Almost Stochastic Dominance: Magnitude Constraints on Risk Aversion

May 24, 2021

Liqun Liu
Private Enterprise Research Center
Texas A&M University
College Station, TX 77843
U.S.A.
lliu@tamu.edu

Jack Meyer
Department of Economics
Michigan State University
East Lansing, MI 48824
U.S.A.
jmeyer@msu.edu

Abstract

Almost stochastic dominance (ASD) extends conventional first and second degree stochastic dominance by placing restrictions on the variability in the first and second derivatives of utility. Such restrictions increase the number of random variables for which a unanimous ranking of one over the other occurs. This paper advances an alternative approach to ASD in which the magnitude of absolute or relative risk aversion is constrained with both an upper bound and a lower bound. Using the results of Meyer (1977b), the paper provides cumulative distribution function (CDF) characterizations of these forms of ASD. Simple closed-form necessary and sufficient conditions for these ASD relations are determined for the special cases where the absolute or relative risk aversion is only bounded on one end or where the pair of random variables under comparison have single-crossing CDFs.

Keywords: stochastic dominance; almost stochastic dominance; risk aversion; relative risk aversion

JEL Classification: D81; G11

*We would like to thank Rachel Huang, Larry Tzeng and Lin Zhao for comments and suggestions that help improve this paper.

1. Introduction

More than fifty years ago Hadar and Russell (1969) and Hanoch and Levy (1969) defined first and second degree stochastic dominance (FSD, SSD). These random variable ranking tools impose minimal assumptions on the risk preferences of decision makers and still are frequently used to provide a partial order over random variables. Random variable $\tilde{x}$ stochastically dominates $\tilde{y}$ in the first degree (second degree) when $\tilde{x}$ is preferred to $\tilde{y}$ by all decision makers who prefer more to less (and are risk averse). While FSD and SSD have proven to be powerful theoretical tools, in practice, there are random variables where it seems obvious that one is preferred to another by all reasonable decision makers yet neither FSD nor SSD provides a unanimous ranking. An example of this is provided by Leshno and Levy (2002) in their paper where the almost stochastic dominance (ASD) concept is first defined. In their example they consider random variables $\tilde{x}$, which yields either \$0 or \$1,000,000 with probabilities 0.01 and 0.99 respectively, and $\tilde{y}$ which yields \$1 with certainty. Neither $\tilde{x}$ nor $\tilde{y}$ dominates the other in either the first or the second degree,[1] yet it seems clear that except for very extreme risk preferences, $\tilde{x}$ would be chosen over $\tilde{y}$. Leshno and Levy provide an example of such extreme preferences using utility function $u(x) = x$ for $x \leq 1$ and $u(x) = 1$ for $x > 1$.

Leshno and Levy's solution to this lack of stochastic dominance when $\tilde{x}$ seems obviously preferable to $\tilde{y}$ is to alter the definition of stochastic dominance so that extreme utility functions, such as the one in the example, are excluded from consideration. They refer to such utility functions as "extreme, pathological or simply unrealistic". Leshno and Levy define almost first and almost second degree stochastic dominance (AFSD, ASSD) by restricting the degree of

[1] Since $\tilde{x}$ has a positive probability mass to the left of the entire distribution of $\tilde{y}$, $\tilde{x}$ does not stochastically dominate $\tilde{y}$ in any degree. On the other hand, since $E(\tilde{x}) > E(\tilde{y})$, $\tilde{y}$ does not stochastically dominate $\tilde{x}$ in any degree either.

variability in the first and the second derivatives of utility, respectively. AFSD or ASSD can be used to rank random variables which FSD and SSD do not rank. The ensuing literature extends and modifies the work of Leshno and Levy. Tzeng et al. (2013) modify the second degree formulation and extend ASSD to the general nth degree. Others, including Guo et al. (2013), Denuit et al. (2014a), Tsetlin et al. (2015), Tsetlin and Winkler (2018) and Chang et al. (2019), offer further refinements and extensions of the original Leshno and Levy AFSD and ASSD definitions.[2]

The original definitions of AFSD and ASSD place restrictions on the magnitude of the variability of derivatives of utility and thereby alter the set of utility functions under consideration. The method used to restrict variability for a derivative of utility is as follows. As is well known, FSD determines necessary and sufficient conditions on cumulative distribution functions (CDFs) F(x) and G(x) so that all decision makers with utility functions in $U_1 = \{u(x): u'(x) \geq 0\}$ are unanimous in preferring F(x) to G(x).[3] Almost first degree stochastic dominance (AFSD (ε)) examines this same unanimous ranking question for the smaller set of utility functions $U_1^\varepsilon = \{u(x) \text{ in } U_1: u'(x) \leq \inf \{u'(x)\}(1/\varepsilon - 1) \text{ for all x in } [a, b]\}$, where $0 < \varepsilon \leq 1/2$. It is the case that the larger the value for ε, the smaller is the set U_1^ε. U_1^ε converges to U_1 as ε approaches zero and to u'(x) equaling a constant as ε approaches 1/2. Similarly, $U_2 = \{u(x): u'(x) \geq 0 \text{ and } u''(x) \leq 0\}$ is the set of utility functions associated with SSD,[4] and almost second degree

[2] Basically, the ASD analysis provides a solution to the so-called "left tail problem". That is, no matter how $\tilde{x}$ appears more desirable than $\tilde{y}$ in other aspects, as long as $\tilde{x}$ has some probability mass to the left of the entire distribution of $\tilde{y}$, $\tilde{x}$ does not stochastically dominate $\tilde{y}$ in any degree. Liu and Meyer (2021) recently propose an alternative approach to the "left tail problem", referred to as stochastic superiority, by introducing a secondary risk reduction decision.

[3] The description of the existing approach to almost stochastic dominance uses notation similar to that in Tzeng et al. (2013).

[4] That is, a pair of random variables that can be ordered by SSD implies uniform preference by all (weakly) risk averse decision makers. According to Liu and Meyer (2017), a somewhat symmetric stochastic order known as the

stochastic dominance (ASSD (ε)) is defined by restricting the variability of -u''(x). Formally, U_2^{ε} = {u(x) in U_2: -u''(x) ≤ inf{-u''(x)}(1/ε - 1) for x in [a, b]}.

We propose in this paper to place restrictions on the degree of risk aversion rather than on the variability of the first and second derivatives of utility. Specifically, we consider a group of decision makers whose Arrow-Pratt absolute (or relative) risk aversion measure is between two constants. This way of defining almost stochastic dominance requires unanimous preference by all decision makers "who are neither too risk averse nor too risk loving". Since restricting the size of absolute risk aversion is equivalent to restricting the size of the percentage rate of change of marginal utility, the approach is similar in spirit to that initiated by Leshno and Levy. The paper goes on to provide CDF-based characterizations of the newly provided definitions of ASD. Much of the analysis relies on results of Meyer (1977a, 1977b).

