (f(x) - h(x))\ dx$

48. 232/3

49. 163/12

50. 264

51. $e^2 - 3$

52. 131/6

53. $\dfrac{e^2 - 3}{2}$

54. 94

55. $2e + (2/e) - 4$

56. 343/6

57. 1/2

58. $\ln(3) - (2/3)$

59. $\dfrac{e^2 - 2e + 1}{2e}$

60. 4/3

61. 136/3

62. 243/10

63. 95/4

64. 1.717

65. 1.083

66. 3.051

67. 1.949

68. 19.666

69. 1.116

70. 84.280

71. (a) \$36,716.42
    (b) \$14,580

72. (a) \$732.33
    (b) \$1830.83

73. (a) \$1972.68
    (b) \$780.28

74. (a) \$11,250
    (b) \$5625

75. (a) \$13,140.37
    (b) \$11,654.93

76. (a) \$50,460
    (b) \$33,640

77. (a) Angie
    (b) \$302.67 per year

78. (a) 2018
    (b) \$393.15

79. (a) Video Frenzy!
    (b) 4 games

80. $-\$40,385.65$

81. 18,400 square feet

82. (a) Object B
    (b) 15.333 meters
    (c) Object A
    (d) 98.667 meters

83. $\int_{-1}^{2} f(x)\,dx - \int_{2}^{A} f(x)\,dx$

84. (a) \$38,400
    (b) \$14,080

85. (a) 11
    (b) 18
    (c) 12

86. 34/3

87. 2

88. Graphing the curves first allows us to see which function is greater than the other on the relevant interval (as well as determine if the curves intersect on the interval). The function that is greater on the interval will be first in the corresponding integrand, and the function that is less will be subtracted from it.

89. Multiple definite integrals are necessary if the curves bound more than one region (because they have multiple intersection points). In this situation, the function that is greater will switch in each bounded region. For each definite integral representing the area of a region, the function that is greater will be first in the integrand.

90. If both companies sell 100 items, Company 2 will earn $\$25,000$ more in revenue than Company 1.

91. No. The consumers' surplus represents the *savings* to consumers who are willing to purchase a commodity at a higher price than the equilibrium price, but they are able to just pay the equilibrium price.

92. The area between the demand curve, $p = D(x)$, and the line representing the equilibrium price, $p = p_0$, on the interval $[0, x_0]$, where x_0 is the equilibrium quantity, represents the consumers' surplus. The area between the supply curve, $p = S(x)$, and the line representing the equilibrium price, $p = p_0$, on the interval $[0, x_0]$, where x_0 is the equilibrium quantity, represents the producers' surplus.