There are several reasons why restricting the magnitude of risk aversion makes good sense. First, placing bounds on the absolute or relative risk aversion measure, denoted A(x) and R(x) (R(x) = xA(x)), respectively, indirectly restricts the variability of u'(x) as well, since A(x) is the percentage rate of change of u'(x). Second, A(x) or R(x) is a unique representation of the risk preferences of an expected utility maximizing decision maker. As is well known, A(x) is the risk aversion measure of u(x) if and only if A(x) is also the risk aversion measure of d·u(x) + c for any d > 0. Pratt (1964) shows how u(x) can be recovered from A(x) except for such a positive linear transformation. Thus, there is a one to one relationship between decision makers and their risk aversion measures. A similar one to one relationship does not hold for any derivative of

"increasing convex order" in mathematical statistics – unanimous preference by decision makers with utility functions in U_2' = {u(x): u'(x) ≥ 0 and u''(x) ≥ 0} – can be used to frame the size versus risk tradeoff for risk averse decision makers. See also Denuit et al. (2014b) and Denuit et al. (2016).

utility.[5] Finally, and likely the most important reason for restricting the magnitude of A(x) or R(x) rather than the variability of u'(x) or u"(x), is that the information concerning risk preferences that is available from experiments or empirical estimation is primarily for the risk aversion measures of decision makers.[6]

Two recent papers also place bounds on risk aversion measures. This paper differs in important ways from each. Huang et al. (2020) place a negative lower bound on the absolute risk aversion measure to form a set of utility functions that are not too risk loving, a set that contains U_2 and is contained in U_1. Their purpose is to define fractional degree stochastic dominance that fills the gap between FSD and SSD. This work does not directly address the issues arising in the ASD literature. The bound on risk aversion needed to provide solutions to the motivating example of Leshno and Levy and similar examples in the ASD literature is an upper rather than lower bound, and, as a result, the relevant group of decision makers should be those who are neither too risk averse nor too risk loving, rather than those who are merely not too risk loving. Another recent paper by Luo and Tan (2020) does consider an upper bound on the absolute risk aversion measure, but does so in conjunction with an upper bound on the variability in the marginal utility. Luo and Tan's work does contribute to the ASD literature, but differs from that presented here. Their method restricts the product of two upper bounds, and does not necessarily constrain the size of either.

The paper is organized as follows. Section 2 provides two new definitions of ASD by imposing both an upper bound and a lower bound on either the absolute or the relative risk

[5] Note, however, that the ratio of first or second derivatives of utility, which is constrained in the conventional ASD definitions, is also invariant with a positive linear transformation of the utility function.
[6] For example, see Eckel and Grossman (2002), Holt and Laury (2002), Callen et al. (2014), Ebert and Wiesen (2014), and Grossman and Eckel (2015). In addition, see the references discussed in Meyer and Meyer (2005a, 2005b) and Liu (2012).

aversion measure. A CDF-based characterization for each of these two forms of ASD is provided, and additional properties of these ASD definitions, dealing with robustness to adding a constant to the random variables or scaling them by a positive constant, are presented. These robustness findings can help determine whether bounds on absolute risk aversion or relative risk aversion are the most appropriate for a specific application. In Section 3, more compact characterization results are obtained for the special cases where the absolute or relative risk aversion is only bounded on one end or where the pair of random variables under comparison have single-crossing CDFs. In Section 4, we show that the conventional AFSD implies the two new notions of ASD, for appropriately chosen parameter values. Finally, the concluding section summarizes the findings presented here and offers some suggestions for additional research.

2. Almost Stochastic Dominance with Bounds on Risk Aversion

In this section, two new definitions of ASD are presented. These result from imposing both an upper bound and a lower bound on either the absolute or relative risk aversion measure of the decision maker. A result from Meyer (1977b) is used to demonstrate a necessary and sufficient condition on F(x) and G(x) for F(x) to almost stochastically dominate G(x). In terms of notation and assumptions, random variables are denoted $\tilde{x}$ and $\tilde{y}$ with corresponding CDFs F(x) and G(x). The supports of these random variables are assumed to lie in bounded interval [a, b] with no probability mass at x = a so that F(a) = G(a) = 0 and F(b) = G(b) = 1. When relative risk aversion is discussed a > 0 is also assumed. First and second derivatives of utility and the absolute risk aversion measure are assumed to exist at all x in [a, b].

We begin with bounds on the absolute risk aversion measure $A(x) = -u''(x)/u'(x)$. The decision makers are assumed to have utility functions u(x) in the set $U(\lambda_1, \lambda_2) = \{u(x)\text{: } u'(x) > 0$

and $\lambda_1 \leq A(x) \leq \lambda_2$}, where λ_1 and λ_2 are two constants with $\lambda_1 \leq 0 < \lambda_2$. The signs of λ_1 and λ_2 allow $U(\lambda_1, \lambda_2)$ to be interpreted as the group of decision makers who are neither too risk averse nor too risk loving according to the absolute risk aversion measure. Note also that the risk neutral decision maker is included in $U(\lambda_1, \lambda_2)$. The decision maker who meets the lower (upper) bound restriction exactly for all x is the constant absolute risk averse utility function $u_1(x) = e^{-\lambda_1 x}$ ($u_2(x) = -e^{-\lambda_2 x}$). The decision makers in $U(\lambda_1, \lambda_2)$ are all more risk averse than $u_1(x)$ and less risk averse than $u_2(x)$.

Definition 1: For λ_1 and λ_2 such that $\lambda_1 \leq 0 < \lambda_2$, $\tilde{x}$ $ASD^A(\lambda_1, \lambda_2)$ $\tilde{y}$ if $\tilde{x}$ is preferred or indifferent to $\tilde{y}$ by all u(x) in $U(\lambda_1, \lambda_2)$.

The decision makers in $U(\lambda_1, \lambda_2)$ are all more (less) risk averse than the constant absolute risk averse decision maker with risk aversion level λ_1 (λ_2). Being more (less) risk averse implies many things, including holding less (more) of the risky asset in a portfolio and having a larger (smaller) risk premium and insurance demand. Because the risk neutral decision maker is included in $U(\lambda_1, \lambda_2)$, it must be the case that $E(\tilde{x}) \geq E(\tilde{y})$ for an $ASD^A(\lambda_1, \lambda_2)$ relation to hold. It is also obvious that the $ASD^A(\lambda_1, \lambda_2)$ relation becomes stronger as λ_1 decreases or λ_2 increases. In addition, the $ASD^A(\lambda_1, \lambda_2)$ relation converges to FSD when λ_1 goes to the negative infinity and λ_2 goes to the positive infinity, and to SSD when $\lambda_1 = 0$ and λ_2 goes to the positive infinity.

Relative risk aversion levels are estimated more often and relative risk aversion estimates from various studies are more comparable.[7] For bounds on the relative risk aversion measure $R(x) = -xu''(x)/u'(x)$, the decision makers are assumed to have utility functions u(x) in the set $U(\rho_1, \rho_2) = \{u(x): u'(x) > 0 \text{ and } \rho_1 \leq R(x) \leq \rho_2\} = \{u(x): u'(x) > 0 \text{ and } \rho_1/x \leq A(x) \leq \rho_2/x\}$, where ρ_1 and ρ_2 are two constants with $\rho_1 \leq 0 < \rho_2$. The signs of ρ_1 and ρ_2 allow $U(\rho_1, \rho_2)$ to be interpreted as the group of decision makers who are neither too risk averse nor too risk loving, this time by the relative risk aversion measure rather than the absolute risk aversion measure. Obviously, the risk neutral decision maker is included in $U(\rho_1, \rho_2)$. The decision maker who meets the lower (upper) bound restriction exactly for all x is the constant relative risk averse utility function $u_1(x) = \frac{x^{1-\rho_1}}{1-\rho_1}$ ($u_2(x) = \frac{x^{1-\rho_2}}{1-\rho_2}$ when $\rho_2 \neq 1$ and $u_2(x) = \ln x$ when $\rho_2 = 1$). The decision makers in $U(\rho_1, \rho_2)$ are all more risk averse than $u_1(x)$ and less risk averse than $u_2(x)$.

Definition 2: For ρ_1 and ρ_2 such that $\rho_1 \leq 0 < \rho_2$, $\tilde{x}\ ASD^R(\rho_1, \rho_2)\ \tilde{y}$ if $\tilde{x}$ is preferred or indifferent to $\tilde{y}$ by all u(x) in $U(\rho_1, \rho_2)$.

As with $ASD^A(\lambda_1, \lambda_2)$, $E(\tilde{x}) \geq E(\tilde{y})$ is a necessary condition for $\tilde{x}\ ASD^R(\rho_1, \rho_2)\ \tilde{y}$, and the $ASD^R(\rho_1, \rho_2)$ relation becomes stronger as ρ_1 decreases or ρ_2 increases. In addition, the $ASD^R(\rho_1, \rho_2)$ relation converges to FSD when ρ_1 goes to the negative infinity and ρ_2 goes to the positive infinity, and to SSD when $\rho_1 = 0$ and ρ_2 goes to the positive infinity.

[7]According to the relevant references cited in Liu (2012), the relative risk aversion coefficient ρ falls within the range of (0, 30). Meyer and Meyer (2005a, 2005b) distinguish between the relative risk aversion measure for the utility function defined over consumption and that for the value function defined over wealth, and find that the former could be 1.25 to 10 times the latter, partially accounting for the widely divergent estimates of ρ.

To determine the condition on F(x) and G(x) that is equivalent to F(x) ASD^A (λ_1, λ_2) G(x) or F(x) ASD^R (ρ_1, ρ_2) G(x), we apply a result in Meyer (1977b), which is provided next.

<u>Theorem 1 (Meyer 1977b):</u> A solution, denoted $u_0(x)$, to the problem of choosing $u(x)$ to minimize $\int_a^b u(x)d\left[F(x)-G(x)\right]=\int_a^b\left[G(x)-F(x)\right]u'(x)dx$ subject to u'(x) > 0 and $r_1(x)\le\frac{-u''(x)}{u'(x)}\le r_2(x)$ for all $x\in[a,b]$ and $u'(a)=1$, is given by

$$\frac{-u_0''(x)}{u_0'(x)}=\begin{cases} r_1(x), & \text{if } \int_x^b\left[G(y)-F(y)\right]u_0'(y)dy<0 \\ r_2(x), & \text{if } \int_x^b\left[G(y)-F(y)\right]u_0'(y)dy\ge 0 \end{cases}.$$

Theorem 1 says that at any point *x*, $u_0(x)$ is chosen in such a way that its absolute risk aversion measure is either the lower bound or the upper bound, depending only on the sign of the objective function from point *x* on to *b*. Therefore, $u_0(x)$ is obtained in a backward fashion. From Theorem 1, the corollary next follows immediately.

<u>Corollary 1 (Meyer 1977b):</u> $\tilde{x}$ is preferred or indifferent to $\tilde{y}$ for all u(x) such that u'(x) > 0 and $r_1(x)\le\frac{-u''(x)}{u'(x)}\le r_2(x)$ for all $x\in[a,b]$ if and only if

$\int_a^b u_0(x)d\left[F(x)-G(x)\right]=\int_a^b\left[G(x)-F(x)\right]u_0'(x)dx\ge 0$.

When F(x), G(x), $r_1(x)$ and $r_2(x)$ are given, $u_0(x)$ can be derived according to Theorem 1. We can then check whether $E\left[u_0(\tilde{x})\right]\ge E\left[u_0(\tilde{y})\right]$ holds, and thus determine whether F(x) is preferred or indifferent to G(x) by all u(x) subject to u'(x) > 0 and $r_1(x) \le \frac{-u''(x)}{u'(x)} \le r_2(x)$ for all $x \in [a,b]$.[8] The next two theorems follow directly from Theorem 1 and Corollary 1.

Theorem 2: For λ_1 and λ_2 such that $\lambda_1 \le 0 < \lambda_2$, $\tilde{x}$ ASD^A (λ_1, λ_2) $\tilde{y}$ if and only if $\int_a^b [G(x)-F(x)]u_0'(x)dx \ge 0$, where $u_0(x)$ is obtained according to

$$\frac{-u_0''(x)}{u_0'(x)} = \begin{cases} \lambda_1, & \text{if } \int_x^b [G(y)-F(y)]u_0'(y)dy < 0 \\ \lambda_2, & \text{if } \int_x^b [G(y)-F(y)]u_0'(y)dy \ge 0 \end{cases}. \qquad (1)$$

Theorem 3: For ρ_1 and ρ_2 such that $\rho_1 \le 0 < \rho_2$, $\tilde{x}$ ASD^R (ρ_1, ρ_2) $\tilde{y}$ if and only if $\int_a^b [G(x)-F(x)]u_0'(x)dx \ge 0$, where $u_0(x)$ is obtained according to

$$\frac{-u_0''(x)}{u_0'(x)} = \begin{cases} \rho_1 / x, & \text{if } \int_x^b [G(y)-F(y)]u_0'(y)dy < 0 \\ \rho_2 / x, & \text{if } \int_x^b [G(y)-F(y)]u_0'(y)dy \ge 0 \end{cases}. \qquad (2)$$

As an illustration of how to apply the characterization results for the two new definitions of ASD, consider the example provided by Leshno and Levy (2002) and described in the introduction. The example is modified slightly by adding an initial wealth w. In the modified example, suppose $\tilde{x}$ yields either \$w or \$(w+1,000,000) with probabilities 0.01 and 0.99 respectively, and $\tilde{y}$ yields \$(w+1) with certainty. Denote the CDF for $\tilde{x}$ and $\tilde{y}$ as F(x) and G(x), respectively. Let a = \$(w-1) and b = \$(w+1,000,001). Then

[8] Although the solution is not in closed form, Meyer (1977b) explains that the solution from an applied standpoint can be calculated with comparative ease (see also Meyer 1977c).

$$G(x)-F(x)=\begin{cases} 0 & a \le x < w \\ -0.01 & w \le x < w+1 \\ 0.99 & w+1 \le x < w+1{,}000{,}000 \\ 0 & w+1{,}000{,}000 \le x \le b \end{cases}$$

Apply Theorem 2 and $u_0(x)$ is obtained according to

$$\frac{-u_0''(x)}{u_0'(x)} = \begin{cases} \lambda_1, & \text{if } \int_x^b [G(y)-F(y)]u_0'(y)dy < 0 \\ \lambda_2, & \text{if } \int_x^b [G(y)-F(y)]u_0'(y)dy \ge 0 \end{cases}$$

For this example, it must be the case that $\int_x^b [G(y)-F(y)]u_0'(y)dy \ge 0$ for all x in [a, b], in order to have $\int_a^b [G(x)-F(x)]u_0'(x)dx \ge 0$. Therefore, $\frac{-u_0''(x)}{u_0'(x)} = \lambda_2$ for all x in [a, b]. This implies that $u_0(x) = -e^{-\lambda_2 x}$, and $\tilde{x}$ ASD^A (λ_1, λ_2) $\tilde{y}$ if and only if $\int_a^b [G(x) - F(x)]e^{-\lambda_2 x}dx \ge 0$, or $\int_0^1 (-0.01)e^{-\lambda_2 x}dx + \int_1^{1,000,000} (0.99)e^{-\lambda_2 x}dx \ge 0$.

Similarly, applying Theorem 3, $\frac{-u_0''(x)}{u_0'(x)} = \rho_2 / x$ for all x in [a, b]. This implies that $u'_0(x) = x^{-\rho_2}$, and $\tilde{x}$ ASD^R (ρ_1, ρ_2) $\tilde{y}$ if and only if $\int_a^b [G(x) - F(x)]x^{-\rho_2}dx \ge 0$, or $\int_w^{w+1} (-0.01)x^{-\rho_2}dx + \int_{w+1}^{w+1,000,000} (0.99)x^{-\rho_2}dx \ge 0$. Risk-aversion estimations are often conducted with respect to the relative risk aversion, and most existing estimates of ρ are smaller than 6.[9] It can be readily checked that $\tilde{x}$ ASD^R (ρ_1, ρ_2) $\tilde{y}$ holds for $\rho_2 = 7$ when w = 1 and for $\rho_2 = 49$ when w = 10.

[9] For example, see Epstein and Zin (1991), Gertner (1993), Metrick (1995), Barsky et al. (1997), Kaplow (2005) and Chetty (2006). On the other hand, the value of relative risk aversion implied by the observed equity premium may be substantially larger than the direct estimates, which is referred to as the equity premium puzzle and is the focus of an extensive literature (Mehra and Prescott 1985, and Kocherlakota 1996).

Therefore, it seems that the notion of ASD^R (ρ_1, ρ_2) captures the "folk" preference of $\tilde{x}$ over $\tilde{y}$ well.

For the random variables $\tilde{x}$ and $\tilde{y}$ given in this example, $\tilde{x}$ ASD^A (λ_1, λ_2) $\tilde{y}$ is characterized by the same condition regardless of the value of λ_1, and $\tilde{x}$ ASD^R (ρ_1, ρ_2) $\tilde{y}$ also has the same necessary and sufficient condition regardless of the value of ρ_1. That this is not a coincidence is demonstrated in Section 3 where one of the special cases analyzed is about pairs of CDFs that are single-crossing, which is the case for this example.

We conclude this section with some additional properties of ASD^A (λ_1, λ_2) and ASD^R (ρ_1, ρ_2), which are concerned with whether each of these ASD notions is invariant to adding a same amount of wealth to the two random variables under comparison, or scaling them by a same positive scalar.

<u>Property 1:</u> For any constant w and positive scalar $\alpha > 0$,

(i) $\tilde{x}+w$ ASD^A (λ_1, λ_2) $\tilde{y}+w$ if and only if $\tilde{x}$ ASD^A (λ_1, λ_2) $\tilde{y}$;

(ii) $\alpha\tilde{x}$ ASD^A ($\alpha\lambda_1$, $\alpha\lambda_2$) $\alpha\tilde{y}$ if and only if $\tilde{x}$ ASD^A (λ_1, λ_2) $\tilde{y}$.

<u>Proof:</u> $Eu(\tilde{x})-Eu(\tilde{y})=\int_a^b u(x)d\left[F(x)-G(x)\right]$.

(i) $Eu(\tilde{x}+w)-Eu(\tilde{y}+w)=\int_a^b u(x+w)d\left[F(x)-G(x)\right]=\int_a^b v(x)d\left[F(x)-G(x)\right]$, where

$v(x)\equiv u(x+w)$. Note that $\dfrac{-v''(x)}{v'(x)}=\dfrac{-u''(x+w)}{u'(x+w)}$ for all x, which means that

$\lambda_1 \le \dfrac{-v''(x)}{v'(x)} \le \lambda_2$ if and only if $\lambda_1 \le \dfrac{-u''(x)}{u'(x)} \le \lambda_2$.

(ii) $Eu(\alpha\tilde{x}) - Eu(\alpha\tilde{y}) = \int_a^b u(\alpha x) d\left[F(x) - G(x)\right] = \int_a^b v(x) d\left[F(x) - G(x)\right]$, where

$v(x) \equiv u(\alpha x)$. Note that $\frac{-v''(x)}{v'(x)} = \alpha \frac{-u''(\alpha x)}{u'(\alpha x)}$ for all x, which means that

$\alpha\lambda_1 \leq \frac{-v''(x)}{v'(x)} \leq \alpha\lambda_2$ if and only if $\lambda_1 \leq \frac{-u''(x)}{u'(x)} \leq \lambda_2$. Q.E.D.

Property 2: For any positive scalar α, $\alpha\tilde{x}$ ASD^R (ρ_1, ρ_2) $\alpha\tilde{y}$ if and only if $\tilde{x}$ ASD^R (ρ_1, ρ_2) $\tilde{y}$.

Proof: $Eu(\tilde{x}) - Eu(\tilde{y}) = \int_a^b u(x) d\left[F(x) - G(x)\right]$, and

$Eu(\alpha\tilde{x}) - Eu(\alpha\tilde{y}) = \int_a^b u(\alpha x) d\left[F(x) - G(x)\right] = \int_a^b v(x) d\left[F(x) - G(x)\right]$, where $v(x) \equiv u(\alpha x)$. Note

that $\frac{-xv''(x)}{v'(x)} = \frac{-(\alpha x)u''(\alpha x)}{u'(\alpha x)}$ for all x, which means that $\rho_1 \leq \frac{-xv''(x)}{v'(x)} \leq \rho_2$ if and only if

$\rho_1 \leq \frac{-xu''(x)}{u'(x)} \leq \rho_2$. Q.E.D.

FSD and SSD have the nice property that they are each invariant to both translations and positive scaling. In contrast, ASD^A (λ_1, λ_2) only inherits invariance to translations whereas ASD^R (ρ_1, ρ_2) only inherits invariance to positive scaling. Thus, it seems that ASD^R (ρ_1, ρ_2) has an advantage over ASD^A (λ_1, λ_2) since the former definition of ASD, based on restricting the relative risk aversion, is invariant to changes in the measuring unit of wealth, e.g., from the dollar to the euro. However, it is important to explicitly specify a reasonable initial wealth when applying ASD^R (ρ_1, ρ_2) because it is not invariant to translations.

3. Simple Characterization Conditions for ASD^A (λ_1, λ_2) and ASD^R (ρ_1, ρ_2): Several Special Cases

The characterizations of ASD^A (λ_1, λ_2) and ASD^R (ρ_1, ρ_2) given in Theorems 2 and 3 for the general case are not closed-form, even though for given CDFs F(x) and G(x), the ASD^A (λ_1, λ_2) or the ASD^R (ρ_1, ρ_2) relation can be checked according to well-defined operational procedures. In this section, simple, closed-form characterization conditions of ASD^A (λ_1, λ_2) and ASD^R (ρ_1, ρ_2) are obtained for several special cases.

3.1. Only an Upper Bound on Risk Aversion

In the general case analyzed in the last section, both an upper bound and a lower bound are placed on the absolute or the relative risk aversion. In other words, extreme risk preferences on both ends of the spectrum are eliminated. In this subsection we consider a special case where only an upper bound is imposed, whereas the symmetric special case with only a lower bound is analyzed in the next subsection. In these special cases, extreme risk preferences on either end of the spectrum, but not on both ends, are eliminated. The closed-form characterization conditions for these special cases can also serve as the sufficient conditions for the corresponding ASD relations in the general case.

For $\lambda_2 > 0$ and $\rho_2 > 0$, denote ASD^A $(-\infty, \lambda_2)$ and ASD^R $(-\infty, \rho_2)$ as the ASD notion associated with sets of utility functions $U(-\infty, \lambda_2) = \{u(x): u'(x) > 0 \text{ and } A(x) \leq \lambda_2\}$ and $U(-\infty, \rho_2) = \{u(x): u'(x) > 0 \text{ and } R(x) \leq \rho_2\} = \{u(x): u'(x) > 0 \text{ and } A(x) \leq \rho_2/x\}$, respectively. In order to determine the condition on F(x) and G(x) so that F(x) ASD^A $(-\infty, \lambda_2)$ G(x) or F(x) ASD^R $(-\infty, \rho_2)$ G(x), the following result is used.

Theorem 4: (Meyer 1977a) Given utility function k(x) with k'(x) > 0, $\tilde{x}$ is preferred or indifferent to $\tilde{y}$ for all u(x) such that u'(x) > 0 and $\frac{-u''(x)}{u'(x)} \leq \frac{-k''(x)}{k'(x)}$ for all x in [a, b] if and only if

$\int_y^b [G(x) - F(x)]dk(x) \geq 0$ for all y in [a, b].

Note that this result reduces to the well-known result characterizing the increasing convex order when k(x) = x.[10] To apply this theorem to $ASD^A(-\infty, \lambda_2)$ and $ASD^R(-\infty, \rho_2)$, observe that the set $U(-\infty, \lambda_2)$ is the set of all u(x) less risk averse than $k(x) = -e^{-\lambda_2 x}$, and the set $U(-\infty, \rho_2)$ is the set of all u(x) less risk averse than k(x) where $k(x) = \frac{x^{1-\rho_2}}{1-\rho_2}$ for $\rho_2 > 0$ and $\rho_2 \neq 1$, and k(x) = ln x for $\rho_2 = 1$. Thus, the following theorem characterizes F(x) $ASD^A(-\infty, \lambda_2)$ G(x) and F(x) $ASD^R(-\infty, \rho_2)$ G(x).

Theorem 5: (i) For $\lambda_2 > 0$, $\tilde{x}$ $ASD^A(-\infty, \lambda_2)$ $\tilde{y}$ if and only if $\int_y^b [G(x) - F(x)]e^{-\lambda_2 x}dx \geq 0$ for all y in [a, b];

(ii) For $\rho_2 > 0$, $\tilde{x}$ $ASD^R(-\infty, \rho_2)$ $\tilde{y}$ if and only if $\int_y^b [G(x) - F(x)]x^{-\rho_2}dx \geq 0$ for all y in [a, b].

Note that the results in Theorem 5 also identify a closed-form sufficient condition for $ASD^A(\lambda_1, \lambda_2)$ and $ASD^R(\rho_1, \rho_2)$, respectively, as summarized below.

Corollary 2: (i) For λ_1 and λ_2 such that $\lambda_1 \leq 0 < \lambda_2$, $\tilde{x}$ $ASD^A(\lambda_1, \lambda_2)$ $\tilde{y}$ if $\int_y^b [G(x) - F(x)]e^{-\lambda_2 x}dx \geq 0$ for all y in [a, b];

[10] See Liu and Meyer (2017) for applications of the increasing convex order in framing the size-for-risk tradeoff.

(ii) For ρ_1 and ρ_2 such that $\rho_1 \leq 0 < \rho_2$, $\tilde{x}$ $ASD^R(\rho_1, \rho_2)$ $\tilde{y}$ if $\int_y^b [G(x) - F(x)]x^{-\rho_2}\,dx \geq 0$ for all y in [a, b].

3.2. Only a Lower Bound on Risk Aversion

For $\lambda_1 \leq 0$ and $\rho_1 \leq 0$, denote $ASD^A(\lambda_1, \infty)$ and $ASD^R(\rho_1, \infty)$ as the ASD notion associated with sets of utility functions $U(\lambda_1, \infty) = \{u(x): u'(x) > 0 \text{ and } \lambda_1 \leq A(x)\}$ and $U(\rho_1, \infty) = \{u(x): u'(x) > 0 \text{ and } \rho_1 \leq R(x)\} = \{u(x): u'(x) > 0 \text{ and } \rho_1/x \leq A(x)\}$, respectively. In order to determine the condition on F(x) and G(x) so that F(x) $ASD^A(\lambda_1, \infty)$ G(x) or F(x) $ASD^R(\rho_1, \infty)$ G(x), the following result is used.

Theorem 6: (Meyer 1977a) Given utility function k(x) with $k'(x) > 0$, $\tilde{x}$ is preferred or indifferent to $\tilde{y}$ for all u(x) such that $u'(x) > 0$ and $\frac{-k''(x)}{k'(x)} \leq \frac{-u''(x)}{u'(x)}$ for all x in [a, b] if and only if $\int_a^y [G(x) - F(x)]dk(x) \geq 0$ for all y in [a, b].

To apply this theorem to $ASD^A(\lambda_1, \infty)$ and $ASD^R(\rho_1, \infty)$, observe that the set $U(\lambda_1, \infty)$ is the set of all u(x) more risk averse than $k(x) = e^{-\lambda_1 x}$, and the set $U(\rho_1, \infty)$ is the set of all u(x) more risk averse than $k(x) = \frac{x^{1-\rho_1}}{1-\rho_1}$. Thus, the following theorem characterizes F(x) $ASD^A(\lambda_1, \infty)$ G(x) and F(x) $ASD^R(\rho_1, \infty)$ G(x).

Theorem 7: (i) For $\lambda_1 \leq 0$, $\tilde{x}$ $ASD^A(\lambda_1, \infty)$ $\tilde{y}$ if and only if $\int_a^y [G(x) - F(x)]e^{-\lambda_1 x}dx \geq 0$ for all y in [a, b].

(ii) For $\rho_1 \leq 0$, $\tilde{x}$ $ASD^R(\rho_1, \infty)$ $\tilde{y}$ if and only if $\int_a^y [G(x) - F(x)]x^{-\rho_1}\,dx \geq 0$ for all y in [a, b].

Huang et al. (2020) present a result identical to (i) in Theorem 7.[11] Their goal is to define and characterize a general notion of (fractional degree) stochastic dominance that fills the gap between FSD and SSD, whereas our purpose here is to define and characterize a notion of stochastic dominance that excludes extremely risk loving decision makers. As the following corollary indicates, nevertheless, the fractional degree stochastic dominance and the notions of ASD proposed here are closely related. Specifically, the characterization conditions for the fractional degree stochastic dominance can serve as sufficient conditions for the corresponding ASD relations.

Corollary 3: (i) For λ_1 and λ_2 such that $\lambda_1 \leq 0 < \lambda_2$, $\tilde{x}$ ASD^A (λ_1, λ_2) $\tilde{y}$ if $\int_a^y [G(x) - F(x)]e^{-\lambda_1 x}dx \geq 0$ for all y in [a, b];

(ii) For ρ_1 and ρ_2 such that $\rho_1 \leq 0 < \rho_2$, $\tilde{x}$ ASD^R (ρ_1, ρ_2) $\tilde{y}$ if $\int_a^y [G(x) - F(x)]x^{-\rho_1} dx \geq 0$ for all y in [a, b].

3.3. Single-Crossing CDFs

Many of the well-known motivating examples used in the ASD literature compare random payoffs whose CDFs are single-crossing. This was true for the original Leshno and Levy example analyzed in Section 2. Another often-cited example, provided by Levy (2006, pp 331-332), also involves a comparison of two CDFs that are single-crossing. In this example, random payoff $\tilde{x}$ yields either \$1 or \$1,000,000 with probabilities 0.1 and 0.9 respectively, and $\tilde{y}$ is either \$2 or \$3 with those same probabilities. As Levy points out, neither $\tilde{x}$ nor $\tilde{y}$ dominates the other in the first degree, yet $\tilde{x}$ appears to be clearly better than $\tilde{y}$. Denote the CDFs of $\tilde{x}$ and $\tilde{y}$

[11] In a working paper version of Huang et al. (2020), they also present a result identical to (ii) in Theorem 7 for a notion of fractional degree stochastic dominance based on the relative risk aversion measure.

as F(x) and G(x), respectively; F(x) is (weakly) above (below) G(x) for $1 \le x < 3$ ($3 \le x \le 1,000,000$)).

These two examples and others involve random variables where F(x) crosses G(x) only once from above. F(x) – G(x) is positive up to the crossing point, so $\tilde{x}$ does not stochastically dominate $\tilde{y}$ in any degree. Additionally, E($\tilde{x}$) > E($\tilde{y}$), so $\tilde{y}$ does not stochastically dominate $\tilde{x}$ in any degree. We summarize these features of the two examples with the following assumption.

Assumption 1: (i) There exists $c \in (a,b)$ such that $F(x) \ge G(x)$ for all $x \in [a,c)$, and $F(x) \le G(x)$ for all $x \in [c,b]$, (ii) $\int_a^c \left(F(x) - G(x)\right) dx > 0$, and (iii) $E(\tilde{x}) > E(\tilde{y})$.

Condition (i) above implies that the CDFs to be compared are single-crossing, and conditions (ii) and (iii) are added to make the comparison interesting. Note that under Assumption 1, neither $\tilde{x}$ nor $\tilde{y}$ stochastically dominates the other in any degree. The following theorem provides closed-form characterizations for ASD^A (λ_1, λ_2) and ASD^R (ρ_1, ρ_2) when F(x) and G(x) are single-crossing.

Theorem 8: Under Assumption 1,

(i) $\tilde{x}$ ASD^A (λ_1, λ_2) $\tilde{y}$ for all $\lambda_1 \le 0$ if and only if $\int_a^b \left(G(x) - F(x)\right) e^{-\lambda_2 x} dx \ge 0$;

(ii) $\tilde{x}$ ASD^R (ρ_1, ρ_2) $\tilde{y}$ for all $\rho_1 \le 0$ if and only if $\int_a^b \left(G(x) - F(x)\right) x^{-\rho_2} dx \ge 0$.

Proof: (i) "If" – Suppose $\int_a^b \left(G(x) - F(x)\right) e^{-\lambda_2 x} dx \ge 0$. This, together with Assumption 1, implies $\int_y^b \left(G(x) - F(x)\right) e^{-\lambda_2 x} dx \ge 0$ for all $y \in [a,b]$. According to (i) in Theorem 5, $\tilde{x}$ ASD^A ($-\infty$, λ_2) $\tilde{y}$, which in turn implies that $\tilde{x}$ ASD^A (λ_1, λ_2) $\tilde{y}$ for all $\lambda_1 \le 0$. "Only if" – Suppose that $\tilde{x}$ ASD^A (λ_1, λ_2) $\tilde{y}$ for all $\lambda_1 \le 0$. By definition, $E\left[u(\tilde{x})\right] \ge E\left[u(\tilde{y})\right]$ for all u(x) in U (λ_1, λ_2). In particular, $E\left[u(\tilde{x})\right] \ge E\left[u(\tilde{y})\right]$ for $u(x) = -e^{-\lambda_2 x}$, or $\int_a^b \left(G(x) - F(x)\right) e^{-\lambda_2 x} dx \ge 0$.

(ii) The proof is similar to that in (i) above, using result (ii) in Theorem 5 instead. Q.E.D.

Theorem 8 applies to the example of Levy (2006) with an initial wealth w added: $\tilde{x}$ is either \$(w+1) or \$(w+1,000,000) with probabilities 0.1 and 0.9 respectively, and $\tilde{y}$ is either \$(w+2) or \$(w+3) with those same probabilities. According to Theorem 8, $\tilde{x}$ ASD^A (λ_1, λ_2) $\tilde{y}$ for all $\lambda_1 \le 0$ if and only if $\int_1^2 (-0.1) e^{-\lambda_2 x} dx + \int_3^{1,000,000} (0.9) e^{-\lambda_2 x} dx \ge 0$ (note that ASD^A (λ_1, λ_2) is invariant to changes in the initial wealth, so the initial wealth can be assumed to be zero), and $\tilde{x}$ ASD^R (ρ_1, ρ_2) $\tilde{y}$ for all $\rho_1 \le 0$ if and only if $\int_{w+1}^{w+2} (-0.1) x^{-\rho_2} dx + \int_{w+3}^{w+1,000,000} (0.9) x^{-\rho_2} dx \ge 0$.

As a final example, consider F(x) and G(x) drawn from the same location-scale family (Meyer 1987, and Wong and Ma 2008). Suppose the mean and standard deviation are μ_F and σ_F for F(x) and μ_G and σ_G for G(x). Let Ψ be the CDF of the seed random variable $\tilde{z}$ with a mean of zero and a standard deviation of 1. Then $F(x) = \Psi\left((x - \mu_F)/\sigma_F\right)$ and $G(x) = \Psi\left((x - \mu_G)/\sigma_G\right)$. Assumption 1 is satisfied when $\sigma_F > \sigma_G$ and $\mu_F > \mu_G$. According to Theorem 8, $\tilde{x}$ ASD^A (λ_1, λ_2) $\tilde{y}$ if and only if $\int_a^b \left(\Psi\left((x - \mu_G)/\sigma_G\right) - \Psi\left((x - \mu_F)/\sigma_F\right)\right) e^{-\lambda_2 x} dx \ge 0$, and $\tilde{x}$ ASD^R (ρ_1, ρ_2) $\tilde{y}$ if and only if $\int_a^b \left(\Psi\left((x - \mu_G)/\sigma_G\right) - \Psi\left((x - \mu_F)/\sigma_F\right)\right) x^{-\rho_2} dx \ge 0$.

The discussion in this subsection has so far been confined to the case in which F(x) crosses G(x) only once from above. What about F(x) crossing G(x) only once from below? Although the latter case is not the focus of the ASD discussions in the literature, we include a quick analysis of it for completeness.

Assumption 2: (i) There exists $c \in (a,b)$ such that $F(x) \le G(x)$ for all $x \in [a,c)$, and $F(x) \ge G(x)$ for all $x \in [c,b]$, (ii) $\int_a^c \big(F(x) - G(x)\big)dx < 0$, and (iii) $\int_c^b \big(F(x) - G(x)\big)dx > 0$

Theorem 9: Under Assumption 2,

(i) If $E(\tilde{x}) \ge E(\tilde{y})$, then $\tilde{x}$ SSD $\tilde{y}$;

(ii) If $E(\tilde{x}) < E(\tilde{y})$, then $\tilde{x}$ does not ASD^A (λ_1, λ_2) $\tilde{y}$ for any (λ_1, λ_2), and $\tilde{x}$ does not ASD^R (ρ_1, ρ_2) $\tilde{y}$ for any (ρ_1, ρ_2).

Proof: (i) Assumption 2 and $E(\tilde{x}) \ge E(\tilde{y})$ together imply $\int_a^y \big(G(x) - F(x)\big)dx \ge 0$ for all y. So $\tilde{x}$ SSD $\tilde{y}$.

(ii) This result holds because $E(\tilde{x}) \ge E(\tilde{y})$ is a necessary condition for $\tilde{x}$ ASD^A (λ_1, λ_2) $\tilde{y}$ or $\tilde{x}$ ASD^R (ρ_1, ρ_2) $\tilde{y}$. Q.E.D.

4. Relation to the Conventional AFSD

Recall that $\tilde{x}$ almost first degree stochastically dominates $\tilde{y}$, or $\tilde{x}$ AFSD (ε) $\tilde{y}$, if $\tilde{x}$ is preferred or indifferent to $\tilde{y}$ by every utility function in the set U_1^{ε} = {u(x) in U_1: u'(x) ≤ inf {u'(x)}(1/ε - 1) for all x in [a, b]}, where $0 < \varepsilon \leq 1/2$. We now show that AFSD (ε) implies ASD^A (λ_1, λ_2) and ASD^R (ρ_1, ρ_2) for appropriately chosen parameter values.

First, for every utility function u(x) in U (λ_1, λ_2) = {u(x): u'(x) > 0 and $\lambda_1 \leq A(x) \leq \lambda_2$}, where $\lambda_1 \leq 0 < \lambda_2$, we have, for all $y \leq z$,

$$e^{\lambda_1 (b-a)} \leq e^{\int_y^z \lambda_1 dx} \leq \frac{u'(y)}{u'(z)} \leq e^{\int_y^z \lambda_2 dx} \leq e^{\lambda_2 (b-a)}. \qquad (3)$$

Choose λ_1 and λ_2 so that

$$\frac{1}{e^{\lambda_1 (b-a)}} = e^{\lambda_2 (b-a)} = \left(\frac{1}{\varepsilon} - 1\right). \qquad (4)$$

Then it is readily seen that u(x) belongs to the set U_1^{ε} .

Similarly, for every utility function u(x) in U (ρ_1, ρ_2) = {u(x): u'(x) > 0 and $\rho_1 \leq R(x) \leq \rho_2$} = {u(x): u'(x) > 0 and $\rho_1/x \leq A(x) \leq \rho_2/x$}, where $\rho_1 \leq 0 < \rho_2$, we have, for all $y \leq z$,

$$\left(\frac{b}{a}\right)^{\rho_1} \leq \left(\frac{z}{y}\right)^{\rho_1} \leq \frac{u'(y)}{u'(z)} \leq \left(\frac{z}{y}\right)^{\rho_2} \leq \left(\frac{b}{a}\right)^{\rho_2}. \qquad (5)$$

Choose ρ_1 and ρ_2 so that

$$\left(\frac{b}{a}\right)^{-\rho_1} = \left(\frac{b}{a}\right)^{\rho_2} = \left(\frac{1}{\varepsilon} - 1\right). \qquad (6)$$

Then it is readily seen that u(x) belongs to the set U_1^{ε} .

Based on the above discussions, we have the following theorem stating the relations between ASD^A (λ_1, λ_2) and ASD^R (ρ_1, ρ_2) on one side and AFSD (ε) on the other.

Theorem 10: If $\tilde{x}$ AFSD (ε) $\tilde{y}$, then

(i) $\tilde{x}$ ASD^A (λ_1, λ_2) $\tilde{y}$ for λ_1 and λ_2 chosen according to (4);

(ii) $\tilde{x}$ ASD^R (ρ_1, ρ_2) $\tilde{y}$ for ρ_1 and ρ_2 chosen according to (6).

The same analysis can be used to establish a connection between two existing notions of fractional degree stochastic dominance that fill the gap between FSD and SSD. Defined by Muller et al. (2017), $\tilde{x}$ (1+γ)th degree stochastically dominates $\tilde{y}$, where $0 \le \gamma \le 1$, if $\tilde{x}$ is preferred or indifferent to $\tilde{y}$ by all utility functions satisfying $u'(x) > 0$ and $\gamma \le \frac{u'(y)}{u'(z)}$ for all $y \le z$.[12] In contrast, Huang et al.'s (2020) notion of fractional degree stochastic dominance based on a negative lower bound on the absolute risk aversion measure is equivalent to ASD^A (λ_1, ∞) which is defined as unanimous preference of $\tilde{x}$ over $\tilde{y}$ on the set of utility functions $U(\lambda_1, \infty) = \{u(x): u'(x) > 0 \text{ and } \lambda_1 \le A(x)\}$. A similar notion of fractional degree stochastic dominance – based on a negative lower bound on the relative risk aversion measure – would be equivalent to ASD^R (ρ_1, ∞) with an associated set of utility functions $U(\rho_1, \infty) = \{u(x): u'(x) > 0 \text{ and } \rho_1 \le R(x)\} = \{u(x): u'(x) > 0 \text{ and } \rho_1/x \le A(x)\}$. With discussions similar to those leading to Theorem 10, we can establish the following theorem.

<u>Theorem 11:</u> If $\tilde{x}$ (1+γ)th degree stochastically dominates $\tilde{y}$ (according to Muller et al. 2017), then

(i) $\tilde{x}$ ASD^A (λ_1, ∞) $\tilde{y}$ for λ_1 such that $e^{\lambda_1(b-a)} = \gamma$;

(ii) $\tilde{x}$ ASD^R (ρ_1, ∞) $\tilde{y}$ for ρ_1 such that $\left(\frac{b}{a}\right)^{\rho_1} = \gamma$.

5. Conclusion

---

[12] The original definition only requires $u'(x) \ge 0$, and we require $u'(x) > 0$ so that the risk aversion measures are well defined.

We advance an alternative approach to almost stochastic dominance (ASD) by considering a group of decision makers whose Arrow-Pratt absolute (or relative) risk aversion measure has both an upper bound and a lower bound, thereby eliminating decision makers with extreme risk preferences. Our ASD definitions can be interpreted as unanimous preference by all decision makers "who are neither too risk averse nor too risk loving". CDF-based characterizations of these two definitions of ASD are provided. We also show that the ASD definition based on bounds on the absolute risk aversion measure is invariant to translations, whereas that based on bounds on the relative risk aversion measure is invariant to positive scaling. In addition, we demonstrate that, for several special cases (including the cases of either an upper bound or a lower bound but not both bounds at the same time, and the case of single-crossing CDFs), each of the two ASD notions can be characterized by a simple closed-form condition. Finally, we provide a connection between the conventional AFSD based on restricting the degree of variability in the first order derivative of utility and the two new ASD concepts.

The Meyer results (1977a, 1977b) only discuss upper and lower bounds on risk aversion and say nothing about the slopes or other properties of the risk aversion measures. As a result, they do not provide a way to extend our ASD definitions to higher orders of stochastic dominance. Finding closed-form CDF characterizations of the two definitions of ASD for the general case and extending the ASD definitions to higher degrees seem to be a worthwhile avenue for future research.

References

Barsky, R.B., Juster, F.T., Kimball, M.S. and Shapiro, M.D. (1997). ‘Preference parameters and behavioral heterogeneity: an experimental approach in the health and retirement study’. *Quarterly Journal of Economics* 112, 537–579.

Callen, M., Isaqzadeh, M., Long, J.D. and Sprenger, C. (2014). ‘Violence and risk preference: Experimental evidence from Afghanistan’. *American Economic Review* 104, 123-148.

Chang J.-R., Liu, W.-H. and Hung, M.-W. (2019). ‘Revisiting generalized almost stochastic dominance’. *Annals of Operations Research* 281, 175-192.

Chetty, R. (2006). ‘A new method of estimating risk aversion’. *American Economic Review* 96, 1821–1834.

Denuit, M., Eeckhoudt, L., Liu, L. and Meyer, J. (2016). ‘Tradeoffs for downside risk-averse decision-makers and the self-protection decision’. *The Geneva Risk and Insurance Review* 41 (1), 19-47.

Denuit, M., Huang R.J. and Tzeng L.Y. (2014a). ‘Bivariate almost stochastic dominance’, *Economic Theory* 57, 377-405.

Denuit, M., Liu, L. and Meyer, J. (2014b). ‘A separation theorem for the weak s-convex orders’. *Insurance: Mathematics and Economics* 59, 279-284.

Ebert, S. and Wiesen, D. (2014). ‘Joint measurement of risk aversion, prudence, and temperance’. *Journal of Risk and Uncertainty* 48, 231-252.

Eckel, C. and Grossman P. (2002). ‘Sex differences and statistical stereotyping in attitudes toward financial risk’. *Evolution and Human Behavior* 23, 281-295.

Epstein, L.G. and Zin, S.E. (1991). ‘Substitution, risk aversion, and the temporal behavior of consumption growth and asset returns: an empirical analysis’. *Journal of Political Economy* 99, 263–286.

Gertner, R. (1993), ‘Game shows and economic behavior: risk-taking on ‘Card Sharks’’. *Quarterly Journal of Economics* 108, 507-521.

Grossman, P. and Eckel C. (2015). ‘Loving the long shot: risk taking with skewed lotteries’. *Journal of Risk and Uncertainty* 51, 195-217.

Guo, X., Zhu, X., Wong, W.K. and Zhu, L. (2013). ‘A note on almost stochastic dominance’, *Economics Letters* 121, 252-256.

Hadar, J. and Russell, W. (1969). ‘Rules for ordering uncertain prospects’. *American Economic Review* 59, 25–34.

Hanoch, G. and Levy, H. (1969). 'The efficiency analysis of choices involving risk'. *Review of Economic Studies* 36, 335–346.

Holt, C. and Laury S. (2002). 'Risk aversion and incentive effects'. *American Economic Review* 92, 1644-1655.

Huang, R.J., Tzeng, L.Y. and Zhao L. (2020). 'Fractional degree stochastic dominance'. *Management Science* 66, 4630-4647.

Kaplow, L. (2005). 'The value of a statistical life and the coefficient of relative risk aversion'. *Journal of Risk and Uncertainty* 31, 23–34.

Kocherlakota, N. R. (1996). 'The equity premium: it's still a puzzle'. *Journal of Economic Literature* 34, 42-71.

Leshno, M. and Levy, H. (2002). 'Preferred by "all" and preferred by "most" decision makers: Almost stochastic dominance'. *Management Science* 48, 1074-1085.

Levy, H. (2006). *Stochastic Dominance: Investment Decision Making under Uncertainty* (2nd edition), Springer Science+Business Media, Inc., New York.

Liu, L. (2012). 'Inferring the rate of pure time preference under uncertainty'. *Ecological Economics* 74, 27-33.

Liu, L. and Meyer, J. (2017). 'The increasing convex order and the tradeoff of size for risk'. *Journal of Risk and Insurance* 84, 881-897.

Liu, L. and Meyer, J. (2021). 'Stochastic superiority'. *Journal of Risk and Uncertainty*, in press.

Luo, C. and Tan, C.H. (2020). 'Almost stochastic dominance for most risk-averse decision makers'. *Decision Analysis* 17, 169-184.

Mehra, R. and Prescott, E.C. (1985). 'The equity premium: a puzzle'. *Journal of Monetary Economics* 15, 145-161.

Metrick, A. (1995). 'A natural experiment in 'Jeopardy!'' *American Economic Review* 85, 240–53.

Meyer, D. and Meyer, J. (2005a). 'Risk preferences in multi-period consumption models, the equity premium puzzle, and habit formation utility'. *Journal of Monetary Economics* 52, 1497-1515.

Meyer, D. and Meyer, J. (2005b). 'Relative risk aversion: What do we know?', *Journal of Risk and Uncertainty* 31, 243-262.

Meyer, J. (1977a). 'Second degree stochastic dominance with respect to a function'. *International Economic Review* 18, 477-487.

Meyer, J. (1977b). ‘Choice among distributions’. *Journal of Economic Theory* 14, 326-336.

Meyer, J. (1977c). ‘Further applications of stochastic dominance to mutual fund performance’. *Journal of Financial and Quantitative Analysis* 12, 235-242.

Meyer, J. (1987). ‘Two-moment decision models and expected utility maximization’. *American Economic Review* 77, 421-430.

Muller, A., Scarsini, M., Tsetlin, I. and Winkler R.L. (2017). ‘Between first- and second-order stochastic dominance’. *Management Science* 63, 2933-2947.

Pratt, J. (1964). ‘Risk aversion in the small and in the large’. *Econometrica*, 32: 122-136.

Tsetlin, I. and Winkler, R.L.(2018). ‘Multivariate almost stochastic dominance’. *Journal of Risk and Insurance* 85, 431-445.

Tsetlin, I., Winkler, R.L, Huang, R.J. and Tzeng, L.Y. (2015). ‘Generalized almost stochastic dominance’. *Operations Research* 63, 363-377.

Tzeng, L.Y., Huang, R.J. and Shih, P.T. (2013). ‘Revisiting almost second-degree stochastic dominance’. *Management Science* 59, 1250-1254.

Wong, W.K. and Ma, C. (2008). ‘Preferences over location-scale family’. *Economic Theory* 37:119–146.